Chapter 12

STATE SPACE PARAMETER PERTURBATIONS

In this chapter we describe some robust parametric results formulated specifically for state space models. We first deal with the robust stability problem for interval matrices. When the parameters appear in the matrices in a unity rank perturbation structure, the characteristic polynomial of the matrix is a multilinear function of the parameters. This allows us to use the Mapping Theorem described in Chapter 11 to develop a computational algorithm based on calculating the phase difference over the vertices of the parameter set. Next we introduce some Lyapunov based methods for parameter perturbations in state space systems. A stability region in parameter space can be calculated using this technique and a numerical procedure for enlarging this region by adjusting the controller parameters is described. We illustrate this algorithm with an example. The last part of the chapter describes some results on matrix stability radius for the real and complex cases and for some special classes of matrices.

12.1 INTRODUCTION

Most of the results given in this book deal with polynomials containing parameter uncertainty. These results can be directly used when the system model is described by a transfer function whose coefficients contain the uncertain parameters. When the system model is described in the state space framework, the parameters appear as entries of the state space matrices. The polynomial theory can then be applied by first calculating the characteristic polynomial of the matrix as a function of its parameters. In this chapter, the aim is to provide some computational procedures which can determine robust stability and compute stability margins for the case in which parameters appear linearly in the state space matrices. Under the technical assumption that the perturbation structure is of unity rank the characteristic polynomial coefficients depend on the parameters in multilinear form. This allows us to use the Mapping Theorem of Chapter 11 to develop an effective computational technique to determine robust stability. This is illustrated with numerical examples.

Next, we describe a Lyapunov based technique to handle perturbations of state space matrices. A stability region in parameter space is determined by this method. While this method does not require us to compute the characteristic polynomial, the stability region obtained from this method is conservative. However, the method gives a direct way to handle perturbations of state space matrices. Furthermore, with these formulas, the stability margin can be increased through optimization over the controller parameter space. This procedure is referred to as robustification. The results are compared with the previously described method, which used the Mapping Theorem, via examples.

In the last part of the chapter we describe some formulas for the matrix stability margin for the real and complex cases and for some special classes of matrices.

12.2 STATE SPACE PERTURBATIONS

Consider the state space description of a linear system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{12.1}$$

with the output feedback control

$$u = Ky. \tag{12.2}$$

The stability of the closed loop system is determined by the stability of the matrix $M := A + BKC$. We suppose that the matrices A, B, K, and C are subject to parameter perturbations. Let

$$\mathbf{p} := [p_1, p_2, \cdots, p_l] \tag{12.3}$$

denote the parameters subject to uncertainty and set

$$\mathbf{p} = \mathbf{p}^0 + \Delta\mathbf{p} \tag{12.4}$$

where $\mathbf{p}^0$ is the nominal parameter and $\Delta\mathbf{p}$ denotes a perturbation. Write

$$\begin{aligned} M(\mathbf{p}) &= M(\mathbf{p}^0 + \Delta\mathbf{p}) \\ &= M(\mathbf{p}^0) + \Delta M(\mathbf{p}^0, \Delta\mathbf{p}). \end{aligned} \tag{12.5}$$

Assuming that the entries of $\Delta M(\mathbf{p}^0, \Delta\mathbf{p})$ are linear functions of $\Delta\mathbf{p}$, we can write

$$\Delta M(\mathbf{p}^0, \Delta\mathbf{p}) = \Delta p_1 E_1 + \Delta p_2 E_2 + \cdots + \Delta p_l E_l. \tag{12.6}$$

We shall say that the perturbation structure is of *unity rank* when each matrix E_i has unity rank. The special attraction of unity rank perturbation structures is the fact that in this case the coefficients of the characteristic polynomial of M are multilinear functions of $\Delta\mathbf{p}$ as shown below. When the Δp_i vary in intervals, this multilinear structure allows us to use the Mapping Theorem of Chapter 11 to develop an effective computational procedure to determine robust stability and stability margins in the parameter space $\mathbf{p}$. The stability margin will be measured as the smallest ℓ_∞ norm of the vector $\Delta\mathbf{p}$ required to make $M(\mathbf{p}^0, \Delta\mathbf{p})$ just unstable.

In the robust stability literature, state space perturbations are often treated by supposing that

$$\Delta M = DUE \tag{12.7}$$

where the matrix U is regarded as a perturbation. In this formulation, one can calculate the smallest induced norm of U for which $M + \Delta M$ just becomes unstable. We remark that the parametric stability margin, defined as the vector norm of the smallest destabilizing vector $\Delta\mathbf{p}$, has a physical significance in terms of the allowable perturbations of the parameter $\mathbf{p}$. Such a direct significance cannot be attached to the matrix norm. Nevertheless, it has become customary to consider matrix valued perturbations and we accordingly define the *matrix stability radius* as the norm of the smallest destabilizing matrix. We give some formulas for the calculation of the matrix stability radius in the real and complex cases.

12.3 ROBUST STABILITY OF INTERVAL MATRICES

We first establish that the unity rank perturbation structure leads to multilinear dependence of the characteristic polynomial coefficients on the parameter $\mathbf{p}$.

12.3.1 Unity Rank Perturbation Structure

Let us suppose that

$$M(\mathbf{p}) := \underbrace{M(\mathbf{p}^0)}_{M_0} + \Delta p_1 E_1 + \Delta p_2 E_2 + \cdots + \Delta p_l E_l. \tag{12.8}$$

Lemma 12.1 *Under the assumption that* $\text{rank}(E_i) = 1$ *for each* i*, the coefficients of the characteristic polynomial of* $M(\mathbf{p})$ *are multilinear functions of* $\mathbf{p}$*.*

Proof. Write

$$\delta(s, \mathbf{p}) = \det\left[sI - M(\mathbf{p})\right].$$

In $\delta(s, \mathbf{p})$, fix all parameters p_j, $j \neq i$ and denote the resulting one parameter function as $\delta(s, p_i)$. To prove the lemma, it is enough to show that $\delta(s, p_i)$ is a linear function of p_i for fixed $s = s^*$. Now since E_i is of unity rank, we write $E_i = b_i c_i^T$ where b_i and c_i are appropriate column vectors. Then

$$\delta(s^*, p_i) = \det\left(s^* I - \underbrace{M_0 - \sum_{j \neq i} p_j E_j}_{\bar{A}} - p_i b_i c_i^T \right)$$

$$
\begin{aligned}
&= \det\left(s^* I - \bar{A} - p_i b_i c_i^T\right) \\
&= \det\left\{ (s^* I - \bar{A}) \left[I - p_i \underbrace{(s^* I - \bar{A})^{-1}}_{\hat{A}(s^*)} b_i c_i^T \right] \right\} \\
&= \det(s^* I - \bar{A}) \det\left(I - p_i \hat{A}(s^*) b_i c_i^T \right) \\
&= \det(s^* I - \bar{A}) p_i^n \det\left(p_i^{-1} I - \hat{A}(s^*) b_i c_i^T \right).
\end{aligned}
$$

