ECEN 605

LINEAR SYSTEMS

Lecture 10

Structure of LTI Systems II
– Observability

Observability

Let

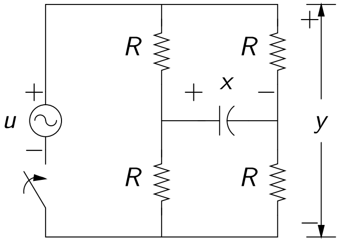
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

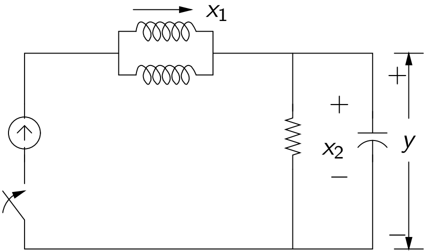
The system in eq. (1) is said to be *observable* if for any unknown initial state $x(0)$, there exists a finite $t_1 > 0$ such that the initial state $x(0)$ is uniquely determined by the input u and the output y over $[0, t_1]$.

Observability (cont.)

Example

Consider the following two circuits.





Observability (cont.)

When the circuit is open (i.e., $u = 0$), the output y will go to zero. However, there is no way to uniquely determine the initial state.

Thus the corresponding state equation is not observable.

Observability (cont.)

Consider the system (1). The response is computed as

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Assume that the output $y(t)$ and input $u(t)$ are known and the initial condition $x(0)$ is not known. Then we write

$$Ce^{At}x(0) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t).$$

Observability (cont.)

The observability problem is reduced to solving the above equation for $x(0)$. For $u = 0$, the problem is further simplified to solving

$$Ce^{At}x(0) = y(t)$$

for $x(0)$. Let $q < n$ that is the case in general, the equation above cannot be solved for $x(0)$ uniquely due to the existence of null vectors of the left-hand side expression.

Therefore, we conclude that knowledge of both $u(t)$ and $y(t)$ over a nonzero time interval is necessary to uniquely determine $x(0)$.

Observability (cont.)

Similar to the case of controllability, we have the following equivalent conditions.

Theorem (Observability)

The system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

is observable if and only if one of the following equivalent conditions hold:

Observability (cont.)

(1)

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

(2)

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \qquad \text{for all } \lambda \in \mathcal{C}$$

(3) $\text{rank}[\Theta_t] = n$ where

$$\Theta_t = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau.$$

Observability (cont.)

Proof

To prove (1), consider the output equation

$$y(t) = Cx(t) + Du(t).$$

To compute $x(0)$ from the knowledge of $y(t)$ and $u(t)$ in finite time, we consider

$$\begin{aligned}
y(t)|_{t=0} &= Cx(t)|_{t=0} + Du(t)|_{t=0} \\
y'(t)|_{t=0} &= C\dot{x}(t)|_{t=0} + Du(t)|_{t=0} = CAx(t)|_{t=0} + (\dot{u}(t) \text{ terms})|_{t=0} \\
&\vdots \\
y^{(n-1)}(t) &= CA^{n-1}x(t)|_{t=0} + (u^{(n-1)}(t) \text{ terms})|_{t=0}
\end{aligned}$$

Observability (cont.)

Then we write

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} - (u(t) \text{ and derivatives terms})|_{t=0}$$

The condition to determine $x(0)$ unqiuely is

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

to be full rank.

Observability (cont.)

To prove (2), we use contradiction. Assume that the condition (1) is true and let the condition (2) is not true. Then there exist $q \neq 0$ such that

$$\mathrm{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} q = 0.$$

This implies that

$$(A - \lambda I)q = 0 \quad \text{and} \quad Cq = 0 \quad \text{for } q \neq 0.$$

Observability (cont.)

Thus, we have

$$\begin{aligned} Aq &= \lambda q \\ A^2 q &= A(Aq) = \lambda(Aq) = \lambda^2 q \\ &\vdots \\ A^{n-1} q &= \lambda^{n-1} q \end{aligned}$$

and

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} q = 0$$

which is contraction. So (2) is proved.

Observability (cont.)

