# ECEN 605

## LINEAR SYSTEMS

Lecture 13

State Feedback and Observers I
– Eigenvalue Assignment by State Feedback

# Multivariable system

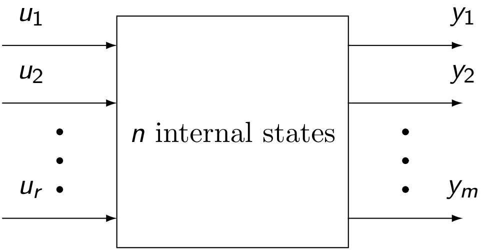


Figure 1: Input - Output System

# Single Input Systems

Consider a single input system of form:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= C^T x. \end{aligned}$$

If we let $u(t) = 0$, then

$$\dot{x} = Ax \qquad y = C^T x.$$

Consequently,

$$x(t) = e^{At}x(0) \qquad \text{and} \qquad y(t) = C^T e^{At} x(0).$$

# Single Input Systems (cont.)

Suppose that the output $y(t)$ is unsatisfactory, then we need a controller to regulate the system. The *state feedback* problem considers the following:

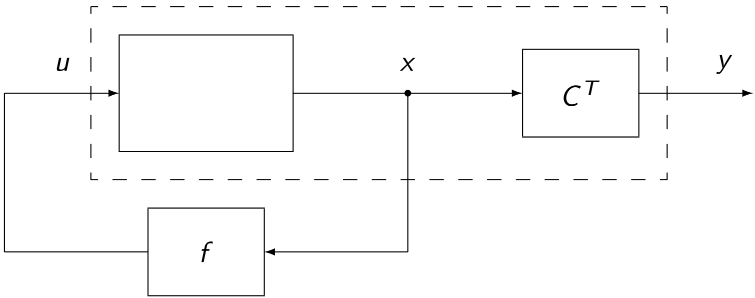


Figure 2: State Feedback Configuration

# Single Input Systems (cont.)

Introducing a state feedback $f$, we have

$$u(t) = fx(t) \implies y(t) = C^T e^{(A+bf)t} x(t)$$

so that $y(t)$ may be satisfactory. For example,

1) If $A$ is unstable, can $A$ be stabilized by $f$?

2) Can we find an $f$ such that eigenvalues of $A + bf$ (i.e., poles of the closed loop system) equal to a prescribed set of eigenvalues $\Lambda$?

This problem is the *pole assignment problem* using state feedback.

# Single Input Systems (cont.)

**Problem**

Given $(A, b)$ and a desired set of eigenvalues, find $f$ so that the eigenvalues of $A + bf$ equal the desired set. Then the *feedback control law*

$$u = fx$$

assigns the eigenvalues of the closed loop system to the desired location.

# Single Input Systems (cont.)

Consider the following special case.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

and

$$\Lambda = \left[\lambda_1^d, \lambda_2^d, \cdots, \lambda_n^d\right].$$

# Single Input Systems (cont.)

Let

$$f = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{n-1} \end{bmatrix},$$

then

$$A + bf = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ a_0 + f_0 & a_1 + f_1 & a_2 + f_2 & \cdots & a_{n-1} + f_{n-1} \end{bmatrix}$$

and

$$\begin{aligned} \det [sI - (A + bf)] &= s^n - (a_{n-1} + f_{n-1})s^{n-1} + \cdots + (a1 + f_1)s + (a_0 + f_0) \\ &= (s - \lambda_1^d)(s - \lambda_2^d)\cdots(s - \lambda_n^d) \\ &= s^n - a_{n-1}^d s^{n-1} + \cdots + a_1^d s + a_0^d. \end{aligned}$$

# Single Input Systems (cont.)

Now we can equate the corresponding coefficients,

$$\begin{aligned} a_0 + f_0 &= a_0^d \\ a_1 + f_1 &= a_1^d \\ &\vdots \\ a_{n-1} + f_{n-1} &= a_{n-1}^d \end{aligned}$$

and solve for $f_i$s.

# Single Input Systems (cont.)

The solution consists of the following:

1) Taking an arbitrary system $(A, b)$ and transforming it to controllable companion form by a coordinate transformation

2) Solve the *easy version* of pole assignment problem in this coordinate system

3) transform back to the original coordinates so that the same eigenvalues are obtained.

