ECEN 605

LINEAR SYSTEMS

Lecture 7

Solution of State Equations

Solution of State Space Equations

Recall from the previous Lecture note, for a system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t), \tag{1b}$$

the solutions, $x(t)$ and $y(t)$, of the state equation are

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau, \tag{2}$$

and

$$\mathbf{y}(t) = \mathbf{C}\,e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\,\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\,\mathbf{u}(t).$$

State Transition Matrix

In the expression of the solution $x(t)$ of the state equation, the term e^{At} is called *state transition matrix* and is commonly notated as

$$\Phi(t) := e^{At}. \tag{3}$$

Let us examine some properties of the state transition matrix that will be used in later chapters. Assume zero input, $u(t) = 0$. Then the solution of the system

$$\dot{x}(t) = Ax(t) \tag{4}$$

becomes

$$x(t) = e^{At}x(0) = \Phi(t)x(0). \tag{5}$$

At $t = 0$, we know $\Phi(0) = I$. Differentiating eq. (5), we have

$$\dot{x}(t) = \dot{\Phi}(t)x(0) = Ax(t). \tag{6}$$

State Transition Matrix (cont.)

At $t = 0$,

$$\dot{x}(0) = \dot{\Phi}(0)x(0) = Ax(0) \tag{7}$$

leads us to have

$$\dot{\Phi}(0) = A. \tag{8}$$

Therefore, the state transition matrix $\Phi(t)$ has the following properties:

$$\Phi(0) = I \quad \text{and} \quad \dot{\Phi}(0) = A. \tag{9}$$

State Transition Matrix (cont.)

In the following, we summarize the property of the state transition matrix.

(a) $\Phi(0) = e^{A0} = I$

(b) $\Phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \Phi^{-1}(-t)$

(c) $\Phi^{-1}(t) = \Phi(-t)$

(d) $\Phi(a+b) = e^{A(a+b)} = e^{Aa}e^{Ab} = \Phi(a)\Phi(b)$

(e) $\Phi^{n}(t) = \left(e^{At}\right)^{n} = e^{Ant} = \Phi(nt)$

(f) $\Phi(a-b)\Phi(b-c) = \Phi(a)\Phi(-b)\Phi(b)\Phi(-c) = \Phi(a)\Phi(-c) = \Phi(a-c)$

State Transition Matrix (cont.)

If

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}.$$

State Transition Matrix (cont.)

If

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 & & \\ 0 & \lambda & 1 & 0 & & \\ 0 & 0 & \lambda & 1 & & \\ 0 & 0 & 0 & \lambda & & \\ & & & & \mu & 1 \\ & & & & 0 & \mu \end{bmatrix},$$

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} & \frac{1}{6}t^3e^{\lambda t} & & \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} & & \\ 0 & 0 & e^{\lambda t} & te^{\lambda t} & & \\ 0 & 0 & 0 & e^{\lambda t} & & \\ & & & & e^{\mu t} & te^{\mu} \\ & & & & 0 & e^{\mu t} \end{bmatrix},$$

The state transition matrix contains all the information about the free motions of the system in eq. (4). However, computing the state transition matrix is in general not easy. To develop some approaches to computation, we first study functions of a square matrix.

Functions of a Square Matrix

Consider a state space model:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

Taking the Laplace transform, we have

$$\begin{aligned}sX(s) - x(0) &= AX(s) + BU(s)\\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

and

$$\begin{aligned}Y(s) &= C\left[(sI-A)^{-1}BU(s) + (sI-A)^{-1}x(0)\right] + DU(s)\\ &= \left[C(sI-A)^{-1}B + D\right]U(s) + C(sI-A)^{-1}x(0).\end{aligned}$$

Functions of a Square Matrix (cont.)

The transfer function $G(s)$ is obtained by letting the initial condition $x(0) = 0$.

$$Y_u(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)} U(s). \tag{10}$$

This shows the relationship between a state space representation of the system and its corresponding transfer function. The following example illustrates how to compute the transfer function from the given state space representation of the system.

Functions of a Square Matrix (cont.)

Example

Consider a 3rd order system with 2 inputs and outputs.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & 2 & 0 \\ -1 & -4 & 1 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t)\end{aligned}$$

Functions of a Square Matrix (cont.)

Example (cont.)

To compute the transfer function,

$$
\begin{aligned}
(sI - A)^{-1} &= \begin{bmatrix} \dfrac{s+4}{s^2+5s+6} & \dfrac{2}{s^2+5s+6} & \dfrac{2}{s^2+5s+6} \\ -\dfrac{1}{s^2+5s+6} & \dfrac{s+1}{s^2+5s+6} & -\dfrac{1}{s^2+5s+6} \\ 0 & 0 & \dfrac{1}{s+1} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{2}{s+2} - \dfrac{1}{s+3} & \dfrac{2}{s+2} - \dfrac{2}{s+3} & \dfrac{1}{s+1} - \dfrac{2}{s+2} + \dfrac{1}{s+3} \\ -\dfrac{1}{s+2} + \dfrac{1}{s+3} & -\dfrac{1}{s+2} + \dfrac{2}{s+3} & -\dfrac{1}{s+2} + \dfrac{1}{s+3} \\ 0 & 0 & \dfrac{1}{s+1} \end{bmatrix}.
\end{aligned}
$$

Functions of a Square Matrix (cont.)

Example (cont.)

Therefore, the state transition matrix is

$$e^{At} = \begin{bmatrix} 2e^{-2t} - e^{-3t} & 2e^{-2t} - 2e^{-3t} & e^{-t} - 2e^{-2t} + e^{-3t} \\ -e^{-2t} + e^{-3t} & -e^{-2t} + 2e^{-3t} & -e^{-2t} + e^{-3t} \\ 0 & 0 & e^{-t} \end{bmatrix}.$$

The transfer function is

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} -\frac{2}{s+2} + \frac{2}{s+3} + 1 & -\frac{1}{s+2} + \frac{2}{s+3} \\ \frac{2}{s+1} & \frac{2}{s+2} - \frac{2}{s+3} \end{bmatrix}$$

Functions of a Square Matrix (cont.)

