ECEN 605
LINEAR SYSTEMS

Lecture 6

Mathematical Representation of Physical Systems II
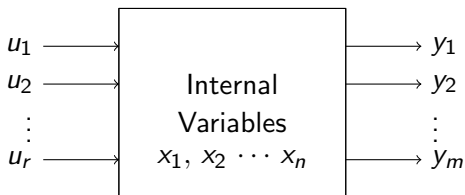
State Variable Models for Dynamic Systems

Figure 1: A conceptual model

The general model of Figure 1 is applicable to dynamic systems as well. The only difference is that time derivative of the internal variables $x_i(t)$ also enter into the mathematical description of the behavior of the system. For linear time invariant (LTI) systems it

State Variable Models for Dynamic Systems (cont.)

will be possible, in many cases, to write the system equations in the standard form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{1b}$$

where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ have appropriate dimensions and $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{x}$ denote the output, input and state vectors respectively. Note that the static model from the previous lecture is a special case of (1), when $\dot{\mathbf{x}}(t) = 0$.

In general the **internal** or **state** variables should be chosen to provide a "complete" description of the system. We illustrate the use of the model (1) with examples in the next section.

Solution of State Space Equations

Let us consider a scalar example:

$$\dot{\mathbf{x}}(t) = a\mathbf{x}(t) + b\mathbf{u}(t) \tag{2}$$

where a, b, c are scalar constants. To solve the differential equation, we take Laplace transforms, and get

$$s\mathbf{X}(s) - \mathbf{x}(0) = a\mathbf{X}(s) + b\mathbf{U}(s), \quad \text{or} \quad \mathbf{X}(s) = \frac{1}{s-a}\mathbf{x}(0) + \frac{b}{s-a}\mathbf{U}(s). \tag{3}$$

Taking inverse Laplace transforms, we have

$$\mathbf{x}(t) = e^{at}\mathbf{x}(0) + \int_0^t e^{a(t-\tau)}b\mathbf{u}(\tau)d\tau. \tag{4}$$

Eq. (4) shows that the solution consists of two parts. The first part is only depending on the initial condition $\mathbf{x}(0)$ and the second part is only depending on the input $\mathbf{u}(t)$. Therefore the first part is

Solution of State Space Equations (cont.)

called the zero input response and the second part is called the zero state response.

The foregoing approach to the solution of the scalar differential equation can be extended to the case of the vector state equations. Let

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{5}$$

where $\mathbf{x}(t)$ is n vector, so $\mathbf{A} \in \mathbb{R}^{n \times n}$. If a system has r inputs and m outputs, $\mathbf{B} \in \mathbb{R}^{n \times r}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathbf{D} \in \mathbb{R}^{m \times r}$. Taking the Laplace transform of both sides of the equation, we have

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)]$, $\mathbf{U}(s) = \mathcal{L}[\mathbf{u}(t)]$ and $\mathbf{x}(t)$ is a vector with size n. Hence

$$(sI - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (sI - \mathbf{A})^{-1}\mathbf{x}(0) + (sI - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Solution of State Space Equations (cont.)

Taking inverse Laplace transform

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(sI - \mathbf{A})^{-1}\right]\mathbf{x}(0) + \mathcal{L}^{-1}\left[(sI - \mathbf{A})^{-1}\right] * \mathbf{u}(t).$$

Note that

$$(sI - \mathbf{A})^{-1} = \frac{I}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \cdot \tag{6}$$

Thus, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left[(sI - \mathbf{A})^{-1}\right] &= I + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdot \\
&= e^{\mathbf{A}t}. \tag{7}
\end{aligned}$$

Therefore, the solution of the state equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{8}$$

Solution of State Space Equations (cont.)

Substituting (8) to

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t),$$

we have

$$\mathbf{y}(t) = \mathbf{C}\,e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\,\mathbf{u}(t).$$

Examples of State Space Models
$R\,L\,C$ **circuits**

Inductors and capacitors have dynamic voltage-current relationships as shown below:
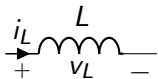

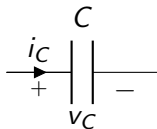Figure 2: $v_L = L\dfrac{d\,i_L}{d\,t}$

Figure 3: $i_C = C\dfrac{d\,v_C}{d\,t}$

It is possible to show, based on the above v-i relationships that an arbitrary $R\,L\,C$ circuit can be "completely" described by choosing all inductor currents and capacitor voltages as components of the state vector **x**.

Examples of State Space Models (cont.)

Example ($R\,L\,C$ circuits)
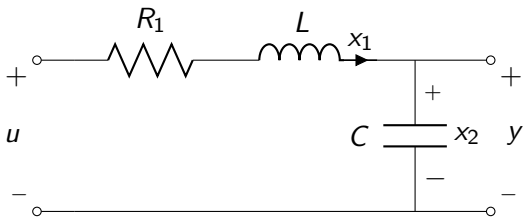
Consider the series $R\,L\,C$ circuit:

Figure 4: An $R\,L\,C$ circuit

Examples of State Space Models (cont.)

Example ($R\,L\,C$ circuits (cont.))

