ECEN 605

LINEAR SYSTEMS

Lecture 2

Laplace Transform I

Linear Time Invariant Systems

A general LTI system may be described by the linear constant coefficient differential equation:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t). \quad (1)$$

Introducing the differentiation operator

$$D^k \{f(t)\} := \frac{d^k f(t)}{dt^k} \quad (2)$$

(1) may be written as

$$a(D)\{y(t)\} = b(D)\{u(t)\} \quad (3)$$

Linear Time Invariant Systems (cont.)

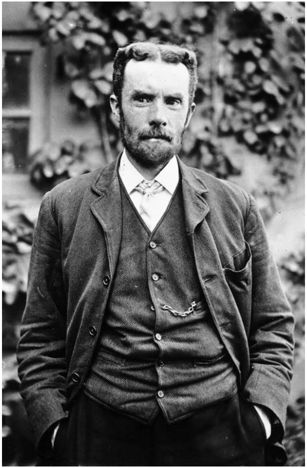
where

$$\begin{aligned} a(D) &= a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \\ b(D) &= b_m D^m + b_{m-1} D^{m-1} + \cdots + b_1 D + b_0. \end{aligned} \tag{4}$$

The solution of (1) or (3) is greatly facilitated by using the Laplace transform. Moreover several fundamental concepts of system theory such as transfer functions, poles and zeros, block diagram algebra, realization theory and frequency response are based on the Laplace Transform. The Laplace Transform was introduced by Pierre-Simon Laplace in 1787. However its use in Engineering was popularized by Oliver Heaviside, one hundred years later. The next few sections describe the Laplace transform and its applications to system analysis.

Linear Time Invariant Systems (cont.)

Pierre-Simon Laplace
(23 March 1749 $\sim$ 5 March 1827)



Oliver Heaviside
(18 May 1850 $\sim$ 3 February 1925)

Definition of the Laplace Transform

The Laplace transform of a function $f(t)$, with the property,

$$f(t) = 0, \quad t < 0 \tag{5}$$

is defined by

$$\mathcal{L}_{-}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st} dt \tag{6}$$

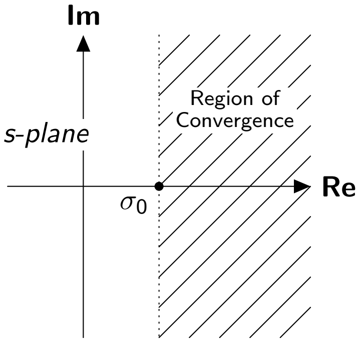
where t is real and s is complex. When (5) is satisfied we will say that $f(t)$ is **causal**. The Laplace transform of a function $f(t)$ **exists** if the integral in (6) exists for some values of s. For example if $f(t) = e^{\alpha t}$, α real, then (6) exists if $\mathbf{Re}\, s > \alpha$. In this case the complex plane region $\mathbf{Re}\, s > \alpha$ is called the region of convergence. More generally if σ_0 is the minimum real value, such that the integral

$$\int_{0^-}^{\infty} |f(t)|\, e^{-\sigma_0 t} dt \tag{7}$$

Definition of the Laplace Transform (cont.)

converges, that is, is finite, then the region

$$\mathbf{Re}\, s > \sigma_0 \tag{8}$$



Definition of the Laplace Transform (cont.)

is a **region of convergence**. If there exists a region of convergence, the Laplace transform is said to exist, otherwise it does not exist. The function e^{t^2}, for example, does not have a Laplace transform as there are no values of s for which (6) is finite.

Remark

The lower limit 0^- in the integral in (6) is used to accommodate, or allow, impulses at the origin as inputs to dynamic systems. In some books the value at 0^+ is also used with corresponding changes in the formulas.

The inverse Laplace transform can be shown to be

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds = f(t) \qquad (9)$$

where $\sigma > \sigma_0$ and is otherwise arbitrary. Although (9) holds true formally, the inverse Laplace transform is most commonly obtained from tables of Laplace transforms rather than the cumbersome

Definition of the Laplace Transform (cont.)

integration involved in evaluating the integral in (9). Implicitly this utilizes the uniqueness of the Laplace transform and its inverses. Thus we denote

$$f(t) \leftrightarrow F(s) \tag{10}$$

as a Laplace transform pair, with the understanding that $f(t)$ and $F(s)$ are alternative mathematical representations of the “same” signal.