Example (cont.)

To solve for $x(t)$ with

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad u(t) = \begin{bmatrix} U(t) \\ 0 \end{bmatrix},$$

we have

$$\begin{aligned} X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \\ &= \begin{bmatrix} \frac{1}{s+1} \\ -\frac{2}{s+2} + \frac{2}{s+3} \\ \frac{1}{s+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ -\frac{1}{s} + \frac{1}{s+2} + \frac{\frac{2}{3}}{s} - \frac{\frac{2}{3}}{s+3} \\ \frac{1}{s} - \frac{1}{s+1} \end{bmatrix} \end{aligned}$$

Functions of a Square Matrix (cont.)

Example (cont.)

so that

$$x(t) = \begin{bmatrix} e^{-t} + 1 - e^{-t} \\ -2e^{-2t} + 2e^{-3t} - \frac{1}{3} + e^{-2t} - \frac{2}{3}e^{-3t} \\ e^{-t} + 1 - e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -\frac{1}{3} - e^{-2t} + \frac{4}{3}e^{-3t} \\ 1 \end{bmatrix}. \qquad (11)$$

Leverrier's Algorithm

This algorithm determines $(sI - A)^{-1}$ without symbolic calculation. First write

$$(sI - A)^{-1} = \frac{\mathrm{Adj}[sI - A]}{\det[sI - A]} = \frac{T(s)}{a(s)} \tag{12}$$

where

$$\begin{aligned} a(s) &= \det[sI - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ T(s) &= \mathrm{Adj}[sI - A] = T_{n-1}s^{n-1} + T_{n-2}s^{n-2} + \cdots + T_1 s + T_0, \\ & \quad T_i \in \mathbb{R}^{n\times n}. \end{aligned}$$

Thus, $(sI - A)^{-1}$ is determined if the matrices T_i and the coefficients a_j are found. Leverrier's algorithm does that as follows. Let

$$a(s) = (a - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) \tag{13}$$

Leverrier's Algorithm (cont.)

where the λ_i are complex numbers and possibly repeated. Then it is easily seen that

$$\begin{aligned} a'(s) &:= \frac{da(s)}{ds} \\ &= (s-\lambda_2)\cdots(s-\lambda_n) + (s-\lambda_1)(s-\lambda_3)\cdots(s-\lambda_n) + \cdots \\ &\quad + (s-\lambda_1)\cdots(s-\lambda_{n-1}). \end{aligned} \tag{14}$$

This leads to the following result.

Lemma (Leverrier I)

$$\frac{a'(s)}{a(s)} = \frac{1}{s-\lambda_1} + \frac{1}{s-\lambda_2} + \cdots + \frac{1}{s-\lambda_n}. \tag{15}$$

To proceed, we note that

$$(sI - A)(sI - A)^{-1} = I \tag{16}$$

Leverrier's Algorithm (cont.)

or

$$sI(sI - A)^{-1} - A(sI - A)^{-1} = I \tag{17}$$

and taking the trace of both sides, we have

$$s\mathrm{Trace}\left[(sI - A)^{-1}\right] - \mathrm{Trace}\left(\frac{AT(s)}{a(s)}\right) = n \tag{18}$$

or

$$s\mathrm{Trace}\left[(sI - A)^{-1}\right] - \mathrm{Trace}\left(\frac{AT_0 + AT_1 s + \cdots + AT_{n-1}s^{n-1}}{a(s)}\right) = n. \tag{19}$$

Lemma (Leverrier II)

$$\mathrm{Trace}(sI - A)^{-1} = \frac{a'(s)}{a(s)} \tag{20}$$

Leverrier's Algorithm (cont.)

Proof

If J denotes the Jordan form of A

$$J = \left[\begin{array}{cccc|cccc|c} \lambda_1 & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ 0 & \cdots & \cdots & \lambda_1 & & & & & \\ \hline & & & & \lambda_2 & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & 0 & \cdots & \cdots & \lambda_2 & \\ \hline & & & & & & & & \ddots \end{array}\right] \tag{21}$$

and

$$A = T^{-1} J T \tag{22}$$

for some T.

Leverrier's Algorithm (cont.)

Proof (cont.)

Thus,

$$(sI - A)^{-1} = T(sI - J)^{-1}T^{-1} \tag{23}$$

and

$$\begin{aligned}
\mathrm{Trace}(sI - A)^{-1} &= \mathrm{Trace}\left(T(sI - J)^{-1}T^{-1}\right) \\
&= \mathrm{Trace}\left(TT^{-1}(sI - J)^{-1}\right) \\
&= \mathrm{Trace}(sI - J)^{-1} \\
&= \frac{1}{s-\lambda_1} + \frac{1}{s-\lambda_2} + \cdots + \frac{1}{s-\lambda_n} \\
&= \frac{a'(s)}{a(s)} \quad \text{(by Lemma (LeverrierI)).} \qquad (24)
\end{aligned}$$

Leverrier's Algorithm (cont.)

Proof (cont.)

Then (19) becomes

$$s\left(\frac{a'(s)}{a(s)}\right) - \mathrm{Trace}\left(\frac{AT_0 + AT_1 s + \cdots + AT_{n-1}s^{n-1}}{a(s)}\right) = n \tag{25}$$

or

$$sa'(s) - \mathrm{Trace}(AT_0) - s\mathrm{Trace}(AT_1) - \cdots - s^{n-1}\mathrm{Trace}(AT_{n-1}) = na(s). \tag{26}$$

Leverrier's Algorithm (cont.)

Proof (cont.)

Now note that

$$
\begin{aligned}
sa'(s) &= na(s) - a_{n-1}s^{n-1} - 2a_{n-2}s^{n-2} - \cdots \\
&\quad -(n-2)a_2 s^2 - (n-1)a_1 s - na_0
\end{aligned}
\tag{27}
$$

so that (26) reduces to

$$
\begin{aligned}
& - \mathrm{Trace}(AT_0) - s\mathrm{Trace}(AT_1) - s^2\mathrm{Trace}(AT_2) \\
& - \cdots - s^{n-1}\mathrm{Trace}(AT_{n-1}) \\
& = a_0 n + sa_1(n-1) + s^2 a_2(n-2) + \cdots + s^{n-1}a_{n-1}.
\end{aligned}
\tag{28}
$$

Leverrier's Algorithm (cont.)

