Chapter 13

ROBUST PARAMETRIC STABILIZATION

This chapter discusses the problem of robust stabilization of plants subject to parametric uncertainty. First, we consider a family of minimum phase plant transfer functions of denominator degree n and numerator degree m. We show that a single feedback compensator of order $n-m-1$ which is stable and minimum phase can always be designed to robustly (simultaneously) stabilize this family under *arbitrarily* large perturbations in the numerator and denominator coefficients. Next we show that it is sometimes possible to use standard synthesis techniques of H_∞ optimal control to design a robustly stabilizing compensator. The obvious technique is to overbound the frequency domain image sets induced by the parametric uncertainty with H_∞ norm bounded uncertainty. This is made possible in the case of linear interval plants by exploiting the extremal property of the generalized Kharitonov segments.We show this by simple numerical examples for single-input single-output interval plants. If the sizes of the intervals are not fixed *apriori*, this technique can always be used to find an appropriate size of interval perturbations for which robust stabilization is possible. The issue of robust performance is not explicitly dealt with but can be handled in the same manner, when it is measured in the H_∞ norm.

13.1 INTRODUCTION

The techniques described thus far are mostly directed towards *analyzing* the robust stability of control systems under various types of uncertainty. The underlying assumption in this type of analysis, is that a controller has already been designed by some means. The purpose of calculating worst case stability and performance margins is therefore to *evaluate* the proposed controller against a competing design. In practice, this analysis tool can itself serve as the basis of controller *design* as shown in various examples.

A different approach to robust controller design is to prespecify a desired level of performance or stability margin and attempt to find a controller from the entire set of stabilizing controllers that attains this objective. This is usually referred to as *synthesis*.

In *optimal* synthesis, one searches for the best controller that optimizes a single criterion of performance. The optimization problem must be formulated so that it is both physically meaningful as well as mathematically tractable. An important optimal synthesis problem closely related to robust control is the H_∞ control problem. Here, one attempts to minimize the H_∞ norm of a "disturbance" transfer function over the set of all stabilizing controllers. An elegant theory of H_∞ based synthesis has been developed. It can be used to design for robust stability and performance by minimizing the norms of suitable transfer functions. The underlying theory is based on the Small Gain Theorem which provides for robust stability under norm bounded perturbations.

At the present time there does not exist a comprehensive and nonconservative theory of synthesis for systems subject to real parameter uncertainty. However partial results are available and design techniques can be developed based on what has been learnt so far. The purpose of this chapter is to explore these ideas in an elementary way.

In the next section we describe a procedure to synthesize a fixed order controller that is guaranteed to robustly stabilize a prescribed family of single-input, single-output minimum phase plants. The parametric perturbations affecting the plant are otherwise essentially *arbitrary*. Moreover, it is shown that the controller itself can always be stable and minimum phase and the order of the controller need be no higher than $n - m - 1$, where the plant has n poles and m zeroes.

Then we describe how H_∞ theory might be exploited to design controllers that provide robustness against parameter uncertainty. The essential link with the parametric theory is that we are able to "tightly" fit the level of unstructured disc-like uncertainty required to cover the actual parametric uncertainty. Once this is done, the H_∞ machinery can take over to produce an answer. We describe the approach using Nevanlinna-Pick interpolation as well as the state space formulation of H_∞ theory with illustrative numerical examples of robust parametric stabilization.

13.2 SIMULTANEOUS STRONG STABILIZATION

In this section stability will mean Hurwitz stability. We show how an infinite family of single input single output (SISO) plants can be simultaneously stabilized by a single stable, minimum phase controller, provided that each member in the family represents a minimum phase plant. Consider the standard unity feedback system of Figure 13.1, where the plant is a single input single output system described by the transfer function

$$G(s) = \frac{n(s)}{d(s)} = \frac{n_0 + n_1 s + \ldots + n_r s^r}{d_0 + d_1 s + \ldots + d_q s^q}.$$

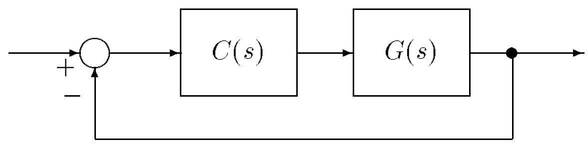

Figure 13.1. Unity feedback system

Let $\mathcal{F}_n$ be a compact set of polynomials $n(s)$ satisfying the following three properties.

Property 13.1.

A1) for all $n(\cdot) \in \mathcal{F}_n$, $n(\cdot)$ is stable.

A2) for all $n(\cdot) \in \mathcal{F}_n$, $n(\cdot)$ is of degree r (fixed degree).

A3) The sign of the highest coefficient of any polynomial $n(\cdot)$ in $\mathcal{F}_n$ is always the same, either always positive or always negative.

Let also $\mathcal{F}_d$ be a family of polynomials satisfying the following three properties:

Property 13.2.

B1) for all $d(\cdot) \in \mathcal{F}_d$, $d(\cdot)$ is of degree q (fixed degree).

B2) $\mathcal{F}_d$ is bounded, that is there exists a constant B such that:

$$\text{for all } d(\cdot) \in \mathcal{F}_d, \quad \text{for all } j \in [0, q], \quad |d_j| \leq B.$$

B3) The coefficient of order q of any polynomial $d(\cdot)$ in $\mathcal{F}_d$ is always of the same sign and bounded from below (or from above). That is,

$$\exists\, b > 0 \text{ such that for all } d(\cdot) \in \mathcal{F}_d, \quad d_q > b > 0,$$

or

$$\exists\, b < 0 \text{ such that for all } d(\cdot) \in \mathcal{F}_d, \quad d_q < b < 0.$$

Now, assuming that $r \leq q$, consider the family $\mathcal{P}$ of proper SISO plants described by their transfer functions

$$\mathcal{P} := \left\{ G(s) = \frac{n(s)}{d(s)}, \text{ where } n(s) \in \mathcal{F}_n, \ d(s) \in \mathcal{F}_d \right\}.$$

Then we have the following result.

Theorem 13.1 (Simultaneous Strong Stabilization)

i) $q = r$: There exists a constant compensator that stabilizes the entire family of plants $\mathcal{P}$.

ii) $q > r$: There exists a proper, stable and minimum phase compensator $C(s)$ of order $q - r - 1$ that stabilizes the entire family of plants $\mathcal{P}$.

First, we can assume without loss of generality that we have,

$$\text{for all } \ n(\cdot) \in \mathcal{F}_n, \quad n_r > 0, \tag{13.1}$$

and

$$\text{for all } \ d(\cdot) \in \mathcal{F}_d, \quad d_q > 0.$$

Then we can also assume, still without loss of generality, that the family $\mathcal{F}_d$ is itself compact, otherwise it would be enough to replace $\mathcal{F}_d$ by the family of interval polynomials $\mathcal{F}'_d$ defined by:

$$d_0 \in [-B, B], \ d_1 \in [-B, B], \ \cdots, \ d_{q-1} \in [-B, B], \ d_q \in [b, B].$$

Given this, the proof of this result now depends on some general properties of such compact stable families as $\mathcal{F}_n$, and of such compact families as $\mathcal{F}_d$.

Property 13.3. Since the family $\mathcal{F}_n$ contains only stable polynomials, and since they all satisfy Property A3 and (13.1), then any coefficient of any polynomial $n(\cdot)$ in $\mathcal{F}_n$ is positive. Moreover, since the set $\mathcal{F}_n$ is compact it is always possible to find two constants a and A such that,

$$\text{for all } \ n(\cdot) \in \mathcal{F}_n, \ \text{for all } \ j \in [0, r], \quad 0 < a \leq n_j \leq A. \tag{13.2}$$

Now, $\mathcal{F}_n$ being compact, it is always possible to find a closed bounded curve $\mathcal{C}$ included in the left-half plane that strictly contains all zeroes of any element in $\mathcal{F}_n$. For example it is well known that in view of (13.2), any zero z_n of an element $n(\cdot)$ of $\mathcal{F}_n$ satisfies:

$$|z_n| \leq 1 + \frac{A}{a}.$$

Hence, it is always possible to choose $\mathcal{C}$ as in Figure 13.2.

Once again by a compacity argument we can write:

$$\inf_{n(\cdot) \in \mathcal{F}_n} \left[\inf_{s \in \mathcal{C}} |n(s)| \right] = \alpha_n > 0.$$

Proof. We proceed by contradiction. If $\alpha_n = 0$, then it is possible to find a sequence of polynomials $n_k(s)$ in $\mathcal{F}_n$, such that for each $k > 0$,

$$\exists \ z_k \in \mathcal{C} \text{ such that } |n_k(z_k)| \leq \frac{1}{k}. \tag{13.3}$$

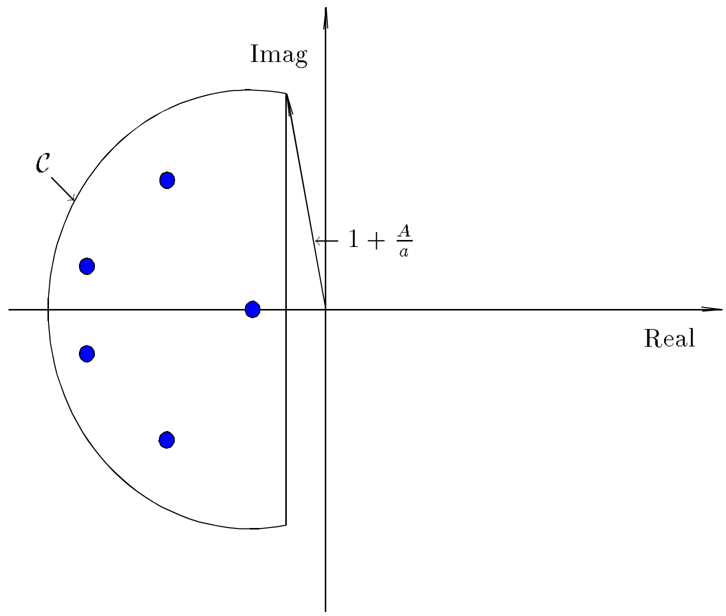

Figure 13.2. A possible choice for $\mathcal{C}$

But, $\mathcal{C}$ being compact in the complex plane, it is possible to find a subsequence $z_{\phi(k)}$ that converges to $z_0 \in \mathcal{C}$. Moreover, $n_{\phi(k)}(\cdot)$ is now a sequence of elements of the compact set $\mathcal{F}_n$, and therefore it is possible to find a subsequence $n_{\phi(\psi(k))}(\cdot)$ that converges to $n_0(\cdot) \in \mathcal{F}_n$. Then we have by (13.3),

$$|n_{\phi(\psi(k))}(z_{\phi(\psi(k))})| \leq \frac{1}{\phi(\psi(k))}. \tag{13.4}$$

Passing to the limit as k goes to infinity in (13.4), one gets:

$$n_0(z_0) = 0.$$

But this is a contradiction because $\mathcal{C}$ is supposed to strictly enclose all the zeroes of any polynomial in $\mathcal{F}_n$. ♣

Property 13.4. Since the family $\mathcal{F}_d$ is bounded we have that for any $d(\cdot)$ in $\mathcal{F}_d$,

$$\text{for all} \;\; s, \;\; |d(s)| \leq B\big(1 + |s| + \cdots + |s|^q\big) = \phi(s).$$

Let

$$\beta_d = \sup_{s \in \mathcal{C}} \phi(s).$$

Then β_d is finite (because $\mathcal{C}$ is compact and $\phi(\cdot)$ is continuous) and we have that

$$\sup_{d(\cdot)\in\mathcal{F}_d}\left[\sup_{s\in\mathcal{C}}|d(s)|\right] \leq \beta_d. \tag{13.5}$$

We can now proceed and prove i) and ii) of Theorem 13.1.

Proof of Theorem 13.1.i) Let $\mathcal{C}$ be a closed bounded curve enclosing every zero of each element in $\mathcal{F}_n$, and let α_n and β_d be defined as in (13.3) and (13.5). Then, if we choose ϵ such that $0 < |\epsilon| < \frac{\alpha_n}{\beta_d}$,

$$\text{for all } \; n(\cdot) \in \mathcal{F}_n, \quad \text{and} \quad \text{for all } \; d(\cdot) \in \mathcal{F}_d$$

we have that

$$\text{for all } \; s \in \mathcal{C}, \; |\epsilon d(s)| \leq |\epsilon|\beta_d < \alpha_n \leq |n(s)|. \tag{13.6}$$

Hence, by Rouché's Theorem, we conclude from (13.6) that, for this choice of ϵ, $n(s) + \epsilon d(s)$ has the same number of zeroes as $n(s)$ in $\mathcal{C}$, namely r. But since $n(s) + \epsilon d(s)$ is itself of degree r it is stable. ♣

Remark 13.1. In this case one can notice that the Property B3 of the family $\mathcal{F}_d$ is not needed.

Proof of Theorem 13.1.ii) Let us first suppose that $q = r + 1$. Again let $\mathcal{C}$ be a closed bounded curve enclosing every zero of each element in $\mathcal{F}_n$, and let α_n and β_d be defined as in (13.3) and (13.5).
If we start by choosing ϵ_1 such that $0 < \epsilon_1 < \frac{\alpha_n}{\beta_d}$, and any μ such that $0 < \mu < \epsilon_1$, then

$$\text{for all } \; n(\cdot) \in \mathcal{F}_n, \; \text{and for all } \; d(\cdot) \in \mathcal{F}_d$$

we have

$$\text{for all } \; s \in \mathcal{C}, \quad |\mu d(s)| \leq \mu\beta_d < \alpha_n \leq |n(s)|.$$

Again we conclude by Rouché's Theorem that for any such μ, $n(s) + \mu d(s)$ has already r zeroes inside $\mathcal{C}$. Moreover it is also possible to find ϵ_2 such that for any μ satisfying $0 < \mu < \epsilon_2$, we have that every coefficient of $n(s) + \mu d(s)$ is positive.
If we now choose any ϵ such that

$$0 < \epsilon < \min(\epsilon_1, \epsilon_2),$$

we have that $n(s) + \epsilon d(s)$ is of degree less than or equal to $r + 1$, has r stable roots, and all its coefficients are positive. But this implies that $n(s) + \epsilon d(s)$ is necessarily stable.

We now proceed by induction on $n = q - r$. Suppose that part ii) of the theorem is true when $q = r + p$, $p \geq 1$. Let $\mathcal{F}_n$ and $\mathcal{F}_d$ be two families of polynomials satisfying Properties A1, A2, A3, and B1, B2, B3, respectively, and let us suppose that $q = r + p + 1$.