Properties of the Laplace transform
1. Linearity

If $f(t)$ and $g(t)$ are causal signals with

$$\begin{aligned} f(t) &\leftrightarrow F(s) \\ g(t) &\leftrightarrow G(s) \end{aligned} \tag{11}$$

then, for arbitrary a, b real or complex,

$$af(t) + bg(t) \leftrightarrow aF(s) + bG(s). \tag{12}$$

Note however that, in general

$$f(t)g(t) \not\leftrightarrow F(s)G(s). \tag{13}$$

The proof of (12) is straightforward using the definition of the Laplace transform.

Properties of the Laplace transform
2. Time Shifting

Let $U(t)$ denote the unit step function:

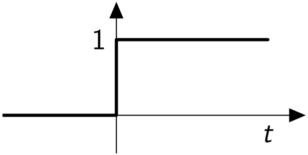


Figure: Unit Step $U(t)$

$$U(t) = \begin{cases} 1, t \geq 0, \\ \\ 0, t \leq 0^{-}. \end{cases} \quad (14)$$

Properties of the Laplace transform (cont.) 2. Time Shifting

Consider a causal function $f(t)\,U(t)$.

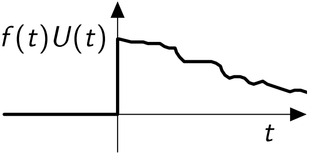


Figure: Causal function

Properties of the Laplace transform (cont.)
2. Time Shifting

If $f(t)$ is shifted to the right by T seconds, we obtain
$f(t-T)\,U(t-T) =: g(t)$, the shifted version of $f(t)$

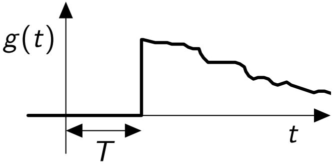


Figure: Shifted function

Properties of the Laplace transform (cont.) 2. Time Shifting

The Laplace transform of the shifted function $g(t)$ is:

$$G(s) = \int_{0^-}^{\infty} f(t-T)U(t-T)e^{-st}dt$$
$$= \int_{T}^{\infty} f(t-T)U(t-T)e^{-st}dt. \quad (15)$$

With $t - T =: \lambda$, we have

$$G(s) = e^{-s\,T} \int_{0^-}^{\infty} f(\lambda)u(\lambda)e^{-s\lambda}d\lambda$$
$$= e^{-s\,T}F(s). \quad (16)$$

Therefore, if

$$f(t)U(t) \leftrightarrow F(s), \quad (17)$$

then

$$f(t-T)U(t-T) \leftrightarrow e^{-s\,T}F(s). \quad (18)$$

Properties of the Laplace transform
3. Differentiation

If $f(t)$ is a continuous signal and $g(t) = \dfrac{d\,f(t)}{dt}$ its derivative, the Laplace transform of $g(t)$ is:

$$\begin{aligned} G(s) &= \int_{0^-}^{\infty} \frac{d\,f(t)}{dt} e^{-s\,t} dt \\ &= f(t)\ e^{-s\,t}\Big|_{0^-}^{\infty} - (-s) \underbrace{\int_{0^-}^{\infty} f(t) e^{-st} dt}_{F(s)} \\ &= -f(0^-) + sF(s). \end{aligned} \tag{19}$$

Therefore

$$\mathcal{L}_- \left\{ \frac{d\,f(t)}{dt} \right\} = sF(s) - f(0^-). \tag{20}$$

Properties of the Laplace transform (cont.) 3. Differentiation

From (20), it follows that

$$\mathcal{L}_-\left\{\frac{d^2 f(t)}{dt^2}\right\} = s\mathcal{L}_-\left\{\frac{d\,f(t)}{dt}\right\} - \dot{f}(0^-)$$
$$= s^2F(s) - sf(0^-) - \dot{f}(0^-) \qquad (21)$$

and similarly, it follows recursively that

$$\mathcal{L}_-\left\{\frac{d^k f(t)}{dt^k}\right\} = s^kF(s) - s^{k-1}f(0^-) - s^{k-2}\dot{f}(0^-)$$
$$- \cdots - \frac{d^{k-1}f(0^-)}{dt^{k-1}} \quad \text{for } k = 0,\ 1,\ 2,\ \cdots. \qquad (22)$$