Choosing x_1 and x_2 as the inductor current and capacitor voltage respectively we see that the vector $(x_1(t), x_2(t))$ provides a complete description of all currents and voltages in the circuit. The system equations are obtained from Kirchhoff's laws:

$$
\begin{aligned}
x_1 &= C\,\dot{x}_2 \quad \text{(current summation)} \\
u &= R\,x_1 + L\,\dot{x}_1 + x_2 \quad \text{(voltage summation)} \\
y &= x_2 \quad \text{(output)}.
\end{aligned}
\tag{9}
$$

Examples of State Space Models (cont.)

Example ($R\,L\,C$ circuits (cont.))

These can be placed in the standard form (1):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{R} \\ \frac{1}{C} & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{R} \\ 0 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u. \tag{10}$$

Examples of State Space Models (cont.)

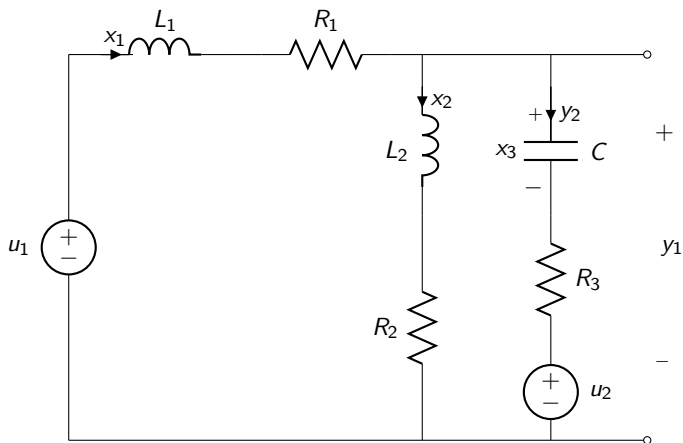Example (Two input Two output $R\,L\,C$ circuit)

Figure 5: 2 input 2 output $R\,L\,C$ circuit

Examples of State Space Models (cont.)

Example (Two input Two output $R\,L\,C$ circuit (cont.))

Choosing inductor currents x_1, x_2, and capacitor voltage x_3 as state variables we can write Kirchhoff's equations:

current summation

$$x_1 = x_2 + C\,\dot{x}_3 \tag{11a}$$

voltage summation

$$u_1 = L_1\,\dot{x}_1 + R_1\,x_1 + x_3 + R_3\,C\,\dot{x}_3 + u_2 \tag{11b}$$

$$L_2\,\dot{x}_2 + R_2\,x_2 = x_3 + R_3\,C\,\dot{x}_3 + u_2 \tag{11c}$$

$$y_1 = x_3 + R_3\,C\,\dot{x}_3 + u_2 \tag{11d}$$

$$y_2 = x_2. \tag{11e}$$

Examples of State Space Models (cont.)

Example (Two input Two output $R\,L\,C$ circuit (cont.))

Equation (11) can be rewritten in the standard state variable form:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{(R_1+R_3)}{L_1} & \frac{R_3}{L_1} & -\frac{1}{L_1} \\ -\frac{R_3}{L_2} & -\frac{(R_2+R_3)}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L_1} & -\frac{1}{L_1} \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R_3 & -R_3 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
$$

$$
\tag{12}
$$

Examples of State Space Models (cont.)

Example

Consider the system,
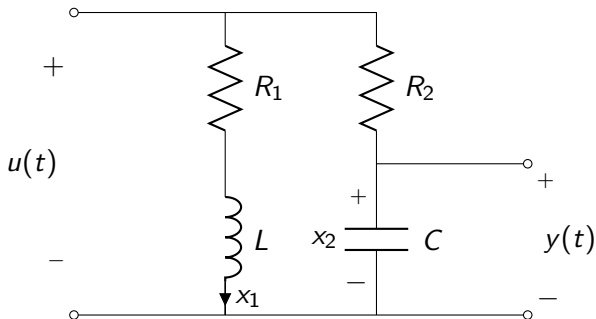
Let x_1 be the current of the inductor L and let x_2 be the voltage across the capacitor C.

Examples of State Space Models (cont.)

Then,

$$u = R_1 x_1 + L \dot{x}_1 \tag{13}$$

$$u = R_2 C \dot{x}_2 + x_2 \tag{14}$$

$$y = x_2. \tag{15}$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2 C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_2 C} \end{bmatrix} u \tag{16}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u. \tag{17}$$
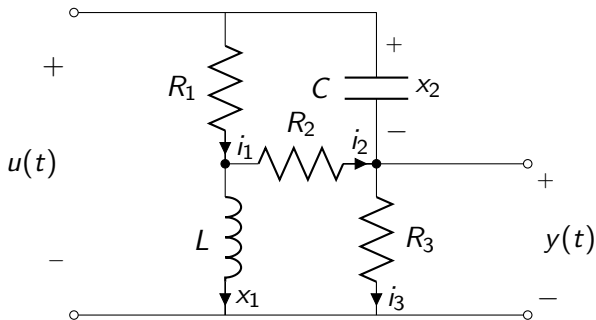
Examples of State Space Models (cont.)