Proof (cont.)

Equating coefficients in (28) we get

$$\begin{aligned} a_0 &= -\frac{1}{n}\mathrm{Trace}(AT_0) \\ a_1 &= -\frac{1}{n-1}\mathrm{Trace}(AT_1) \\ &\vdots \\ a_k &= -\frac{1}{n-k}\mathrm{Trace}(AT_k) \\ &\vdots \\ a_{n-1} &= -\mathrm{Trace}(AT_{n-1}). \end{aligned} \tag{29}$$

Leverrier's Algorithm (cont.)

Proof (cont.)

Now from

$$(sI - A)(sI - A)^{-1} = I$$

we obtain

$$\begin{aligned} a(s)I &= (sI - A)\left(T_0 + T_1 s + \cdots + T_{n-1}s^{n-1}\right) \\ &= a_0 I + a_1 Is + \cdots + a_{n-1}Is + Is^n \end{aligned}$$

or

$$\begin{aligned} &-AT_0 + (T_0 - AT_1)s + (T_1 - AT_2)s^2 \\ &+ \cdots + (T_{n-2} - AT_{n-1})s^{n-1} + T_{n-1}s^n \\ &= a_0 I + a_1 Is + a_2 Is^2 + \cdots + a_{n-1}Is^{n-1} + Is^n. \end{aligned}$$

Leverrier's Algorithm (cont.)

Proof (cont.)

Equating the matrix coefficients, we get

$$\begin{aligned}
T_{n-1} &= I \\
T_{n-2} &= AT_{n-1} + a_{n-1}I \\
T_{n-3} &= AT_{n-2} + a_{n-2}I \\
&\vdots \\
T_0 &= AT_1 + a_1 I \\
0 &= AT_0 + a_0 I.
\end{aligned} \tag{30}$$

Leverrier's Algorithm (cont.)

Proof (cont.)

The relations (29) and (30) suggest the following sequence of calculations.

$$
\begin{aligned}
T_{n-1} &= I, & a_{n-1} &= -\mathrm{Trace}(AT_{n-1}) \\
T_{n-2} &= AT_{n-1} + a_{n-1}I, & a_{n-2} &= -\frac{1}{2}\mathrm{Trace}(AT_{n-2}) \\
T_{n-3} &= AT_{n-2} + a_{n-2}I, & a_{n-3} &= -\frac{1}{3}\mathrm{Trace}(AT_{n-3}) \\
&\vdots & &\vdots \\
T_1 &= AT_2 + a_2I, & a_1 &= -\frac{1}{n-1}\mathrm{Trace}(AT_1) \\
T_0 &= AT_1 + a_1I, & a_0 &= -\frac{1}{n}\mathrm{Trace}(AT_0) \quad \square
\end{aligned}
\tag{31}
$$

The computation sequence above constitutes Leverrier's algorithm.

Leverrier's Algorithm (cont.)

Example

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Let

$$(sI - A)^{-1} = \frac{T_3 s^3 + T_2 s^2 + T_1 s + T_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Leverrier's Algorithm (cont.)

Example

$$T_3 = I \quad \rightarrow \quad a_3 = -\text{Trace}[A] = -4$$

$$T_2 = AT_3 + a_3 I = A - 4I = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix} \quad \rightarrow \quad a_2 = -\frac{1}{2}\text{Trace}\left[AT_2\right] = 2$$

$$T_1 = AT_2 + a_2 I = \begin{bmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 5 \end{bmatrix} \quad \rightarrow \quad a_1 = -\frac{1}{3}\text{Trace}\left[AT_1\right] = 5$$

$$T_0 = AT_1 + a_1 = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 1 & 5 & -2 & -4 \\ -1 & -7 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{bmatrix} \quad \rightarrow \quad a_0 = -\frac{1}{4}\text{Trace}\left[AT_0\right] = 2$$

Leverrier's Algorithm (cont.)

Example

Therefore,

$$\det[sI - A] = s^4 - 4s^3 + 2s^2 + 5s + 2$$

$$\mathrm{Adj}[sI - A] = Is^3 + \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix} s^2 + \begin{bmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 5 \end{bmatrix} s + \begin{bmatrix} 0 & 2 & 0 & -2 \\ 1 & 5 & -2 & -4 \\ -1 & -7 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{bmatrix}$$

Leverrier's Algorithm (cont.)

Example

and

$$(sI - A)^{-1} = \frac{1}{s^4 - 4s^3 + 2s^2 + 5s + 2}$$

$$\begin{bmatrix} s^3 - 2s^2 - s & -s^2 + 4s + 2 & s^2 & 2s^2 - 3s - 2 \\ -s + 1 & s^3 - 3s^2 + 5 & s^2 - 2s - 2 & s - 4 \\ -s^2 + 2s - 1 & s^2 - 7 & s^3 - 3s^2 + 2 & s^2 - 5s + 4 \\ s^2 - 3s - 2 & s^2 - 3s + 4 & s^2 - s - 2 & s^3 - 4s^2 + 5s - 2 \end{bmatrix}$$

Cayley-Hamilton Theorem

An eigenvalue of the $n \times n$ real matrix is a real or complex number λ such that $Ax = \lambda x$ with a nonzero vector x. Any nonzero vector x satisfying $Ax = \lambda x$ is called an eigenvector of A associated with eigenvalue λ. To find the eigenvalues of A, we solve

$$(A - \lambda I)x = 0 \tag{32}$$

In order for eq. (32) to have a nonzero solution x, the matrix $(A - sI)$ must be singular. Thus, the eigenvalues of A are just the solutions of the following n^{th} order polynomial equation:

$$\Delta(s) = \det(sI - A) = 0. \tag{33}$$

$\Delta(s)$ is called the *characteristic polynomial* of A.