Now consider the new family $\mathcal{F}'_n$ of polynomials $n'(s)$ of the form,

$$n'(s) = (s+1)n(s), \text{ where } n(s) \in \mathcal{F}_n.$$

Obviously $\mathcal{F}'_n$ is also a compact set and each element of $\mathcal{F}'_n$ satisfies Properties A1, A2, A3, but now with $r' = r+1$. Hence by the induction hypothesis it is possible to find a stable polynomial $n'_c(s)$ of degree less than or equal to $p-1$, and a stable polynomial $d'_c(s)$ of degree $p-1$ such that,

$$\text{for all } \; n(\cdot) \in \mathcal{F}_n, \text{ and for all } \; d(\cdot) \in \mathcal{F}_d$$

we have that

$$n(s)(s+1)n'_c(s) + d(s)d'_c(s) \text{ is stable }. \tag{13.7}$$

Now, let $n_c(s) = (s+1)n'_c(s)$, and consider the new family of polynomials $\mathcal{F}'_n$ described by (13.7). That is $\mathcal{F}'_n$ consists of all polynomials $n'(\cdot)$ of the form

$$n'(s) = n(s)n_c(s) + d(s)d'_c(s)$$

where $n(s)$ is an arbitrary element in $\mathcal{F}_n$ and $d(s)$ is an arbitrary element in $\mathcal{F}_d$. The new family of polynomials $\mathcal{F}'_d$ consists of all polynomials $d'(\cdot)$ of the form

$$d'(s) = sd'_c(s)d(s),$$

where $d(s)$ is an element of $\mathcal{F}_d$. Clearly the family $\mathcal{F}'_n$ is compact and satisfies Properties A1, A2, A3 with $r' = r+2p$, and $\mathcal{F}'_d$ is also a compact family of polynomials satisfying Properties B1, B2, B3 with $q' = r+2p+1$. Hence, by applying our result when $n = 1$, we can find an $\epsilon > 0$ such that

$$\text{for all } \; n'(\cdot) \in \mathcal{F}'_n, \quad \text{and} \quad \text{for all } \; d'(\cdot) \in \mathcal{F}'_d,$$

$$n'(s) + \epsilon d'(s) \text{ is stable.}$$

But, in particular, this implies that,

$$\text{for all } \; n(\cdot) \in \mathcal{F}_n, \quad \text{and} \quad \text{for all } \; d(\cdot) \in \mathcal{F}_d,$$

$$n(s)n_c(s) + d'_c(s)d(s) + \epsilon s d'_c(s)d(s) = n(s)n_c(s) + (\epsilon s+1)d'_c(s)d(s) \text{ is stable.}$$

Therefore, the controller defined by

$$C(s) = \frac{n_c(s)}{d_c(s)} = \frac{n_c(s)}{(\epsilon s+1)d'_c(s)},$$

is an answer to our problem and this ends the proof of Theorem 13.1. ♣

In the following sections we discuss first the problem of robust stabilization against unstructured perturbations, the related interpolation problems, and the use of this theory to solve robust stabilization problems under parameter uncertainty. We begin with a description of the Q parametrization.

13.3 INTERNAL STABILITY AND THE Q PARAMETERIZATION

In this section we derive some conditions for the internal stability of a feedback system in terms of the so-called Q parameter. This convenient parametrization allows stabilizing controllers to be determined for the nominal system, and also for perturbed systems.

Consider the standard single loop feedback configuration shown in Figure 13.3.

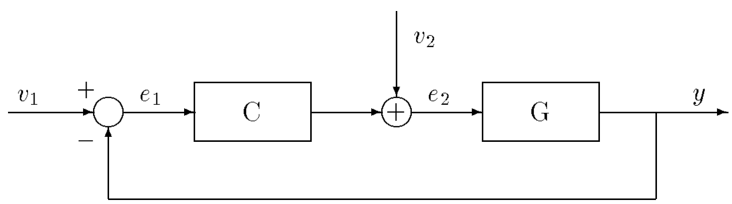

Figure 13.3. General closed-loop system

We assume that the controller and plant are both single-input single-output systems and are represented by the proper, real, rational transfer functions $C(s)$ and $G(s)$ respectively. We first discuss the problem of stabilizing a fixed plant $G(s) = G_0(s)$ with some $C(s)$. Throughout this chapter stability refers to Hurwitz stability unless specified otherwise. As usual we assume that there are no hidden pole-zero cancellations in the closed right half plane (RHP) in $G_0(s)$ as well as in $C(s)$. Let H denote the 2×2 transfer matrix between the *virtual input* $v = [v_1, v_2]'$ and $e = [e_1, e_2]'$. We have

$$H = \begin{bmatrix} H_{e_1,v_1} & H_{e_1,v_2} \\ H_{e_2,v_1} & H_{e_2,v_2} \end{bmatrix} = \begin{bmatrix} (1+G_0C)^{-1} & -G_0(1+CG_0)^{-1} \\ C(1+G_0C)^{-1} & (1+CG_0)^{-1} \end{bmatrix} \tag{13.8}$$

The closed loop system in Figure 13.3 is *internally stable* if and only if all the elements of H are stable and proper, or equivalently, H_∞ functions. This condition for stability is completely equivalent to that of stability of the characteristic polynomial. It guarantees that all signals in the loop remain bounded as long as the external virtual inputs at v_1 or v_2 are bounded.

The controller appears nonlinearly in H. A "linear" parametrization of the feedback loop can be obtained by introducing the *parameter*

$$Q(s) := \frac{C(s)}{1 + G_0(s)C(s)}. \tag{13.9}$$

The controller

$$C(s) = \frac{Q(s)}{1 - G_0(s)Q(s)} \tag{13.10}$$

and it can be seen that $C(s)$ is proper if and only if $Q(s)$ is proper. In terms of the parameter $Q(s)$

$$H = \begin{bmatrix} 1 - G_0 Q & -G_0(1 - G_0 Q) \\ Q & 1 - G_0 Q \end{bmatrix} \tag{13.11}$$

and therefore the necessary and sufficient conditions for internal stability of the feedback system are the following:

Stability Conditions

1) $Q(s) \in H_\infty$;

2) $G_0(s)Q(s) \in H_\infty$;

3) $G_0(s)(1 - G_0(s)Q(s)) \in H_\infty$.

To proceed, we make the simplifying standing assumptions:

i) $G_0(s)$ has no poles on the $j\omega$ axis;

ii) The RHP poles $\alpha_i, \; i = 1, \ldots l$ of $G_0(s)$ are nonrepeated.

The stability conditions 1)-3) can then be translated into the equivalent conditions:

1′) $Q(s) \in H_\infty$;

2′) $Q(\alpha_i) = 0, \quad i = 1, \cdots, l$;

3′) $G_0(\alpha_i)Q(\alpha_i) = 1, \quad i = 1, \cdots, l$.

The search for a proper, rational stabilizing compensator $C(s)$ is thus reduced to the *equivalent* problem of finding a real, rational, proper function $Q(s)$ which satisfies the above conditions. Once a suitable $Q(s)$ is found, the compensator can be recovered from the inverse relation (13.10) above. Therefore we can parametrize all stabilizing compensators for $G_0(s)$ by parametrizing all functions $Q(s)$ which satisfy the above conditions. This is what we do next.

Let $\bar{\alpha}$ denote the conjugate of α and introduce the *Blaschke product*

$$B(s) = \Pi \left[\frac{\alpha_i - s}{\bar{\alpha}_i + s} \right] \tag{13.12}$$

and let

$$Q(s) := B(s)\tilde{Q}(s). \tag{13.13}$$

With this choice of $Q(s)$ condition 2′) is automatically satisfied for every stable $\tilde{Q}(s)$. Now let

$$\tilde{G}_0(s) := B(s)G_0(s) \tag{13.14}$$

so that

$$G_0(s)Q(s) = \frac{\tilde{G}_0(s)}{B(s)} B(s)\tilde{Q}(s) = \tilde{G}_0(s)\tilde{Q}(s). \tag{13.15}$$

We note that $B(s) \in H_\infty$ and $\tilde{G}_0(s) \in H_\infty$ so that conditions 1) and 2) for internal stability are satisfied if and only if $\tilde{Q}(s) \in H_\infty$. The remaining stability condition 3′) now becomes

$$\tilde{Q}(\alpha_i) = \frac{1}{\tilde{G}_0(\alpha_i)} := \tilde{\beta}_i. \tag{13.16}$$

If $G_0(s)$ is given, so are α_i and $\tilde{\beta}_i$ and therefore the problem of stabilizing the nominal system by some $C(s)$ is reduced to the *interpolation problem*:

Find a function $\tilde{Q}(s) \in H_\infty$ satisfying

$$\tilde{Q}(\alpha_i) = \tilde{\beta}_i, \quad i = 1, \cdots, l. \tag{13.17}$$

Once a $\tilde{Q}(s) \in H_\infty$ satisfying (13.17) is found, $Q(s) = B(s)\tilde{Q}(s)$ satisfies the stability conditions 1), 2) and 3), and hence the corresponding $C(s)$, which can be determined uniquely from (13.10), is guaranteed to be proper and stabilizing. It is straightforward to find a stable proper $Q(s)$ that satisfies the interpolation conditions (13.17). We also remark that the assumptions regarding the poles of $G_0(s)$ can be relaxed by placing some more interpolation conditions on $\tilde{Q}(s)$.

We shall see in the next section that in the problem of *robust stabilization*, additional restrictions in the form of a norm bound will have to be imposed on $\tilde{Q}(s)$.

13.4 ROBUST STABILIZATION: UNSTRUCTURED PERTURBATIONS

We continue with the problem setup and notation established in the last section, but suppose now that the plant transfer function is subject to unstructured norm bounded perturbations which belong to an additive or multiplicative class.

Additive and Multiplicative Perturbations

Write $G(s) = G_0(s) + \Delta G(s)$ with $\Delta G(s)$ specified by a frequency dependent magnitude constraint as follows. Let $r(s)$ be a given, real, rational, minimum phase (no zeroes in the closed RHP), H_∞ function. We introduce the definitions:

Definition 13.1. (Additive Perturbations) A transfer function $G(s)$ is said to be in the class $A(G_0(s), r(s))$ if

i) $G(s)$ has the same number of poles as $G_0(s)$;
ii) $|G(j\omega) - G_0(j\omega)| \leq |r(j\omega)|$, $|r(j\omega)| > 0$, for all $\omega \in \mathbb{R}$.

Similarly we can also consider multiplicative perturbations. Here we have $G(s) = (1 + \Delta G(s))G_0(s)$ and a frequency dependent magnitude constraint is placed on $\Delta G(j\omega)$ using a suitable real, rational, minimum phase H_∞ function $r(s)$.

Definition 13.2. **(Multiplicative Perturbations)** A transfer function $G(s)$ is said to be in the class $M(G_0(s), r(s))$ if:

1) $G(s)$ has the same number of unstable poles as $G_0(s)$;
2) $G(s) = (1 + M(s))G_0(s)$ with $|M(j\omega)| < |r(j\omega)|$, for all $\omega \in \mathbb{R}$.

Suppose now that a compensator $C(s)$ which stabilizes $G_0(s)$ is given. We first establish the conditions for $C(s)$ to be a robust stabilizer for all plants in the class $A(G_0(s), r(s))$.

If $C(s)$ stabilizes $G_0(s)$ we have

$$G_0(j\omega)C(j\omega) + 1 \neq 0, \quad \text{for all} \quad \omega \in \mathbb{R} \tag{13.18}$$

and we also know that the Nyquist plot of $G_0(s)C(s)$ has the correct number of encirclements of the -1 point.

Now let us consider what happens to the stability of the closed loop system when $G_0(s)$ is replaced by its perturbed version $G(s) = G_0(s) + \Delta G(s)$. Since $G_0(s)$ and $G(s)$ have the same number of unstable poles, the only way that the closed loop can become unstable is by a change in the number of encirclements of the $-1 + j0$ point by the Nyquist plot of $G(s)C(s)$. In terms of frequency domain plots, this means that the plots of $G_0(j\omega)C(j\omega)$ and $G(j\omega)C(j\omega)$ lie on opposite sides of -1. Thus, stability of the perturbed system can be ensured if the perturbation $\Delta G(s)$ is of small enough size that the plot of $G(j\omega)C(j\omega)$ does not pass through the -1 point. This can be stated as the condition

$$1 + G(j\omega)C(j\omega) \neq 0, \qquad \text{for all} \ \ \omega \in \mathbb{R} \tag{13.19}$$

or, equivalently as

$$(1 + G_0(j\omega)C(j\omega))(1 + (1 + G_0(j\omega)C(j\omega))^{-1}C(j\omega)\Delta G(j\omega)) \neq 0, \quad \text{for all} \ \ \omega \in \mathbb{R}.$$

The above condition will hold as long as

$$\sup_{\omega} |(1 + G_0(j\omega)C(j\omega))^{-1}C(j\omega)\Delta G(j\omega)| < 1 \tag{13.20}$$

It follows from this analysis that a sufficient condition for robust stability with respect to the class $A(G_0(s), r(s))$ is given by

$$||(1 + G_0(s)C(s))^{-1}C(s)r(s)||_{\infty} < 1 \tag{13.21}$$

If the above condition is violated, a real rational admissible perturbation $\Delta G(s)$ can be constructed for which the closed loop system with $G(s) = G_0(s) + \Delta G(s)$ is unstable. Thus, the condition (13.21) is also necessary for robust stability.

Analogous results hold for multiplicative unstructured uncertainty. By arguing exactly as in the additive case we can show that a compensator $C(s)$ which stabilizes $G_0(s)$, stabilizes all plants $G(s)$ in the class $M(p_0(s), r(s))$ if

$$||G_0(s)C(s)(1 + G_0(s)C(s))^{-1}r(s)||_{\infty} < 1. \tag{13.22}$$

Henceforth we focus on the additive case since similar results hold for the multiplicative case. We see that the robustness condition (13.21) is equivalent to

$$\|Q(s)r(s)\|_\infty < 1. \tag{13.23}$$

Using the fact that $\|B(s)\|_\infty = 1$ we can write (13.23) as

$$\|\tilde{Q}(s)r(s)\|_\infty < 1. \tag{13.24}$$

Now introduce the function

$$u(s) := \tilde{Q}(s)r(s). \tag{13.25}$$

The robust stability condition can now be written as

$$\|u(s)\|_\infty < 1. \tag{13.26}$$

The interpolation conditions (13.17) on $\tilde{Q}(s)$ translate to corresponding ones on $u(s)$:

$$u(\alpha_i) = \tilde{Q}(\alpha_i)r(\alpha_i) = \frac{r(\alpha_i)}{\tilde{G}_0(\alpha_i)} = \beta_i, \quad i = 1, ...l. \tag{13.27}$$

In these terms, the robust stabilization problem is reduced to the following:

Interpolation Problem

Given complex numbers $\alpha_i, \beta_i,\ i = 1, \cdots, l$ find, if possible, a real rational function $u(s)$ which satisfies the conditions

a) $\|u(s)\|_\infty < 1$
b) $u(\alpha_i) = \beta_i,\ i = 1, \cdots, l.$

The solution of the above problem is described in the next section (Nevanlinna-Pick Interpolation).