Notice that $\hat{A}(s^*)b_i c_i^T$ is of unity rank. Let λ denote the only nonzero eigenvalue of this matrix. We have

$$
\begin{aligned}
\delta(s^*, p_i) &= p_i^n \left[p_i^{-n+1} \left(p_i^{-1} - \lambda \right) \right] \det\left(s^* I - \bar{A}\right) \\
&= p_i^n \left[p_i^{-n} - \lambda p_i^{-n+1} \right] \det\left(s^* I - \bar{A}\right) \\
&= (1 - \lambda p_i) \det\left(s^* I - \bar{A}\right).
\end{aligned}
$$

Thus we have proved that $\delta(s, p_i)$ is a linear function of p_i. ♣

12.3.2 Interval Matrix Stability via the Mapping Theorem

The objectives are to solve the following problems:

Problem 1 Determine if each matrix $M(\mathbf{p})$ remains stable as the parameter $\mathbf{p}$ ranges over given perturbation bounds $p_i^- \le p_i \le p_i^+, \quad i = 1, \cdots, l$.

Problem 2 With a stable $M(\mathbf{p}^0)$, determine the maximum value of ϵ so that the matrix $M(\mathbf{p})$ remains stable under all parameter perturbations ranging over $p_i^0 - w_i\epsilon \le p_i \le p_i^0 + w_i\epsilon$ for predetermined weights $w_i > 0$.

These problems may be effectively solved by using the fact that the characteristic polynomial of the matrix is a multilinear function of the parameters. This will allow us to use the algorithm developed in Chapter 11 for testing the robust stability of such families.

The problem 1 can be solved by the following algorithm:

Step 1: Determine the eigenvalues of the matrix $M(\mathbf{p})$ with $\mathbf{p}$ fixed at each vertex of $\mathbf{\Pi}$. With this generate the characteristic polynomials corresponding to the vertices of $\mathbf{\Pi}$.

Step 2: Verify the stability of the line segments connecting the vertex characteristic polynomials. This may be done by checking the Bounded Phase Condition or the Segment Lemma.

We remark that the procedure outlined above does not require the determination of the characteristic polynomial as a *function* of the parameter $\mathbf{p}$. It is enough to know that the function is multilinear. To determine the maximum value of ϵ which solves the second problem, we may simply repeat the previous steps for incremental values of ϵ. In fact, an upper bound $\bar{\epsilon}$ can be found as that value of ϵ for which one of the *vertices* becomes just unstable. A lower bound $\underline{\epsilon}$ can be determined as the value of ϵ for which a *segment* joining the vertices becomes unstable as follows:

Step 1: Set $\underline{\epsilon} = \bar{\epsilon}/2$

Step 2: Check the maximal phase differences of the vertex polynomials over the parameter box corresponding to $\underline{\epsilon}$.

Step 3: If the maximal phase difference is less than π radians, then increase $\underline{\epsilon}$ to $\underline{\epsilon} + (\bar{\epsilon} - \underline{\epsilon})/2$ for example, and repeat Step 2.

Step 4: If the maximal phase difference is π radians or greater, then decrease $\underline{\epsilon}$ to $\underline{\epsilon} - (\bar{\epsilon} - \underline{\epsilon})/2$ and repeat Step 2.

Step 5: This iteration stops when the incremental step or decremental step becomes small enough. This gives a lower bound $\underline{\epsilon}$ and an upper bound $\bar{\epsilon}$.

If $\underline{\epsilon}$ and $\bar{\epsilon}$ are not close enough, we can refine the iteration by partitioning the interval uncertainty set into smaller boxes as in Chapter 10.

The following examples illustrate this algorithm.

12.3.3 Numerical Examples

Example 12.1. Consider the interval matrix:

$$A(\mathbf{p}) = \begin{bmatrix} p_1 & p_2 \\ p_3 & 0 \end{bmatrix}$$

where

$$\mathbf{p}^0 = [p_1^0, p_2^0, p_3^0] = [-3, -2, 1]$$

and

$$p_1 \in [p_1^-, p_1^+] = [-3 - \epsilon, -3 + \epsilon], \quad p_2 \in [p_2^-, p_2^+] = [-2 - \epsilon, -2 + \epsilon],$$

$$p_3 \in [p_3^-, p_3^+] = [1 - \epsilon, 1 + \epsilon].$$

The problem is to find the maximum value ϵ^* so that the matrix $A(\mathbf{p})$ remains stable for all $\epsilon \in [0, \epsilon^*]$. Although the solution to this simple problem can be worked out analytically we work through the steps in some detail to illustrate the calculations involved.

The characteristic polynomial of the matrix is:

$$\delta(s, \mathbf{p}) = s(s - p_1) - p_2 p_3.$$

In general this functional form is not required since only the vertex characteristic polynomials are needed and they can be found from the eigenvalues of the corresponding vertex matrices. Let us now compute the upper bound for ϵ. We have eight vertex polynomials parametrized by ϵ:

$$\mathbf{\Delta}_{\mathrm{V}}(s) := \{\delta_i(s, \mathbf{p}) : \mathbf{p} \in \mathbf{V}\}$$

where

$$\begin{aligned}\mathbf{V} := \big\{&(p_1^-, p_2^+, p_3^+), (p_1^-, p_2^-, p_3^+), (p_1^-, p_2^+, p_3^-), (p_1^-, p_2^-, p_3^-), (p_1^+, p_2^+, p_3^+),\\ &(p_1^+, p_2^-, p_3^+), (p_1^+, p_2^+, p_3^-), (p_1^+, p_2^-, p_3^-)\big\}\end{aligned}$$