Cayley-Hamilton Theorem (cont.)

Theorem (**Cayley-Hamilton Theorem**)

Let A be a $n \times n$ matrix and let

$$\Delta(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

be the characteristic polynomial of A. Then

$$\Delta(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0. \qquad (34)$$

Cayley-Hamilton Theorem (cont.)

Proof

$(sI - A)^{-1}$ can always be written as

$$(sI - A)^{-1} = \frac{1}{\Delta(s)} \left(R_{n-1}s^{n-1} + R_{n-2}s^{n-2} + \cdots + R_1 s + R_0 \right)$$

where

$$\Delta(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

and R_i, $i = 0, 1, \cdots, n-1$ are constant matrices formed from the adjoint of $(sI - A)$.

Cayley-Hamilton Theorem (cont.)

Proof (cont.)

Write

$$\begin{aligned}
\Delta(s)I &= (sI - A)\left(R_{n-1}s^{n-1} + R_{n-2}s^{n-2} + \cdots + R_1 s + R_0\right) \\
&= (sI - A)R_{n-1}s^{n-1} + (sI - A)R_{n-2}s^{n-2} \\
&\qquad + \cdots + (sI - A)R_1 s + (sI - A)R_0 \\
&= R_{n-1}s^n + (R_{n-2} - AR_{n-1})\, s^{n-1} + (R_{n-3} - AR_{n-2})\, s^{n-2} \\
&\qquad + \cdots + (R_0 - AR_1)\, s - AR_0.
\end{aligned}$$

Since

$$\Delta(s)I = Is^n + a_{n-1}Is^{n-1} + \cdots + a_1 Is + a_0 I,$$

Cayley-Hamilton Theorem (cont.)

Proof (cont.)

we have by matching the coefficients,

$$\begin{aligned}
R_{n-1} &= I \\
R_{n-2} &= AR_{n-1} + a_{n-1}I \\
R_{n-3} &= AR_{n-2} + a_{n-2}I \\
&\vdots \\
R_0 &= AR_1 + a_1 I \\
0 &= AR_0 + a_0 I.
\end{aligned}$$

Cayley-Hamilton Theorem (cont.)

Proof (cont.)

Substituting R_i's successively from the bottom, we have

$$\begin{aligned}
0 &= AR_0 + a_0 I \\
&= A^2 R_1 + a_1 A + a_0 I \\
&= A^3 R_2 + a_2 A^2 + a_1 A + a_0 I \\
&\vdots \\
&= A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0 I = \Delta(A).
\end{aligned}$$

Therefore, $\Delta(A) = 0$. □

Cayley-Hamilton Theorem

The Cayley-Hamilton theorem implies that a square matrix A^n can be expressed as a linear combination of $I, A, A^2, \cdots, A^{n-1}$. Furthermore, multiplying A to (34), we have

$$A^{n+1} + a_{n-1}A^n + a_{n-2}A^{n-1} + \cdots + a_1 A^2 + a_0 A = 0$$

which implies that A^{n+1} can also be expressed as a linear combination of $I, A, A^2, \cdots, A^{n-1}$. In other words, any matrix polynomial $f(A)$ of arbitrary degree can always be expressed as a linear combination of $I, A, A^2, \cdots, A^{n-1}$, i.e.,

$$f(A) = \beta_{n-1}A^{n-1} + \beta_{n-2}A^{n-2} + \cdots + \beta_1 A + \beta_0 I$$

with appropriate values of β_i.

Application of C-H Theorem
Computing A^{-1}

Let $\Delta(s)$ be the characteristic polynomial of a matrix $A \in \mathbb{R}^{n \times n}$.

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$$

From the Cayley-Hamilton theorem, we have

$$\Delta(A) = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = 0. \quad (35)$$

Multiplying A^{-1} both sides, we have after rearrangement,

$$A^{-1} = -\frac{1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I\right). \quad (36)$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Then the characteristic polynomial is

$$\Delta(s) = (s-3)(s-2) - 1 = s^2 - 5s + 5.$$

Therefore,

$$A^{-1} = -\frac{1}{5}(A-5I) = -\frac{1}{5}\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right) = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Proceeding forward, we consider any polynomial $f(s)$, then we can write as

$$f(s) = q(s)\Delta(s) + h(s).$$

It implies that

$$f(\lambda_i) = h(\lambda_i), \quad \text{for all eigenvalues of } A.$$

Applying Cayley-Hamilton theorem, $\Delta(A) = 0$, we have

$$f(A) = q(A)\Delta(A) + h(A) = h(A).$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example

Using A given in the previous example, compute

$$f(A) = A^4 + 3A^3 + 2A^2 + A + I.$$

Calculation can be simplified by following. Consider

$$\begin{aligned} f(s) &= q(s)\Delta(s) + h(s) \\ &= \left(s^2 + 8s + 37\right)\Delta(s) + \underbrace{(146s - 184)}_{h(s)}. \end{aligned}$$

Thus,

$$f(A) = h(A) = 146A - 184.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Remark

An obvious way of computing $h(s)$ is to carry out long division. However, when long division requires lengthy computation, $h(s)$ can be directly calculated by letting

$$h(s) = \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0.$$

The unknowns β_i can be easily calculated by using the relationship

$$f(s)|_{s=\lambda_1} = h(s)|_{s=\lambda_i}, \quad \text{for } i = 1, 2, \cdots, n$$

where λ_i are eigenvalues of A.

This idea is extended to any analytic function $f(s)$ and A with eigenvalues with m multiplicities.