In summary the robustly stabilizing controller $C(s)$ is determined from the steps:

Step 1 Determine the RHP poles α_i of $G_0(s)$, $B(s)$ and $\tilde{G}_0(s)$;

Step 2 Determine

$$\beta_i = \frac{r(\alpha_i)}{\tilde{G}_0(\alpha_i)}, \quad i = 1, \cdots, l;$$

Step 3 Calculate a real rational H_∞ function $u(s)$ with $\|u(s)\|_\infty < 1$ solving the Interpolation Problem

$$u(\alpha_i) = \beta_i, \quad i = 1, \cdots, l$$

by using the Nevanlinna-Pick theory, described in the next section;

Step 4 Calculate $Q(s)$ from

$$Q(s) = \frac{B(s)u(s)}{r(s)};$$

Step 5 Determine $C(s)$ from

$$C(s) = \frac{Q(s)}{1 - G_0(s)Q(s)}.$$

In Step 3, $Q(s)$ must be an H_∞ function and this requires, since $B(s) \in H_\infty$, that

$$\frac{u(s)}{r(s)} \in H_\infty. \tag{13.28}$$

Since $u(s) \in H_\infty$ it follows that $Q(s) \in H_\infty$ if $\frac{1}{r(s)} \in H_\infty$. We have already assumed that $r(s)$ has no finite RHP zeros. The condition $\frac{1}{r(s)} \in H_\infty$ means that $r(s)$ should have relative degree zero, equivalently no zeros at infinity. If it is necessary to let $r(s)$ have relative degree 1, say, we need then to let $u(s)$ also have relative degree 1, so that we have $\frac{u(s)}{r(s)} \in H_\infty$. This translates to an additional interpolation condition

$$u(\infty) = 0.$$

on $u(s)$.

In the next section we describe the solution of the Interpolation Problem formulated above.

13.5 NEVANLINNA-PICK INTERPOLATION

In the previous section we established that robust stabilization against additive unstructured perturbations can be achieved provided we find a real rational function $u(s)$ with H_∞ norm $\|u\|_\infty < 1$ which satisfies the interpolation condition

$$u(\alpha_i) = \beta_i, \quad i = 1, \ldots, l; \quad \mathrm{Re}[\alpha_i] > 0, \ |\beta_i| < 1. \tag{13.29}$$

Once $u(s)$ is obtained, we can find the robustly stabilizing controller $C(s)$ using the Steps outlined in the previous section.

A real, rational, function $u(s)$ with $\|u\|_\infty < 1$ is also called a *strictly bounded real* (SBR) function. To get a proper controller, $u(s)$ needs to have relative degree at least as large as $r(s)$. Thus $u(s)$ needs to be proper if $r(s)$ has relative degree 0 and strictly proper ($u(\infty) = 0$) when the relative degree of $r(s)$ is 1.

A complex, stable, proper, rational function $u(s)$ with $\|u(s)\|_\infty < 1$ is called a Schur function. The problem of finding a Schur function $u(s)$ satisfying the interpolation conditions given above is known as the *Nevanlinna-Pick problem* (NP problem) and is outlined below without proofs. Once a Schur function $u(s)$ is found, an SBR function satisfying the same interpolation conditions can easily be found. First, we have the condition for the existence of a solution.

Theorem 13.2 *The Nevanlinna-Pick problem admits a solution if and only if the Pick matrix P, whose elements p_{ij} are given by*

$$p_{ij} = \frac{1 - \beta_i \overline{\beta}_j}{\alpha_i + \overline{\alpha}_j} \tag{13.30}$$

is positive definite.

If the Pick matrix is positive definite a solution exists. The solution is generated by successively reducing an interpolation problem with k interpolation points to one with $k-1$ points. The problem with one point has an obvious solution. This is the *Nevanlinna algorithm* and is described next.

Consider the linear fractional transformation mapping $u_i(s)$ to $u_{i-1}(s)$

$$u_{i-1}(s) = \frac{\rho_{i-1} + u_i(s)\left(\dfrac{s-\alpha_{i-1}}{s+\overline{\alpha}_{i-1}}\right)}{1 + \overline{\rho}_{i-1}\left(\dfrac{s-\alpha_{i-1}}{s+\overline{\alpha}_{i-1}}\right)u_i(s)}. \tag{13.31}$$

The inverse transformation is

$$u_i(s) = \frac{u_{i-1}(s) - \rho_{i-1}}{1 - \overline{\rho}_{i-1} u_{i-1}(s)}\left(\frac{s+\overline{\alpha}_{i-1}}{s-\overline{\alpha}_{i-1}}\right). \tag{13.32}$$

Let us denote these transformations compactly as

$$u_i(s) = T_{\rho_{i-1}} u_{i-1}(s); \quad u_{i-1}(s) = T^{-1}_{\rho_{i-1}} u_i(s). \tag{13.33}$$

It can be seen that

$$u_{i-1}(\alpha_{i-1}) = \rho_{i-1}, \quad \text{for all } \ u_i(s). \tag{13.34}$$

Moreover, it can be shown that $u_{i-1}(s)$ is Schur if $u_i(s)$ is Schur and conversely that $u_{i-1}(s)$ Schur and $u_{i-1}(\alpha_{i-1}) = \rho_{i-1}$ imply that $u_i(s)$ is Schur.

Now suppose that $i = 2$, $u_1(s) = u(s)$, $\rho_1 = \beta_1$ in (13.31). We see that $u(\alpha_1) = \beta_1$ regardless of $u_2(s)$ and the remaining $l-1$ interpolation conditions on $u_1(s)$ are transferred to $u_2(s)$:

$$\begin{aligned}
u_2(\alpha_2) &= T_{\rho_1}(u_1(\alpha_2)) = T_{\rho_1}(\beta_2) := \rho_2 \\
u_2(\alpha_3) &= T_{\rho_1}(u_1(\alpha_3)) = T_{\rho_1}(\beta_3) := \rho_{2,3} \\
&\vdots \\
u_2(\alpha_l) &= T_{\rho_1}(u_1(\alpha_l)) = T_{\rho_1}(\beta_l) := \rho_{2,l}.
\end{aligned}$$

Thus, the original problem of interpolating l with a Schur function $u_1(s)$ is reduced to the problem of interpolating $l-1$ points with a Schur function $u_2(s)$. In this way, a family of solutions $u(s)$, parametrized in terms of an arbitrary initial Schur function $u_{l+1}(s)$, can be obtained.

The above calculations can be organized by forming the *Fenyves array* as shown in Table 13.1.

Table 13.1. Fenyves array

α_1	α_2	α_3	$\dots$	α_l	
ρ_1	$\rho_{1,2}$	$\rho_{1,3}$	$\dots$	$\rho_{1,l}$	$u_1(s)$
	ρ_2	$\rho_{2,3}$	$\dots$	$\rho_{2,l}$	$u_2(s)$
		$\dots$	$\dots$	$\dots$	$\dots$
				ρ_l	$u_l(s)$

where $u_1(s) = u(s)$, $\rho_1 = \beta_1$ and $\rho_{1,j} = \beta_j$

$$\rho_i = \frac{\rho_{i-1,i} - \rho_{i-1}}{1 - \overline{\rho}_{i-1}\rho_{i-1,i}} \quad \frac{\alpha_i + \overline{\alpha}_{i-1}}{\alpha_i - \alpha_{i-1}}; \quad 1 < i \leq l \tag{13.35}$$

and

$$\rho_{i,j} = \frac{\rho_{i-1,j} - \rho_{i-1}}{1 - \overline{\rho}_{i-1}\rho_{i-1,j}} \quad \frac{\alpha_j + \overline{\alpha}_{i-1}}{\alpha_j - \alpha_{i-1}}; \quad 1 < i < j \leq l \tag{13.36}$$

An existence condition equivalent to that of the positive definiteness of the Pick matrix is the following.

Theorem 13.3 *The Nevanlinna-Pick problem admits a solution if and only if the modulus of all the elements of the Fenyves array is less than one:* $|\rho_i| < 1$, $|\rho_{i,j}| < 1$.

The algorithm given generates a Schur function $u(s) = u_R(s) + ju_I(s)$ which satisfies the interpolation conditions. It can be shown that if the complex elements in the set α_i appear along with their conjugates and the corresponding β_i are also conjugated, then $u_R(s)$ is a real H_∞ function with $\|u_R(s)\|_\infty < 1$ which also satisfies the interpolation conditions. The above algorithm can also be suitably modified to handle the case when the nominal model has a pole at the origin. We show this in the examples.

In the next section we discuss how robust stabilization against parametric uncertainty might be attempted using the theory for unstructured perturbations.

13.6 OVERBOUNDING PARAMETRIC UNCERTAINTY BY UNSTRUCTURED UNCERTAINTY

In this section our objective is to show how the frequency domain uncertainty induced by parametric uncertainty can be "covered" by overbounding with a suitable bounding function $r(s)$. Once this is accomplished, the preceding theory of robust synthesis under norm bounded perturbations (Section 13.4) can be applied to the problem. If the procedure is successful the resulting controller robustly stabilizes the system under the given parametric uncertainty set.

Consider the feedback system shown in Figure 13.4.

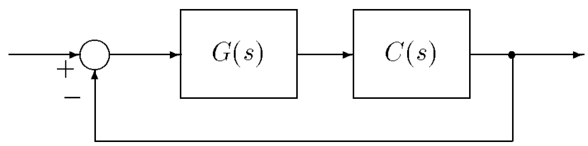

Figure 13.4. A unity feedback system

Suppose the system consists of a plant $G(s, \mathbf{p})$ containing a parameter $\mathbf{p}$ which varies in an uncertainty set $\Omega(\epsilon)$ about a nominal value $\mathbf{p}^0$. The parametric uncertainty set $\Omega(\epsilon)$ is defined by:

$$\Omega(\epsilon) := \left\{ \mathbf{p} : \|\mathbf{p} - \mathbf{p}^0\| \leq \epsilon \right\}.$$

Thus, the size of the uncertainty set Ω is parametrized by ϵ, and we shall loosely refer to $\Omega(\epsilon)$ as a set of plants. Ideally one would like to find the largest such set in the parameter space of the plant that can be robustly stabilized. This means that the free parameter ϵ needs to be increased and the set Ω enlarged until the maximum ϵ, $\epsilon_{\max}$ is reached for which the entire family of plants is stabilizable. It is also necessary to find at least one compensator $C(s)$ that will robustly stabilize the family of plants for any $\epsilon \leq \epsilon_{\max}$. This problem of finding $\epsilon_{\max}$ is as *yet unsolved*. However, we shall show that by using the extremal properties of the generalized Kharitonov segments and the techniques from H_∞ synthesis, we can determine an $\epsilon^* \leq \epsilon_{\max}$ such that a robustly stabilizing controller $C^*(s)$ can be found for the family of plants $\Omega(\epsilon^*)$.

Norm-bounded Uncertainty and Parametric Uncertainty

Our objective is to determine a bounding function $r(s)$ which bounds the frequency domain uncertainty induced by parametric uncertainty. To be specific, let us consider the parametric uncertainty to be modelled by an interval plant denoted by $\mathbf{G}^\epsilon(s)$:

$$\mathbf{G}^\epsilon(s) := \left\{ G(s) \ : \ G(s) = \frac{n(s,\epsilon)}{d(s,\epsilon)} = \frac{n_q(\epsilon)s^q + \cdots + n_0(\epsilon)}{d_q(\epsilon)s^q + \cdots + d_0(\epsilon)} \right\}, \tag{13.37}$$

where

$$n_i(\epsilon) \in [n_i^0 - w_i\epsilon, n_i^0 + w_i\epsilon], \qquad d_i(\epsilon) \in [d_j^0 - w_j\epsilon, d_j^0 + w_j\epsilon].$$

Here q is the plant order and $i, j = 0, \ldots, q$. The size of the plant coefficient perturbations are parametrized by the free parameter ϵ and the weighting factors w_i and w_j which are chosen to reflect scaling factors and the relative importance of the perturbations. The perturbation set $\Omega(\epsilon)$ is a box in the coefficient space of the plant, each side of which has a length of $2w_i\epsilon$. The center of this box corresponds to the nominal coefficient values, $n_i^{p_o}$ and $d_i^{p_o}$.