We found that the vertex polynomial $\delta_3(s)$ has a $j\omega$ root at $\epsilon = 1$. Thus we set $\bar{\epsilon} = 1$. Using the multilinear version of GKT (Theorem 11.1), $\mathbf{\Delta}(s)$ is robustly Hurwitz stable if and only if the following sets are Hurwitz stable:

$$\begin{aligned}
\mathbf{L}_1 &= \left\{s\left[\lambda(s - p_1^-) + (1-\lambda)(s - p_1^+)\right] - p_2^+ p_3^+ : \lambda \in [0,1]\right\}\\
\mathbf{L}_2 &= \left\{s\left[\lambda(s - p_1^-) + (1-\lambda)(s - p_1^+)\right] - p_2^- p_3^+ : \lambda \in [0,1]\right\}\\
\mathbf{L}_3 &= \left\{s\left[\lambda(s - p_1^-) + (1-\lambda)(s - p_1^+)\right] - p_2^+ p_3^- : \lambda \in [0,1]\right\}\\
\mathbf{L}_4 &= \left\{s\left[\lambda(s - p_1^-) + (1-\lambda)(s - p_1^+)\right] - p_2^- p_3^- : \lambda \in [0,1]\right\}\\
\mathbf{M}_5 &= \big\{s(s - p_1^-) - \left[\lambda_1 p_2^- + (1-\lambda_1)p_2^+\right]\left[\lambda_2 p_3^- + (1-\lambda_2)p_3^+\right] :\\
&\qquad (\lambda_1, \lambda_2) \in [0,1]\times[0,1]\big\}\\
\mathbf{M}_6 &= \big\{s(s - p_1^+) - \left[\lambda_1 p_2^- + (1-\lambda_1)p_2^+\right]\left[\lambda_2 p_3^- + (1-\lambda_2)p_3^+\right] :\\
&\qquad (\lambda_1, \lambda_2) \in [0,1]\times[0,1]\big\}.
\end{aligned}$$

The $\mathbf{L}_i$, $i = 1, 2, 3, 4$ are line segments of polynomials, so we rewrite them as follows:

$$\begin{aligned}
L_1 &= \lambda(s - p_1^- - p_2^+ p_3^+) + (1-\lambda)(s - p_1^+ - p_2^+ p_3^+)\\
L_2 &= \lambda(s - p_1^- - p_2^- p_3^+) + (1-\lambda)(s - p_1^+ - p_2^- p_3^+)\\
L_3 &= \lambda(s - p_1^- - p_2^+ p_3^-) + (1-\lambda)(s - p_1^+ - p_2^+ p_3^-)\\
L_4 &= \lambda(s - p_1^- - p_2^- p_3^-) + (1-\lambda)(s - p_1^+ - p_2^- p_3^-)
\end{aligned}$$

Now we need to generate the set of line segments that constructs the convex hull of the image sets of $\mathbf{M}_5$ and $\mathbf{M}_6$. This can be done by connecting every pair of vertex polynomials. The vertex set corresponding to $\mathbf{M}_5$ is:

$$\mathbf{M}_{5\mathrm{V}}(s) := \{M_5 : (\lambda_1, \lambda_2) \in \{(0,0), (0,1), (1,0), (1,1)\}\}.$$

If we connect every pair of these vertex polynomials, we have the line segments:

$$\begin{aligned}
L_5 &= s(s - p_1^-) - \left[\lambda p_2^+ p_3^+ + (1-\lambda)p_2^+ p_3^-\right]\\
&= \lambda(s^2 - p_1^- s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^- s - p_2^+ p_3^-)\\
L_6 &= s(s - p_1^-) - \left[\lambda p_2^+ p_3^+ + (1-\lambda)p_2^- p_3^+\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lambda(s^2 - p_1^- s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^- s - p_2^- p_3^+) \\
L_7 &= s(s - p_1^-) - \left[\lambda p_2^- p_3^+ + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^- s - p_2^- p_3^+) + (1-\lambda)(s^2 - p_1^- s - p_2^- p_3^-) \\
L_8 &= s(s - p_1^-) - \left[\lambda p_2^+ p_3^- + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^- s - p_2^+ p_3^-) + (1-\lambda)(s^2 - p_1^- s - p_2^- p_3^-) \\
L_9 &= s(s - p_1^-) - \left[\lambda p_2^+ p_3^- + (1-\lambda) p_2^- p_3^+\right] \\
&= \lambda(s^2 - p_1^- s - p_2^+ p_3^-) + (1-\lambda)(s^2 - p_1^- s - p_2^- p_3^+) \\
L_{10} &= s(s - p_1^-) - \left[\lambda p_2^+ p_3^+ + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^- s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^- s - p_2^- p_3^-).
\end{aligned}
$$

Similarly, for $\mathbf{M}_6$ we have the line segments:

$$
\begin{aligned}
L_{11} &= s(s - p_1^+) - \left[\lambda p_2^+ p_3^+ + (1-\lambda) p_2^+ p_3^-\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^+ s - p_2^+ p_3^-) \\
L_{12} &= s(s - p_1^+) - \left[\lambda p_2^+ p_3^+ + (1-\lambda) p_2^- p_3^+\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^+ s - p_2^- p_3^+) \\
L_{13} &= s(s - p_1^+) - \left[\lambda p_2^- p_3^+ + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^- p_3^+) + (1-\lambda)(s^2 - p_1^+ s - p_2^- p_3^-) \\
L_{14} &= s(s - p_1^+) - \left[\lambda p_2^+ p_3^- + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^+ p_3^-) + (1-\lambda)(s^2 - p_1^+ s - p_2^- p_3^-) \\
L_{15} &= s(s - p_1^+) - \left[\lambda p_2^+ p_3^- + (1-\lambda) p_2^- p_3^+\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^+ p_3^-) + (1-\lambda)(s^2 - p_1^+ s - p_2^- p_3^+) \\
L_{16} &= s(s - p_1^+) - \left[\lambda p_2^+ p_3^+ + (1-\lambda) p_2^- p_3^-\right] \\
&= \lambda(s^2 - p_1^+ s - p_2^+ p_3^+) + (1-\lambda)(s^2 - p_1^+ s - p_2^- p_3^-).
\end{aligned}
$$

The total number of line segments joining vertex pairs in the parameter space is 28. However the actual number of segments we checked is 16. This saving is due to the multilinear version of GKT (Theorem 11.1), which reduces the set to be checked. By testing these segments we find that they are all stable for $\epsilon < 1$. Since $\epsilon = 1$ corresponds to the instability of the vertex polynomial $\delta_3(s)$ we conclude that the exact value of the stability margin is $\epsilon = 1$. The stability check of the segments can be carried out using the Segment Lemma of Chapter 2.

We can also solve this problem by using the Bounded Phase Condition. The set of vertices is

$$\mathbf{\Delta}_{\mathrm{V}}(s) = \{V_i, i = 1, 2, \cdots, 8\}$$

where

$$
\begin{aligned}
V_1(s) &= s^2 - p_1^+ s - p_2^+ p_3^+ = s^2 - 2s + 2 \\
V_2(s) &= s^2 - p_1^+ s - p_2^- p_3^+ = s^2 - 2s + 6
\end{aligned}
$$

$$V_3(s) = s^2 - p_1^+ s - p_2^+ p_3^- = s^2 - 2s$$
$$V_4(s) = s^2 - p_1^+ s - p_2^- p_3^- = s^2 - 2s$$
$$V_5(s) = s^2 - p_1^- s - p_2^+ p_3^+ = s^2 - 4s + 2$$
$$V_6(s) = s^2 - p_1^- s - p_2^- p_3^+ = s^2 - 4s + 6$$
$$V_7(s) = s^2 - p_1^- s - p_2^+ p_3^- = s^2 - 4s$$
$$V_8(s) = s^2 - p_1^- s - p_2^- p_3^- = s^2 - 4s.$$

From the vertex set we see that the difference polynomials $V_i(s) - V_j(s)$ are either constant, first order, antiHurwitz or of the form cs and each of these forms satisfy the conditions of the Vertex Lemma in Chapter 2. Thus the stability of the vertices implies that of the edges. Thus the first encounter with instability can only occur on a vertex. This implies that the smallest value already found of $\epsilon = 1$ for which a vertex becomes unstable is the correct value of the margin.