The transfer function is

$$
\begin{aligned}
G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & 0 \\ 0 & s + \frac{1}{R_2\,C} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_2\,C} \end{bmatrix} + 0 \\
&= \frac{\frac{1}{R_2\,C}}{s + \frac{1}{R_2\,C}}.
\end{aligned}
\tag{18}
$$

Examples of State Space Models (cont.)

Example

Consider the system,

Examples of State Space Models (cont.)

From Kirchhoff current law, we have

$$i_1 = i_2 + x_1 \quad \text{(node on the left)} \tag{19}$$

$$i_2 + C\,\dot{x}_2 = i_3 \quad \text{(node on the right)} \tag{20}$$

$$x_1 + i_3 = i_1 + C\,\dot{x}_2 \quad \text{(current from/to } u(t)). \tag{21}$$

Rewriting above with respect to i_1, i_2 and i_3, we get

$$i_1 = \frac{u - L\,\dot{x}_1}{R_1} \tag{22}$$

$$i_2 = \frac{u - L\,\dot{x}_1}{R_1} - x_1 \tag{23}$$

$$i_3 = \frac{u - L\,\dot{x}_1}{R_1} - x_1 + C\,\dot{x}_2. \tag{24}$$

Examples of State Space Models (cont.)

From Kirchhoff voltage law, we have

$$u - L\dot{x}_1 + R_2\left(\frac{u - L\dot{x}_1}{R_1} - x_1\right) - x_2 = 0 \quad \text{(Top loop)} \tag{25}$$

$$L\dot{x}_1 = R_2\left(\frac{u - L\dot{x}_1}{R_1} - x_1\right) + R_3\left(\frac{u - L\dot{x}_1}{R_1} - x_1 + C\dot{x}_2\right) \tag{26}$$

(Bottom loop).

Thus,

$$u - L\dot{x}_1 + R_2\left(\frac{u - L\dot{x}_1}{R_1} - x_1\right) - x_2 = 0 \tag{27}$$

$$\Leftrightarrow u + \frac{R_2\, u}{R_1} - R_2\, x_1 - x_2 = L\dot{x}_1 + \frac{R_2}{R_1}L\dot{x}_1 \tag{28}$$

$$\Leftrightarrow \left(L\left(1 + \frac{R_2}{R_1}\right)\right)\dot{x}_1 = -R_2\, x_1 - x_2 + \left(1 + \frac{R_2}{R_1}\right)u \tag{29}$$

Examples of State Space Models (cont.)

Example (cont.)

$$
\begin{aligned}
\dot{x}_1 &= \frac{-R_2}{\frac{L(R_1+R_2)}{R_1}}x_1 - \frac{R_1}{L(R_1+R_2)}x_2 + \frac{1}{L}u \\
&= \frac{-R_1 R_2}{L(R_1+R_2)}x_1 - \frac{R_1}{L(R_1+R_2)}x_2 + \frac{1}{L}u
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
R_1 L \dot{x}_1 &= R_2 u - R_2 L \dot{x}_1 - R_1 R_2 x_1 + R_3 u - R_3 L \dot{x}_1 \\
&\quad - R_1 R_3 x_1 + R_1 R_3 C \dot{x}_2 \tag{32} \\
\Leftrightarrow R_1(R_2 + R_3)x_1 &- (R_2 + R_3)u + L(R_1 + R_2 + R_3)\dot{x}_1 \\
&= R_1 R_3 C \dot{x}_2 \tag{33}
\end{aligned}
$$

Examples of State Space Models (cont.)

Example (cont.)

$$R_1 R_3 C \dot{x}_2 = (R_1 + R_2 + R_3) \left\{ \frac{-R_1 R_2}{R_1 + R_2} x_1 - \frac{R_1}{R_1 + R_2} x_2 + u \right\}$$
$$+ R_1(R_2 + R_3)x_1 - (R_2 + R_3)u \tag{34}$$

$$\Leftrightarrow R_3 C \dot{x}_2 = \left(\frac{-R_2(R_1 + R_2 + R_3)}{R_1 + R_2} + (R_2 + R_3) \right) x_1$$
$$+ \frac{-(R_1 + R_2 + R_3)}{R_1 + R_2} x_2 + u \tag{35}$$

$$\Leftrightarrow R_3 C \dot{x}_2 = \frac{R_1 R_3}{R_1 + R_2} x_1 - \frac{R_1 + R_2 + R_3}{R_1 + R_2} x_2 + u \tag{36}$$

$$\Leftrightarrow \dot{x}_2 = \frac{R_1}{C(R_1 + R_2)} x_1 - \left(\frac{1}{R_3 C} + \frac{1}{C(R_1 + R_2)} \right) x_2 + u \tag{37}$$

Examples of State Space Models (cont.)