Properties of the Laplace transform
4. Final Value Theorem

If $f(t) \leftrightarrow F(s)$ and if $f(\infty)$ exists, then we show below that

$$f(\infty) = \lim_{s \to 0} s\, F(s). \tag{23}$$

To prove (23) consider

$$\mathcal{L}_{-}\left\{\frac{d\, f(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{d\, f(t)}{dt} e^{-s\,t} dt = sF(s) - f(0^-) \tag{24}$$

so that

$$\int_{0^-}^{\infty} \frac{d\, f(t)}{dt} dt = \lim_{s \to 0} s\, F(s) - f(0^-) \tag{25}$$

and

$$f(\infty) - f(0^-) = \lim_{s \to 0} s\, F(s) - f(0^-) \tag{26}$$

which implies (23).
Note that it is important to be sure that $f(\infty)$ exists otherwise the right hand side of (23) may yield a value that is not equal to $f(\infty)$. To see this, consider the following example.

Properties of the Laplace transform (cont.)
4. Final Value Theorem

Example

If $f(t) = e^t$, $F(s) = \dfrac{1}{s-1}$ and

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s-1} = 0 \tag{27}$$

whereas

$$\lim_{t\to\infty} e^t = \infty. \tag{28}$$

Properties of the Laplace transform
5. Initial Value Theorem

If $f(t) \leftrightarrow F(s)$ is a Laplace transform pair the initial value theorem states that

$$f(0^+) = \lim_{s \to \infty} s\, F(s) \tag{29}$$

provided the left hand side of (29) exists. To prove (29) note that the Laplace transform of $\dfrac{d\, f(t)}{dt}$ is:

$$\begin{aligned} s\, F(s) - f(0^-) &= \int_{0^-}^{\infty} \frac{d\, f(t)}{dt}\, e^{-s\,t}\, dt \\ &= \int_{0^-}^{0^+} \frac{d\, f(t)}{dt}\, e^{-s\,t}\, dt + \int_{0^+}^{\infty} \frac{d\, f(t)}{dt}\, e^{-s\,t}\, dt \\ &= f(0^+) - f(0^-) + \int_{0^+}^{\infty} \frac{d\, f(t)}{dt}\, e^{-s\,t}\, dt \end{aligned} \tag{30}$$

Properties of the Laplace transform (cont.) 5. Initial Value Theorem

so that

$$s\,F(s) = f(0^+) + \int_{0^+}^{\infty} \frac{d\,f(t)}{dt}\, e^{-s\,t}\, dt. \qquad (31)$$

Taking limits in (31) as $s \to \infty$, we have

$$\begin{aligned} \lim_{s\to\infty} s\,F(s) &= f(0^+) + \int_{0^+}^{\infty} \frac{df}{dt} \left(\lim_{s\to\infty} e^{-st} \right) dt \\ &= f(0^+) \qquad (32) \end{aligned}$$

proving (29). Note that if $F(s)$ is rational the right hand side exists only if $F(s)$ is strictly proper.

Properties of the Laplace transform (cont.) 5. Initial Value Theorem

Example

If

$$F(s) = \frac{1}{s} + \frac{1}{s-1} \tag{33}$$

we have

$$s\,F(s) = 1 + \frac{s}{s-1} \tag{34}$$

$$\lim_{s\to\infty} s\,F(s) = 2 \tag{35}$$

and

$$f(t) = U(t) + e^{t} \tag{36}$$

giving

$$f(0^{+}) = 1 + 1 = 2 \tag{37}$$

verifying (29).

Properties of the Laplace transform
6. Convolution

Let

$$f_1(t) \leftrightarrow F_1(s)$$
$$f_2(t) \leftrightarrow F_2(s) \tag{38}$$

then

$$F_1(s)\,F_2(s) = \int_{0^-}^{\infty} \left\{ \int_{0^-}^{t} f_1(\tau)\,f_2(t-\tau)\,d\tau \right\} e^{-s\,t}\,dt. \tag{39}$$

The function

$$g(t) := \int_{0^-}^{t} f_1(\tau)\,f_2(t-\tau)\,d\tau \tag{40}$$

is called the **convolution** of $f_1(t)$ and $f_2(t)$, and is usually denoted as $f_1(t) * f_2(t)$. Thus, (39) states that

$$\mathcal{L}_- \{f_1(t) * f_2(t)\} = \mathcal{L}_- \{g(t)\} = F_1(s)\,F_2(s). \tag{41}$$

