ECEN 605

LINEAR SYSTEMS

Lecture 9

Structure of LTI Systems I
– Controllability

Introduction

The realization problem is: Given a transfer function $G(s)$, construct a circuit, using standard components, so that the transfer function of the circuit is precisely equal to the given $G(s)$. In the "old" days, the standard circuit components used to be R, L, C elements plus transformers.

The modern version of the problem uses *integrators*, *multipliers*, and *summers* as standard components for continuous time systems and *delays*, *multipliers*, and *summers* for discrete time (sampled data) systems.

Throughout this unit, we assume that $G(s)$ is rational and proper unless specifically stated.

Controllability

Definition

A dynamic system is said to be completely (state) controllable if every initial state can be transferred to any final prescribed state in a finite time T by some input $u(t)$, $0 \leq t \leq T$.

Controllability (cont.)

Theorem (Controllability)

The system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ *is controllable if and only if one of the following equivalent conditions hold:*

(1) $\mathrm{rank}[W_c] = n$ *where*

$$W_c(t) := \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau,$$

(2) $\mathrm{rank}[B, AB, \cdots, A^{n-1}B] = n$

(3) $\mathrm{rank}[A - \lambda I, B] = n$ *for all eigenvalue* $\lambda \in \mathbb{C}$ *of* A

Controllability (cont.)

Proof

We first show that if W_c is nonsingular, then Equation (1) is controllable.

Consider the response of the system in Equation (1) at time t_1, that is

$$x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau. \tag{2}$$

We claim that any $x(0) = x_0$ and any $x(t_1) = x_1$. Select

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \left[e^{At_1}x_0 - x_1 \right].$$

Controllability (cont.)

Before we proceed, we establish

$$W_c(t) = \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau. \quad (3)$$

Let $\sigma := t - \tau$,

$$\int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau = \int_0^t e^{A\sigma} B B^T e^{A^T \sigma} d\sigma$$

and we obtain (3) by letting $\tau := \sigma$.

Controllability (cont.)

Then

$$
\begin{aligned}
x_1 &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \\
&= e^{At_1}x_0 - \int_0^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}W_c(t_1)^{-1}\left[e^{At_1}x_0 - x_1\right]d\tau \\
&= e^{At_1}x_0 - \underbrace{\int_0^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}d\tau}_{W_c(t_1)}\, W_c(t_1)^{-1}\left[e^{At_1}x_0 - x_1\right] \\
&= e^{At_1}x_0 - e^{At_1}x_0 + x_1 = x_1.
\end{aligned}
$$

Since the selected $u(t)$ transfers any x_0 to any x_1 at time t_1, Equation (1) is controllable.

Controllability (cont.)

Now we show the converse by contradiction. Note that the expression of W_c shows that W_c is symmetric. Suppose that Equation (1) is controllable but W_c is singular (not positive definite). It means that there exists a nonzero vector v such that

$$\begin{aligned} v^T W_c(t_1) v &= \int_0^{t_1} v^T e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} v dt \\ &= \int_0^{t_1} \left\| B^T e^{A^T(t_1-\tau)} v \right\|^2 d\tau = 0 \end{aligned}$$

which implies that

$$B^T e^{A^T(t_1-\tau)} v = 0 \quad \text{or} \quad v^T e^{A(t_1-\tau)} B = 0 \tag{4}$$

for all $\tau \in [0, t_1]$.

Controllability (cont.)

Now select the initial state and the final state:

$$x_0 = e^{-At_1} v \qquad \text{and} \qquad x_1 = 0.$$

If the system is controllable, **we should be able to transfer x_0 to x_1 in a finite time**. So Equation (2) becomes

$$\begin{aligned} x_1 &= e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau. \\ 0 &= e^{At_1} e^{-At_1} v + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau \\ &= v + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau. \end{aligned}$$

Controllability (cont.)

Premultiplying by v^T yields

$$0 = v^T v + \int_0^{t_1} \underbrace{v^T e^{A(t_1-\tau)} B}_{=0 \text{ from Equation}(4)} u(\tau) d\tau = \|v\|^2 + 0$$

which contradicts $v \neq 0$. This proves (1).

Controllability (cont.)

To prove (2), let the initial state be x_0 and the final state be $x_1 = 0$. then

$$
\begin{aligned}
0 &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \\
&= e^{At_1}x_0 + e^{At_1}\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau
\end{aligned}
$$

yields

$$
x_0 = -\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau.
$$

Controllability (cont.)