Example (cont.)

$$y = R_3 \, i_3 = R_3 \left(\frac{u - L \, \dot{x}_1}{R_1} - x_1 + C \, \dot{x}_2 \right) \tag{38}$$

$$= -R_3 \, x_1 + \frac{R_3}{R_1} u - \frac{R_3}{R_1} L \left(\frac{-R_1 \, R_2}{L(R_1 + R_2)} x_1 - \frac{R_1}{L(R_1 + R_2)} x_2 + \frac{1}{L} u \right)$$

$$+ R_3 \, C \left\{ \frac{R_1}{C(R_1 + R_2)} x_1 - \left(\frac{1}{R_3 \, C} + \frac{1}{C(R_1 + R_2)} \right) x_2 + \frac{1}{R_3 \, C} u \right\} \tag{39}$$

$$= \left(-R_3 + \frac{R_2 \, R_3}{R_1 + R_2} + \frac{R_1 \, R_3}{R_1 + R_2} \right) x_1 + \left(\frac{R_3}{R_1 + R_2} - 1 - \frac{R_3}{R_1 + R_2} \right) x_2$$

$$+ \left(\frac{R_3}{R_1} - \frac{R_3}{R_1} + 1 \right) u \tag{40}$$

$$= -x_2 + u. \tag{41}$$

Examples of State Space Models (cont.)

(41) can also be obtained from the fact that

$$u = x_2 + R_3\, i_3. \tag{42}$$

The state variable equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1\, R_2}{L(R_1+R_2)} & -\frac{R_1}{L(R_1+R_2)} \\ \frac{R_1}{C(R_1+R_2)} & -\left(\frac{1}{R_3\, C} + \frac{1}{C(R_1+R_2)}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_3\, C} \end{bmatrix} u \tag{43}$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u. \tag{44}$$

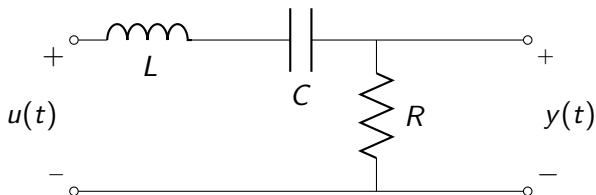# Examples of State Space Models (cont.)

Example (cont.)

The transfer function is

$$G(s) = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} s + \frac{R_1 R_2}{L(R_1+R_2)} & \frac{R_1}{L(R_1+R_2)} \\ -\frac{R_1}{C(R_1+R_2)} & s + \left(\frac{1}{R_3 C} + \frac{1}{C(R_1+R_2)} \right) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_3 C} \end{bmatrix} + 1 \tag{45}$$

Examples of State Space Models (cont.)

Example

Consider the system,

Examples of State Space Models (cont.)

By Kirchhoff current law, we have

$$x_1 = C \, \dot{x}_2. \tag{46}$$

By Kirchhoff voltage law, we have

$$u = L \, \dot{x}_1 + x_2 + R \, x_1. \tag{47}$$

The state variable form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \tag{48}$$

$$y = \begin{bmatrix} R & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u. \tag{49}$$

Examples of State Space Models (cont.)

Example (cont.)

The transfer function is

$$
\begin{aligned}
G(s) &= \begin{bmatrix} R & 0 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{L} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} + 0 \\
&= \frac{\begin{bmatrix} R & 0 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{L} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \\
&= \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.
\end{aligned}
\tag{50}
$$

Examples of State Space Models (cont.) Mechanical Systems

The common elements of mechanical systems are masses, springs and dampers. If a linear spring is stretched (compressed) from its relaxed position by a certain amount, it produces a restoring force $K\,x$ in the opposite direction. A **linear** damper as shown below produces a force $B\,\dot{x}$ opposing the motion,
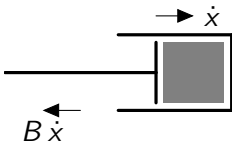

Figure 6: A linear damper (dashpot)

Examples of State Space Models (cont.) Mechanical Systems

with $\dot{x}$ being the relative velocity of the plunger with respect to the housing. The main law of physics governing the behavior of mechanical systems is Newton's equation of motion which can be stated as follows, for linear motion:

The algebraic sum of forces acting on a mass m, in the positive x direction $= m\ddot{x}$, where $\ddot{x}$ is the acceleration of m in the positive x direction.

A similar law holds for rotational motion with force, mass and linear acceleration replaced by torque, moment of inertia and angular acceleration, respectively.

Examples of State Space Models (cont.) Mechanical Systems

Example (Two input, two output mechanical system)
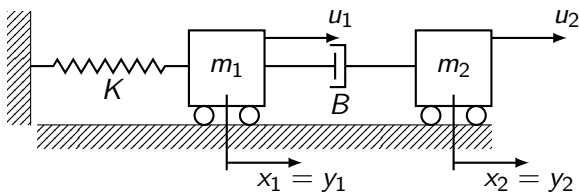
Consider the mechanical system below:

Figure 7: 2 input 2 output mechanical system I

Examples of State Space Models (cont.) Mechanical Systems

where x_1, x_2 denote the displacements from the equilibrium positions of the masses m_1 and m_2 respectively. A "free body" diagram of each mass is shown below:
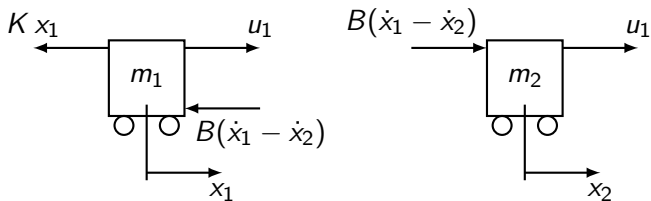

Figure 8: Free Body Diagrams

Examples of State Space Models (cont.) Mechanical Systems

Example (Two input, two output mechanical system (cont.))

