ECEN 605

LINEAR SYSTEMS

Lecture 8

Invariant Subspaces

State Space Structure Using Invariant Subspaces

Let

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{1a}$$
$$y(t) = C\,x(t) \tag{1b}$$

denote a dynamic system where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ denote n, r and m dimensional vector spaces, and A, B, C are linear operators:

$$A : \mathcal{X} \to \mathcal{X}, \quad B : \mathcal{U} \to \mathcal{X}, \quad C : \mathcal{X} \to \mathcal{Y}. \tag{2}$$

The structure of the system can be effectively explored and displayed in several different coordinate systems using various invariant subspaces.
A subspace $\mathcal{V} \subset \mathcal{X}$ is A-invariant if $A\mathcal{V} \subset \mathcal{V}$. If $\mathcal{V}$ has dimension K and $\{v_1, v_2, ...v_K\}$ is a basis for $\mathcal{V}$, one can choose any subspace

State Space Structure Using Invariant Subspaces (cont.)

$\mathcal{W} \subset \mathcal{X}$ such that $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$, a basis $\{w_{K+1}, ...w_n\}$ for $\mathcal{W}$ and form the coordinate transformation

$$T = [v_1, ..., v_K, w_{K+1}, ..., w_n] \tag{3}$$

and set

$$x = T z. \tag{4}$$

The system equation (1) is then transformed into

$$\dot{z}(t) = A_n z(t) + B_n u(t) \tag{5a}$$

$$y(t) = C_n z(t) \tag{5b}$$

where

$$A_n = T^{-1}AT, \quad B_n = T^{-1}B, \quad C_n = CT. \tag{6}$$

State Space Structure Using Invariant Subspaces (cont.)

Writing (6) as

$$AT = TA_n, \quad TB_n = B, \quad C_n = CT \tag{7}$$

it is easy to see that, the A-invariance of $\mathcal{V}$ implies that

$$A_n = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}. \tag{8}$$

Thus (5) can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{9a}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{9b}$$

If $\mathcal{W}$ also happens to be A-invariant then $A_3 = 0$ in (9a).

State Space Structure Using Invariant Subspaces (cont.)

Now we construct two special A-invariant subspaces for the dynamic system (1). These are the **controllable subspace** and **unobservable subspace**. For this, let $\mathcal{B}$ denote the image of B:

$$\mathcal{B} := \{ Bu | \, u \in \mathcal{U} \} \tag{10}$$

and let

$$\mathcal{V}_u := \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}. \tag{11}$$

Let $\mathbf{Ker}\, C$ denote the null space of C:

$$\mathbf{Ker}\, C := \{ x | \, Cx = 0 \} \tag{12}$$

and

$$\mathcal{V}_y = \bigcap_{i=0}^{n-1} \mathbf{Ker}\, CA^i. \tag{13}$$

State Space Structure Using Invariant Subspaces (cont.)

$\mathcal{V}_u$ is called the controllable subspace and $\mathcal{V}_y$ is called the unobservable subspace of the system (1).

Lemma

$$A\mathcal{V}_u \subset \mathcal{V}_u \tag{14a}$$

$$\mathcal{B} \subset \mathcal{V}_u \tag{14b}$$

State Space Structure Using Invariant Subspaces (cont.)

proof

That $\mathcal{B} \subset \mathcal{V}_u$ follows from (11). To prove (14a) consider an arbitrary vector $v \in \mathcal{V}_u$. Then

$$v = Bu_0 + ABu_1 + \cdots + A^{n-1}Bu_{n-1} \tag{15}$$

for some u_i, $i = 0, 1, \ldots, n-1$ Therefore

$$Av = ABu_0 + A^2Bu_1 + \cdots + A^nBu_{n-1}. \tag{16}$$

Since A^n can be expressed as a linear combination of lower powers of A, it follows that

$$Av = B\bar{u}_0 + A^2B\bar{u}_1 + \cdots + A^nB\bar{u}_{n-1}. \tag{17}$$

for some $\bar{u}_i$, $i = 0, 1, 2, \ldots, n-1$.