Without loss of generality, we let

$$e^{-A\tau} = \sum_{k=0}^{n-1} f_k(\tau) A^k.$$

Then

$$x_0 = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} f_k(\tau) u(\tau) d\tau. \tag{5}$$

Controllability (cont.)

Now let

$$\beta_k := \int_0^{t_1} f_k(\tau)u(\tau)d\tau,$$

then Equation (5) becomes

$$\begin{aligned} x_0 &= -\sum_{k=0}^{n-1} A^k B \beta_k \\ &= -\begin{bmatrix} B & AB & A^B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}. \end{aligned} \tag{6}$$

If the system is completely controllable, Equation (6) must be satisfied for any given initial state x_0. This requires that the condition (2) in the theorem.

Controllability (cont.)

To prove (3), we consider the following. If (2) is true and (3) is not true, there exists an eigenvalue λ^* and a nonzero vector $v \neq 0$ such that

$$v[A - \lambda^* I \quad B] = 0.$$

It implies that

$$v(A - \lambda^* I) = 0 \quad \text{and} \quad vB = 0$$

or

$$vA = \lambda^* v \quad \text{and} \quad vB = 0.$$

Controllability (cont.)

Note that v is a left eigenvector of A associated with λ^*. Now consider

$$\begin{aligned} vA^2 &= (vA)A = (\lambda^* v)A = \lambda^*(vA) = (\lambda^*)^2 v \\ vA^3 &= (vA^2)A = ((\lambda^*)^2 v)A = (\lambda^*)^2(vA) = (\lambda^*)^3 v \\ &\vdots \\ vA^{n-1} &= (\lambda^*)^{n-1} v \end{aligned}$$

Now write

$$v[B\ AB\ A^2B\ \cdots\ A^{n-1}B] = [vB\ \lambda^* vB\ (\lambda^*)^2 vB\ \cdots\ (\lambda^*)^{n-1} vB] = 0.$$

This contracdicts (3). Thus,

$$\text{rank}[A - \lambda I \quad B] = n \quad \text{for all eigenvalues of } A.$$

□

Controllability (cont.)

Remark

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B & A^nB \end{bmatrix} = \text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

This is due to the Cayley Hamilton Theorem.

$$A^n = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}.$$

Controllability (cont.)

If we let $x(0)$ and x^* be the initial condition and the desired final state of $x(t)$, respectively, it is easy to show that

$$u(\tau) = B^T e^{A^T(t_1-\tau)} W_c(t_1)^{-1}(x^* - e^{AT}x(0)), \quad 0 \le \tau \le t_1$$

transfers $x(0)$ to $x^* = x(t_1)$ in t_1 seconds. Recall the solution of the state equation.

$$\begin{aligned} x(t_1) &= e^{AT}x(0) + \int_0^T e^{A(t_1-\tau)}Bu(\tau)d\tau \\ &= e^{AT}x(0) + \underbrace{\int_0^T e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}d\tau}_{W_c(t_1)} W_c(t_1)^{-1}(x^* - e^{AT}x(0)) \\ &= e^{AT}x(0) + (x^* - e^{AT}x(0)) \\ &= x^*. \end{aligned}$$

Controllability (cont.)

Theorem

(A, B) is controllable if and only if the following Lyapunov eqaution,

$$AW_c + W_c A^T = -BB^T$$

with real part of all eigenvalues of A being negative, has the unique solution that is positive definite. The solution is

$$W_c = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau$$

and is called the controllability Gramian.

Controllability (cont.)

Example

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t).$$

Since

$$\mathrm{Rank}[B \quad AB] = \mathrm{Rank} \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix} = 2,$$

the system is controllable. Let

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

Controllability (cont.)

We want to find $u(t)$ that moves the state $x(0)$ to $x(t) = [0 \;\; 0]^T$ within 2 seconds. Recall

$$u_{t_1}(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \left[e^{At_1} x(0) - x(t_1) \right]$$

where

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau.$$

Controllability (cont.)

We compute

$$
\begin{aligned}
W_c(2) &= \int_0^2 e^{A\tau} BB^T e^{A^T\tau}\, d\tau \\
&= \int_0^2 \left(\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} [0.5 \;\; 1] \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) d\tau \\
&= \int_0^2 \begin{bmatrix} e^{0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau \\
&= \int_0^2 \begin{bmatrix} 0.25e^{-0.5\tau} & 0.5e^{-0.5\tau} \\ 0.5e^{-\tau} & e^{-\tau} \end{bmatrix} \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau \\
&= \int_0^2 \begin{bmatrix} 0.25e^{-\tau} & 0.5e^{-1.5\tau} \\ 0.5e^{-1.5\tau} & e^{-2\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} -0.25e^{-\tau} & -\frac{1}{3}e^{-1.5\tau} \\ -\frac{1}{3}e^{-1.5\tau} & -0.5e^{-2\tau} \end{bmatrix}_0^2 = \begin{bmatrix} -0.25\left(e^{-2}-1\right) & -\frac{1}{3}\left(e^{-3}-1\right) \\ -\frac{1}{3}\left(e^{-3}-1\right) & -0.5\left(e^{-4}-1\right) \end{bmatrix} \\
&= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}.
\end{aligned}
$$