The Bode uncertainty magnitude band induced by parametric uncertainty is

$$|G^\epsilon(j\omega) - G_0(j\omega)| = |\Delta G^\epsilon(j\omega)|. \tag{13.38}$$

We seek a stable, proper, real rational, minimum phase function $r(s)$ for which

$$|\Delta G^\epsilon(j\omega)| < |r(j\omega)|, \quad \text{for all } \omega \in \mathbb{R} \tag{13.39}$$

We know from Chapter 8 that at each frequency, the maximum magnitude in the complex plane template $\mathbf{G}^\epsilon(j\omega)$ will correspond to a point on one of the extremal segments of $\mathbf{G}^\epsilon(s)$ which we denote as $\mathbf{G}_{\mathrm{E}}^\epsilon(s)$. We can therefore search over the extremal segments $\mathbf{G}_{\mathrm{E}}^\epsilon(s)$ at all frequencies and calculate exactly the maximum perturbation $\delta(\epsilon,\omega)$ induced at the frequency ω. In other words

$$\delta(\epsilon,\omega) = \max_{G\in\mathbf{G}^\epsilon} |\Delta G^\epsilon(j\omega)|.$$

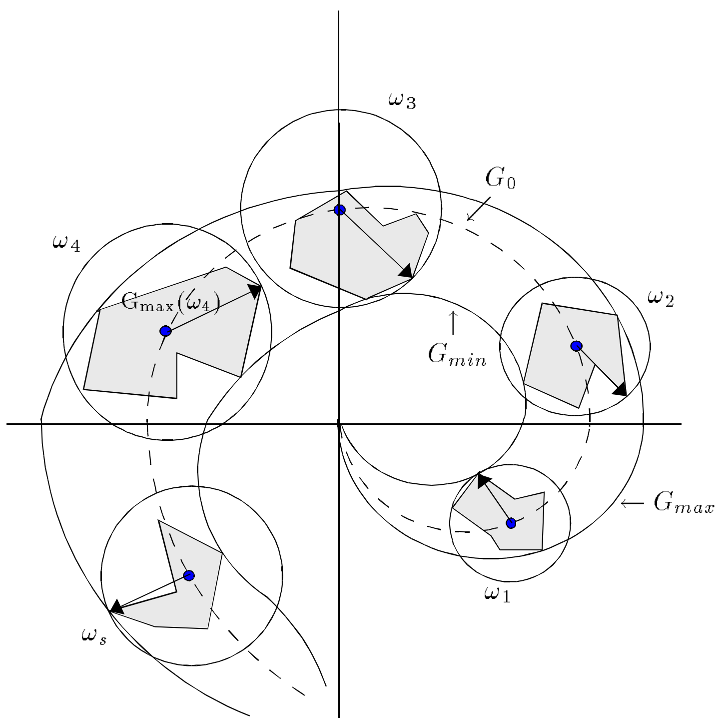

Figure 13.5. Parametric uncertainty converted to H_∞ uncertainty for an interval plant $G(s)$

Figure 13.5, which shows the polar plot for an arbitrary $G(s)$, is drawn to illustrate this overbounding for a prescribed value of ϵ.

In Figure 13.5 the uncertainty image set is induced by the variation of parameters within the ϵ-sized box in parameter space. These image sets are enclosed within a band which represents the minimum and maximum magnitudes at each frequency. The curves G_{min} and G_{max} are formed by the minimum and maximum magnitude points of $\mathbf{G}(s)$ at each frequency. The curve G_0 denotes the plot for the nominal plant. At each frequency, the largest distance between the nominal point and a point on the boundary of the uncertainty set is the maximum additive unstructured perturbation magnitude, $\delta(\omega)$ at that frequency. A circle drawn with its center at the nominal point and of radius $\delta(\omega)$ represents "the H_∞ uncertainty at that frequency and ϵ." The figure shows such sets for five frequencies. At frequency ω_4, for example, the center of the circle is named $G_0(\omega_4)$ and its radius is $\delta(\omega_4)$ which is equal to $\Delta G_{\text{max}}(\omega_4)$. The conservativeness introduced by replacing the parametric uncertainty by the H_∞ uncertainty circles is obvious from Figure 13.5. Figure 13.6 shows the bigger and more conservative uncertainty band. It covers the uncertainty circles with radius $\delta(\omega)$ at each frequency.

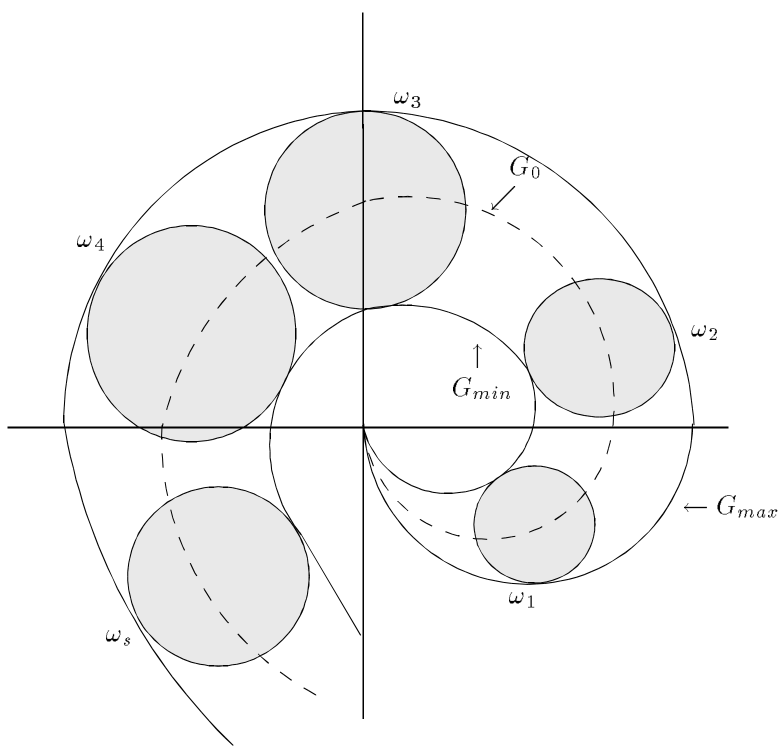

Figure 13.6. H_∞ uncertainty circles of radius $\delta(\omega)$

With this computation in hand we can proceed to the choice of a bounding function $r(s)$. The simplest choice of $r(s)$ is a constant

$$r(s) = r = \delta(\epsilon) = \max_{\omega \in \mathbb{R}} \delta(\epsilon, \omega) \tag{13.40}$$

which equals the radius of the largest such uncertainty circle over all frequencies. Figure 13.7 shows the uncertainty band for the above choice of $r(s)$.

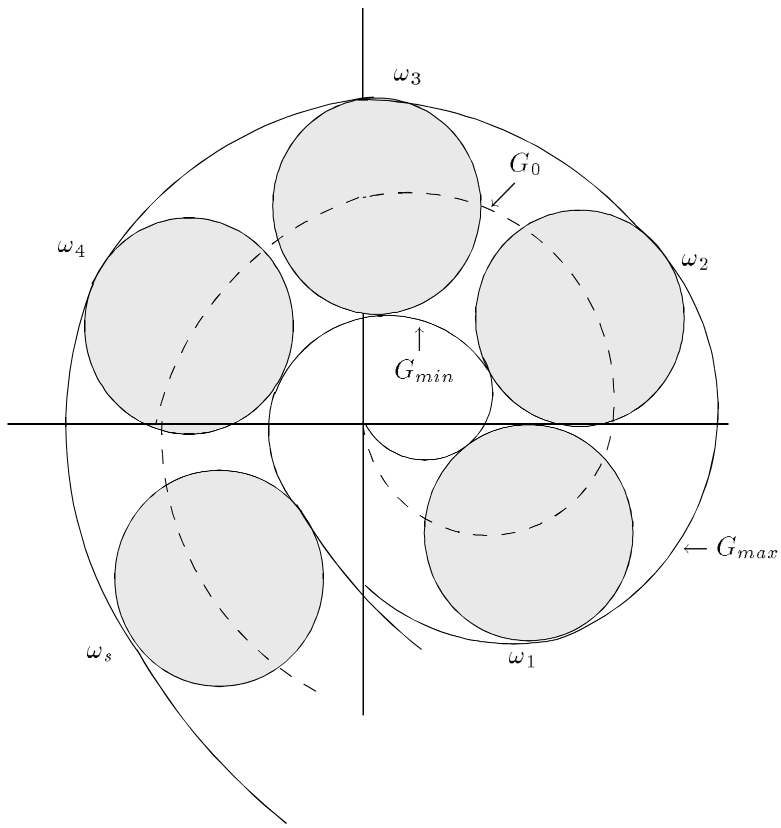

Figure 13.7. Uncertainty band when $r = \delta(\epsilon)$

Here, the circles of different sizes at different frequencies are all replaced by the biggest circle. In Figure 13.7 this is the circle at frequency ω_4. The choice of constant $r(s)$, although simple, gives a more conservative bounding of $\Delta G(s)$, which eventually means a lower value of $\epsilon_{\max}$. This conservativeness can often be minimized by introducing poles and zeros in $r(s)$ and shaping it such that $|r(j\omega)|$ approximates $\delta(\epsilon, \omega)$ at each ω as closely as possible from above:

$$|r(j\omega)| > \delta(\epsilon, \omega), \quad \text{for all } \ \omega \in \mathbb{R}.$$

We describe by examples how the H_∞ synthesis methods may be applied to interval plants. Let us start with an example for a stable nominal plant.

Example 13.1. Let the nominal plant be

$$G_0(s) = \frac{s+1}{s^3+8s^2+22s+20}$$

with poles at $-2, -3 \pm j$, all in the LHP. The perturbed plant is written as

$$G(s) = \frac{s+a}{s^3+bs^2+cs+d},$$

and the perturbation of the coefficients about the nominal is given by

$$a \in [1-\epsilon, 1+\epsilon], \quad b \in [8-\epsilon, 8+\epsilon], \quad c \in [22-\epsilon, 22+\epsilon], \quad d \in [20-\epsilon, 20+\epsilon].$$

Since it is assumed that the number of unstable poles of the plant should remain unchanged, it is required that ϵ be less than some ϵ_1 which is such that the entire family of plants is stable. This initial bound, ϵ_1 on ϵ can be found by checking the Hurwitz stability of the denominator Kharitonov polynomials:

$$\begin{aligned}
K_{d1}(s) &= s^3 + (8+\epsilon)s^2 + (22-\epsilon)s + (20-\epsilon) \\
K_{d2}(s) &= s^3 + (8+\epsilon)s^2 + (22+\epsilon)s + (20-\epsilon) \\
K_{d3}(s) &= s^3 + (8-\epsilon)s^2 + (22-\epsilon)s + (20+\epsilon) \\
K_{d4}(s) &= s^3 + (8-\epsilon)s^2 + (22+\epsilon)s + (20+\epsilon).
\end{aligned}$$

From the Routh-Hurwitz table for $K_{d1}(s)$,$K_{d2}(s)$,$K_{d3}(s)$, $K_{d4}(s)$, it can be shown that $\epsilon_1 = 6.321$, i.e., the entire family of plants will be stable as long as $\epsilon < 6.321$.

Now, for each ϵ, all the extremal segments will have to be searched at each ω to get $\delta(\epsilon,\omega) = \max\limits_{G \in \mathbf{G}^\epsilon} |\Delta G^\epsilon(j\omega)|$. Then

$$\delta(\epsilon) := \max_{\omega \in \mathbf{R}} \delta(\epsilon, \omega).$$

The next task is to find a suitable bounding function $r(s)$. Once an $r(s)$ is obtained, a robustly stabilizing compensator can be synthesized for the interval plant with parameter perturbations in that $\epsilon -$ box. This can be done for any $\epsilon < 6.321$.

For a constant $r(s)$ we must have

$$r(s) = r = \delta(\epsilon).$$

To proceed let us set $\epsilon = 6$. For this choice of ϵ the numerator and denominator Kharitonov polynomials are:

$$\begin{aligned}
K_{n1}(s) = K_{n2}(s) &= s - 5, & K_{n3}(s) &= K_{n4}(s) = s + 7 \\
K_{d1}(s) &= s^3 + 14s^2 + 16s + 14, & K_{d2}(s) &= s^3 + 14s^2 + 28s + 14 \\
K_{d3}(s) &= s^3 + 2s^2 + 16s + 26, & K_{d4}(s) &= s^3 + 2s^2 + 28s + 26.
\end{aligned}$$

The numerator and denominator Kharitonov segments are

$$
\begin{aligned}
S_n(s,\lambda) &= (1-\lambda)K_{n1}(s) + \lambda K_{n3}(s) = s + (12\lambda - 5) \\
S_{d1}(s,\lambda) &= (1-\lambda)K_{d1}(s) + \lambda K_{d2}(s) = s^3 + 14s^2 + (12\lambda + 16)s + 14 \\
S_{d2}(s,\lambda) &= (1-\lambda)K_{d1}(s) + \lambda K_{d3}(s) = s^3 + (14 - 12\lambda)s^2 + 16s + (14 + 12\lambda) \\
S_{d3}(s,\lambda) &= (1-\lambda)K_{d2}(s) + \lambda K_{d4}(s) = s^3 + (14 - 12\lambda)s^2 + 28s + (14 + 12\lambda) \\
S_{d4}(s,\lambda) &= (1-\lambda)K_{d3}(s) + \lambda K_{d4}(s) = s^3 + 2s^2 + (16 + 12\lambda)s + 26,
\end{aligned}
$$

where $\lambda \in [0,1]$. The extremal plants are

$$
\begin{array}{lll}
G_1(s,\lambda) = \dfrac{K_{n1}(s)}{S_{d1}(s,\lambda)}, & G_2(s,\lambda) = \dfrac{K_{n1}(s)}{S_{d2}(s,\lambda)}, & G_3(s,\lambda) = \dfrac{K_{n1}(s)}{S_{d3}(s,\lambda)}, \\
G_4(s,\lambda) = \dfrac{K_{n1}(s)}{S_{d4}(s,\lambda)}, & G_5(s,\lambda) = \dfrac{K_{n3}(s)}{S_{d1}(s,\lambda)}, & G_6(s,\lambda) = \dfrac{K_{n3}(s)}{S_{d2}(s,\lambda)}, \\
G_7(s,\lambda) = \dfrac{K_{n3}(s)}{S_{d3}(s,\lambda)}, & G_8(s,\lambda) = \dfrac{K_{n3}(s)}{S_{d4}(s,\lambda)}, & G_9(s,\lambda) = \dfrac{S_n(s,\lambda)}{K_{d1}(s)}, \\
G_{10}(s,\lambda) = \dfrac{S_n(s,\lambda)}{K_{d2}(s)}, & G_{11}(s,\lambda) = \dfrac{S_n(s,\lambda)}{K_{d3}(s)}, & G_{12}(s,\lambda) = \dfrac{S_n(s,\lambda)}{K_{d4}(s)}.
\end{array}
$$