Properties of the Laplace transform (cont.) 6. Convolution

Proof.
To prove (41) observe that, since $f_1(t-\tau) = 0$ for $\tau > t$,

$$\begin{aligned} \mathcal{L}_-\{g(t)\} =: G(s) &= \int_{0-}^{\infty} \left(\int_{0-}^{t} f_1(t-\tau)\, f_2(\tau)\, d\tau \right) e^{-s\,t}\, dt \\ &= \int_{0-}^{\infty} \left(\int_{0-}^{t} f_1(t-\tau)\, e^{-s\,t}\, dt \right) f_2(\tau)\, d\tau. \end{aligned} \tag{42}$$

Now

$$\int_{0-}^{\infty} f_1(t-\tau)\, e^{-s\,t}\, dt = e^{-s\,\tau}\, F_1(s) \tag{43}$$

so that substituting (43) in (42)

$$\begin{aligned} G(s) &= \int_{0-}^{\infty} e^{-s\,\tau}\, F_1(s)\, f_2(\tau)\, d\tau \\ &= F_1(s) \int_{0-}^{\infty} f_2(\tau)\, e^{-s\,\tau}\, d\tau \\ &= F_1(s)\, F_2(s) \end{aligned} \tag{44}$$

proving (41). □

Properties of the Laplace transform (cont.) 6. Convolution

Example

If $f_1(t) = e^t$ and $f_2(t) = e^{-t}$ we have

$$\begin{aligned} f_1(t) * f_2(t) &= \int_0^t e^{\tau}\, e^{-(t-\tau)}\, d\tau \\ &= \left[\int_0^t e^{2\,\tau}\, d\tau\right] e^{-t} \\ &= \left[\left.\frac{e^{2\,\tau}}{2}\right|_0^t \right] e^{-t} \\ &= \left[\frac{1}{2}e^{2\,t} - \frac{1}{2}\right] e^{-t} \\ &= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}. \end{aligned} \tag{45}$$

Properties of the Laplace transform (cont.) 6. Convolution

Example (cont.)

On the other hand

$$\begin{aligned}
\mathcal{L}_-\left\{\frac{1}{2}e^{t}-\frac{1}{2}e^{-t}\right\} &= \frac{1}{2}\left[\frac{1}{s-1}-\frac{1}{s+1}\right] \\
&= \frac{1}{2}\frac{2}{(s+1)(s-1)} \\
&= \frac{1}{s+1}\,\frac{1}{s-1} \\
&= \mathcal{L}_-\left\{f_1(t)\right\}\mathcal{L}_-\left\{f_2(t)\right\} \\
&= F_1(s)\,F_2(s)
\end{aligned} \tag{46}$$

verifying (39).

Properties of the Laplace transform (cont.) 6. Convolution

Example

Let $f_1(t) = \delta(t)$, and let $f_2(t)$ be arbitrary. Then

$$\begin{aligned} f_1(t) * f_2(t) &= \mathcal{L}_-^{-1}\{F_1(s)\,F_2(s)\} \\ &= \mathcal{L}_-^{-1}\{1 \cdot F_2(s)\} \\ &= f_2(t) \end{aligned} \tag{47}$$

The left hand side of (47) is

$$\int_0^t \delta(\tau)\, f_2(t-\tau)\, d\tau = f_2(t). \tag{48}$$

Thus, convolution with a unit impulse **leaves the function unchanged**.

Properties of the Laplace transform (cont.) 6. Convolution

Example

Let $f_1(t) = U(t)$(unit step) and $f_2(t)$ be arbitrary. Then

$$
\begin{aligned}
f_1(t) * f_2(t) &= \int_0^t U(\tau)\, f_2(t-\tau)\, d\tau \\
&= \int_0^t 1\, f_2(t-\tau)\, d\tau \\
&= \int_t^0 f_2(\lambda)\,(-d\lambda) \\
&= \int_0^t f_2(\lambda)\, d\lambda. \qquad (49)
\end{aligned}
$$

Thus convolution with a unit step is equivalent to integrating the function for $t \geq 0$.