From Figure 8, Newton's equations for the two masses are:

$$u_1 - K x_1 - B(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1 \tag{51a}$$

$$u_2 + B(\dot{x}_1 - \dot{x}_2) = m_2 \ddot{x}_2. \tag{51b}$$

To rewrite (51) in the standard first order state variable form we **introduce** two additional variables

$$\begin{aligned} x_3 &:= \dot{x}_1 \\ x_4 &:= \dot{x}_2. \end{aligned} \tag{52}$$

Examples of State Space Models (cont.) Mechanical Systems

Example (Two input, two output mechanical system (cont.))

We may now write the system equations in the standard first order state variable form,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/m_1 & 0 & -B/m_1 & B/m_1 \\ 0 & 0 & B/m_2 & -B/m_2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

(53)

with matrices $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ as shown.

Examples of State Space Models (cont.) Mechanical Systems

Example

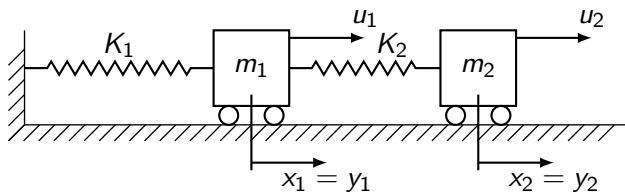Consider the system in Figure 9,

Figure 9: 2 input 2 output mechanical system II

Examples of State Space Models (cont.)
Mechanical Systems

Example (cont.)

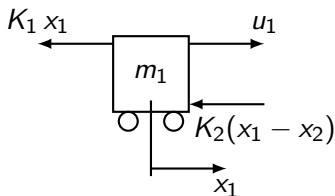The free body diagrams for m_1 and m_2 are:
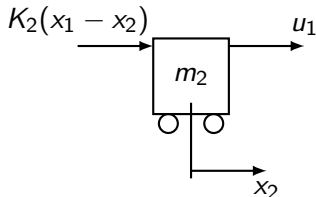

Figure 10: The free body diagram for m_1

Figure 11: The free body diagram for m_2

Examples of State Space Models (cont.) Mechanical Systems

Newton's equation of motion are:

$$u_1 - K_1 x_1 - K_2(x_1 - x_2) = m_1 \ddot{x}_1 \tag{54}$$

$$u_2 + K_2(x_1 - x_2) = m_2 \ddot{x}_2 \tag{55}$$

They can be cast in the standard state variable form:

$$\dot{x}_1 =: x_3 \tag{56}$$

$$\dot{x}_2 =: x_4 \tag{57}$$

$$\dot{x}_3 = -\frac{K_1 + K_2}{m_1}x_1 + \frac{K_2}{m_1}x_2 + \frac{1}{m_1}u_1 \tag{58}$$

$$\dot{x}_4 = \frac{K_2}{m_2}x_1 - \frac{K_2}{m_2}x_2 + \frac{1}{m_2}u_2 \tag{59}$$

Examples of State Space Models (cont.)
Mechanical Systems

Example (cont.)

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1+K_2}{m_1} & \frac{K_2}{m_1} & 0 & 0 \\ \frac{K_2}{m_2} & -\frac{K_2}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{60}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

Examples of State Space Models (cont.) Mechanical Systems
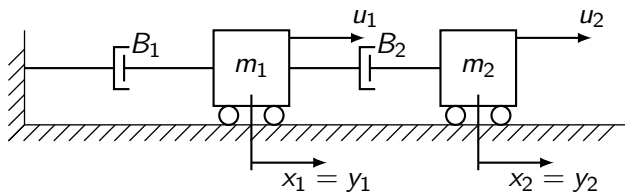
Example
Consider the system in Figure 12,

Figure 12: 2 input 2 output mechanical system III

Examples of State Space Models (cont.)
Mechanical Systems

Example (cont.)

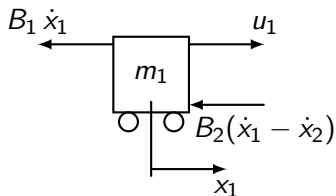The free body diagrams and Newton's equations are:

Figure 13: The free body diagram for m_1

$$u_1 - B_1 \dot{x}_1 - B_2(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1 \tag{61}$$

Examples of State Space Models (cont.)
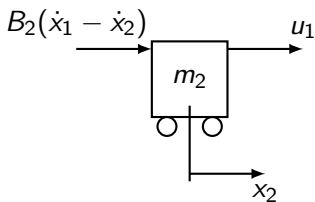## Mechanical Systems

Example (cont.)