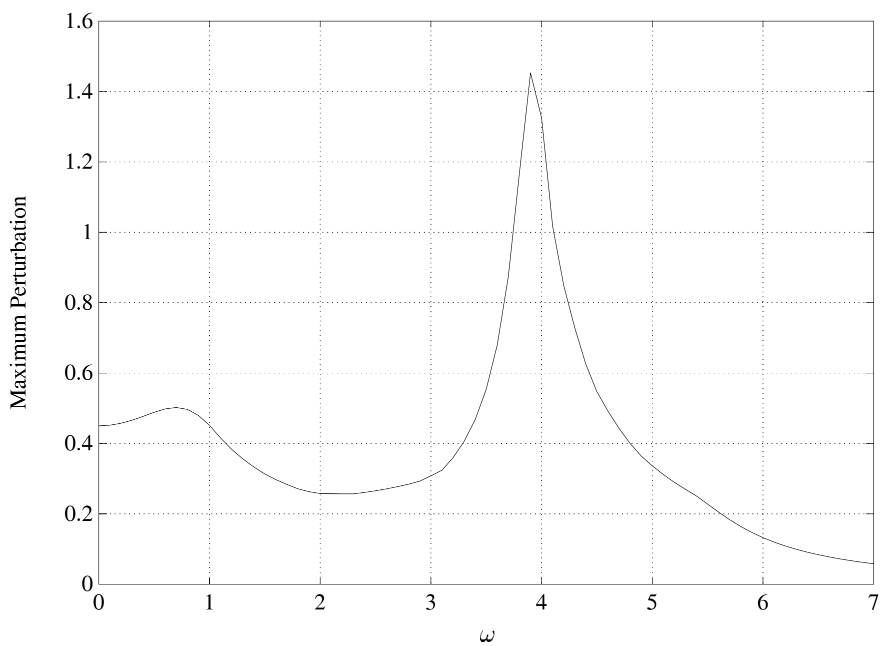

Figure 13.8. Maximum perturbation $\delta(\epsilon,\omega)$ vs ω for $\epsilon = 6$ (Example 13.1)

By searching through these extremal segment plants we get the plot of the maximum perturbation $\delta(\epsilon, \omega)$ at each ω as shown in Figure 13.8. The maximum value $\delta(6)$ of $\delta(6, \omega)$ is found to be 1.453. Therefore we choose $r = 1.453$ to proceed with our synthesis. Since there are no unstable poles in $G_0(s)$ there are no interpolation constraints. So, for the stable plant case, the controller can be parametrized in terms of an arbitrary Schur function $u(s)$. Let us pick $u(s) = u = 0.3 < 1$. Then

$$Q = \frac{u}{r} = \frac{0.3}{1.453} = 0.207$$

and

$$\begin{aligned} C(s) &= \frac{Q}{1 - G_0 Q} = \frac{u(s)}{r(s) - G_0(s)u(s)} \\ &= \frac{0.207(s^3 + 8s^2 + 22s + 20)}{s^3 + 8s^2 + 21.793s + 19.793}. \end{aligned}$$

This $C(s)$ stabilizes the interval plant $G(s)$ with coefficients varying in the intervals

$$a \in [-5, 7], \quad b \in [2, 14], \quad c \in [16, 28], \quad d \in [14, 26].$$

The above example shows that for an interval plant which is built around a *stable* nominal plant it is *always* possible to come up with a robustly stabilizing compensator so long as the coefficient perturbation, ϵ, is less than ϵ_1.

Next we consider an example of a nominal plant with a single unstable pole.

Example 13.2. Let the nominal plant be

$$G_0(s) = \frac{5s + 4}{(s-3)(s+5)} = \frac{5s + 4}{s^2 + 2s - 15},$$

and the interval plant be

$$G(s) = \frac{5s + a}{s^2 + bs + c},$$

where

$$a \in [4 - \epsilon, 4 + \epsilon], \quad b \in [2 - \epsilon, 2 + \epsilon], \quad c \in [-15 - \epsilon, -15 + \epsilon].$$

Using the Routh-Hurwitz criterion on the denominator Kharitonov polynomials, it can be shown that the number of unstable zeros of $K_{di}(s)$, $i = 1, 2, 3, 4$ (and hence the number of unstable poles of the family of plants) does not change for any $\epsilon < 15$. The Blaschke product is

$$B(s) = \frac{3 - s}{3 + s}.$$

$$\begin{aligned} \tilde{G}_0(s) &= B(s)P_0(s) = -\frac{5s + 4}{(s+3)(s+5)} \\ \tilde{G}_0(3) &= -\frac{5(3) + 4}{(3+3)(3+5)} = -0.396 \end{aligned}$$

We have to find a Schur function $u(s)$ such that

$$u(3) = \frac{r(3)}{\tilde{G}_0(3)} = \frac{-r(3)}{0.396},$$

and it is necessary that

$$|u(3)| < 1.$$

Here, if the design is to be done with a constant r, it is necessary that $r < 0.396$ for a robust stabilizer to exist. Therefore $\delta(\epsilon) = 0.396$.

As before we now generate the $\delta - \epsilon$ graph relating unstructured perturbations to structured perturbations. This involves generating the extremal set and searching over it at each frequency for the largest perturbation $\delta(\epsilon, \omega)$ for each value of ϵ. The details of this calculation are left as an exercise (Exercise 13.3). By further maximizing this perturbation over ω, we obtain $\delta(\epsilon)$. The plot $\delta(\epsilon)$ vs ϵ is shown in Figure 13.9.

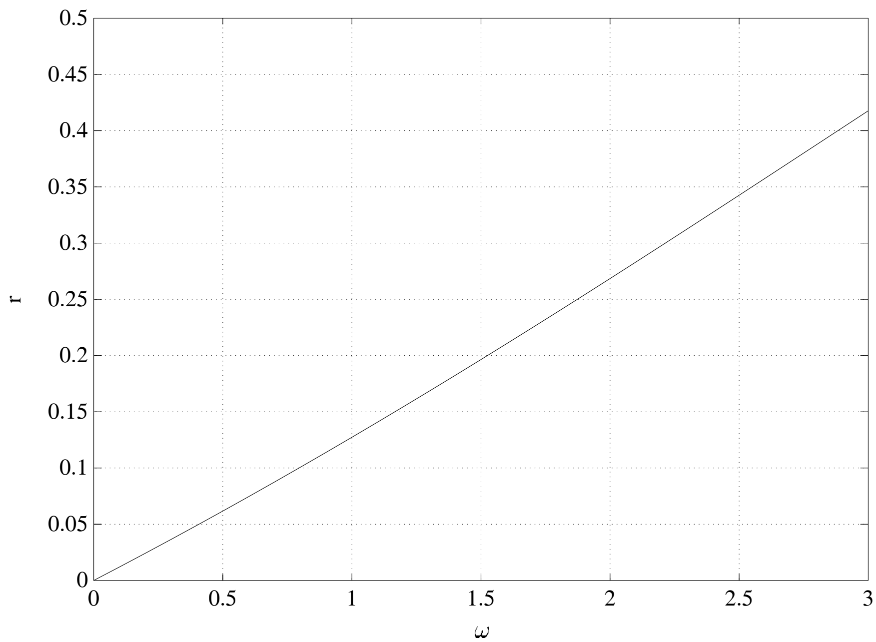

Figure 13.9. $\delta(\epsilon)$ vs ϵ (Example 13.2)

The value of ϵ corresponding to $\delta(\epsilon) = 0.396$ comes out to be equal to 2.83. Hence the maximum ϵ for which the plant can be robustly stabilized, with a constant r is 2.83.

To proceed let us choose the constant function $r(s) = r = \delta(2.83) = 0.396$. Hence $u(3) \approx 1 = u(s)$. Now

$$\tilde{Q} = \frac{u}{r} = -\frac{1}{0.396} = -2.532$$

$$Q(s) = B(s)\tilde{q}(s) = \frac{2.532(s-3)}{(s+3)}$$

and therefore,

$$C(s) = \frac{q(s)}{1 - G_0(s)Q(s)} = \frac{2.532(s+5)}{(s-1.584)}.$$

This $C(s)$ stabilizes $G(s)$ with

$$a \in [1.2, 6.8], \quad b \in [-0.8, 4.8], \quad c \in [-17.8, -12.2].$$

This can be verified by checking if $C(s)$ stabilizes the extremal segments of $\mathbf{G}(s)$.

Now suppose that we wish to tolerate a larger amount of parametric uncertainty, corresponding to, say $\epsilon = 4$. For this value of ϵ we verify that the maximum perturbation over all frequencies is $\delta = 0.578$. If a constant $r(s) = r$ were attempted we immediately fail to design a robust controller because of the required constraint $r \leq 0.396$.

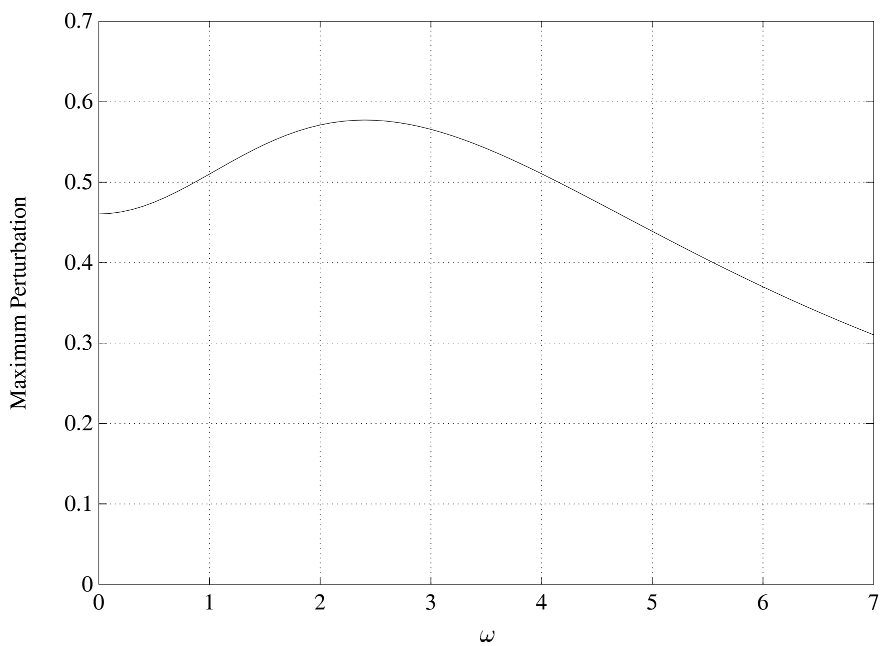

Figure 13.10. Maximum perturbation vs frequency for $\epsilon = 4$ (Example 13.2)

In this case it is obviously advisable to take the frequency information into account and attempt to design a rational function $r(s)$ that is loop-shaped to approximate $\delta(\epsilon, \omega)$ from above. This can be done from the plot of $\delta(4, \omega)$, as in Figure 13.10. One such $r(s)$ is:

$$r(s) = \frac{2.52(s+0.6)}{(s+1.3)(s+2.4)}.$$

Figure 13.11 shows the plot of both $|r(j\omega)|$ and $\delta(4, \omega)$ and $|r(j\omega)|$ is seen to approximate $\delta(4, \omega)$ from above.

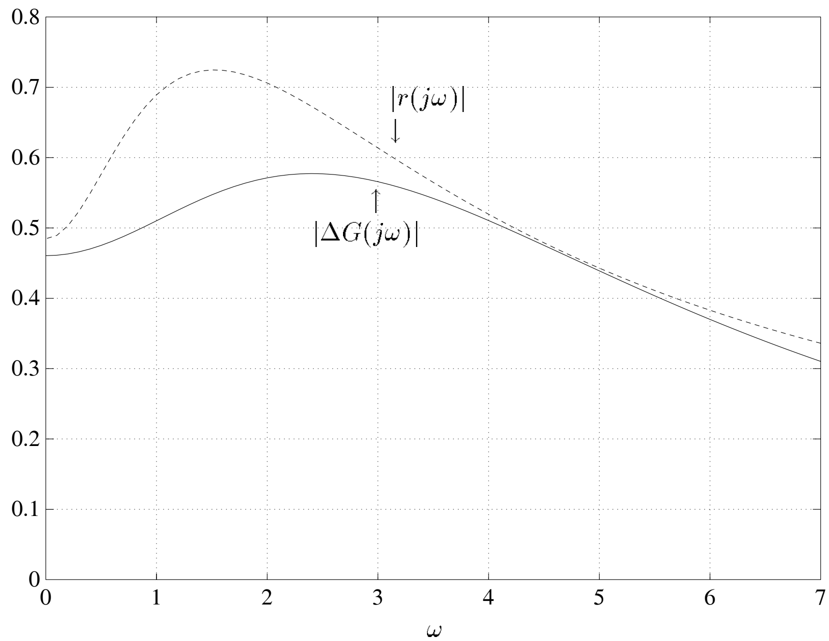

Figure 13.11. Maximum perturbation and $r(s)$ vs frequency for $\epsilon = 4$ (Example 13.2)

Then

$$u(3) = \frac{r(3)}{\tilde{G}_0(3)} = \frac{-2.52 \times 3.6}{4.3 \times 5.4 \times 0.395} = -0.987.$$

Since $r(s)$ is of relative degree 1, another interpolation condition on $u(s)$ is $u(\infty) = 0$ so that a proper controller is obtained.

Now, the second row of the Fenyves array will be

$$u_2(\infty) = \frac{u(\infty) + 0.987}{1 + 0.987u(\infty)} \left[\frac{s+3}{s-3}\right]_{s \to \infty} = 0.987.$$

Hence the Fenyves array is as shown in Table 13.2.

Table 13.2. Fenyves array

3	∞	
-0.987	0	$u(s)$
	0.987	$u_2(s)$

Since the elements of the Fenyves array are of modulus less than 1, hence an SBR function $u(s)$ exists which interpolates to

$$u(3) = -0.987, \quad u(\infty) = 0.$$

This can also be verified by checking for the non-negative definiteness of the Pick matrix. Here

$$\begin{aligned}\alpha_1 &= 3, \quad \alpha_2 = \infty, \quad \text{and} \\ \beta_1 &= -0.987, \quad \beta_2 = 0\end{aligned}$$

so that the Pick matrix is

$$P = \begin{bmatrix} \dfrac{1-0.987^2}{6} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4.305 \times 10^{-3} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since P is symmetric and has non-negative eigenvalues, it is non-negative definite. This also confirms the existence of $u(s)$.

Now, $u_2(s)$ can be parametrized in terms of an arbitrary Schur function $u_3(s)$, i.e.,

$$u_2(s) = \frac{0.987 + u_3(s)\left(\frac{s-\infty}{s+\infty}\right)}{1 + 0.987\left(\frac{s-\infty}{s+\infty}\right)u_3(s)} = \frac{0.987 - u_3(s)}{1 - 0.987u_3(s)}.$$

Choosing $u_3(s) = 0$, we get $u_2(s) = 0.987$. Then the next interpolation condition is $u(3) = -0.987$. This is satisfied by taking $u(s)$ as given below

$$u(s) = \frac{-0.987 + 0.987\left(\frac{s-3}{s+3}\right)}{1 - 0.987\left(\frac{s-3}{s+3}\right)u_2(s)} = \frac{-229.535}{s + 229.574}.$$

Then

$$\begin{aligned}\tilde{q}(s) &= \frac{u(s)}{r(s)} = \frac{-91.0853(s+1.3)(s+2.4)}{(s+0.6)(s+229.574)}, \\ q(s) &= B(s)\tilde{q}(s) = \frac{91.0853(s+1.3)(s+2.4)(s-3)}{(s+0.6)(s+3)(s+229.574)}.\end{aligned}$$

Therefore, the robust stabilizer is

$$C(s) = \frac{Q(s)}{1 - G_0(s)Q(s)} = \frac{91.0853(s+1.3)(s+2.4)(s+5)}{(s-217.47)(s+2.687)(s+0.53)}$$

for the interval plant

$$G(s) = \frac{5s + a}{s^2 + bs + c}; \quad a \in [0,8], \quad b \in [-2,6], \quad c \in [-19,-11].$$

In general larger values of ϵ can be obtained by increasing ϵ in steps, and for each ϵ, an attempt can be made to find a $u(s)$ which is Schur and satisfies the interpolation constraints. Then a robust stabilizer can be obtained from this $u(s)$.