Properties of the Laplace transform
Summary

Property	Time Domain	Laplace Domain
Linearity	$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$
Time Shifting	$f(t-T)\,U(t-T)$	$e^{-s\,T}\,F(s)$
Differentiation	$\dfrac{d^k\,f(t)}{dt^k}$	$s^k\,F(s) - s^{k-1} f(0^-) - \cdots - \dfrac{d^{k-1}\,f(0^-)}{dt^{k-1}}$
Final Value	$f(\infty)$	$\lim_{s\to 0} s\,F(s)$
Initial Value	$f(0^+)$	$\lim_{s\to\infty} s\,F(s)$
Convolution	$f(t) * g(t)$	$F(s)\,G(s)$

Laplace Transforms of Some Common Signals
1. Step

A step of height A can be written as $A\,U(t)$.

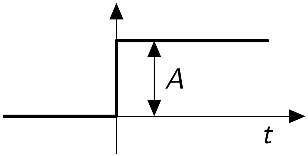


Figure: A step function

The Laplace transform of $A\,U(t)$ is

$$\begin{aligned}\mathcal{L}_{-}\left\{A\,U(t)\right\} &= \int_{0^-}^{\infty} Ae^{-st}dt \\ &= A\left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} \\ &= \frac{A}{s}.\end{aligned} \tag{50}$$

Laplace Transforms of Some Common Signals
2. Ramp

A ramp signal with slope R is shown below

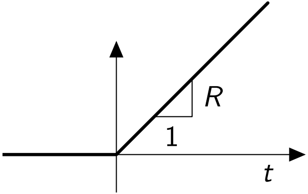


Figure: A ramp function

and can be written as $R\,t\,U(t)$.

$$\mathcal{L}_{-}\left\{R\,t\,U(t)\right\} = R\int_{0^-}^{\infty} t\,U(t)\,e^{-st}\,dt$$
$$= R\,t\,\frac{e^{-st}}{-s}\Bigg|_{0}^{\infty} + \frac{R}{s}\int_{0^-}^{\infty} e^{-st}\,dt = \frac{R}{s^2}. \qquad (51)$$

Laplace Transforms of Some Common Signals
3. Exponential

If $f(t) = e^{\alpha\, t}\, U(t)$

$$\mathcal{L}_{-}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{\alpha t}\, e^{-st}\, dt$$
$$= \frac{1}{s-\alpha} = F(s). \tag{52}$$

We note that α in (52) may be real or complex and this fact can be utilized to obtain the Laplace transform of exponentially weighted sinusoids. For instance, consider

$$\begin{aligned} f(t) &= e^{\sigma\, t}\, cos\, \omega\, t \\ &= e^{\sigma\, t}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] \\ &= \frac{1}{2}\left[e^{(\sigma + j\omega)\, t} + e^{(\sigma - j\omega)\, t}\right]. \end{aligned} \tag{53}$$

Laplace Transforms of Some Common Signals (cont.)
3. Exponential

Using (52) with $\alpha = \sigma + j\omega$ and $\alpha = \sigma - j\omega$ respectively, we get

$$F(s) = \frac{1}{2}\left[\frac{1}{s-\sigma-j\omega} + \frac{1}{s-\sigma+j\omega}\right] = \frac{(s-\sigma)}{(s-\sigma)^2+\omega^2}. \quad (54)$$

Similarly, it can be shown that

$$\mathcal{L}_-\left\{e^{\sigma t}\sin\omega\, t\right\} = \frac{\omega}{(s-\sigma)^2+\omega^2}. \quad (55)$$

Laplace Transforms of Some Common Signals
4. Time weighted exponential

Consider the function

$$f(t) = t\,e^{\alpha t}\,U(t). \tag{56}$$

When α is real and $\alpha = -2$, for example,

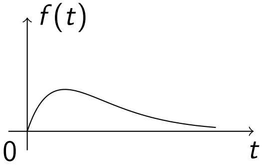


Figure: $f(t) = t\,e^{-2\,t}\,U(t)$

Laplace Transforms of Some Common Signals (cont.)
4. Time weighted exponential

The Laplace transform of $f(t)$ is:

$$
\begin{aligned}
F(s) &= \int_{0^-}^{\infty} t\, e^{\alpha t}\, e^{-st}\, dt \\
&= \int_{0^-}^{\infty} t\, e^{-(s-\alpha)t}\, dt \\
&= \left. \frac{t\, e^{-(s-\alpha)t}}{-(s-\alpha} \right|_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-(s-\alpha)t}}{-(s-\alpha)}\, dt \\
&= 0 - 0 + \frac{1}{s-\alpha} \int_{0^-}^{\infty} e^{-(s-\alpha)t}\, dt \\
&= \frac{1}{(s-\alpha)^2} \qquad (57)
\end{aligned}
$$