Application of C-H Theorem (cont.) Computing A^{-1}

Theorem

Let A be a $n \times n$ square matrix with characteristic polynomial

$$\Delta(\lambda) = \Pi_{i=1}^{m} (s - \lambda_i)^{n_i}$$

where $n = \Pi_{i=1}^{m} n_i$. Let $f(s)$ be any function and $h(s)$ be a polynomial of degree $n-1$

$$h(s) = \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0.$$

then

$$f(A) = h(A)$$

if the coefficients, β_is, of $h(s)$ are chosen such that

$$\left.\frac{d^l f(s)}{ds^l}\right|_{s=\lambda_i} = \left.\frac{d^l h(s)}{ds^l}\right|_{s=\lambda_i}, \qquad \text{for } l = 0, 1, \cdots, n_i - 1; \;\; i = 1, 2, \cdots, m.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example

Compute A^{100} with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

Define

$$\begin{aligned} f(s) &= s^{100} \\ h(s) &= \beta_1 s + \beta_0. \end{aligned}$$

Note that

$$\Delta(s) = s(s+2) + 1 = s^2 + 2s + 1 = (s+1)^2 = 0$$

and the eigenvalues of A are $\{-1, -1\}$.

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

To determine β_i,

$$\begin{aligned} f(-1) = h(-1) \quad &\Rightarrow \quad (-1)^{100} = -\beta_1 + \beta_0 \\ f'(-1) = h'(-1) \quad &\Rightarrow \quad 100(-1)^{99} = \beta_1. \end{aligned}$$

Thus, we have $\beta_1 = -100$ and $\beta_0 = 1 - 100 = -99$, and

$$h(s) = -100s - 99.$$

Therefore, from Cayley-Hamilton theorem

$$A^{100} = h(A) = -100A - 99I = \begin{bmatrix} 0 & -100 \\ 100 & 200 \end{bmatrix} - \begin{bmatrix} 99 & 0 \\ 0 & 99 \end{bmatrix} = \begin{bmatrix} -99 & -100 \\ 100 & 101 \end{bmatrix}.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

To compute e^{At}, we let $f(s) = e^{st}$ and

$$h(s) = \beta_2 s^2 + \beta_1 s + \beta_0.$$

Since the eigenvalues of A are $\{1,\ 1,\ 2\}$, we evaluate the following.

$$\begin{aligned} f(s)|_{s=1} = h(s)|_{s=1} \quad &\Rightarrow \quad e^t = \beta_2 + \beta_1 + \beta_0 \\ f'(s)|_{s=1} = h'(s)|_{s=1} \quad &\Rightarrow \quad te^t = 2\beta_2 + \beta_1 \\ f(s)|_{s=2} = h(s)|_{s=2} \quad &\Rightarrow \quad e^{2t} = 4\beta_2 + 2\beta_1 + \beta_0. \end{aligned}$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

From these, we have

$$\begin{aligned}\beta_0 &= -2te^t + e^{2t}\\ \beta_1 &= 3te^t + 2e^t - 2e^{2t}\\ \beta_2 &= e^{2t} - e^t - te^t\end{aligned}$$

and

$$h(s) = \left(e^{2t} - e^t - te^t\right)s^2 + \left(3te^t + 2e^t - 2e^{2t}\right)s - 2te^t + e^{2t}.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

Therefore,

$$\begin{aligned} e^{At} &= f(A) \\ &= h(A) \\ &= \left(e^{2t} - e^t - te^t\right) A^2 + \left(3te^t + 2e^t - 2e^{2t}\right) A + \left(-2te^t + e^{2t}\right) I \\ &= \begin{bmatrix} -e^{2t} + 2e^t & 2te^t & -2e^{2t} + 2e^t \\ 0 & e^t & 0 \\ e^{2t} - e^t & -te^t & 2e^{2t} - e^t \end{bmatrix}. \end{aligned}$$

Remark

Consider two square matrices A_1 and A_2 that are similar. Then

$$f(A_1) = f(A_2).$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example

Compute e^{At} with

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

The characteristic polynomial is

$$\Delta(s) = (s - \lambda_1)^2 (s - \lambda_2)$$

and the eigenvalues of A are $\{\lambda_1, \lambda_1, \lambda_2\}$. Define

$$f(s) = e^{st} \quad \text{and} \quad h(s) = \beta_2 s^2 + \beta_1 s + \beta_0.$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

To determine β_i,

$$
\begin{aligned}
f(s)|_{s=\lambda_1} = h(s)|_{s=\lambda_1} \quad &\Rightarrow \quad e^{\lambda_1 t} = \beta_2\lambda_1^2 + \beta_1\lambda_1 + \beta_0 \\
f'(s)|_{s=\lambda_1} = h'(s)|_{s=\lambda_1} \quad &\Rightarrow \quad te^{\lambda_1 t} = 2\beta_2\lambda_1 + \beta_1 \\
f(s)|_{s=\lambda_2} = h(s)|_{s=\lambda_2} \quad &\Rightarrow \quad e^{\lambda_2 t} = \beta_2\lambda_2^2 + \beta_1\lambda_2 + \beta_0.
\end{aligned}
$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

$$\begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & \lambda_1 & 1 \\ 2\lambda_1 & 1 & 0 \\ \lambda_2^2 & \lambda_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ \lambda_1 e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{1}{(\lambda_1-\lambda_2)^2} & \dfrac{1}{\lambda_1-\lambda_2} & \dfrac{1}{(\lambda_1-\lambda_2)^2} \\ \dfrac{2\lambda_1}{(\lambda_1-\lambda_2)^2} & -\dfrac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} & -\dfrac{2\lambda_1}{(\lambda_1-\lambda_2)^2} \\ -\dfrac{2\lambda_1\lambda_2-\lambda_2^2}{(\lambda_1-\lambda_2)^2} & \dfrac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} & \dfrac{\lambda_1^2}{(\lambda_1-\lambda_2)^2} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ te^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-e^{\lambda_1 t} + (\lambda_1-\lambda_2)te^{\lambda_1 t} + e^{\lambda_2 t}}{(\lambda_1-\lambda_2)^2} \\ \dfrac{2\lambda_1 e^{\lambda_1 t} - (\lambda_1^2-\lambda_2^2)te^{\lambda_1 t} - 2\lambda_1 e^{\lambda_2 t}}{(\lambda_1-\lambda_2)^2} \\ \dfrac{-(2\lambda_1\lambda_2-\lambda_2^2)e^{\lambda_1 t} + (\lambda_1-\lambda_2)\lambda_1\lambda_2 te^{\lambda_1 t} + \lambda_1^2 e^{\lambda_2 t}}{(\lambda_1-\lambda_2)^2} \end{bmatrix}$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

$$\begin{aligned} h(s) &= \frac{-e^{\lambda_1 t} + (\lambda_1 - \lambda_2)te^{\lambda_1 t} + e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} s^2 + \frac{2\lambda_1 e^{\lambda_1 t} - (\lambda_1^2 - \lambda_2^2)te^{\lambda_1 t} - 2\lambda_1 e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} s \\ &\quad + \frac{-(2\lambda_1\lambda_2 - \lambda_2^2)e^{\lambda_1 t} + (\lambda_1 - \lambda_2)\lambda_1\lambda_2 te^{\lambda_1 t} + \lambda_1^2 e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} \end{aligned}$$