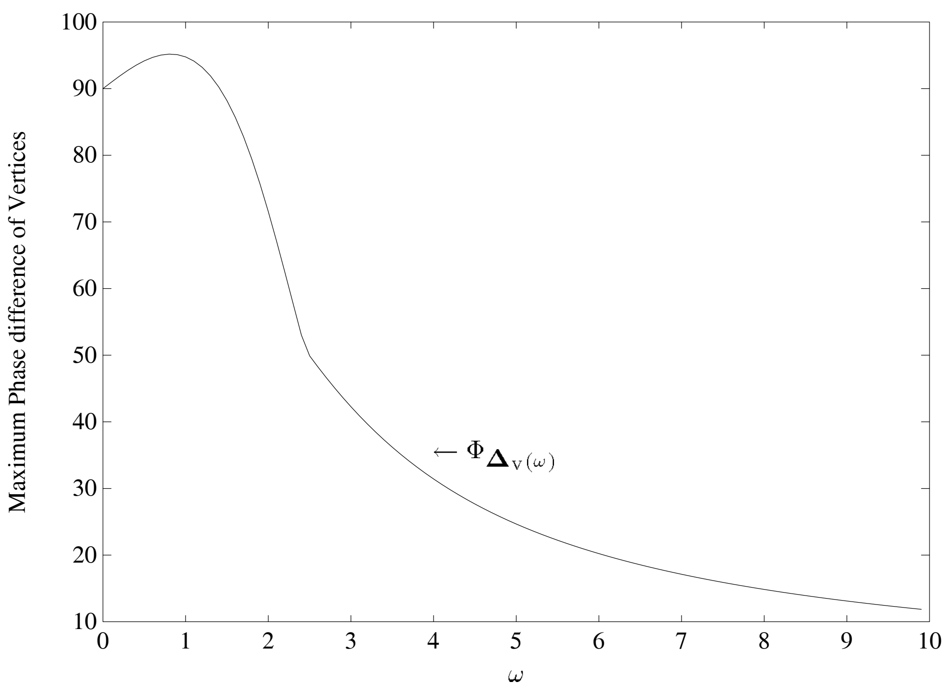

Figure 12.1. Maximum Phase Differences $\Phi_{\mathbf{\Delta}_{\rm V}(\omega)}$ (in degrees) of Vertex Polynomials (Example 12.1)

This conclusion can also be verified by checking the phases of the vertices. Since there are duplicated vertices, we simply plot phase differences of six distinct vertices of $\mathbf{\Delta}(j\omega)$ with $\epsilon = 1$. Figure 12.1 shows that the maximum phase difference plot as a function of frequency. This plot shows that the maximal phase difference never reaches 180 degrees confirming once again that $\epsilon = 1$ is indeed the true margin.

Remark 12.1. The phase of a vertex which touches the origin cannot be determined and only the phase difference over the vertex set is meaningful. The phase condition can therefore only be used to determine whether the line segment excluding endpoints, intersects the origin. Thus, the stability of all the vertices must be verified independently.

Example 12.2. Let

$$\frac{dx}{dt} = (A + BKC)x$$
$$= \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 + k_1 & 0 \\ 0 & -1 + k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) x$$

where

$$k_1 \in [k_1^-, k_1^+] = [-\epsilon, \epsilon] \qquad k_2 \in [k_2^-, k_2^+] = [-\epsilon, \epsilon].$$

We first find all the vertex polynomials.

$$\mathbf{\Delta}_{\mathrm{V}}(s) := \{\delta_i(s, \mathbf{k}_i) : \mathbf{k}_i \in \mathbf{V}, \quad i = 1, 2, 3, 4\}$$

where

$$\mathbf{V} := \left\{(k_1, k_2) : (k_1^+, k_2^+), (k_1^-, k_2^-), (k_1^-, k_2^+), (k_1^+, k_2^-)\right\}.$$

We found that the minimum value of ϵ such that a vertex polynomial just becomes unstable is 1.75. Thus, $\bar{\epsilon} = 1.75$. Then we proceed by checking either the phase condition or the Segment Lemma. If the Segment Lemma is applied, one must verify the stability of six segments joining the four vertices in $\mathbf{\Delta}_{\mathbf{V}}(s)$. If the phase condition is applied, one must evaluate the phases of the four vertex polynomials and find the maximum phase difference at each frequency to observe whether it reaches 180°. Again, the calculation shows that the smallest value of ϵ that results in a segment becoming unstable is 1.75. Thus $\underline{\epsilon} = 1.75$. This shows that the value obtained $\epsilon = 1.75$ is the true margin.

The algorithm can also be applied to the robust stability problem for nonHurwitz regions. The following example illustrates the discrete time case.

Example 12.3. Consider the discrete time system:

$$\underline{x}(k+1) = \begin{bmatrix} -0.5 & 0 & k_2 \\ 1 & 0.50 & -1 \\ k_1 & k_1 & 0.3 \end{bmatrix} \underline{x}(k)$$

For the nominal values of $k_1^0 = k_2^0 = 0$, the system is Schur stable. We want to determine the maximum value of ϵ^* so that for all parameters lying in the range

$$k_1 \in (-\epsilon^*, \epsilon^*) \qquad k_2 \in (-\epsilon^*, \epsilon^*)$$

the system remains Schur stable. Using the procedure, we find the upper bound $\bar{\epsilon} = 0.2745$ which is the minimum value of ϵ which results in a vertex polynomial just becoming unstable. Figure 12.2 shows that the maximum phase difference over all vertices at each $\theta \in [0, 2\pi)$ with $\epsilon = \bar{\epsilon}$ is less than $180°$. Thus we conclude from the Mapping Theorem that the exact parametric stability margin of this system is 0.2745.