To prove (3), consider

$$Ce^{At}x(0) = \underbrace{y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)}_{=:\hat{y}(t)}.$$

Premultiplying $e^{A^T t}C^T$ and integrate it over $(0, t_1)$, we have

$$\underbrace{\int_0^{t_1} e^{A^T t}C^T Ce^{At}dt}_{\Theta_{t_1}}\, x(0) = \int_0^{t_1} e^{A^T t}C^T \hat{y}(t)dt.$$

Clearly, if Θ_{t_1} has full rank, then $x(0)$ is uniquely determined by

$$x(0) = \Theta_{t_1}^{-1}\int_0^{t_1} e^{A^T t}C^T \hat{y}(t)dt.$$

Observability (cont.)

To prove sufficiency, we use contradiction. Assume that Θ_t is singular. Then there exists $v \neq 0$ such that

$$v^T \Theta_t v = \int_0^{t_1} v^T e^{A^T t} C^T C e^{At} v dt = \int_0^{t_1} \|Ce^{At} v\|^2 dt = 0.$$

It implies that

$$Ce^{At} v = 0 \quad \text{for all } t \in [0, t_1].$$

Under this condition, we know that $x(0)$ is not a unique solution of

$$Ce^{At} x(0) = \hat{y}(t).$$

Therefore, it is proved. $\square$

Observability (cont.)

As seen, for a given system (A, B, C, D), controllability depends on the pair (A, B) and observability depends on the pair (C, A). This leads the following.

Theorem (Duality Theorem)

The pair (A, B) is controllable if and only if the pair (B^T, A^T) is observable.

Observability (cont.)

Proof.

The pair (A, B) controllable implies that

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

is nonsingular for all t. On the other hand, the pair (B^T, A^T) observable implies that

$$\Theta(t) = \int_0^t e^{(A^T)^T \tau} (B^T)^T (B^T) e^{A^T \tau} d\tau = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

is nonsingular for all t. Since $W_c(t) = \Theta(t)$, they are equivalent. $\square$

Observability (cont.)

Consider the following two system:

$$\mathcal{S}_\infty \quad : \quad \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

$$\mathcal{S}_\in \quad : \quad \begin{cases} \dot{z}(t) &= A^T z(t) + C^T u(t) \\ y(t) &= B^T z(t) + D^T u(t) \end{cases}$$

System $\mathcal{S}_2$ is called the dual of system $\mathcal{S}_1$ and vice versa. In transfer function representation, we have the following.

$$\mathcal{S}_1 \quad : \quad C(sI - A)^{-1}B + D$$

$$\mathcal{S}_2 \quad : \quad B^T(sI - A^T)^{-1}C^T + D^T = B^T\left[(sI - A)^T\right]^{-1}C^T + D^T = \left(C(sI - A)^{-1}B + D\right)^T$$

As seen, the dual system is obtained by exchanging input and output of the system.

Observability (cont.)

The observability indices are also defined similarly. Consider a pair (C, A) and let ν_i be the number of linearly independent rows associated with the i^{th} c_i of C. If the rank of observability matrix is n where

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

then

$$\nu_1 + \nu_2 + \cdots + \nu_q = n.$$

The set $\{\nu_1, \nu_2, \cdots, \nu_q\}$ is called the observability indices and

$$\nu := \max(\nu_1, \nu_2, \cdots, \nu_q)$$

is called the observability index.

Observability (cont.)

Corollary

The pair (C, A) is observable if and only if

$$\operatorname{rank}\left[\mathcal{O}_{n-q+1}\right] = n$$

where

$$\mathcal{O}_{n-q+1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-q} \end{bmatrix}$$

and $\operatorname{rank}[C] = q$ *or* $\mathcal{O}_{n-q+1}^T \mathcal{O}_{n-q+1}$ *is nonsingular.*

Corollary

The observability property is invariant under any equivalence transformation.

Observability (cont.)

Example

Consider the circuit with variables as shown in Figure 1.

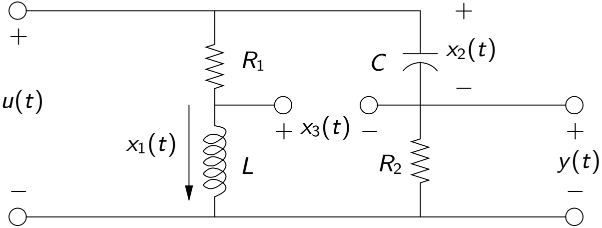


Figure 1: An electrical circuit (Example 2)

Observability (cont.)