Laplace Transforms of Some Common Signals (cont.)
4. Time weighted exponential

Similarly it can be shown that

$$\mathcal{L}_{-}\left\{t^2 e^{\alpha t}\right\} = \frac{2!}{(s-\alpha)^3} \tag{58}$$

and, indeed, by induction, that

$$\mathcal{L}_{-}\left\{t^k\, e^{\alpha t}\right\} = \frac{k\,!}{(s-\alpha)^{k+1}}, \quad k = 0,\, 1,\, 2,\, \cdots . \tag{59}$$

As an application of (56) consider the function

$$\begin{aligned} f(t) &= (t \cos \omega\, t)\; U(t) \\ &= \left(\frac{t}{2} e^{j\omega t} + \frac{t}{2} e^{-j\omega t}\right)\; U(t) \end{aligned} \tag{60}$$

when $\omega = 1$, for example,

Laplace Transforms of Some Common Signals (cont.)
4. Time weighted exponential

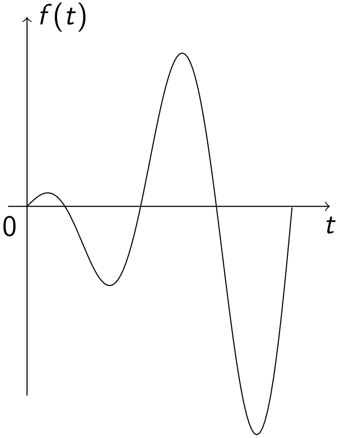


Figure: $f(t) = (t\, cos\, t)\ U(t)$

Laplace Transforms of Some Common Signals (cont.)
4. Time weighted exponential

so that

$$F(s) = \frac{1}{2}\left[\frac{1}{(s-j\omega)^2} + \frac{1}{(s+j\omega)^2}\right] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}. \qquad (61)$$

Similarly, it can be seen that

$$\begin{aligned} \mathcal{L}_-\left\{(t\,sin\,\omega\,t)\,U(t)\right\} &= \mathcal{L}_-\left\{\left(\frac{t\,e^{j\omega t}}{2j} - \frac{t\,e^{-j\omega t}}{2j}\right)U(t)\right\} \\ &= \frac{1}{2j}\left[\frac{1}{(s-j\omega)^2} - \frac{1}{(s+j\omega)^2}\right] \\ &= \frac{2\,\omega\,s}{(s^2+\omega^2)^2}. \qquad (62) \end{aligned}$$

The above formulas are related to the phenomena of resonance which occurs when a system with $j\,\omega$ axis poles is excited by a sinusoidal signal of the same frequency.

Laplace Transforms of Some Common Signals
5. Pulse

Consider the pulse shown below.

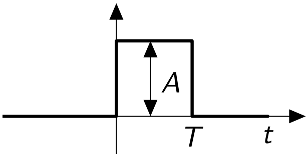


Figure:

It is described by

$$f(t) = A\left[U(t) - U(t-T)\right] \tag{63}$$

and we have

$$\mathcal{L}_{-}\left\{f(t)\right\} =: F(s) = \frac{A}{s}\left[1 - e^{-sT}\right]. \tag{64}$$

Laplace Transforms of Some Common Signals
6. Impulse

Consider the pulse of area $A\,T$

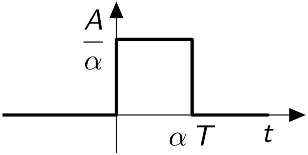
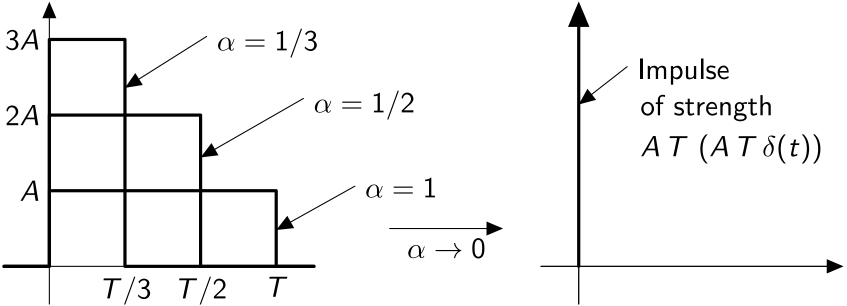


Figure: Pulse with area $A\,T$

and the family of pulses of height $\dfrac{A}{\alpha}$ and width αT with α decreasing from $1 \downarrow 0$.