# Single Input Systems (cont.)

**When is it possible?**

Lemma

*If $(A, b)$ is controllable, there exists a coordinate transformation $T$ such that*

$$A_n = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \qquad b_n = T^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

# Single Input Systems (cont.)

Theorem

*Pole assignment by state feedback is possible iff* $(A, b)$ *is a controllable pair.*

# Single Input Systems (cont.)

## Proof

Suppose that $(A, b)$ is not controllable, then we know we can separate controllable and uncontrollable parts as follows.

$$\begin{aligned} \dot{x} &= Ax + bu \\ &\Downarrow \quad x = Tz \\ \dot{z} &= T^{-1}ATz + T^{-1}bu \end{aligned}$$

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} z_c \\ z_u \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ 0 \end{bmatrix}}_{\hat{b}} u$$

# Single Input Systems (cont.)

Since

$$u = fx = fTz = \hat{f}z = \begin{bmatrix} \hat{f}_1 & \hat{f}_2 \end{bmatrix} \begin{bmatrix} z_c \\ z_u \end{bmatrix},$$

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 + b_1\hat{f}_1 & A_3 + b_1\hat{f}_2 \\ 0 & A_2 \end{bmatrix}}_{\hat{A}+\hat{b}\hat{f}} \begin{bmatrix} z_c \\ z_u \end{bmatrix}.$$

The eigenvalues of $\hat{A} + \hat{b}\hat{f}$ are the roots of the polynomial

$$\begin{aligned} \det\left[sI - (\hat{A} + \hat{b}\hat{f})\right] &= \det\begin{bmatrix} sI - (A_1 + b_1\hat{f}_1) & -(A_3 + b_1\hat{f}_2) \\ 0 & sI - A_2 \end{bmatrix} \\ &= \det\left[sI - (A_1 + b_1\hat{f}_1)\right]\det\left[sI - A_2\right]. \end{aligned}$$

As seen $f$ has no effect on the uncontrollable part of eigenvalues that is, the eigenvalues of $A_2$ are fixed and independent of $f$. □

# Single Input Systems (cont.)

**How to make the controllable companion transformation**

The following procedure constructs a transformation matrix that coordinate transforms an arbitrary controllable system to the controllable companion form.

1)

$$L = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

2) take the last row of $L^{-1}$ and call it $q^T$

3) Construct

$$T^{-1} = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$

# Single Input Systems (cont.)

Proceeding, let a state feedback $\hat{f}$ assign the eigenvalues of $\hat{A} + \hat{b}\hat{f}$ to the desired locations. The last step is to find the solution

$$f = \hat{f}\,T^{-1}$$

which is valid in the original coordinates, because from

$$A + bf = T(\hat{A} + \hat{b}\hat{f})T^{-1}$$

$\hat{A} + \hat{b}\hat{f}$ and $A + bf$ have the same eigenvalues.

# Multi Input Systems

Consider a multi input system,

$$\dot{x} = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}.$$

In some cases (which cases?) we can reduce this system to a single input system by introducing a new signal

$$u = gv \qquad g \in \mathbb{R}^m$$

and retain controllability of the system from the new input $v$.

# Multi Input Systems (cont.)

Then we have the new system which has a single input,

$$\dot{x} = Ax + Bgv := Ax + bv.$$

We now design a state feedback for this system. In general, we may have to use coordinate transformation of $z = Tx$.

$$\begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}bv := \hat{A}z + \hat{b}v; \qquad v = \hat{f}z \\ &= (\hat{A} + \hat{b}\hat{f})z. \end{aligned}$$

Consequently, since

$$v = \hat{f}z = \hat{f}T^{-1}x = fx,$$

we have

$$\dot{x} = Ax + bv = (A + b\underbrace{\hat{f}T^{-1}}_{f})x = (A + B\underbrace{g\hat{f}T^{-1}}_{F})x. \qquad (1)$$

# Multi Input Systems (cont.)

Remark

*This approach of using controllable companion form can be numerically unreliable, because the controllable companion form transformation is sometimes numerically ill conditioned.*

# Solution Using Sylvester's Equation

An attractive alternative method of solution is as follows.

Consider the equation

$$X^{-1}(A+BF)X = \tilde{A}; \qquad \tilde{A} \text{ has the desired set of eigenvalues.}$$

Then,

$$\begin{aligned} AX + BFX &= X\tilde{A} \\ AX - X\tilde{A} &= -BFX. \end{aligned}$$

This leads the following matrix equations:

$$\begin{aligned} AX - X\tilde{A} &= -BG; \qquad \text{given } A \text{ and } \tilde{A}, \text{a choice of } G && (2) \\ F &= GX^{-1}. && (3) \end{aligned}$$

# Solution Using Sylvester's Equation (cont.)

**The questions that arise are:**

1) Does the solution of Eq. (2) always exist? (perhaps, unique?)

2) Is the solution $X$ invertible?

3) How to choose $G$?

# Solution Using Sylvester's Equation (cont.)

Lemma

*If $(A, B)$ is controllable, and $(G, \tilde{A})$ is observable, then the unique solution $X$ of eq. (2) is "almost always" nonsingular.*

Based on this we develop the procedure:

Procedure:

1) Pick $\tilde{A}$ such that it has the desired eigenvalues.

2) Pick $G$ with $G, \tilde{A}$ and solve eq. (2). If $X$ is singular, choose a different $G$ and repeat the process.

3) If $X$ is nonsingular, solve for $F$ from eq. (3).