Figure 14: The free body diagram for m_2

$$u_2 + B_2(\dot{x}_1 - \dot{x}_2) = m_2 \ddot{x}_2 \qquad (62)$$

Examples of State Space Models (cont.) Mechanical Systems

Example (cont.)

The standard state variable form is:

$$\dot{x}_1 =: x_3 \tag{63}$$

$$\dot{x}_2 =: x_4 \tag{64}$$

$$\dot{x}_3 = -\frac{B_1 + B_2}{m_1}x_3 + \frac{B_2}{m_1}x_4 + \frac{1}{m_1}u_1 \tag{65}$$

$$\dot{x}_4 = \frac{B_2}{m_2}x_3 - \frac{B_2}{m_2}x_4 + \frac{1}{m_2}u_2 \tag{66}$$

Examples of State Space Models (cont.)
Mechanical Systems

Example (cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{B_1+B_2}{m_1} & \frac{B_2}{m_1} \\ 0 & 0 & \frac{B_2}{m_2} & -\frac{B_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(67)

Examples of State Space Models (cont.) Electromechanical System

An armature controlled, separately excited DC motor is shown below driving a rotational load with a resistive load torque T_L.
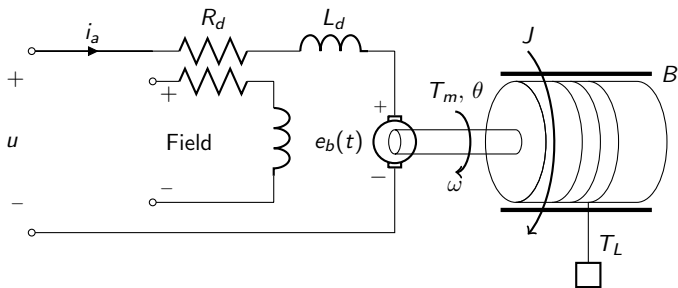

Figure 15: Separately excited DC motor

Examples of State Space Models (cont.) Electromechanical System

The armature resistance, inductance and current are denoted R_a, L_a and i_a, e_b denotes the back electromotive force (emf), T_m the motor torque, θ and ω the motor shaft angular position and velocity and J and B the moment of inertia of the load (reflected to the motor drive shaft) and the viscous damping constant "seen" by the load.

The motor equations are

$$T_m = K_t\, i_a \qquad (68a)$$
$$e_b = K_b\, \omega \qquad (68b)$$

where K_t, K_b are constants provided by the manufacturer. We write the system equations by summing voltages around the loop (electrical balance) and sum of mechanical torques resulting in acceleration (mechanical balance).

Examples of State Space Models (cont.) Electromechanical System

Electrical Balance

$$u = R_a \, i_a + L_a \, \dot{i}_a + e_b. \tag{69}$$

Mechanical Balance

$$T_m = T_L + B \, \dot{\theta} + J \, \ddot{\theta}. \tag{70}$$

Assuming that angular position and velocity are the outputs of interest we have

$$y_1 = \theta \tag{71a}$$

$$y_2 = \omega := \dot{\theta}. \tag{71b}$$

Examples of State Space Models (cont.) Electromechanical System

We may now choose $x_1 = \theta_1$, $x_2 = \omega$ and $x_3 = i_a$ as the state variables to obtain the DC motor model in the standard form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K_t/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -1/J \\ 1/L_a & 0 \end{bmatrix}}_{B} \begin{bmatrix} u \\ T_L \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} u \\ T_L \end{bmatrix}.$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart)

Consider the inverted pendulum below:
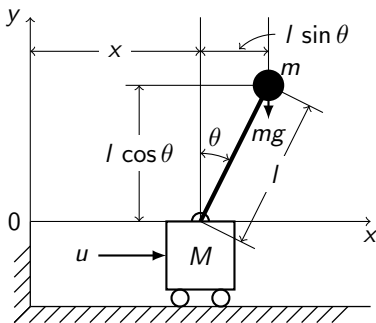

Figure 16: Inverted Pendulum

Examples of State Space Models (cont.)

There are two balance equations; 1) force balance and 2) torque balance. Firstly we consider the force balance. Since the force in the horizon is the force of the cart and the force of the bob, we have

$$M \frac{d^2}{d\,t^2} x + m \frac{d^2}{d\,t^2} x_m = u \tag{73}$$

where the coordinate of the point mass, bob, is given by (x_m, y_m). Using the reference coordinate, length of massless rod and the angle we have

$$\begin{aligned} x_m &= x + l \sin\theta \\ y_m &= l \cos\theta. \end{aligned} \tag{74}$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Thus, we can rewrite (73) as

$$M \frac{d^2}{d\,t^2}x + m\frac{d^2}{d\,t^2}(x + l\,\sin\theta) = u \qquad (75)$$

Noting that $\frac{d^2}{d\,t^2}\sin\theta = -(\sin\theta)(\dot\theta)^2 + (\cos\theta)\ddot\theta$ we get

$$(M + m)\,\ddot{x} - m\,l\,(\sin\theta)(\dot\theta)^2 + m\,l\,(\cos\theta)\ddot\theta = u. \qquad (76)$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Secondly, for the torque balance, refer to Figure 17 for the notation of directions.
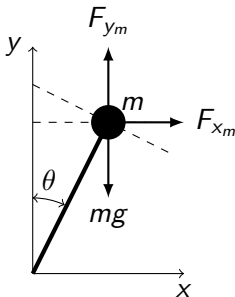

