ECEN 605

LINEAR SYSTEMS

Lecture 22

Characteristics of Feedback Control Systems IV
– Rejecting Disturbances

Rejecting Step Disturbances

Let us now consider what happens when disturbances are present. The plant is now represented as

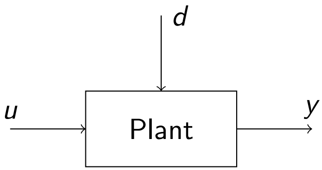

Figure 1: A plant with disturbance

with transfer function representation

$$y(s) = P(s)\,U(s) + Q(s)\,D(s) \tag{1a}$$

$$= \frac{n_P(s)}{d_P(s)} U(s) + \frac{n_Q(s)}{d_Q(s)} D(s). \tag{1b}$$

Rejecting Step Disturbances (cont.)

To simplify our analysis, and without loss of generality, we assume that

$$d_P(s) = d_Q(s) \tag{2}$$

and consider the closed loop system

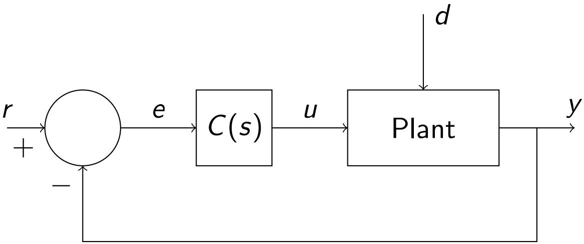

Figure 2: A closed loop system with disturbance

with $C(s) = \dfrac{n_C(s)}{d_C(s)}$.

Rejecting Step Disturbances (cont.)

The Laplace transform of the error $e(t)$ is now given by

$$\begin{aligned} E(s) &= \frac{1}{1 + P(s)\,C(s)} R(s) - \frac{Q(s)}{1 + P(s)\,C(s)} D(s) && (3) \\ &= \underbrace{\frac{d_P(s)\,d_C(s)}{d_{cl}(s)} R(s)}_{E_r(s)} + \underbrace{\frac{-n_Q(s) d_C(s)}{d_{cl}(s)} D(s)}_{E_d(s)} && (4) \end{aligned}$$

where

$$d_{cl}(s) = d_P(s)\,d_C(s) + n_P(s)\,n_C(s). \qquad (5)$$

Rejecting Step Disturbances (cont.)

If $d(t)$ is a step of height d_0, we have

$$E_d(s) = \frac{-n_Q(s)\, d_C(s)}{d_{cl}(s)} \cdot \frac{d_0}{s} \tag{6}$$

and it follows that the steady state error due to disturbances is

$$E_d = \frac{-n_Q(0)\, d_C(0)}{d_{cl}(0)} \cdot d_0. \tag{7}$$

Clearly E_d can be **zeroed** if $d_C(0) = 0$, that is if the controller has an integrator.

Rejecting Step Disturbances (cont.)

We conclude that by
1) including an integrator in the controller,
2) driving it with the error signal and
3) stabilizing the closed loop,
we obtain zero steady state error for arbitrary step references and arbitrary step disturbances.

Moreover the zero steady state error condition holds for all perturbations in the plant and controller parameters as long as the condition $d_C(0) = 0$ holds, that is as long as the controller contains an accurate integrator.

Rejecting Step Disturbances (cont.)

Example

Consider the plant described by

$$Y(s) = \frac{s-1}{(s-2)(s+3)}U(s) + \frac{1}{s-2}D(s) \qquad (8)$$

which is required track step references and reject step disturbances with **zero** steady state error.

Rejecting Step Disturbances (cont.)

We see that the controller must be of form

$$C(s) = \frac{\beta(s)}{s\,\alpha(s)} \tag{9}$$

where the polynomials $\beta(s)$, $\alpha(s)$ are to be chosen so that the closed loop is stable or equivalently

$$d_{cl}(s) = s\,\alpha(s)\,(s-2)(s+3) + \beta(s)\,(s-1) \tag{10}$$

is Hurwitz. It is easy to see that the simple choice $\alpha(s) = 1$, $\beta(s) = K$, which is an integral controller, does not work. (Verify this.)