State Space Structure Using Invariant Subspaces (cont.)

proof (cont.)

Therefore

$$Av \in \mathcal{V}_u. \tag{18}$$

Since v was an arbitrary vector in $\mathcal{V}$, it follows that

$$A\mathcal{V}_u \subset \mathcal{V}_u. \tag{19}$$

□

State Space Structure Using Invariant Subspaces (cont.)

Remark

Consider the collection of subspaces

$$\underline{\mathcal{V}} = \{\mathcal{V} | \, A\mathcal{V} \subset \mathcal{V}, \mathcal{B} \subset \mathcal{V}\}. \quad (20)$$

It is easy to show that $\mathcal{V}_u$ *is the "smallest" element of* $\underline{\mathcal{V}}$*, that is, is contained in every subspace in* $\mathcal{V}_u$*.*

State Space Structure Using Invariant Subspaces (cont.)

Lemma

$$A\mathcal{V}_y \subset \mathcal{V}_y \tag{21a}$$

$$\mathcal{V}_y \subset \mathbf{Ker}\, C. \tag{21b}$$

State Space Structure Using Invariant Subspaces (cont.)

Proof.
That (21b) is true follows from (13). To prove (21a), pick an arbitrary $v \in \mathcal{V}_y$. Then

$$CA^i v = 0, \quad i = 0, 1, \ldots, n-1. \tag{22}$$

Now, it is easy to verify that

$$CA^i(Av) = 0, \quad i = 0, 1, \ldots, n-1 \tag{23}$$

again using the Cayley-Hamilton Theorem. Thus $Av \in \mathcal{V}_y$ and this proves (21a). □

State Space Structure Using Invariant Subspaces (cont.)

Remark

Consider the collection of all subspaces which are A-invariant and contained in **Ker** C*:*

$$\underline{\mathcal{V}} = \{ \mathcal{V} | \, A\mathcal{V} \subset \mathcal{V}, \mathcal{V} \subset \textbf{Ker}\, C \} \, . \tag{24}$$

It is easy to see that $\mathcal{V}_y$ *is the "largest" element of* $\underline{\mathcal{V}}$*, that is, every element* $\mathcal{V} in \underline{\mathcal{V}}$ *satisfies*

$$\mathcal{V} \subset \mathcal{V}_y . \tag{25}$$

State Space Structure Using Invariant Subspaces (cont.)

Lemma

$$A(\mathcal{V}_u \cap \mathcal{V}_y) \subset (\mathcal{V}_u \cap \mathcal{V}_y) \tag{26a}$$

$$A(\mathcal{V}_u + \mathcal{V}_y) \subset (\mathcal{V}_u + \mathcal{V}_y). \tag{26b}$$

State Space Structure Using Invariant Subspaces (cont.)

Proof.

If $v \in \mathcal{V}_u \cap \mathcal{V}_y$, $Av \in \mathcal{V}_u$ and $Av \in \mathcal{V}_y$ and so $Av \in \mathcal{V}_u \cap \mathcal{V}_y$. If $v \in \mathcal{V}_u + \mathcal{V}_y$, then $v = v_1 + v_2$ with $v_1 \in \mathcal{V}_u$ and $v_2 \in \mathcal{V}_y$. Then $Av = Av_1 + Av_2 \in \mathcal{V}_u + \mathcal{V}_y$. □

State Space Structure Using Invariant Subspaces (cont.)

Now consider a decomposition of the state space as follows:

$$\mathcal{X} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4 \tag{27}$$

where

$$\mathcal{V}_1 := \mathcal{V}_u \cap \mathcal{V}_y \tag{28}$$

and $\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$ satisfy

$$\mathcal{V}_2 \oplus \mathcal{V}_1 = \mathcal{V}_u \tag{29}$$

$$\mathcal{V}_3 \oplus \mathcal{V}_1 = \mathcal{V}_y \tag{30}$$

and

$$\mathcal{V}_4 \oplus \mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_1 = \mathcal{X} \tag{31}$$

but are otherwise arbitrary.

State Space Structure Using Invariant Subspaces (cont.)