Example 13.3. (Robust Stabilization of Two-Phase Servomotor) A two-phase servomotor is very rugged and reliable and is commonly used for instrument servomechanisms. Its transfer function, with the control voltage E_c as the input and the displacement of the motor shaft Θ as the output, is given by

$$\frac{\Theta(s)}{E_c(s)} = \frac{K_c}{Js^2 + (f + K_n)s} \tag{13.41}$$

where

J is the equivalent moment of inertia of the motor shaft,

f is the equivalent viscous-friction coefficient of the motor shaft,

K_c and K_n are positive constants.

K_n is the negative of the slope of the torque-speed curve and K_c gives the variation of the torque with respect to E_c for a given speed. In the low-speed region, the variation of torque with respect to speed and control voltage, is linear. However, at higher speeds this linearity is not maintained. The above transfer function of the servomotor is derived for low-speed operating regions. The following example shows the robust stabilization of a servomotor whose parameters K_c and K_n vary due to the nonlinearity.

$$G(s) = \frac{\Theta(s)}{E(s)} = \frac{K_c}{s[Js + (f + K_n)]}$$

is the transfer function of the servomotor. Let the nominal values of the parameters be

$$\begin{aligned} K_c &= 0.0435 \text{ [oz} - \text{in./V]}, \\ K_n &= 0.0119 \text{ [oz} - \text{in./rad/sec]}, \\ J &= 7.77 \times 10^{-4} \text{ [oz} - \text{in.} - \text{sec}^2\text{]}, \text{ and} \\ f &= 0.005 \text{ [oz} - \text{in./rad/sec]}. \end{aligned}$$

These values are scaled up by 10^3, so that the nominal transfer function is

$$G_0(s) = \frac{43.5}{s(0.777s + 16.9)}.$$

Now let K_c and K_n vary in the intervals

$$K_c \in [43.5 - \epsilon, 43.5 + \epsilon],$$
$$K_n \in [11.9 - 0.7\epsilon, 11.9 + 0.7\epsilon].$$

If $K = f + K_n$, then

$$K \in [16.9 - 0.7\epsilon, 16.9 + 0.7\epsilon],$$

and

$$G(s) = \frac{K_c}{s(Js + K)}.$$

The block diagram in Figure 13.12 shows the closed-loop system with the interval plant $G(s)$ being the servomotor, $C(s)$ the stabilizing controller to be designed, and a position sensor of gain K_p in the feedback loop. Let us choose $K_p = 1$ so that we have a unity feedback system.

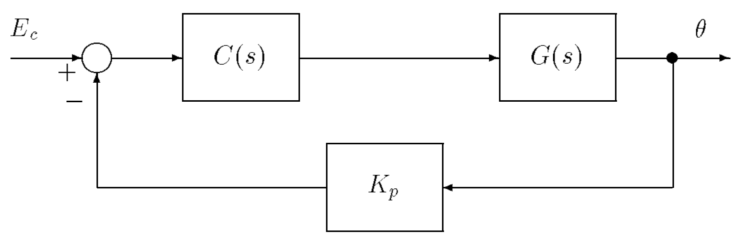

Figure 13.12. A feedback system (Example 13.3)

Since $G_0(s)$ does not have a pole in the RHP, it is necessary that $16.9 - 0.7\epsilon > 0$. This implies $\epsilon < 24.143$ in order to guarantee that no plant in the family has a pole in the RHP. The pole at the origin is assumed to be preserved under perturbations. Hence the uncertainty bound $r(s)$ must also have a pole at $s = 0$. So, we can write

$$r(s) = \frac{r'(s)}{s};$$

where $r'(s)$ is a minimum phase H_∞ function. Here $B(s) = 1$. Let

$$\tilde{G}_0(s) = sG_0(s) = \frac{43.5}{0.777s + 16.9}; \text{ and}$$
$$\tilde{q}(s) = \frac{q(s)}{s}.$$

Now an SBR function $u(s)$ needs to be found such that $u(s)$ interpolates to

$$\begin{aligned} u(0) &= \frac{r'(0)}{\tilde{G}_0(0)} = \frac{r'(0)}{2.574}, \\ u(\infty) &= 0, \end{aligned}$$

and it is necessary that $|u(0)| < 1$ which implies $|r'(0)| < 2.574$. For a constant r' we will have $r' = 2.574$. From the plot of δ vs ϵ as shown in Figure 13.13, we have $\epsilon_{\max} = 9.4$.

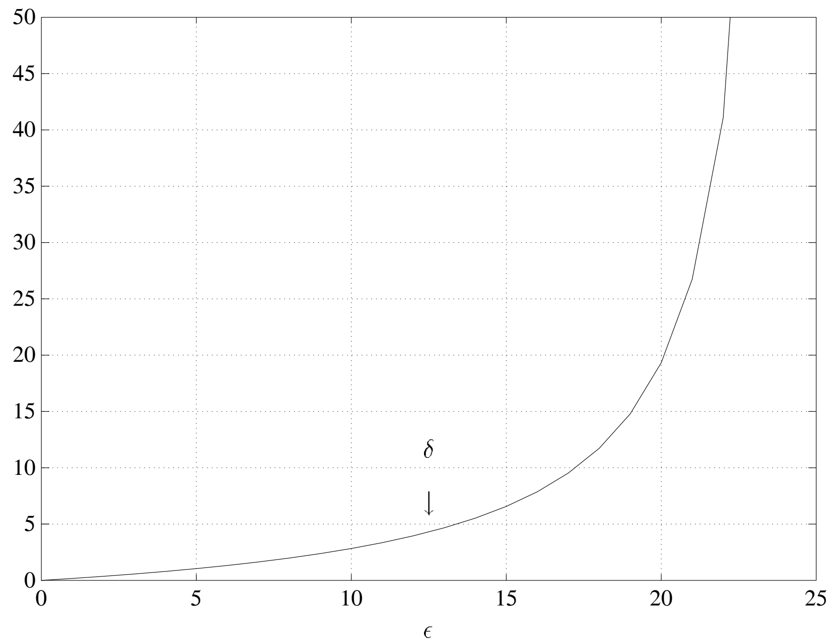

Figure 13.13. δ vs ϵ (Example 13.3)

Therefore, choosing $r' = 2.574$, we have $u(0) = 1$. A $u(s)$ which will interpolate to these two points can be chosen as

$$u(s) = \frac{1}{s+1}.$$

So

$$\begin{aligned} \tilde{q}(s) &= \frac{u(s)}{r'} = \frac{1}{2.574(s+1)}, \text{ and} \\ Q(s) &= s\tilde{Q}(s) = \frac{s}{2.574(s+1)}. \end{aligned}$$

Therefore

$$C(s) = \frac{q(s)}{1 - G_0(s)Q(s)} = \frac{0.777s + 16.9}{2s + 45.5},$$

which should stabilize $G(s)$ for $\epsilon = 9.4$, i.e.,

$$K_c \in [34.1, 52.9], \qquad \text{and} \qquad K_n \in [5.32, 18.48].$$

A controller for perturbations greater than $\epsilon = 9.4$ can be obtained by choosing poles and zeros in $r'(s)$ that is, loop-shaping $r'(s)$ appropriately and redoing the interpolation.

The controller that results on applying the techniques described above depends on the procedure and is not guaranteed to succeed if the level of parametric uncertainty ϵ is specified apriori. However, it always produces a controller which robustly stabilizes the system against parametric uncertainty for small enough ϵ. The technique can be extended, in an obvious way to linear interval systems as well as multilinear interval systems using the extremal image set generating properties of the generalized Kharitonov segments, established in Chapters 8 and 11. We leave the details of this to the reader.

13.7 ROBUST STABILIZATION: STATE SPACE SOLUTION

In this section, we describe, without proof, the state-space approach to solving a standard H_∞ problem, which is to find an output feedback controller so that the H_∞ norm of the closed-loop transfer function is (strictly) less than a prescribed positive number γ. The existence of the controller depends upon the unique stabilizing solutions to two algebraic Riccati equations being positive definite and the spectral radius of their product being less than γ^2.

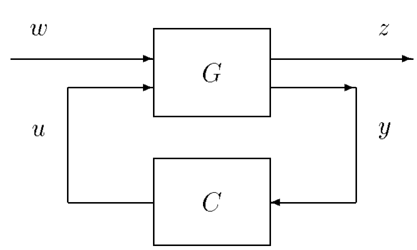

Figure 13.14. A feedback system

The feedback configuration is shown in Figure 13.14 where G is a linear system described by the state space equation:

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1x + D_{11}w + D_{12}u$$
$$y = C_2x + D_{21}w + D_{22}u$$

The following assumptions are made:

Assumption 13.1.

1) (A, B_1) is stabilizable and (C_1, A) is detectable.

2) (A, B_2) is stabilizable and (C_2, A) is detectable.

3) $D_{12}^T[C_1 \; D_{12}] = [0 \; I]$.

4) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Assumptions 1 and 2 together simplify the theorem statement and proof and also imply that internal stability is essentially equivalent to input-output stability ($T_{zw} \in \mathcal{RH}_\infty$). Assumptions 3 and 4 simplify the controller formula. It is also assumed that $D_{11} = D_{22} = 0$. This assumption is also made to simplify the formulas substantially.

Considering the Riccati equation

$$A^TX + XA + XRX - Q = 0, \tag{13.42}$$

where R and Q are real symmetric matrices and $X = Ric(H)$ denotes the solution of the Riccati equation associated with the Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix}. \tag{13.43}$$

The solution to the H_∞ optimal control problem is given by the following theorem.

Theorem 13.4 *There exists a compensator $C(s)$ such that*

$$\|T_{zw}(s)\|_\infty < \gamma$$

if and only if

1) $X_\infty = Ric(H_\infty) \geq 0$; *2)* $Y_\infty = Ric(J_\infty) \geq 0$; *3)* $\rho(X_\infty Y_\infty) < \gamma^2$;

where

$$H_\infty = \begin{bmatrix} A & \dfrac{B_1B_1^T}{\gamma^2} - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}; \tag{13.44}$$

$$J_\infty = \begin{bmatrix} A^T & \dfrac{C_1C_1^T}{\gamma^2} - C_2C_2^T \\ -B_1B_1^T & -A \end{bmatrix}; \tag{13.45}$$

and $\rho(\cdot)$ denotes the spectral radius of a matrix.

When the above condition holds, a controller is given by

$$C(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right] \tag{13.46}$$

where

$$\begin{aligned} \hat{A}_\infty &= A + \frac{B_1 B_1^T X_\infty}{\gamma^2} + B_2 F_\infty + Z_\infty L_\infty C_2; \\ L_\infty &= -Y_\infty C_2^T; \\ F_\infty &= -B_2^T X_\infty; \\ Z_\infty &= \left(I - \frac{Y_\infty X_\infty}{\gamma^2}\right)^{-1}. \end{aligned}$$

The following example uses the above method to synthesize a robust controller.

Example 13.4. Given the augmented plant

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & -1 \\ \hline 0 & 0 & 0.2 \\ -1 & 1 & 0 \end{array}\right].$$

Let γ be chosen as 1. In the MATLAB Robust Control ToolBox, the function *hinfkgjd*, uses the above method to check stabilizability and generate the compensator. Using this function,

$$H = \begin{bmatrix} 1 & 25 \\ 0 & -1 \end{bmatrix},$$

$$J = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

and their eigenvalues,

$$X = 0.08 \geq 0, \quad Y = 2 \geq 0,$$

and the spectral radius of their product,

$$\rho(XY) = 0.16 < \gamma^2 = 1.$$

Hence, the given system is stabilizable and the MATLAB function *hinfkgjd* gives the following compensator

$$C(s) := \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} -3.381 & -2 \\ \hline 2.381 & 0 \end{array}\right],$$

or equivalently

$$C(s) = \frac{-4.762}{s + 3.381}.$$

It is found that

$$T_{zw} = \frac{0.952(1 - s)}{s^2 + 2.381s + 1.381},$$

and

$$\|T_{zw}\|_\infty = 0.69 < 1.$$

We know that the closed loop system configuration shown in Figure 13.4 is a special case of the general feedback configuration shown in Figure 13.14. Figure 13.15 shows the same closed loop system, but with the plant block being broken into the nominal plant G_0 block and the perturbation block ΔG. The notation for signals used in Figure 13.15 is the same as in Figure 13.14. Here, x is the state vector of the nominal plant; u is the controlled input; z is the error signal and y is the measured variable vector which is contaminated by the disturbance w.

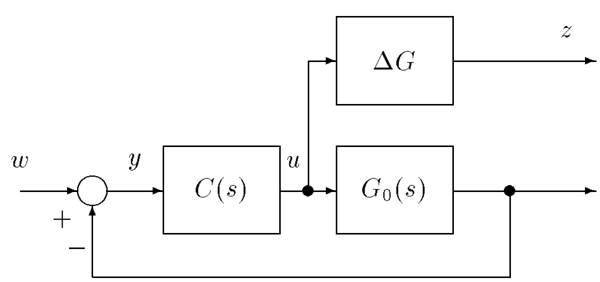

Figure 13.15. Closed-loop system configuration

Although Figure 13.15 shows the nominal plant and the perturbation as separate blocks, the perturbation is basically inherent to the plant and the error signal z is due to the plant perturbation. It is assumed that the compensator $C(s)$ stabilizes the nominal system. In order to minimize the effect of external disturbances, $C(s)$ should be such that the error signal z should not respond excessively to the external disturbance w. This translates to $\|T_{zw}(s)\|_\infty < \gamma$, where γ is some given number.