Rejecting Step Disturbances (cont.)

This prompts us to try the higher order, Proportional-Integral (PI), controller

$$C_1(s) = \frac{\beta_0 + \beta_1\, s}{s} = \underbrace{\beta_1}_{\text{Proportional}} + \underbrace{\frac{\beta_0}{s}}_{Integral} . \tag{11}$$

It is seen that the 3^{rd} degree polynomial

$$d_{cl}(s) = s(s-2)(s+3) + (\beta_0 + \beta_1\, s)(s-1) \tag{12}$$

must be rendered Hurwitz, by adjusting the two parameters β_0 and β_1.

Rejecting Step Disturbances (cont.)

Rewriting (12)

$$d_{cl}(s) = s^3 + (1+\beta_1)s^2 + (\beta_0 - 6)s - \beta_0 - \beta_1 \tag{13}$$

we see from the Routh Criterion that for stability

$$\begin{aligned} 1 + \beta_1 &> 0 && \text{(14a)} \\ \beta_0 - 6 &> 0 && \text{(14b)} \\ -\beta_0 - \beta_1 &> 0 && \text{(14c)} \\ (1+\beta_1)(\beta_0 - 6) + \beta_0 + \beta_1 &> 0. && \text{(14d)} \end{aligned}$$

In this example it is easy to see that (14a), (14b) and (14c) are incompatible, that is the intersection of the half planes satisfying (14a), (14b) and (14c) is empty. (Verify this!) Thus a PI controller cannot stabilize the plant in (8).

Rejecting Step Disturbances (cont.)

The next step is to try a controller of higher order with more free parameters. For instance

$$C_2(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2}{s(s + \alpha_0)} \tag{15}$$

has an integrator, is proper and has 4 adjustable parameters. Note that it is undesirable to admit improper controllers as they are equivalent to differentiation of the input and thus are susceptible to high frequency noise.

Rejecting Step Disturbances (cont.)

With the controller (15)

$$d_{cl}(s) = s(s+\alpha_0)(s-2)(s+3) + (\beta_0 + \beta_1\, s + \beta_2\, s^2)(s-1). \quad (16)$$

The four free parameters (α_0, β_0, β_1, β_2) can adjust the 4 roots of (16) arbitrarily. For example choosing closed loop characteristic roots

$$\lambda_1 = -1, \quad \lambda_2 = -2 \quad \lambda_3 = -1-j, \quad \lambda_4 = -1+j \qquad (17)$$

we have

$$\begin{aligned} d_{cl}(s) &= s^4 + (\alpha_0 + \beta_2 + 1)s^3 + (\beta_1 - \beta_2 + \alpha_0 - 6)s^2 \\ &\quad + (\beta_0 - 6\alpha_0 - \beta_1)s - \beta_0 \\ &= (s+1)(s+2)((s+1)^2 + 1^2) \\ &= s^4 + 5s^3 + 10s^2 + 10s + 4. \qquad (18) \end{aligned}$$

Rejecting Step Disturbances (cont.)

By equating coefficients in (18) we obtain the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -6 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 10 \\ 4 \end{bmatrix}. \tag{19}$$

The coefficient matrix in (19) is nonsingular and thus (19) is solved by

$$\alpha_0 = -7.25, \quad \beta_0 = -4, \quad \beta_1 = 29.5, \quad \beta_2 = 12.25 \tag{20}$$

giving the controller

$$C_2(s) = \frac{-4 + 29.5s + 12.25s^2}{s(s - 7.25)}. \tag{21}$$

Rejecting Step Disturbances (cont.)

To summarize, we note that the controller $C_2(s)$ provides closed loop stability with closed loop characteristic roots

$$\lambda_1 = -1, \quad \lambda_2 = -2 \quad \lambda_3 = -1 - j, \quad \lambda_4 = -1 + j \tag{22}$$

and provides **zero** steady state errors to arbitrary step reference input and arbitrary step disturbances.

In the following section we extend these results to more general classes of reference and disturbance signals beyond constants (steps).