Figure 17: Torque directions

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Since the torque due to the acceleration force is balanced by the torque due to the gravity force we have

$$F_{x_m} \cos\theta - F_{y_m} \sin\theta = m\,g\,\sin\theta \tag{77}$$

where the forces are

$$
\begin{aligned}
F_{x_m} &= m\frac{d^2}{d\,t^2}x_m = m\left(\ddot{x} - l(\sin\theta)(\dot{\theta})^2 + l(\cos\theta)\ddot{\theta}\right), \\
F_{y_m} &= m\frac{d^2}{d\,t^2}y_m = -m\left(l(\cos\theta)(\dot{\theta})^2 + l(\sin\theta)\ddot{\theta}\right).
\end{aligned}
\tag{78}
$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Substituting (77) in (78) gives

$$
\begin{aligned}
m\,g\,&\sin\theta \\
&= F_{x_m}\cos\theta - F_{y_m}\sin\theta \\
&= m\left\{\ddot{x} - l\,(\sin\theta)(\dot{\theta})^2 + l\,(\cos\theta)\ddot{\theta}\right\}\cos\theta \\
&\quad - m\left\{l\,(\cos\theta)(\dot{\theta})^2 + l\,(\sin\theta)\ddot{\theta}\right\}\sin\theta \\
&= m\,\ddot{x}\,\cos\theta + m\,l\,\ddot{\theta}.
\end{aligned}
\tag{79}
$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Using small signal approximation about $\theta = 0$, we have $\cos\theta \cong 1$, $\sin\theta \cong \theta$, and $\dot{\theta}^2 \cong 0$; thus the inverted pendulum equations become

$$(M + m)\ddot{x} + m\,l\,\ddot{\theta} = u \tag{80}$$

$$m\,\ddot{x} + m\,l\,\ddot{\theta} = m\,g\,\theta. \tag{81}$$

Now we define the state vector as

$$z = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}. \tag{82}$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Since $\dot{z}$ has elements $\ddot{x}$ and $\ddot{\theta}$ in the vector we derive these expressions using (80) and (81). First, from (81), we have

$$m\, l\, \ddot{\theta} = m\, g\, \theta - m\, \ddot{x} \tag{83}$$

and substituting this into (80),

$$\begin{aligned}
&(M+m)\ddot{x} + m\, l\, \ddot{\theta} = u \\
\Leftrightarrow\ &(M+m)\ddot{x} + m\, g\, \theta - m\, \ddot{x} = u \\
\Leftrightarrow\ &M\, \ddot{x} + m\, g\, \theta = u \\
\Leftrightarrow\ &\ddot{x} = -\frac{m}{M}g\, \theta + \frac{1}{M}u.
\end{aligned} \tag{84}$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Next, substituting (84) back into (83) we get

$$m\,l\,\ddot{\theta} = m\,g\,\theta - m\,\ddot{x}$$

$$\Leftrightarrow l\,\ddot{\theta} = g\,\theta + \left\{\frac{m}{M}g\,\theta - \frac{1}{M}u\right\}$$

$$\Leftrightarrow \ddot{\theta} = \frac{g}{l}\,\theta + \frac{1}{l}\left\{\frac{m}{M}g\,\theta - \frac{1}{M}u\right\}$$

$$\Leftrightarrow \ddot{\theta} = \frac{g}{l}\left(1 + \frac{m}{M}\right)\theta - \frac{1}{l\,M}u. \tag{85}$$

Now fit these to the state space representation of the form:

$$\dot{z} = \mathbf{A}\,z + \mathbf{B}\,u \tag{86}$$

$$y = \mathbf{C}\,z + \mathbf{D}\,u. \tag{87}$$

Examples of State Space Models (cont.)

Example (Inverted Pendulum on a Cart (cont.))

Then, the final state space representation is

$$
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{l}(1 + \frac{m}{M}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -g\,\theta(\frac{m}{M}) & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1/l\,M \\ 0 \\ 1/M \end{bmatrix}}_{\mathbf{B}} u
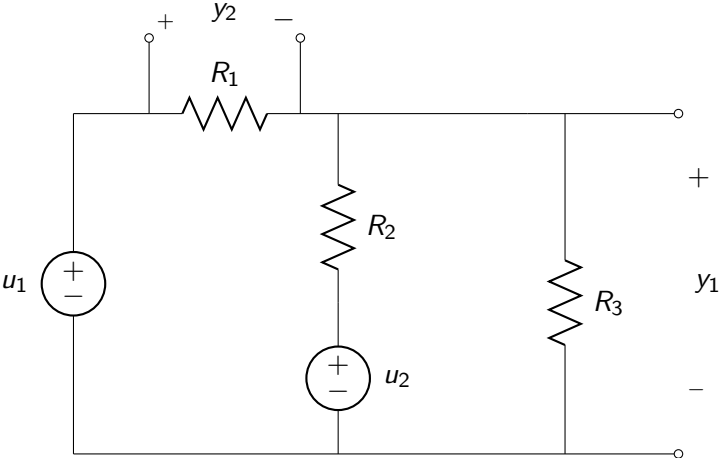$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{\mathbf{D}} u
$$