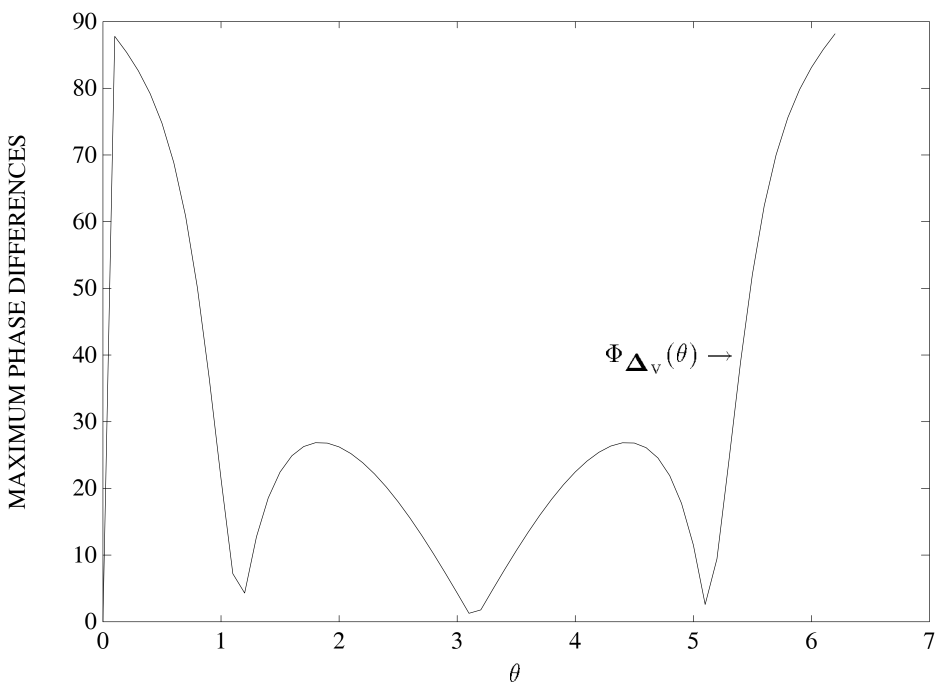

Figure 12.2. $\Phi_{\mathbf{\Delta}_v}(\theta)$ vs θ (Example 12.3)

In the next section we describe a Lyapunov function based approach to parameter perturbations in state space systems which avoids calculation of the characteristic polynomial.

12.4 ROBUSTNESS USING A LYAPUNOV APPROACH

Suppose that the plant equations in the state space form are

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{12.9}$$

The controller, of order t, is described by

$$\begin{aligned} \dot{x_c} &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y. \end{aligned} \tag{12.10}$$

The closed-loop system equation is

$$\left[\begin{array}{c} \dot{x} \\ \dot{x_c} \end{array}\right] = \left[\begin{array}{cc} A + BD_cC & BC_c \\ B_cC & A_c \end{array}\right] \left[\begin{array}{c} x \\ x_c \end{array}\right]$$

$$= \left(\underbrace{\left[\begin{array}{cc} A & 0 \\ 0 & 0_t \end{array}\right]}_{A_t} + \underbrace{\left[\begin{array}{cc} B & 0 \\ 0 & I_t \end{array}\right]}_{B_t} \underbrace{\left[\begin{array}{cc} D_c & C_c \\ B_c & A_c \end{array}\right]}_{K_t} \underbrace{\left[\begin{array}{cc} C & 0 \\ 0 & I_t \end{array}\right]}_{C_t} \right) \left[\begin{array}{c} x \\ x_c \end{array}\right]. \tag{12.11}$$

Now (12.10) is a stabilizing controller if and only if $A_t + B_t K_t C_t$ is stable. We consider the compensator order to be fixed at each stage of the design process and therefore drop the subscript t. Consider then the problem of robustification of $A + BKC$ by choice of K when the plant matrices are subject to parametric uncertainty.

Let $\mathbf{p} = [p_1, p_2, \cdots, p_r]$ denote a parameter vector consisting of physical parameters that enter the state-space description *linearly*. This situation occurs frequently since the state equations are often written based on physical considerations. In any case combination of primary parameters can always be defined so that the resulting dependence of A, B, C on $\mathbf{p}$ is linear. We also assume that the nominal model (12.9) has been determined with the nominal value $\mathbf{p}^0$ of $\mathbf{p}$. This allows us to treat $\mathbf{p}$ purely as a perturbation with nominal value $\mathbf{p}^0 = 0$. Finally, since the perturbation enters at different locations, we consider that $A + BKC$ perturbs to

$$A + BKC + \sum_{i=1}^{r} p_i E_i$$

for given matrices E_i which prescribe the structure of the perturbation.

We now state a result that calculates the radius of a spherical stability region in the parameter space $\mathbf{p} \in \mathbb{R}^r$. Let the nominal asymptotically stable system be

$$\dot{x}(t) = Mx(t) = (A + BKC)x(t) \tag{12.12}$$

and the perturbed equation be

$$\dot{x}(t) = \left(M + \sum_{i=1}^{r} p_i E_i \right) x(t) \tag{12.13}$$

where the p_i, $i = 1, \cdots, r$ are perturbations of parameters of interest and the E_i, $i = 1, \cdots, r$ are matrices determined by the structure of the parameter perturbations. Let $Q > 0$ be a positive definite symmetric matrix and let P denote the unique positive definite symmetric solution of

$$M^T P + PM + Q = 0. \tag{12.14}$$

Theorem 12.1 *The system (12.13) is stable for all p_i satisfying*

$$\sum_{i=1}^{r} |p_i|^2 < \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2} \tag{12.15}$$

where $\mu_i := \|E_i^T P + PE_i\|_2$.

Proof. Under the assumption that M is asymptotically stable with the stabilizing controller K, choose as a Lyapunov function

$$V(x) = x^T P x \tag{12.16}$$

where P is the symmetric positive definite solution of (12.14). Since M is an asymptotically stable matrix, the existence of such a P is guaranteed by Lyapunov's Theorem. Note that $V(x) > 0$ for all $x \neq 0$ and $V(x) \to \infty$ as $\|x\| \to \infty$. We require $\dot{V}(x) \leq 0$ for all trajectories of the system, to ensure the stability of (12.13). Differentiating (12.16) with respect to x along solutions of (12.14) yields

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (M^T P + PM) x + x^T \left(\sum_{i=1}^{n} p_i E_i^T P + \sum p_i P E_i \right) x. \end{aligned} \tag{12.17}$$

Substituting (12.14) into (12.17) we have

$$\dot{V}(x) = -x^T Q x + x^T \left(\sum_{i=1}^{r} p_i E_i^T P + \sum_{i=1}^{r} p_i P E_i \right) x. \tag{12.18}$$

The stability requirement $\dot{V}(x) \leq 0$ is equivalent to

$$x^T \left(\sum_{i=1}^{r} p_i E_i^T P + \sum_{i=1}^{r} p_i P E_i \right) x \leq x^T Q x. \tag{12.19}$$