Laplace Transforms of Some Common Signals (cont.)

6. Impulse



Laplace Transforms of Some Common Signals (cont.)
6. Impulse

The Laplace transform of a typical pulse in this sequence is:

$$\begin{aligned}
&\mathcal{L}_{-}\left\{\frac{A}{\alpha}\,[U(t) - U(t-\alpha\,T)]\right\} \\
&= \frac{A}{\alpha}\frac{1}{s}\left[1 - e^{-\alpha\,s\,T}\right] \\
&= \frac{A}{\alpha\,s}\left[1 - 1 + \alpha\,T\,s + \frac{\alpha^2\,T^2\,s^2}{2!} - \frac{\alpha^3\,T^3\,s^3}{3!} + \cdots\right] \\
&= A\,T + \frac{A}{2!}\,T^2\,\alpha\,s - \frac{A}{3!}\,T^3\,\alpha^2\,s^2 + \cdots \qquad (65)
\end{aligned}$$

As $\alpha \to 0$, we obtain an impulse of strength $A\,T$:

$$\lim_{\alpha\to 0}\frac{A}{\alpha}\,[U(t) - U(t-\alpha\,T)] := A\,T\,\delta(t). \qquad (66)$$

Laplace Transforms of Some Common Signals (cont.)
6. Impulse

From (65), with $\alpha \to 0$, we have

$$\mathcal{L}_- \{A\,T\,\delta(t)\} = A\,T. \tag{67}$$

$\delta(t)$ denotes an impulse of unit strength and we have from (67), with $A\,T = 1$,

$$\mathcal{L}_- \{\delta(t)\} = 1. \tag{68}$$

From the above analysis it is easy to see that

$$\begin{aligned}
&\int_{0^-}^{\infty} f(t)\,\delta(t - t_0)dt \\
&\quad = \lim_{T \to 0} \int_{t_0}^{t_0+T} f(t) \left[\frac{U(t_0) - U(t_0 - T)}{T} \right] dt \\
&\quad = f(t_0),
\end{aligned} \tag{69}$$

which is called the **time-sifting property** of the impulse function.

Laplace Transforms of Some Common Signals
7. Periodic Functions

Consider the periodic function

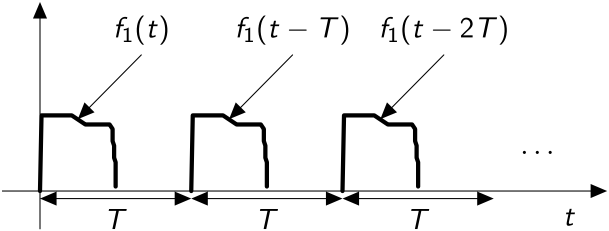


Figure: A periodic function

Laplace Transforms of Some Common Signals (cont.)
7. Periodic Functions

which can be described as

$$f(t) = f_1(t) + f_1(t - T)U(t - T) + f_1(t - 2T)U(t - 2T) + \cdots + f_1(t - kT)U(t - k\,T) + \cdots . \quad (70)$$

Then

$$\begin{aligned} F(s) &= F_1(s) + e^{-s\,T}F_1(s) + e^{-2\,s\,T}F_1(s) + \cdots + e^{-k\,T\,s}F_1(s) + \cdots \\ &= F_1(s)\left[1 + e^{-s\,T} + e^{-2\,s\,T} + \cdots + e^{-k\,T\,s} + \cdots\right] \\ &= F_1(s)\frac{1}{1 - e^{-s\,T}}. \end{aligned} \quad (71)$$

Laplace Transforms of Some Common Signals (cont.)
7. Periodic Functions

Example

Consider the periodic triangular wave

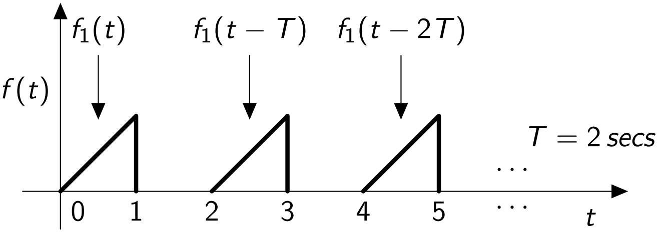


Figure: Periodic triangular wave

Laplace Transforms of Some Common Signals (cont.)
7. Periodic Functions

Example (cont.)