Application of C-H Theorem (cont.) Computing A^{-1}

Example (cont.)

$$
\begin{aligned}
f(A) &= e^{At} = h(A) \\
&= \frac{-e^{\lambda_1 t} + (\lambda_1 - \lambda_2)te^{\lambda_1 t} + e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} \begin{bmatrix} \lambda_1^2 & 2\lambda_1 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix} \\
&\quad + \frac{2\lambda_1 e^{\lambda_1 t} - (\lambda_1^2 - \lambda_2^2)te^{\lambda_1 t} - 2\lambda_1 e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \\
&\quad + \frac{-(2\lambda_1\lambda_2 - \lambda_2^2)e^{\lambda_1 t} + (\lambda_1 - \lambda_2)\lambda_1\lambda_2 te^{\lambda_1 t} + \lambda_1^2 e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix}.
\end{aligned}
$$

Similarity Transformation

The state variables describing a system are not unique and the state space representation of a given system depends on the choice of state variables. By choosing different bases for the underlying n ddimensional state space we obtain different nth order representations of the same system. Such systems are called *similar*. Similar systems have the same transfer function although their state space representations are different. This fact can be exploited to develop equivalent state space representations consists of matrices that display various structural properties or simplify computation.
Now consider a state space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Nu(t)\\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

Similarity Transformation (cont.)

Define $x(t) = Tz(t)$ where T is a square invertible matrix. Then

$$\begin{aligned} T\dot{z}(t) &= ATz(t) + Bu(t) \\ y(t) &= CTz(t) + Du(t). \end{aligned}$$

or

$$\begin{cases} \dot{z}(t) = T^{-1}ATz(t) + T^{-1}Bu(t) \\ y(t) = CTz(t) + Du(t) \end{cases} \Rightarrow \begin{cases} \dot{z} = A_n z(t) + B_n u(t) \\ y(t) = C_n z(t) + Du(t). \end{cases}$$

The transfer function of the system may be computed

$$\begin{aligned} CT\left(sI - T^{-1}AT\right)T^{-1}B + D &= CT\left[T^{-1}(sI - A)T\right]^{-1}T^{-1}B + D \\ &= CT\left[T^{-1}(sI - A)^{-1}T\right]T^{-1}B + D \\ &= CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= C(sI - A)^{-1}B + D \end{aligned}$$

which shows that the representations in eqs. (37) and (37) have the same transfer function.

Diagonalization

Theorem

Consider an $n \times n$ matrix A with distinct eigenvalues λ_1, λ_2, $\cdots$, λ_n. Let x_i, $i = 1, 2, \cdots, n$ be eigenvectors associated with λ_i. Define a $n \times n$ matrix

$$T = [x_1 \;\; x_2 \;\; x_3 \;\; \cdots \;\; x_n].$$

Then

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

Diagonalization (cont.)

Proof

Since x_i is an eigenvector of A associated with λ_i, $Ax_i = \lambda_i x_i$. So we write

$$Ax_i = \lambda_i x_i = [x_1 \quad x_2 \quad \cdots \quad x_i \quad \cdots \quad x_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Diagonalization (cont.)

Proof cont.

Consequently,

$$A\underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_i & \cdots & x_n \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_i & \cdots & x_n \end{bmatrix}}_{T} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

and

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

□

Jordan Form

We have seen in the previous section that an $n \times n$ matrix A can be converted through a similarity transformation into a diagonal matrix when the eigenvalues of A are distinct. When the eigenvalues of A are not distinct, that is, some are repeated, it is not always possible to "diagonalize" A, as before. In this subsection, we deal with this case, and develop the so-called *Jordan form* of A which is the "maximally" diagonal form that can be obtained.
The characteristic polynomial of the $n \times n$ real or complex matrix A, denoted $\Pi(s)$, is:

$$\begin{aligned} \Pi(s) &= \det(sI - A) \\ &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \end{aligned} \tag{37}$$

Write (37) in the factored form

$$\Pi(s) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2} + \cdots + (s - \lambda_p)^{n_p} \tag{38}$$

Jordan Form (cont.)

where the λ_i are distinct, that is,

$$\lambda_i \neq \lambda_j, \quad \text{for } i \neq j$$

and

$$n_1 + n_2 + \cdots + n_p = n.$$

It is easy to establish the following.

Jordan Form (cont.)

Theorem

There exists an $n \times n$ nonsingular matrix T such that

$$T^{-1}AT = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix}$$

where A_i is $n_i \times n_i$ and

$$\det(sI - A_i) = (s - \lambda_i)^{n_i}, \quad \text{for } i = 1, 2, \cdots, p.$$

Jordan Form (cont.)

In the remaining part of the section, we develop the Jordan decomposition of the matrices A_i. Therefore, we now consider, without loss of generality, an $n \times n$ matrix A such that

$$\Pi(s) = \det(sI - A) = (s - \lambda)^n. \tag{39}$$

The Jordan structure of such an A matrix may be found by forming the matrices

$$(A - \lambda I)^k, \quad \text{for } k = 0, 1, 2, \cdots, n. \tag{40}$$

Let $\mathcal{N}_k$ denote the null space of $(A - \lambda I)^k$:

$$\mathcal{N}_k = \left\{ x \mid (A - \lambda I)^k x = 0 \right\} \tag{41}$$

and denote the dimension of $\mathcal{N}_k$ by ν_k:

$$\text{dimension } \mathcal{N}_k = \nu_k, \quad \text{for } k = 0, 1, 2, \cdots, n+1. \tag{42}$$

Jordan Form (cont.)