Using the so-called Rayleigh principle,

$$\sigma_{\min}(Q) \leq \frac{x^T Q x}{x^T x} \leq \sigma_{\max}(Q), \quad \text{for all} \quad x \neq 0 \tag{12.20}$$

and we have

$$\sigma_{\min}(Q) x^T x \leq x^T Q x. \tag{12.21}$$

Thus, (12.19) is satisfied if

$$x^T \left(\sum_{i=1}^{r} p_i E_i^T P + \sum_{i=1}^{r} p_i P E_i \right) x \leq \sigma_{\min}(Q) x^T x. \tag{12.22}$$

Since

$$\begin{aligned} \left| x^T \left(\sum_{i=1}^{r} p_i E_i^T P + \sum_{i=1}^{r} p_i P E_i \right) x \right| &\leq \|x^T\|_2 \left\| \left(\sum_{i=1}^{r} p_i E_i^T P + \sum_{i=1}^{r} p_i P E_i \right) \right\|_2 \|x\|_2 \\ &\leq \|x\|_2^2 \left(\sum_{i=1}^{r} |p_i| \, \|E_i^T P + P E_i\|_2 \right), \end{aligned} \tag{12.23}$$

(12.22) is satisfied if

$$\sum_{i=1}^{r} \left(|p_i| \, \|E_i^T P + P E_i\|_2 \right) \leq \sigma_{\min}(Q). \tag{12.24}$$

Let

$$\mu_i := \|E_i^T P + P E_i\|_2 = \sigma_{\max}(E_i^T P + P E_i).$$

Then (12.24) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{r} \left(|p_i| \, \|E_i^T P + P E_i\|_2 \right) \\ &= \underbrace{\left(|p_1| \; |p_2| \; \cdots \; \cdots \; |p_r| \right)}_{\underline{\mathbf{p}}} \underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \vdots \\ \mu_r \end{pmatrix}}_{\underline{\mu}} \leq \sigma_{\min}(Q) \end{aligned} \tag{12.25}$$

which is satisfied if

$$\|\underline{\mathbf{p}}\underline{\mu}\|_2^2 \leq \|\underline{\mathbf{p}}\|_2^2 \; \|\underline{\mu}\|_2^2 \leq \sigma_{\min}^2(Q). \tag{12.26}$$

Using the fact that

$$\|\underline{\mathbf{p}}\|_2^2 = \sum_{i=1}^{r} |p_i|^2$$

$$\|\underline{\mu}\|_2^2 = \sum_{i=1}^{r} |\mu_i|^2$$

we obtain

$$\sum_{i=1}^{r} |p_i|^2 \leq \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2}. \tag{12.27}$$

♣

This theorem determines for the given stabilizing controller K, the quantity

$$\rho(K,Q) := \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2} = \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \|E_i^T P + PE_i\|_2^2} \tag{12.28}$$

which determines the range of perturbations for which stability is guaranteed and this is therefore the radius of a stability hypersphere in parameter space.

12.4.1 Robustification Procedure

Using the index obtained in (12.28) we now give an iterative design procedure to obtain the optimal controller K^* so that (12.28) is as large as possible. For a given K, the largest stability hypersphere we can obtain is

$$\max_{Q} \rho^2(K,Q) = \max_{Q} \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2} \tag{12.29}$$

and the problem of designing a robust controller with respect to structured parameter perturbations can be formulated as follows:

Find K to maximize (12.29), i.e.

$$\max_{K} \left\{ \max_{Q} \rho^2(K,Q) \right\} = \max_{K} \left\{ \max_{Q} \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2} \right\} \tag{12.30}$$

subject to all the eigenvalues of $A + BKC$ lying in the left half plane, i.e.

$$\lambda(A + BKC) \subset \mathbb{C}^-.$$

Equivalently

$$\max_{K,Q} \rho^2(K,Q) = \max_{K,Q} \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2} \tag{12.31}$$

subject to

$$\lambda(A + BKC) \subset \mathbb{C}^-.$$

Thus the following constrained optimization problem is formulated: For the given (A, B, C) with the nominal stabilizing controller K define

$$(A+BKC)^T P + P(A+BKC) = -Q := -L^T L, \tag{12.32}$$

and the optimization problem

$$\min_{K,L} J := \min_{K,L} \frac{\sum_{i=1}^{r} \|E_i^T P + P E_i\|_2^2}{\sigma_{\min}^2(L^T L)} \tag{12.33}$$

subject to

$$J_c := \max_{\lambda(A+BKC)} \mathrm{Re}[\lambda] < 0.$$

Note that the positive definite matrix Q has been replaced without loss of generality by $L^T L$. For any square full rank matrix L, $L^T L$ is positive definite symmetric. This replacement also reduces computational complexity.

A gradient based descent procedure to optimize the design parameters K and L can be devised. The gradient of J with respect to K and L is given below. Before we state this result, we consider a slightly more general class of perturbations by letting

$$A = A_0 + \sum_{i=1}^{r} p_i A_i, \qquad B = B_0 + \sum_{i=1}^{r} p_i B_i. \tag{12.34}$$

Then we get

$$M = A_0 + B_0 KC \quad \text{and} \quad E_i = A_i + B_i KC. \tag{12.35}$$

Theorem 12.2 *Let J be defined as in (12.33) and let (12.34) and (12.35) hold. Then*

a)

$$\frac{\partial J}{\partial L} = \frac{2}{\sigma_{\min}^3(L^T L)} L$$

$$\left\{ \sigma_{\min}(L^T L) V^T - \sum_{i=1}^{r} \sigma_{\max}^2 \left(E_i^T P + P E_i\right)\left(u_m v_m^T + v_m u_m^T\right) \right\} \tag{12.36}$$

where V satisfies

$$(A_0 + B_0 KC)V + V(A_0 + B_0 KC)^T =$$

$$-\sum_{i=1}^{r} \sigma_{\max}\left(E_i^T P + P E_i\right)\left[E_i(u_{ai} v_{ai}^T + v_{ai} u_{ia}^T) + (u_{ai} v_{ai}^T + v_{ai} u_{ai}^T) E_i^T\right] \tag{12.37}$$

v_{ai} and u_{ai} are left and right singular vectors corresponding to $\sigma_{\max}(E_i^T P + P E_i)$, respectively, and v_m and u_m are left and right singular vectors corresponding to $\sigma_{\min}(L^T L)$.

b)

$$\frac{\partial J}{\partial K} = \frac{2}{\sigma_{\min}^2(L^T L)}$$

$$\left\{ \sum_{i=1}^{r} \sigma_{\max}\left(E_i^T P + P E_i\right) B_i^T P \left(v_{ai} u_{ai}^T + u_{ai} v_{ai}^T\right) + B^T P^T V^T \right\} C^T \quad (12.38)$$

c)

$$\frac{\partial J_c}{\partial K_{ij}} = \mathrm{Re}\left\{ \frac{v^T B_0 \left(\frac{\partial K}{\partial K_{ij}}\right) C w}{v^T w} \right\} \quad (12.39)$$

where v and w are the corresponding left and right eigenvectors of $(A_0 + B_0 K C)$ corresponding to $\lambda_{\max}$ the eigenvector with $\max\{\mathrm{Re}\,(\lambda)\}$.