We write
$f(t) = f_1(t) + f_1(t - T)U(t - T) + f_1(t - 2T)U(t - 2T) + \cdots,$
where

$$\begin{aligned} f_1(t) &= t\,[U(t) - U(t-1)] \\ &= t\,U(t) - t\,U(t-1). \end{aligned} \tag{72}$$

To Laplace transform (72) we rewrite it as

$$f_1(t) = t\,U(t) - (t-1)\,U(t-1) - U(t-1) \tag{73}$$

so that

$$F_1(s) = \left[\frac{1}{s^2}\left[1 - e^{-s}\right] - e^{-s}\frac{1}{s}\right]. \tag{74}$$

Laplace Transforms of Some Common Signals (cont.)
7. Periodic Functions

Example (cont.)

From (71) it follows that

$$\mathcal{L}_{-}\left\{f(t)\right\} =: F(s) = F_1(s)\frac{1}{1-e^{-2\,s}} \tag{75}$$

with $F_1(s)$ given by (74).

Laplace Transforms of Some Common Signals
Summary

Function	Symbol	Laplace Transform
Unit Step	$U(t)$	$\frac{1}{s}$
Unit Impulse	$\delta(t)$	1
Ramp	t	$\frac{1}{s^2}$
Exponential	$e^{\alpha t}\, U(t)$	$\frac{1}{s-\alpha}$
Time Weighted Exponential	$t^k\, e^{\alpha t}\, U(t)$	$\frac{k!}{(s-\alpha)^{k+1}}$
Pulse	$A\,[U(t)-U(t-T)]$	$\frac{A}{s}\left(1-e^{-sT}\right)$
Periodic Functions	$f_1(t) + f_1(t-T) + f_1(t-2T) + \cdots$	$F_1(s)\frac{1}{1-e^{-sT}}$

Exercises

Exercise 1

Find the region of convergence of the Laplace transforms of the following causal functions:

a) $e^t + e^{2\,t} + e^{3\,t}$

b) $sin\,t$

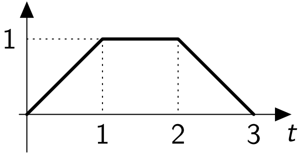
c) $U(t) - U(t-1)$ ($U(t)$: unit step.)

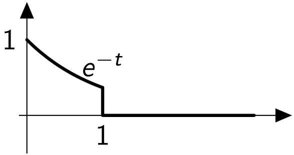
d) $t^5\,e^t$.

Exercises

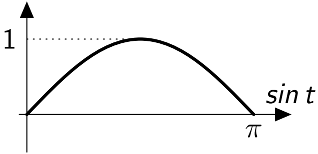
Exercise 2

Find the Laplace transforms of the functions:

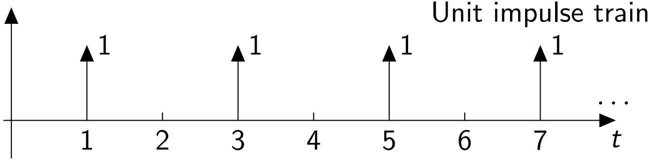
a) 

b) 

Exercises (cont.)



c)



d)

Exercises

Exercise 3

Find the inverse Laplace transforms of:

a) $\frac{1}{s}\left[1 - e^{-s}\right]$

b) $\frac{1}{(1 - e^{-s})}$

c) $\frac{e^{-2\,s}}{(s+1)(s+2)}$

d) $\frac{s^3}{(s+1)^2}$.

Exercises

Exercise 4

Find the Laplace transforms of:

a) $(e^t)^2$

b) $\frac{1}{(e^t)^2}$

c) $sin^2 t$

d) $t^2 cos^2 t$

e) $sin\, t\, cos\, t$

f) $t\, sin\, t\, cos\, t$.