We remark that

$$0 = \nu_0 \leq \nu_1 \leq \cdots \leq \nu_{n-1} \leq \nu_n = n = \nu_{n+1}. \qquad (43)$$

The following theorem gives the Jordan structure of A.

Jordan Form (cont.)

Theorem

Given the $n \times n$ matrix A with

$$\det(sI - A) = (s - \lambda)^n, \tag{44}$$

there exists an $n \times n$ matrix T with $\det(T) \neq 0$, such that

$$T^{-1}AT = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix}$$

where

$$A_j = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}, \quad \text{for } j = 1, 2, \cdots, r$$

is called a Jordan block $n_j \times n_j$.

Jordan Form (cont.)

The number r and sizes n_j of the Jordan blocks A_i can be found from the formula given in the following Lemma.

Lemma

Under the assumption of the previous theorem, the number of Jordan blocks of size $k \times k$ is given by

$$2\nu_k - \nu_{k-1} - \nu_{k+1}, \qquad \text{for } k = 1, 2, \cdots, n.$$

Jordan Form (cont.)

Example

Let A be an 11×11 matrix with

$$\det(sI - A) = (s - \lambda)^{11}.$$

Suppose that

$$\nu_1 = 6,\ \nu_2 = 9,\ \nu_3 = 10,\ \nu_4 = 11 = \nu_5 = \nu_6 = \cdots = v_{11} = \nu_{12}.$$

Jordan Form (cont.)

Example (cont.)

Then the number of Jordan blocks:

$$\begin{aligned}
&\text{a) of size } \; 1 \times 1 = 2\nu_1 - \nu_0 - \nu_2 = 3\\
&\text{b) of size } \; 2 \times 2 = 2\nu_2 - \nu_1 - \nu_3 = 2\\
&\text{c) of size } \; 3 \times 3 = 2\nu_3 - \nu_2 - \nu_4 = 0\\
&\text{d) of size } \; 4 \times 4 = 2\nu_4 - \nu_3 - \nu_5 = 1\\
&\text{e) of size } \; 5 \times 5 = 2\nu_5 - \nu_4 - \nu_6 = 0\\
&\vdots\\
&\text{k) of size } \; 11 \times 11 = 2\nu_{11} - \nu_{12} - \nu_{10} = 0.
\end{aligned}$$

Jordan Form (cont.)

Example (cont.)

Therefore, the Jordan form of A is as shown below:

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 & & & & & & & \\ 0 & \lambda & 1 & 0 & & & & & & & \\ 0 & 0 & \lambda & 1 & & & & & & & \\ 0 & 0 & 0 & \lambda & & & & & & & \\ & & & & \lambda & 1 & & & & & \\ & & & & 0 & \lambda & & & & & \\ & & & & & & \lambda & 1 & & & \\ & & & & & & 0 & \lambda & & & \\ & & & & & & & & \lambda & & \\ & & & & & & & & & \lambda & \\ & & & & & & & & & & \lambda \end{bmatrix}$$

Jordan Form (cont.)

This result can now be applied to each of the blocks A_i in the previous theorem to obtain the complete Jordan decomposition in the general case. It is best to illustrate the procedure with an example.

Example

Let A be a 15×15 matrix with

$$\det(sI - A) = (s - \lambda_1)^8 (s - \lambda_2)^4 (s - \lambda_3)^3$$

with λ_i being distinct.

Jordan Form (cont.)

Example (cont.)

We compute

$$(A - \lambda_i I)^k, \qquad k = 0, 1, 2, \cdots, 8, 9$$

and set

$$h_k = \dim \mathcal{N} (A - \lambda_1 I)^k, \qquad k = 0, 1, 2, \cdots, 9.$$

Similarly, let

$$i_j = \dim \mathcal{N} (A - \lambda_2 I)^j, \qquad j = 0, 1, 2, \cdots, 5$$

and

$$l_s = \dim \mathcal{N} (A - \lambda_3 I)^s, \qquad s = 0, 1, 2, 3, 4.$$

Jordan Form (cont.)

Example (cont.)

Suppose, for example,

$$h_0 = 0, \quad h_1 = 2, \quad h_2 = 4, \quad h_3 = 6, \quad h_4 = 7, \quad h_5 = h_6 = h_7 = h_8 = 8$$
$$i_0 = 0, \quad i_1 = 3, \quad i_2 = 4, \quad i_3 = i_4 = i_5 = 4$$
$$l_0 = 0, \quad l_1 = 1, \quad l_2 = 2, \quad l_3 = l_4 = 3.$$

By Theorem 29, the Jordan form of A has the following structure.

$$T^{-1}AT = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

with $A_1 \in \mathbb{R}^{8\times 8}$, $A_2 \in \mathbb{R}^{4\times 4}$ and $A_3 \in \mathbb{R}^{3\times 3}$. Furthermore, the detailed structure of A_1, by Theorem 30, consists of the following numbers of Jordan blocks.

Jordan Form (cont.)

Example (cont.)

a) of size $1 \times 1 = 2h_1 - h_0 - h_2 = 0$ blocks

b) of size $2 \times 2 = 2h_2 - h_1 - h_3 = 0$ blocks

c) of size $3 \times 3 = 2h_3 - h_2 - h_4 = 1$ blocks

d) of size $4 \times 4 = 2h_4 - h_3 - h_5 = 0$ blocks

e) of size $5 \times 5 = 2h_5 - h_4 - h_6 = 1$ blocks

f) of size $6 \times 6 = 2h_6 - h_5 - h_7 = 0$ blocks

g) of size $7 \times 7 = 2h_7 - h_6 - h_8 = 0$ blocks

h) of size $8 \times 8 = 2h_8 - h_7 - h_9 = 0$ blocks

Jordan Form (cont.)

Example (cont.)