The proof of this theorem is omitted. The gradient can be computed by solving the two equations (12.37) and (12.38). A gradient based algorithm for enlarging the radius of the stability hypersphere $\rho(K, Q)$ by iteration on (K, Q) can be devised using these gradients. This procedure is somewhat ad hoc but nevertheless it can be useful.

Example 12.4. A VTOL helicopter is described as the linearized dynamic equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & p_1 & -0.7070 & p_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0.4422 & 0.1761 \\ p_3 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

where

x_1 horizontal velocity, knots
x_2 vertical velocity, knots
x_3 pitch rate, degrees/sec
x_4 pitch angle, degrees
u_1 collective pitch control
u_2 longitudinal cyclic pitch control

The given dynamic equation is computed for typical loading and flight conditions of the VTOL helicopter at the airspeed of 135 knots. As the airspeed changes all the elements of the first three rows of both matrices also change. The most significant changes take place in the elements p_1, p_2, and p_3. Therefore, in the following all the other elements are assumed to be constants. The following bounds on the parameters are given:

$$\begin{aligned} p_1 &= 0.3681 + \Delta p_1, & |\Delta p_1| &\leq 0.05 \\ p_2 &= 1.4200 + \Delta p_2, & |\Delta p_2| &\leq 0.01 \\ p_3 &= 3.5446 + \Delta p_3, & |\Delta p_3| &\leq 0.04. \end{aligned}$$

Let

$$\Delta \mathbf{p} = [\Delta p_1, \Delta p_2, \Delta p_3]$$

and compute

$$\max \|\Delta \mathbf{p}\|_2 = 0.0648. \tag{12.40}$$

The eigenvalues of the open-loop plant are

$$\lambda(A) = \begin{pmatrix} 0.27579 \pm j0.25758 \\ -0.2325 \\ -2.072667 \end{pmatrix}.$$

The nominal stabilizing controller is given by

$$K_0 = \begin{bmatrix} -1.63522 \\ 1.58236 \end{bmatrix}.$$

Starting with this nominal stabilizing controller, we performed the robustification procedure. For this step the initial value is chosen as

$$L_0 = \begin{bmatrix} 1.0 & 0.0 & -0.50 & 0.06 \\ 0.5 & 1.0 & -0.03 & 0.00 \\ -0.1 & 0.4 & 1.00 & 0.14 \\ 0.2 & 0.6 & -0.13 & 1.50 \end{bmatrix}$$

The nominal values gave the stability margin

$$\rho_0 = 0.02712 < 0.0648 = \|\Delta \mathbf{p}\|_2$$

which does not satisfy the requirement (12.40). After 26 iterations we have

$$\rho^* = 0.12947 > 0.0648 = \|\Delta \mathbf{p}\|_2$$

which does satisfy the requirement. The robust stabilizing 0^{th} order controller computed is

$$K^* = \begin{bmatrix} -0.996339890 \\ 1.801833665 \end{bmatrix}$$

and the corresponding optimal L^*, P^* and the closed-loop eigenvalues are

$$L^* = \begin{bmatrix} 0.51243 & 0.02871 & -0.13260 & 0.05889 \\ -0.00040 & 0.39582 & -0.07210 & -0.35040 \\ 0.12938 & 0.08042 & 0.51089 & -0.01450 \\ -0.07150 & 0.34789 & -0.02530 & 0.39751 \end{bmatrix}$$

$$P^* = \begin{bmatrix} 2.00394 & -0.38940 & -0.50010 & -0.49220 \\ -0.38940 & 0.36491 & 0.46352 & 0.19652 \\ -0.50010 & 0.46352 & 0.61151 & 0.29841 \\ -0.49220 & 0.19652 & 0.29841 & 0.98734 \end{bmatrix}$$

$$\lambda(A + BK^*C) = \begin{pmatrix} -18.396295 \\ -0.247592 \pm j1.2501375 \\ -0.0736273 \end{pmatrix}.$$

This example can also be solved by the Mapping Theorem technique previously described. With the controller K^* obtained by the robustification procedure given above we obtained a stability margin of $\epsilon^* = 1.257568$ which is much greater than the value obtained by the Lyapunov stability based method. In fact, the Lyapunov stability based method gives $\rho^* = 0.12947$ which is equivalent to $\epsilon = 0.07475$. This comparison shows that the Mapping Theorem based technique gives a much better approximation of the stability margin than the Lyapunov based technique.

12.5 THE MATRIX STABILITY RADIUS PROBLEM

In this section, we suppose that the perturbations of the system model can be represented in the feedback form shown in Figure 12.3.

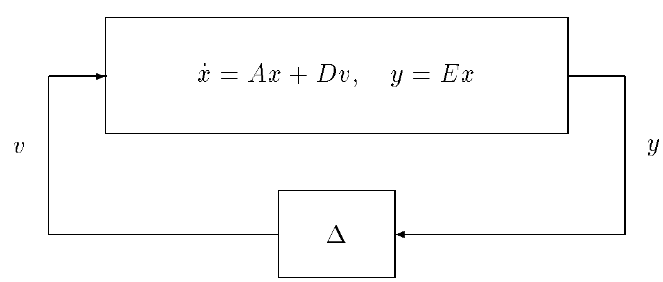

Figure 12.3. Feedback Interpretation of the Perturbed System

Thus we have

$$\begin{aligned} \dot{x} &= Ax + Dv \\ y &= Ex \\ v &= \Delta y. \end{aligned} \tag{12.41}$$

In the perturbed system, we therefore have

$$\dot{x} = (A + D\Delta E)x. \tag{12.42}$$

We regard the nominal system matrix to be A. Under perturbations it becomes $A + D\Delta E$ where E and D are given matrices defining the structure of the perturbations and Δ is an unstructured uncertainty block. In the following Δ will be allowed to be complex or real. In each case the size of the "smallest" destabilizing matrix Δ will be of interest. This problem is therefore a generalization of the gain and phase margin problems. In general, the signals v and y are artificially introduced and do not have any significance as inputs or outputs.