(88)

where

$$
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \dot{\mathbf{z}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \mathbf{z} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}.
$$

(89)

Exercise

Exercise 1

Exercise (cont.)

a) Write equations for the circuit in the matrix form :

$$
\begin{aligned}
\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} &= \mathbf{0}, \\
\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} &= \mathbf{y}.
\end{aligned}
\tag{90}
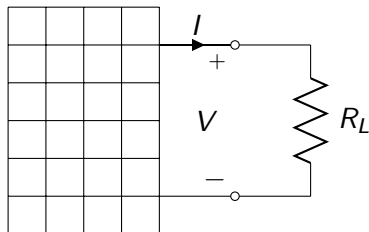$$

b) Solve for the gain matrix $\mathbf{G}$, where

$$
\mathbf{y} = \mathbf{G}\mathbf{u}
\tag{91}
$$

in terms of R_i, $i = 1, 2, 3$.

Exercise (cont.)

Exercise 2

A photovoltaic (PV) array is connected to a load R_L as shown:

Exercise (cont.)

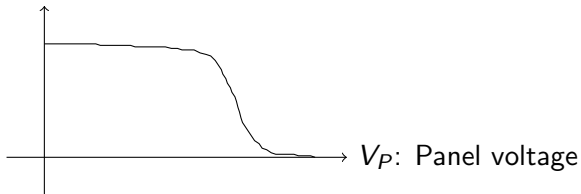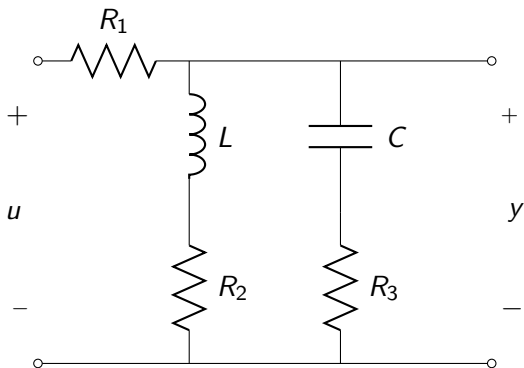The nonlinear $V - I$ characteristic of the PV panel is shown

I_P: Panel current

V_P: Panel voltage

Figure 18: $V - I$ characteristic of the PV array

a) Find the current and voltage in the circuit as a "function" of R_L.

b) Find R_L to obtain the maximum power output.

Exercise (cont.)

Exercise 3

Exercise (cont.)

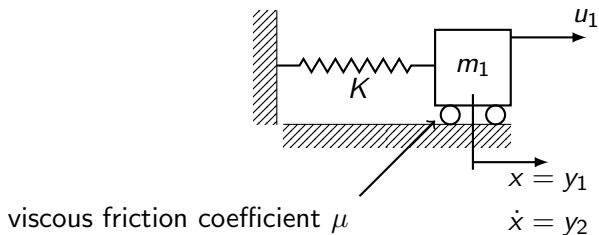a) Write state variable equations for the circuit in the standard form:
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \tag{92}$$

b) Find the transfer function using the state space model

c) Solve for the state $\mathbf{x}(t)$, and output $\mathbf{y}(t)$, assuming zero initial conditions on the capacitor voltage and inductor current, $u(t) = U(t)$ and $R_i = 1$, $i = 1, 2, 3$, $L = 1$, $C = 1$.

d) Verify that the "steady state" values for a constant input u satisfy
$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = \mathbf{0},$$
$$\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} = \mathbf{y}. \tag{93}$$

Exercise (cont.)

Exercise 4

viscous friction coefficient μ

$x = y_1$

$\dot{x} = y_2$

a) Write state variable equations for the mass-spring system.

b) Calculate the transfer function y_1/u and y_2/u using the state space model.

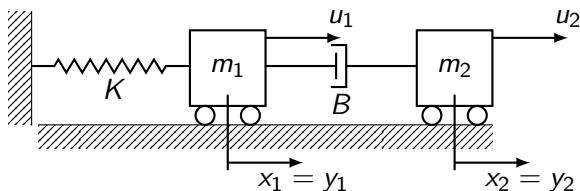c) Find the poles of the system (or eigenvalues of A) as a function of m, k, μ.

Exercise (cont.)

d) Solve for $y_1(t)$, $y_2(t)$, for zero initial condition and $u(t) = U(t)$, as functions of m, k, μ.

e) Comment on the solution in d) considering the cases $\mu \to 0$, and $K \to 0$.

Exercise (cont.)

Exercise 5

a) Write state variable equations for the system.

b) Find the transfer functions

$$\frac{Y_i(s)}{U_j(s)}, \quad i = 1, 2, \quad j = 1, 2. \tag{94}$$

c) Find the impulse response matrix $\mathbf{G}(t)$