The transfer function $T_{zw}(s)$ is given by

$$T_{zw}(s) = \frac{C(s)(\Delta G)}{1 + C(s)G_0(s)}. \tag{13.47}$$

For this configuration, the block ΔG is uncontrollable and it makes sense to minimize the H_∞ norm of the transfer function from w to u:

$$T_{uw}(s) = \frac{C(s)}{1 + C(s)P_0(s)}. \tag{13.48}$$

Then, assuming a strictly proper plant, the state space equations will be

$$\dot{x} = Ax + B_2 u \tag{13.49}$$
$$y = C_2 x + w, \tag{13.50}$$

Here we have B_1 and C_1 as zero vectors, $D_{12} = 1$ and D_{21} as unit vectors so assumptions 2, 3 and 4 hold here.

We already know that $\|\Delta G(s)\|_\infty \leq r$. From the Small Gain Theorem, a necessary and sufficient condition for robust stability of our system is

$$\|(1 + G_0(s)C(s))^{-1} C(s) r\|_\infty < 1 \tag{13.51}$$
$$\Longrightarrow \|T_{uw}(s)\|_\infty < \frac{1}{r} = \gamma. \tag{13.52}$$

Now, with the help of Theorem 13.4, we can find $\gamma_{\min}$, the minimum value of γ that corresponds to the maximum value of r, $r_{\max}$, for which the system is robustly stabilizable. Once $r_{\max}$ is found, the corresponding $\epsilon_{\max}$ can be found from the r vs. ϵ plot of the plant. The controller $C^*(s)$ obtained for $r_{\max}$ is one that stabilizes the family of plants $\mathbf{G}_{\epsilon_{\max}}$.

Example 13.5. Let the interval plant be

$$G(s) = \frac{20(s^2 + as - 9)}{s^3 + bs^2 + s + c},$$

where

$$a \in [8 - \epsilon, 8 + \epsilon], \quad b \in [-4 - \epsilon, -4 + \epsilon], \quad c \in [6 - \epsilon, 6 + \epsilon],$$

and the nominal plant

$$G_0(s) = \frac{20(s^2 + 8s - 9)}{s^3 - 4s^2 + s + 6}.$$

Checking the Kharitonov polynomials of the denominator of $G(s)$, it is found that the number of unstable poles remains unchanged as long as $\epsilon < \epsilon_1 = 4$.

The state-space matrices for G_0 are

$$A = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$
$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$
$$C_1 = [0\ 0\ 0]; \quad C_2 = [-20\ \ -160\ 180];$$
$$D_{11} = 0; \quad D_{12} = 1; \quad D_{21} = 1; \quad D_{22} = 0.$$

Now, an initial value of γ is chosen arbitrarily and γ is decreased in steps with the stabilizability condition being checked at each step, until a $\gamma = \gamma_{\min}$ is reached after which one of the conditions of Theorem 13.4 fail. Here $\gamma_{\min}$ is found to be 0.333 so that $r_{\max}$ is 3. It was found that the spectral radius ρ for $\gamma = 0.333$ was

$$\rho = 0.105 < \gamma^2 = 0.111.$$

The compensator corresponding to $\gamma = 0.333$ is

$$C^*(s) = \frac{2.794s^2 + 19.269s + 16.475}{s^3 + 24.531s^2 + 202.794s + 544.101}.$$

Now $\epsilon_{\max}$ needs to be found from the r vs. ϵ plot. Again, the extremal segments of $G(s)$ are searched to obtain r for each ϵ. From the plot of r vs. ϵ in Figure 13.16, we have $\epsilon_{\max} = 0.4$.

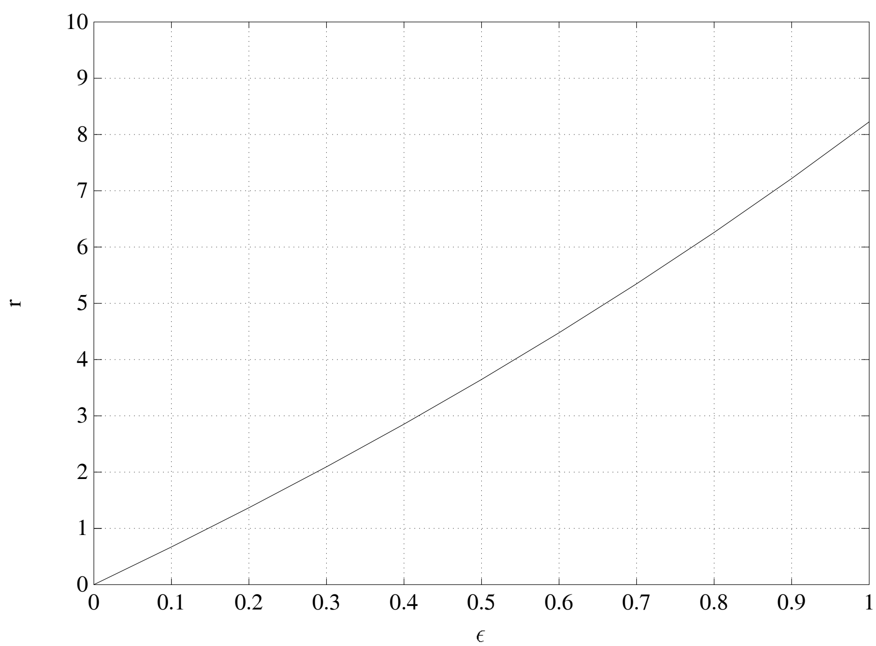

Figure 13.16. r vs ϵ (Example 13.5)

Therefore, $C^*(s)$ stabilizes the entire family $G(s)$ where

$$a \in [7.6, 8.4], \quad b \in [-4.4, -3.6], \quad c \in [5.6, 6.4].$$

The following example repeats Example 13.2 using the 2-Riccati equation.

Example 13.6.

$$G_0(s) = \frac{5s+4}{(s-3)(s+5)} = \frac{5s+4}{s^2+2s-15}.$$

The interval plant is

$$G(s) = \frac{5s+a}{s^2+bs+c}$$

with the intervals being

$$a \in [4-\epsilon, 4+\epsilon], \quad b \in [2-\epsilon, 2+\epsilon], \quad c \in [-15-\epsilon, -15+\epsilon].$$

Here the system and input-output matrices are:

$$\begin{aligned}
A &= \begin{bmatrix} -2 & 15 \\ 1 & 0 \end{bmatrix}; \\
B_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \\
C_1 &= [0 \quad 0]; \quad C_2 = [-5 \quad -4]; \\
D_{11} &= 0; \quad D_{12} = 1; \quad D_{21} = 1; \quad D_{22} = 0.
\end{aligned}$$

Since we already know from Example 13.2 that $r_{\max} = 0.395$, we put $\gamma_{\min} = 1/0.395 = 2.532$ and check if the conditions of the Theorem are satisfied. The conditions were found to be satisfied with the spectral radius $\rho(X_\infty Y_\infty) = 6.382 < \gamma^2 = 6.386$. Since we require that

$$\rho(X_\infty Y_\infty) < \gamma^2,$$

and the value $\frac{\rho}{\gamma}$ is close to 1 we may conclude that $r_{\max} = 0.395$. The controller obtained is

$$C^*(s) = 28000 \, \frac{s+5}{s^2+11090s-18070}.$$

$C^*(s)$ was found to stabilize all the extremal segments for the family of plants corresponding to $\epsilon = \epsilon_{\max} = 2.83$.

13.8 A ROBUST STABILITY BOUND FOR INTERVAL SYSTEMS

In this section we derive a bound on the parameter excursions allowed in an interval plant for which closed loop stability is preserved, with a given fixed controller. The result described here does not require the restriction that the number of unstable poles of the open loop system remain invariant under perturbations. The derivation is tailored to exploit an H_∞ technique which can be used to maximize this bound over the set of all stabilizing controllers.

Consider the feedback configuration of Figure 13.17.

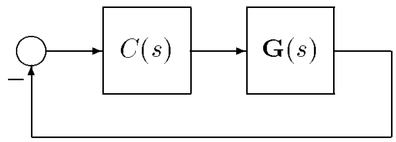

Figure 13.17. Interval Control System

Let $L(s) = C(s)G(s)$ denote the *loop gain*, $G_0(s)$ the nominal plant and $C(s)$ the fixed controller stabilizing $G_0(s)$. Let

$$C(s) = \frac{N_c(s)}{D_c(s)}, \qquad G_0(s) = \frac{N_p^0(s)}{D_p^0(s)} \qquad G(s) = \frac{N_p(s)}{D_p(s)} \tag{13.53}$$

$$N(s) = N_c(s)N_p(s), \qquad N_0(s) = N_c(s)N_p^0(s) \tag{13.54}$$

and

$$D(s) = D_c(s)D_p(s), \qquad D_0(s) = D_c(s)D_p^0(s). \tag{13.55}$$

We investigate the stability of the feedback system when the loop gain is perturbed from its nominal value L_0. Writing

$$L(s) = \frac{N(s)}{D(s)}, \qquad L_0(s) = \frac{N_0(s)}{D_0(s)}. \tag{13.56}$$

We make the following standing assumption:

Assumption 13.2. The polynomials $D(s)$ and $N(s)$ have the same degrees as the nominal polynomials $D_0(s)$ and $N_0(s)$, and $L_0(s)$ is proper.

Let

$$S_0(s) = \frac{D_0(s)}{D_0(s) + N_0(s)} \quad \text{and} \quad T_0(s) = \frac{N_0(s)}{D_0(s) + N_0(s)} \tag{13.57}$$

denote the sensitivity and complementary sensitivity functions respectively.

Theorem 13.5 (Robust Stability Criterion)
The closed loop system remains stable under all perturbations satisfying

$$\left| \frac{(D(j\omega) - D_0(j\omega))/D_0(j\omega)}{W_1(j\omega)} \right|^2 + \left| \frac{(N(j\omega) - N_0(j\omega))/N_0(j\omega)}{W_2(j\omega)} \right|^2 < \frac{1}{\delta} \tag{13.58}$$

for all $\omega \in [-\infty, \infty]$ *if*

$$|W_1(j\omega)S_0(j\omega)|^2 + |W_2(j\omega)T_0(j\omega)|^2 \le \delta, \qquad \textit{for all } \ \omega \in [-\infty, \infty]$$

where $W_1(s)$ *and* $W_2(s)$ *are given stable proper rational functions.*

Proof. The perturbed and nominal closed loop characteristic polynomials are respectively,

$$D(s)+N(s) \quad \text{and} \quad D_0(s)+N_0(s).$$

Applying the Principle of the Argument to the proper rational function

$$\sigma(s) := \frac{D(s)+N(s)}{D_0(s)+N_0(s)}$$

leads to the conclusion that the perturbed closed loop system is stable if and only if the image of $\sigma(s)$ does not encircle the origin as s traverses the imaginary axis $j\omega$. Now let us rewrite $\sigma(s)$ in the form

$$\begin{aligned}
\sigma(s) &= \frac{D(s)+N(s)}{D_0(s)+N_0(s)} \\
&= \frac{D(s)}{D_0(s)} \underbrace{\frac{D_0(s)}{D_0(s)+N_0(s)}}_{S_0(s)} + \frac{N(s)}{N_0(s)} \underbrace{\frac{N_0(s)}{D_0(s)+N_0(s)}}_{T_0(s)} \\
&= 1 + \left[\frac{D(s)}{D_0(s)} - 1\right] S_0(s) + \left[\frac{N(s)}{N_0(s)} - 1\right] T_0(s).
\end{aligned}$$

Obviously a sufficient condition for the image of this function to not encircle the origin is that

$$\left| \left[\frac{D(j\omega)}{D_0(j\omega)} - 1\right] S_0(j\omega) + \left[\frac{N(j\omega)}{N_0(j\omega)} - 1\right] T_o(j\omega) \right| < 1$$

for $\omega \in [-\infty, \infty]$ or equivalently

$$\begin{aligned}
&\Bigg| \frac{(D(j\omega)-D_0(j\omega))/D_0(j\omega)}{W_1(j\omega)} W_1(j\omega) S_0(j\omega) + \\
&\qquad \frac{(N(j\omega)-N_0(j\omega))/N_0(j\omega)}{W_2(j\omega)} W_2(j\omega) T_0(j\omega) \Bigg| < 1
\end{aligned} \tag{13.59}$$

From the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\Bigg| \frac{(D(j\omega)-D_0(j\omega))/D_0(j\omega)}{W_1(j\omega)} W_1(j\omega) S_0(j\omega) + \\
&\qquad \frac{(N(j\omega)-N_0(j\omega))/N_0(j\omega)}{W_2(j\omega)} W_2(j\omega) T_0(j\omega) \Bigg|^2 \\
\leq &\left(\left| \frac{(D(j\omega)-D_0(j\omega))/D_0(j\omega)}{W_1(j\omega)} \right|^2 + \left| \frac{(N(j\omega)-N_0(j\omega))/N_0(j\omega)}{W_2(j\omega)} \right|^2 \right) \\
&\qquad\qquad \cdot \left(|W_1(j\omega) S_0(j\omega)|^2 + |W_2(j\omega) T_0(j\omega)|^2 \right).
\end{aligned}$$

Thus the conditions given in the theorem guarantee that (13.59) holds. This prevents any encirclements of the origin by the Nyquist locus of $\sigma(s)$ which proves that stability is preserved. ♣

Suppose now that a controller stabilizing the nominal plant is given and that

$$\sup_{\omega \in \mathbf{R}} \left(|W_1(j\omega)S_0(j\omega)|^2 + |W_2(\omega)T_0(j\omega)|^2 \right) = \delta. \tag{13.60}$$

The theorem derived above shows that the smaller δ is, the more robust the feedback system is. An interpretation of this is that robust stability is obtained if a weighted sum of the input and output signals (y_1 and y_2) as in Figure 13.18 can be tightly bounded.

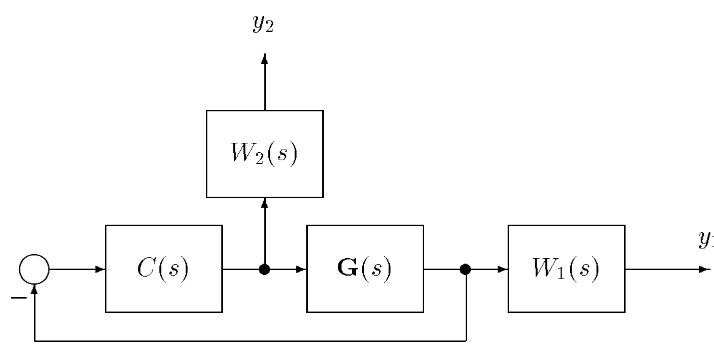

Figure 13.18. Interpretation of robust stability

We shall now relate δ to the allowable parameter excursions in the plant family $\mathbf{G}(s)$, assuming specifically that $\mathbf{G}(s)$ is an interval plant.