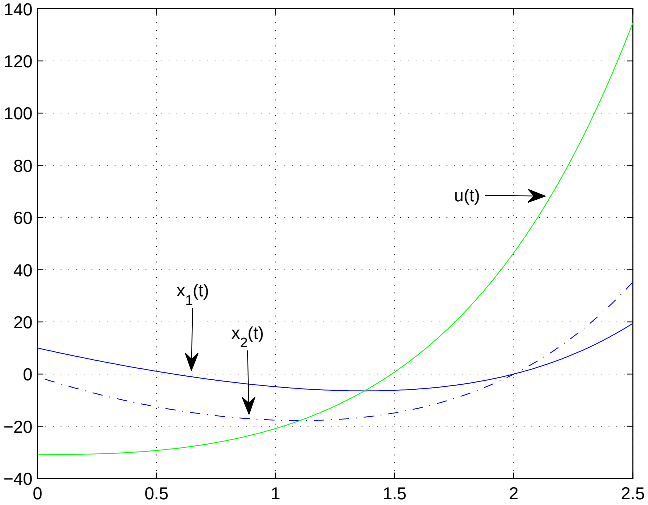
Controllability (cont.)

$$
\begin{aligned}
u_{t_1}(t) &= -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\left[e^{At_1}x(0) - x(t_1)\right] \\
&= -[0.5 \quad 1]\begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} \underbrace{\begin{bmatrix} 84.4450 & -54.4901 \\ -54.4901 & 37.1985 \end{bmatrix}}_{W^{-1}} \\
&\quad \left(\begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix}\begin{bmatrix} 10 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\
&= \left[-0.5e^{-1}e^{0.5t} \quad -e^{-2}e^{t}\right]\begin{bmatrix} 84.4450 & -54.4901 \\ -54.49011 & 37.1985 \end{bmatrix}\begin{bmatrix} 10e^{-1} \\ -e^{-2} \end{bmatrix} \\
&= \left[-0.5e^{-1}e^{0.5t} \quad -e^{-2}e^{t}\right]\begin{bmatrix} 844.45e^{-1} + 54.4901e^{-2} \\ -544.901e^{-1} - 37.1985e^{-2} \end{bmatrix} \\
&= \left(-422.225e^{-2} - 27.245e^{-3}\right)e^{0.5t} + \left(544.901e^{-3} + 37.1985e^{-4}\right)e^{t} \\
&= -58.50e^{0.5t} + 27.81e^{t}.
\end{aligned}
$$

Controllability (cont.)

```
clear
x0=[10    -1];
a=[-0.5    0;    0    -1];
b=[0.5    1]';
c=[1    1];
d=0;
sys=ss(a,b,c,d);
t=0:0.01:2.5;
u=-58.50*exp(0.5*t)+27.81*exp(t);
 [y,t,x]=lsim(sys,u,t,x0);
plot(t,x(:,1),'-b',t,x(:,2),'-.b',t,u,'-g'), grid
```

Controllability (cont.)



Controllability (cont.)

Remark

In the above example, the states have been transfered to the final states in 2 seconds. This was possible because no restruction was imposed on the control effort $u(t)$*. However, this assumption is often no longer valid in practice. For example, in the above example, if we restrict the control effort to be restricted, i.e.,* $|u(t)| \leq M$ *for all* t*, the transfer of the states to the final states may not be achieved in 2 seconds. Nevertheless, controllability imples the existence of control effort* $u(t)$ *that transfers any state to any arbitrary state in finite time.*

Controllability (cont.)

Example

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

Controllability (cont.)

Clearly,

$$\mathrm{Rank}[B \quad AB] = \mathrm{Rank}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 1$$

and the system is not completely controllable. From the theorem, we know that $W_c(t)$ will not be invertible and therefore, no input can transfer $x(0)$ to any states in a finite time.

Controllability (cont.)