Similarly, the numbers of Jordan blocks in A_2 are:

$$\begin{aligned}
&\text{a) of size } 1\times 1 = 2i_1 - i_0 - i_2 = 2 \quad \text{blocks}\\
&\text{b) of size } 2\times 2 = 2i_2 - i_1 - i_3 = 1 \quad \text{blocks}\\
&\text{c) of size } 3\times 3 = 2i_3 - i_2 - i_4 = 0 \quad \text{blocks}\\
&\text{d) of size } 4\times 4 = 2i_4 - i_3 - i_5 = 0 \quad \text{blocks}
\end{aligned}$$

and the numbers of Jordan blocks in A_3 are:

$$\begin{aligned}
&\text{a) of size } 1\times 1 = 2l_1 - l_0 - l_2 = 0 \quad \text{blocks}\\
&\text{b) of size } 2\times 2 = 2l_2 - l_1 - l_3 = 0 \quad \text{blocks}\\
&\text{c) of size } 3\times 3 = 2l_3 - l_2 - l_4 = 1 \quad \text{blocks}
\end{aligned}$$

Jordan Form (cont.)

Example (cont.)

With this information, we know that the Jordan form of A is given by

$$
\begin{aligned}
T^{-1}AT &= \left[\begin{array}{ccccccccccccc}
\lambda_1 & 1 & 0 & 0 & 0 & & & & & & & & \\
0 & \lambda_1 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & \lambda_1 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & \lambda_1 & 1 & & & & & & & & \\
0 & 0 & 0 & 0 & \lambda_1 & & & & & & & & \\
 & & & & & \lambda_1 & 1 & 0 & & & & & \\
 & & & & & 0 & \lambda_1 & 1 & & & & & \\
 & & & & & 0 & 0 & \lambda_1 & & & & & \\
 & & & & & & & & \lambda_2 & 1 & & & \\
 & & & & & & & & 0 & \lambda_2 & & & \\
 & & & & & & & & & & \lambda_2 & & \\
 & & & & & & & & & & & \lambda_2 & \\
 & & & & & & & & & & & & \lambda_3 \; 1 \; 0 \\
 & & & & & & & & & & & & 0 \; \lambda_3 \; 1 \\
 & & & & & & & & & & & & 0 \; 0 \; \lambda_3
\end{array}\right] \\
&= A_J
\end{aligned}
$$

Finding the Transformation Matrix T

Once the Jordan form is found, it is relatively easy to find the transformation matrix. We illustrate using the previous example. Write T in terms of its columns.

$$T = \left[\underbrace{t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5}_{\text{block 1}} \quad \underbrace{t_6 \quad t_7 \quad t_8}_{\text{block 2}} \quad \underbrace{t_9 \quad t_{10}}_{\text{block 3}} \quad \underbrace{t_{11}}_{\text{block 4}} \quad \underbrace{t_{12}}_{\text{block 5}} \quad \underbrace{t_{13} \quad t_{14} \quad t_{15}}_{\text{block 6}} . \right]$$

Finding the Transformation Matrix T (cont.)

Then we have

$$\begin{aligned} At_1 &= \lambda_1 t_1 & \text{or} & \quad (A - \lambda_1 I)\, t_1 = 0 \\ At_2 &= \lambda_1 t_2 + t_1 & \text{or} & \quad (A - \lambda_1 I)\, t_2 = t_1 \\ At_3 &= \lambda_1 t_3 + t_2 & \text{or} & \quad (A - \lambda_1 I)\, t_3 = t_2 \\ At_4 &= \lambda_1 t_4 + t_3 & \text{or} & \quad (A - \lambda_1 I)\, t_4 = t_3 \\ At_5 &= \lambda_1 t_5 + t_4 & \text{or} & \quad (A - \lambda_1 I)\, t_5 = t_4 \end{aligned} \tag{45}$$

as well as

$$\begin{aligned} At_6 &= \lambda_1 t_6 & \text{or} & \quad (A - \lambda_1 I)\, t_6 = 0 \\ At_7 &= \lambda_1 t_7 + t_6 & \text{or} & \quad (A - \lambda_1 I)\, t_7 = t_6 \\ At_8 &= \lambda_1 t_8 + t_7 & \text{or} & \quad (A - \lambda_1 I)\, t_8 = t_7 \end{aligned} \tag{46}$$

Finding the Transformation Matrix T (cont.)

Therefore, we need to find two linearly independent vectors t_1, t_6 in $\mathcal{N}(A - \lambda_1 I)$ such that the chains of vectors

$$t_1, \quad t_2, \quad t_3, \quad t_4, \quad t_5 \quad \text{and} \quad t_6, \quad t_7, \quad t_8$$

are linearly independent. Alternatively, we can attempt to find t_5 such that

$$(A - \lambda_i I)^5 \, t_5 = 0$$

and find t_4, t_3, t_2, t_1 from the set in eq. (45). Similarly, we can find t_8, such that

$$(A - \lambda_i I)^3 \, t_8 = 0$$

and t_6, t_7, t_8 found from eq. (46) are linearly independent.

Finding the Transformation Matrix T (cont.)

Moving to the blocks associated with λ_2, we see that we need to find t_9, t_{11}, t_{12} so that

$$\begin{aligned}(A - \lambda_2 I)\, t_9 &= 0 \\ (A - \lambda_2 I)\, t_{11} &= 0 \\ (A - \lambda_2 I)\, t_{12} &= 0\end{aligned}$$

and $(t_9, \quad t_{10}, \quad t_{11}, \quad t_{12})$ are independent with

$$At_{10} = \lambda_2 t_{10} + t_9.$$

Likewise, we find t_{13}, such that

$$\begin{aligned}(A - \lambda_3 I)\, t_{13} &= 0 \\ (A - \lambda_3 I)\, t_{14} &= t_{13} \\ (A - \lambda_3 I)\, t_{15} &= t_{14}\end{aligned}$$

with $(t_{13}, \quad t_{14}, \quad t_{15})$ linearly independent. In each of the above cases, the existence of the vectors t_{ij} are guaranteed by the existence of the Jordan forms.