We write *state equations* as follows:

$$
\begin{aligned}
u(t) &= R_1 x_1(t) + L\dot{x}_1(t) && (2) \\
u(t) &= R_2 C\dot{x}_2(t) + x_2(t) && (3) \\
x_2(t) &= R_1 x_1(t) + x_3(t) && (4)
\end{aligned}
$$

Observability (cont.)

so that with

$$\tau_1 = \frac{R_1}{L} \quad \text{and} \quad \tau_2 = \frac{1}{R_2 C}$$

$$\begin{aligned}
\dot{x}_1(t) &= -\tau_1 x_1(t) + \frac{1}{L} u(t) \\
\dot{x}_2(t) &= -\tau_2 x_2(t) + \tau_2 u(t) \\
\dot{x}_3(t) &= -R_1 \dot{x}_1(t) + \dot{x}_2(t) \\
&= R_1 \tau_1 x_1(t) - \tau_2 x_2(t) + (\tau_2 - \tau_1)\, u(t)
\end{aligned}$$

and

$$y(t) = -x_2(t) + u(t).$$

Observability (cont.)

In standard notation

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} &= \begin{bmatrix} -\tau_1 & 0 & 0 \\ 0 & -\tau_2 & 0 \\ R_1\tau_1 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ \tau_2 \\ \tau_2 - \tau_1 \end{bmatrix} u(t) \qquad (5) \\
y(t) &= [0 \quad -1 \quad 0]x(t) + [1]u(t).
\end{aligned}
$$

Observability (cont.)

The controllability matrix

$$C_1 := \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \dfrac{1}{L} & -\dfrac{\tau_1}{L} & \dfrac{\tau_1^2}{L} \\ \tau_2 & -\tau_2^2 & \tau_2^3 \\ \tau_2 - \tau_1 & \tau_1^2 - \tau_2^2 & -\tau_1^3 + \tau_2^3 \end{bmatrix} \tag{6}$$

and it is easy to verify that

$$\mathrm{Rank}\,[C_1] \leq 2, \qquad \text{for all } \tau_1, \tau_2 \tag{7}$$

and

$$\mathrm{Rank}\,[C_1] = 1, \qquad \text{for } \tau_1 = \tau_2. \tag{8}$$

This means that the system in (5) is generically uncontrollable.

Observability (cont.)

This is expected since eq. (4) shows that *independent* control over $x_1(t)$, $x_2(t)$ and $x_3(t)$ is impossible. This means that eqs. (2) - (4) have redundant variables and a more meaningful model could be the second order model:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -\tau_1 & 0 \\ 0 & -\tau_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \tau_2 \end{bmatrix} u(t) && (9) \\ y(t) &= [0 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [1]u(t). && (10) \end{aligned}$$

Observability (cont.)

The controllability matrix of (9) is

$$C_2 := \begin{bmatrix} \frac{1}{L} & \frac{-\tau_1}{L} \\ \tau_2 & -\tau_2^2 \end{bmatrix} \tag{11}$$

and it is easy to see that

$$\mathrm{Rank}\,[C_2] = 2, \qquad \text{for } \tau_1 \neq \tau_2 \tag{12}$$

and

$$\mathrm{Rank}\,[C_2] = 1, \qquad \text{for } \tau_1 = \tau_2. \tag{13}$$

Thus, (9) is generically controllable and loses controllability only when eq. (13) holds. In fact, it can be seen that when eq. (13) holds two inputs are necessary to render (9) controllable.

Observability (cont.)

It is easy to check that both the 3rd and 2nd order models are unobservable independent of the circuit parameter values. Indeed it is possible to check that the system transfer function is

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{-\tau_2 s\,(s+\tau_1)}{s\,(s+\tau_2)\,(s+\tau_1)} + 1 \\ &= -\frac{\tau_2}{s+\tau_2} + 1 \end{aligned} \tag{14}$$

showing that it is possible to realize it as a first order system. This is due to the fact that $y(t)$ is determined independent of $x_1(t)$.