Let

$$\dim \mathcal{V}_i = K_i \tag{32}$$

and let T_i denote an $n \times K_i$ matrix whose columns form a bases for $\mathcal{V}_i$, $i = 1, 2, 3, 4$. Let

$$T = [T_1, T_2, T_3, T_4] \tag{33}$$

denote a coordinate transformation and set

$$x = Tz. \tag{34}$$

In this coordinate

$$\dot{z} = A_n z + B_n u \tag{35a}$$
$$y = C_n z \tag{35b}$$

State Space Structure Using Invariant Subspaces (cont.)

and

$$A_n = T^{-1}AT \tag{36a}$$

$$B_n = T^{-1}B \tag{36b}$$

$$C_n = CT. \tag{36c}$$

State Space Structure Using Invariant Subspaces (cont.)

Theorem (Kalman Canonical Decomposition)

In the coordinate system defined by (28)-(36); (A_n, B_n, C_n) have the following structure:

$$
\begin{aligned}
A_n &= \begin{bmatrix} A_1 & A_3 & A_5 & A_7 \\ 0 & A_2 & 0 & A_8 \\ 0 & 0 & A_4 & A_9 \\ 0 & 0 & 0 & A_6 \end{bmatrix} \\
B_n &= \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \\
C_n &= \begin{bmatrix} 0 & C_2 & 0 & C_6 \end{bmatrix}.
\end{aligned}
\tag{37}
$$

State Space Structure Using Invariant Subspaces (cont.)

Proof.

The proof follows from the facts:

$$A\mathcal{V}_1 \subset \mathcal{V}_1 \tag{38a}$$

$$A(\mathcal{V}_1 \oplus \mathcal{V}_2) \subset \mathcal{V}_1 \oplus \mathcal{V}_2 \tag{38b}$$

$$A(\mathcal{V}_1 \oplus \mathcal{V}_3) \subset \mathcal{V}_1 \oplus \mathcal{V}_3 \tag{38c}$$

and

$$B \subset \mathcal{V}_1 \oplus \mathcal{V}_2 \tag{39a}$$

$$\mathcal{V}_1 \oplus \mathcal{V}_3 \subset \mathbf{Ker}\, C, \tag{39b}$$

and the relations (36). □

State Space Structure Using Invariant Subspaces (cont.)

Remark

The structure (37) specifies only the zero blocks in the matrices. The non-zero blocks will depend on the actual subspaces $\mathcal{V}_2$, $\mathcal{V}_3$ *and* $\mathcal{V}_4$ *and the bases for these subspaces.*

State Space Structure Using Invariant Subspaces (cont.)

Example (Kalman Canonical Decomposition)

Consider the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \tag{40}$$

with

State Space Structure Using Invariant Subspaces (cont.)

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
B &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}
\tag{41}
$$

State Space Structure Using Invariant Subspaces (cont.)

The controllable subspace $\mathcal{V}_u$ is generated by

$$\left\{\begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{matrix}\right\} = \mathcal{V}_u. \quad (42)$$

The unobservalbe subspace

$$\mathcal{V}_y = \textbf{Ker} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \left\{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}\right\}. \quad (43)$$

State Space Structure Using Invariant Subspaces (cont.)

Then

$$\mathcal{V}_u \cap \mathcal{V}_y = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} =: \mathcal{V}_1. \tag{44}$$

State Space Structure Using Invariant Subspaces (cont.)

Define

$$\mathcal{V}_2 = \left\{ \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \right\}, \quad \mathcal{V}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\} \textit{and} \quad \mathcal{V}_4 = \left\{ \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right\} \tag{45}$$

to satisfy

$$\mathcal{X} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4 \tag{46}$$

and let

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{47}$$

State Space Structure Using Invariant Subspaces (cont.)

It is easy to verify that

$$\begin{aligned} T^{-1}AT = A_n &= \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \\ T^{-1}B = B_n &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \\ CT = C_n &= \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \end{array}\right] \end{aligned} \tag{48}$$

which is the Kalman Canonical Decomposition of the system.