We first introduce an obvious but useful property of interval polynomials from which our result will follow. Let

$$p(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_q s^q$$

denote a real polynomial, $\mathbf{p}$ the vector of coefficients and $\mathbf{\Pi}$ be the box in the coefficient space defined as

$$p_i \in [p_i^0 - w_i\epsilon, p_i^0 + w_i\epsilon], \quad i = 0, 1, 2, \cdots, q$$

where p_i^0 and w_i are the nominal value and a weight, respectively. Then interval family of polynomials is defined as

$$\mathbf{p}(s,\epsilon) = \left\{ p(s,\epsilon) = p_0 + p_1 s + p_2 s^2 + \cdots + p_q s^q : \mathbf{p} \in \mathbf{\Pi} \right\} \tag{13.61}$$

Let $p^0(s)$ be the nominal polynomial with coefficients p_i^0 for $i = 0, 1, 2, \cdots, q$ and let $K_p^k(s,\epsilon)$, $i = 1, 2, 3, 4$ denote the Kharitonov polynomials associated with the interval family $\mathbf{p}(s,\epsilon)$.

Lemma 13.1 *For each ω,*

$$\begin{aligned}\sup_{\mathbf{p}\in\mathbf{\Pi}} \left|p(j\omega,\epsilon)-p^0(j\omega)\right| &= \max_{k=1,2,3,4}\left|K_p^k(j\omega,\epsilon)-p^0(j\omega)\right| \\ &= \left|K_p^i(j\omega,\epsilon)-p^0(j\omega)\right| \quad \text{for any } i=1,2,3,4 \\ &= \epsilon\left|K_p^i(j\omega,1)-p^0(j\omega)\right| \end{aligned} \tag{13.62}$$

Proof. The proof of this lemma follows from the fact that the image set $\mathbf{p}(j\omega,\epsilon)$, at each ω is a rectangle with corners equal to the images of the Kharitonov polynomials and moreover the distance from the center to each corner is equal. ♣

In fact it is easy to see that

$$\sup_{\mathbf{p}\in\mathbf{\Pi}}\left|p(j\omega,\epsilon)-p^0(j\omega)\right| = \epsilon\left|w_0+w_2\omega^2+w_4\omega^4\cdots+j\omega(w_1+w_3\omega^2+w_5\omega^4\cdots\right|$$

Now let us bring in the interval plant parametrized by μ:

$$\begin{aligned}\mathbf{G}(s) &= \frac{\mathbf{N}_{\mathrm{p}}(s)}{\mathbf{D}_{\mathrm{p}}(s)} \\ &= \left\{G(s)=\frac{n_0+n_1s+\cdots+n_qs^q}{d_0+d_1s+\cdots+d_qs^q}:\right. \\ &\qquad\left. n_i\in[n_i^-,n_i^+],\, d_i\in[d_i^-,d_i^+],\ i=0,1,2,\cdots,q\right\}\end{aligned}$$

where

$$n_i\in[n_i^0-w_n^i\mu,\, n_i^0+w_n^i\mu], \quad \text{and} \quad d_i\in[d_i^0-w_d^i\mu,\, d_i^0+w_d^i\mu]$$

and d_i^0, n_i^0 are nominal values. For convenience, we let $K_{D_p}(s,\mu)$ and $K_{N_p}(s,\mu)$ denote any of the Kharitonov polynomials of $\mathbf{D}_p(s)$ and $\mathbf{N}_p(s)$, respectively. Our problem is to determine how large μ can be without losing stability.

Suppose that the weighting functions $W_1(s)$ and $W_2(s)$ are selected to satisfy

$$\left|\frac{K_{D_p}(j\omega,1)-D_p^0(j\omega)}{D_p^0(j\omega)}\right| < |W_1(j\omega)| \tag{13.63}$$

$$\left|\frac{K_{N_p}(j\omega,1)-N_p^0(j\omega)}{N_p^0(j\omega)}\right| < |W_2(j\omega)|\,. \tag{13.64}$$

Theorem 13.6 *Let weighting functions $W_1(s)$ and $W_2(s)$ be selected satisfying (13.63) and (13.64), and let $C(s)$ be a stabilizing compensator for the nominal plant $G_0(s)$. If*

$$\left\|\ |W_1(s)S_0(s)|^2+|W_2(s)T_0(s)|^2\ \right\|_\infty=\delta \tag{13.65}$$

then $C(s)$ robustly stabilizes the interval plant $\mathbf{G}(s)$ for all $\mu < \mu_{\max}$ where

$$\mu_{\max} = \frac{1}{\sqrt{2\delta}}. \tag{13.66}$$

Proof. Recall that

$$C(s) = \frac{N_c(s)}{D_c(s)}, \qquad G_0(s) = \frac{N_p^0(s)}{D_p^0(s)} \qquad G(s) = \frac{N_p(s)}{D_p(s)} \tag{13.67}$$

and

$$N(s) = N_c(s)N_p(s), \qquad N_0(s) = N_c(s)N_p^0(s), \tag{13.68}$$
$$D(s) = D_c(s)D_p(s), \qquad D_0(s) = D_c(s)D_p^0(s). \tag{13.69}$$

From Lemma 13.1 we know that for a fixed ω

$$\left|\frac{D(j\omega,\mu) - D_0(j\omega)}{D_0(j\omega)}\right| \le \mu \left|\frac{\left[K_{D_p}(j\omega,1) - D_p^0(j\omega)\right]}{D_p^0(j\omega)}\right|$$
$$\left|\frac{N(j\omega,\mu) - N_0(j\omega)}{N_0(j\omega)}\right| \le \mu \left|\frac{\left[K_{N_p}(j\omega,1) - N_p^0(j\omega)\right]}{N_p^0(j\omega)}\right|.$$

for $D_p(s) \in \mathbf{D}_p(s)$ and $N_p(s) \in \mathbf{N}_p(s)$. Now

$$\sup_{D_p(s)\in\mathbf{D}_p(s)} \left|\frac{D(j\omega) - D_0(j\omega)}{D_0(j\omega)}\right| = \mu \left|\frac{K_{D_p}(j\omega,1) - D_p^0(j\omega)}{D_p^0(j\omega)}\right| < \mu \left|W_1(j\omega)\right|$$
$$\sup_{N_p(s)\in\mathbf{N}_p(s)} \left|\frac{N(j\omega) - N_0(j\omega)}{N_0(j\omega)}\right| = \mu \left|\frac{K_{N_p}(j\omega,1) - N_p^0(j\omega)}{N_p^0(j\omega)}\right| < \mu \left|W_2(j\omega)\right|.$$

Therefore the condition (13.58) for robust stability given in Theorem 13.5 is satisfied if

$$\mu^2 + \mu^2 < \frac{1}{\delta}. \tag{13.70}$$

It follows that

$$\mu_{\max} = \frac{1}{\sqrt{2\delta}}.$$

♣

Remark 13.2. It is clear that $\mu_{\max}$ can be increased (and therefore larger parameter excursions allowed) by choosing $C(s)$ to make δ small. There are techniques available in the H_∞ control literature for designing the controller $C(s)$ to minimize δ over the set of all stabilizing controllers for $G_0(s)$. The details of this procedure are omitted and we refer the reader to the literature.

Summary

In this chapter we have presented two synthesis results concerning the robust stability problem. These results are but two steps towards the solution to the general robust synthesis problem, but so little is known about this solution that every step is of interest. We have shown here that a minimum phase interval plant can always be stabilized by a stable controller of order $n - m - 1$ regardless of the magnitude of the perturbations. Many problems are still to be solved in this domain. For example, it would be interesting to extend the robust state feedback stabilization of Chapter 5 to the case when (A, B) are not in controllable companion form. Another open problem of particular interest is the following. In view of the generalization of Kharitonov's theorem given in Chapter 7, consider a Single-Input Single-Output interval plant and find the necessary and sufficient conditions for the existence of a stabilizing controller. Moreover, if stabilization is possible, give a constructive method for finding the controller. If the four Kharitonov polynomials associated with the numerator of the interval plant are stable, then every plant in the family is minimum phase, and the problem is reduced to that of Section 13.2 of this chapter and it is always possible to find a solution. On the other hand, if these Kharitonov polynomials are not all stable, there will not always be a solution and the solution if it exists certainly need not be a 'high-gain' controller as in Section 13.2.

The multivariable versions of the problems of robust stabilization under additive and multiplicative unstructured uncertainty described in this chapter are fully developed and well documented in the literature. This theory is based on the YJBK parametrization of all stabilizing controllers, inner outer factorization of transfer matrices and on minimal H_∞ norm solutions to matrix interpolation problems. These results are thoroughly treated in textbooks on H_∞ control theory. Moreover they relate only indirectly to robustness under parametric uncertainty and since our purpose is only to demonstrate by examples the use of H_∞ techniques in robust parametric problems we omit their treatment here. We have also omitted various other techniques such as robustness under coprime factor perturbations as developed in the H_∞ literature, which can certainly be profitably used in parametric uncertainty problems. In the specific case of coprime factor perturbations, we could relax the assumption of constant number of RHP poles in the family of uncertain plants, and the absence of $j\omega$ axis poles in the nominal plant $G_0(s)$. Establishing connections between parametric and nonparametric uncertainty, and extending robust parametric results to multivariable systems, is an area of ongoing research. The simple examples given here may be the beginnings of a much more sophisticated framework.

13.9 EXERCISES

13.1 Find a robust stabilizing controller that stabilizes the following interval plant.

$$G(s) = \frac{n(s)}{d(s)} = \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}$$

where the parameters vary as follows:

$$\begin{array}{llll} \alpha_2 \in [1,2], & \alpha_1 \in [2,4], & \alpha_0 \in [1,3], & \\ \beta_3 \in [1,2], & \beta_2 \in [-1,2], & \beta_1 \in [0.5,1], & \beta_0 \in [-1.5,1]. \end{array}$$

13.2 Repeat Exercise 13.1 with the following interval plant.

$$G(s) = \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{\beta_4 s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}$$

where

$$\begin{array}{lll} \alpha_2 \in [1,3], & \alpha_1 \in [0.5,1.5], & \alpha_0 \in [2.5,3.5], \\ \beta_4 \in [-3,-1], & \beta_3 \in [-2,2], & \beta_2 \in [-1.5,1.5], \\ \beta_1 \in [0.5,-0.5], & \beta_0 \in [-1.5,-0.5]. & \end{array}$$

13.3 Consider the plant

$$G(s) = \frac{5s + a}{s^2 + bs + c}$$

and suppose that the parameters vary within the intervals

$$a \in [4-\epsilon, 4+\epsilon], \quad b \in [2-\epsilon, 2+\epsilon], \quad c \in [-15-\epsilon, -15+\epsilon].$$

Plot the relationship between parametric uncertainty ϵ and maximal unstructured uncertainty $\delta(\epsilon, \omega)$. What is the constant $\delta(\epsilon)$ which corresponds to the parametric uncertainty $\epsilon = 2$.
Answer: For $\epsilon = 2$, $\delta = 0.27$.

13.4 Example 13.1 used a constant r to represent the unstructured perturbation $\|\Delta G(j\omega)\|$. Repeat this example with a rational function $r(s)$ that loopshapes the uncertainty function $\delta(\epsilon, \omega)$.

13.5 Using the same nominal plant and controller transfer functions as in Example 13.1, solve the following problem:

1) Compute the maximum allowable perturbation $\epsilon_{\max}$ by using the method described in Section 13.8. Choose the weight functions $W_1(s)$ and $W_2(s)$ to be both 1.

2) Using GKT (Chapter 7), compute $\epsilon_{\max}$.

3) Compare these two results and discuss the difference.

13.6 Repeat Example 13.1 by using the 2 Riccati equation method described in Section 13.7.

Hint: MATLAB function *hinfkgjd* is needed to solve this problem. The function is a part of MATLAB Robust Control ToolBox.

13.7 In Example 13.1 the constant r was used to bound the structured uncertainty.

1) Rework this problem with a rational function $r(s)$ that is loopshaped to bound the uncertainty function $\delta(\omega)$.

2) Let us call the controller obtained from Example 13.1 $C_1(s)$ and the solution of 1) $C_2(s)$. With these respective controllers, compute the true maximum structured uncertainty bound $\epsilon_{\max}$.

Hint: Once the controller is given, the true $\epsilon_{\max}$ can be found by applying GKT.

13.8 Let the nominal plant be

$$G_0(s) = \frac{30s+10}{(s+2)(s-3)(s-2)} = \frac{30s+10}{s^3-3s^2-4s+12}$$

and the interval plant

$$G(s) = \frac{30s+a}{s^3+bs^2+cs+d}$$

with the intervals being

$$a \in [10-\epsilon, 10+\epsilon], \quad b \in [-3-\epsilon, -3+\epsilon], \quad c \in [-4-\epsilon, -4+\epsilon], \quad d \in [12-\epsilon, 12+\epsilon].$$

Find a robust stabilizing controller and the corresponding value of ϵ.

13.10 NOTES AND REFERENCES

The results on simultaneous strong stabilization given in this chapter are based on Chapellat and Bhattacharyya [60]. The problem treated in Section 13.2 has also been treated by Barmish and Wei [18]. The problem of simultaneous stabilization of a discrete set of plants has been considered by Vidyasagar and Viswanadham [235] Saeks and Murray [200] and Blondel [41]. Property 13.3 may be found in the book of Marden [175]. Rantzer and Megretski have given a convex parametrization of robust parametric stabilizers [196]. The section on Nevanlinna-Pick interpolation is adapted from Dorato, Fortuna and Muscato [85]. A proof of the Nevanlinna algorithm for the matrix case, as well as multivariable versions of the unstructured, additive, multiplicative and coprime factor perturbation problems are thoroughly treated in Vidyasagar [231]. The application of H_∞ techniques for robust parametric synthesis is adapted from S. Bhattacharya, L.H. Keel and S.P. Bhattacharyya [28]. The state space solution of the H_∞ optimal control problem via Riccati equations, Theorem 13.4, is due to Doyle, Glover, Khargonekar and Francis [91]. The proof of Theorem 13.2 can be found in Walsh [236]. Theorem 13.5 is due to Kwakernaak [157] and the result of Theorem 13.6 is due to Patel and Datta [186].