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$. Also let

$$\mathrm{rank}[B] = p.$$

Then

$$\begin{aligned}\mathcal{C} &= \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 & \cdots b_p & | & Ab_1 & Ab_2 & \cdots Ab_p & | & \cdots & | A^{n-1}b_1 & A^{n-1}b_2 & \cdots & A^{n-1}b_p \end{bmatrix}.\end{aligned}$$

Let μ_i be the number of linearly independent columns associated with b_i. If $\mathrm{rank}[\mathcal{C}] = n$, then

$$\mu_1 + \mu_2 + \cdots + \mu_p = n.$$

$$\begin{aligned}\text{Controllability Indices} &: \{\mu_1, \mu_2, \cdots, \mu_p\} \\ \text{Controllability Index} &: \mu := \max(\mu_1, \mu_2, \cdots, \mu_p)\end{aligned}$$

Controllability (cont.)

Corollary

Let $\operatorname{rank}[B] = p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$. *The n-dimensional pair* (A, B) *is controllable if and only if*

$$\operatorname{rank}\left[\mathcal{C}_{n-p+1}\right] = n$$

where

$$\mathcal{C}_{n-p+1} := \begin{bmatrix} B & AB & A^2B & \cdots A^{n-p}B \end{bmatrix},$$

or $\mathcal{C}_{n-p+1}\mathcal{C}_{n-p+1}^T \in \mathbb{R}^{n \times n}$ *is nonsingular.*

Controllability (cont.)

Proof.

If $\mathcal{C}_{n-p+1}^T$ has full rank, then $\mathcal{C}_{n-p+1}^T x = 0$ for all $x \neq 0$. It follows that

$$\mathcal{C}_{n-p+1}\mathcal{C}_{n-p+1}^T x = 0 \quad \text{for all } x \neq 0.$$

This implies that $\mathcal{C}_{n-p+1}\mathcal{C}_{n-p+1}^T \in \mathbb{R}^{n\times n}$ is nonsingular (or has full rank). □

Controllability (cont.)

Theorem

The controllability property is invariant under any equivalent similarity transformation.

Controllability (cont.)

Before we prove, we establish the following rank formulas.

Lemma

Let $M \in \mathbb{R}^{n \times m}$, $N \in \mathbb{R}^{m \times p}$.

$$\mathrm{rank}[MN] \leq \min(\mathrm{rank}[M], \mathrm{rank}[N]).$$

Proof.

Let $\mathrm{rank}[M] = \alpha$. Then M has α linearly independent columns. In MN, N operates on the columns of M. Thus, MN has at most α linearly independent columns. Similarly, let $\mathrm{rank}[N] = \beta$. In MN, M operates on the rows of N. Then the row rank of MN is at most β. □

Controllability (cont.)

Lemma

Let $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times m}$ *where* Y, Z *are nonsingular. Then*

$$\mathrm{rank}[XY] = \mathrm{rank}[X] = \mathrm{rank}[ZX].$$

This means that the rank of a matrix will not change after pre or postmultiplying by a nonsingular matrix.

Controllability (cont.)

Proof

Note that

$$\mathrm{rank}[X] \leq \min(m, n), \quad \text{and} \quad \mathrm{rank}[Y] = n.$$

It follows that

$$\mathrm{rank}[X] \leq \mathrm{rank}[Y].$$

From the previous lemma,

$$\mathrm{rank}[XY] \leq \min(\mathrm{rank}[X], \mathrm{rank}[Y]) = \mathrm{rank}[X].$$

We now consider

$$\mathrm{rank}[X] = \mathrm{rank}[XYY^{-1}] \leq \min(\mathrm{rank}[XY], \mathrm{rank}[Y^{-1}]) = \mathrm{rank}[XY].$$

These imply that

$$\mathrm{rank}[XY] = \mathrm{rank}[X].$$

Controllability (cont.)

Proof of [Theorem]

For the pair (A, B),

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

For the pair $(\hat{A}, \hat{B})$ where

$$\hat{A} := T^{-1}AT, \quad \hat{B}T^{-1}B,$$

$$\begin{aligned} \hat{\mathcal{C}} &= \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \cdots & \hat{A}^{n-1}\hat{B} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}B & T^{-1}AB & T^{-1}A^2B & \cdots & T^{-1}A^{n-1}B \end{bmatrix} \\ &= T^{-1}\mathcal{C}. \end{aligned}$$

From the second stated lemma,

$$\operatorname{rank}[T^{-1}\mathcal{C}] = \operatorname{rank}[\mathcal{C}].$$

Controllability (cont.)

Theorem

The set of the controllability indices of (A, B) *is invariant under any equivalence transformation and any reordering of the columns of* B.