In the literature, it has become customary to measure the size of the perturbation using the operator norm of the matrix Δ. Let $\mathbb{K}$ denote a field. We will consider the two cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $y \in \mathbb{K}^q$, $v \in \mathbb{K}^l$, and $\|\cdot\|_{\mathbb{K}^q}$ and $\|\cdot\|_{\mathbb{K}^l}$ denote given norms in $\mathbb{K}^q$ and $\mathbb{K}^l$, respectively. We measure the size of Δ by the corresponding operator norm

$$\|\Delta\| = \sup\left\{\|\Delta y\|_{\mathbb{K}^l} : y \in \mathbb{K}^q, \|y\|_{\mathbb{K}^q} \leq 1\right\}. \tag{12.43}$$

Let $\mathcal{S}$ denote the stability region in the complex plane as usual. Let $\Lambda(\cdot)$ denote the eigenvalues of the matrix $(\cdot)$. We will suppose that the nominal matrix A is stable. This means that $\Lambda(A) \subset \mathcal{S}$.

Definition 12.1. The stability radius, in the field $\mathbb{K}$, of A with respect to the perturbation structure (D, E) is defined as:

$$\mu = \inf\left\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q}, \Lambda(A + D\Delta E) \cap \mathcal{U} \neq \emptyset\right\}. \tag{12.44}$$

The operator norm of Δ is most often measured by its maximum singular value, denoted $\bar{\sigma}(\Delta)$:

$$\|\Delta\| = \bar{\sigma}(\Delta).$$

In this case it can easily be established by continuity of the eigenvalues of $A + D\Delta E$ on Δ, and the stability of A, that

$$\begin{aligned}
\mu &= \inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \text{ and } \Lambda(A + D\Delta E) \cap \mathcal{U} \neq \emptyset\right\} \\
&= \inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \text{ and } \Lambda(A + D\Delta E) \cap \partial\mathcal{S} \neq \emptyset\right\} \\
&= \inf_{s\in\partial\mathcal{S}} \inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \text{ and } \det(sI - A - D\Delta E) = 0\right\} \qquad (12.45) \\
&= \inf_{s\in\partial\mathcal{S}} \inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \text{ and } \det\left[(I - \Delta \underbrace{E(sI - A)^{-1}D}_{G(s)}\right] = 0\right\} \\
&= \inf_{s\in\partial\mathcal{S}} \inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \text{ and } \det\left[I - \Delta G(s)\right] = 0\right\}.
\end{aligned}$$

For a fixed $s \in \partial\mathcal{S}$, write $G(s) = M$. Then the calculation above reduces to solving the optimization problem:

$$\inf\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{K}^{l\times q} \quad \text{and} \quad \det\left[I - \Delta M\right] = 0\right\}. \tag{12.46}$$

When Δ is allowed to be a complex matrix, it can be shown easily that the solution to this optimization problem is given by

$$\bar{\sigma}(\Delta) = [\bar{\sigma}(M)]^{-1}. \tag{12.47}$$

When Δ is constrained to be a real matrix the solution is much more complicated due to the fact that M is a complex matrix. In the following subsections, we give the calculation of μ for the complex and real cases. These are called the *complex matrix stability radius* and *real matrix stability radius* respectively.

12.5.1 The Complex Matrix Stability Radius

Consider the case where (A, D, E) are regarded as complex matrices, $\mathbb{K} = \mathbb{C}$, and Δ is a complex matrix. In this case, we denote μ as μ_C and call it the complex matrix stability radius.

The theorem given below shows how μ_C can be determined. The proof is an immediate consequence of equations (12.45)-(12.47) above. As before, let the transfer matrix associated with the triple (A, D, E) be

$$G(s) = E(sI - A)^{-1}D. \tag{12.48}$$

Theorem 12.3 *If A is stable with respect to $\mathcal{S}$, then*

$$\mu_C = \frac{1}{\sup_{s\in\partial\mathcal{S}} \|G(s)\|} \tag{12.49}$$

where $\|G(s)\|$ denotes the operator norm of $G(s)$ and, by definition, $0^{-1} = \infty$.

When $\|\Delta\| = \bar{\sigma}(\Delta)$, the complex matrix stability radius is given by

$$\mu_C = \frac{1}{\sup_{s\in\partial\mathcal{S}} \bar{\sigma}(G(s))}. \tag{12.50}$$

The special case of this formula corresponding to the case where D and E are square, and $D = E = I$, is referred to as the unstructured matrix stability radius. In this case,

$$\mu_C = \frac{1}{\sup_{s\in\partial\mathcal{S}} \|(sI - A)^{-1}\|}. \tag{12.51}$$

In the case of Hurwitz stability, (12.49) becomes

$$\mu_C = \frac{1}{\|G(s)\|_\infty}. \tag{12.52}$$

12.5.2 The Real Matrix Stability Radius

In this section we give the results on the matrix stability radius problem when A, D, E are real matrices and Δ is constrained to be a real matrix. The main result shows that the real matrix stability radius can be computed by means of a two-parameter optimization problem. Let $\sigma_2(\cdot)$ denote the second largest singular value of the matrix $(\cdot)$, and $\mathrm{Re}[G(s)]$ and $\mathrm{Im}[G(s)]$ denote the real and imaginary parts of the matrix $G(s)$.

Theorem 12.4 *The real matrix stability radius is given by*

$$\mu_R = \inf_{s\in\partial\mathcal{S}} \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \mathrm{Re}[G(s)] & -\gamma\ \mathrm{Im}[G(s)] \\ \gamma^{-1}\ \mathrm{Im}[G(s)] & \mathrm{Re}[G(s)] \end{bmatrix}\right) \tag{12.53}$$

The proof of this formula is rather lengthy and is omitted. An important feature of this formula is the fact that the function

$$\sigma_2\left(\begin{bmatrix} \mathrm{Re}[G(s)] & -\gamma\ \mathrm{Im}[G(s)] \\ \gamma^{-1}\ \mathrm{Im}[G(s)] & \mathrm{Re}[G(s)] \end{bmatrix}\right) \tag{12.54}$$

is unimodal over $\gamma \in (0, 1]$.

We conclude this section with a simple example which emphasizes the fundamental difference between the complex and real stability radii.

12.5.3 Example

Example 12.5. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -B \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ -B \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

This triple could describe a linear oscillator with small damping coefficient $B > 0$ and perturbed restoring force 1. The associated transfer function

$$G(s) = E(j\omega I - A)^{-1}D = \frac{-B}{1-\omega^2+jB\omega}.$$

The real stability radius is easily seen to be,

$$\mu_R = \frac{1}{B}.$$

To compute the complex stability radius we determine

$$|G(j\omega)|^2 = \frac{B^2}{(1-\omega^2)^2 + B^2\omega^2}.$$

If $B < \sqrt{2}$, a simple calculation shows that $|G(j\omega)|^2$ is maximized when $\omega^2 = 1 - \frac{B^2}{2}$ so that

$$\mu_C^2 = \frac{\frac{B^4}{4} + B^2\left(1-\frac{B^2}{2}\right)}{B^2} = 1 - \frac{B^2}{4}.$$

Now regarding B as a parameter we can see that μ_C is always bounded by 1 whereas μ_R can be made *arbitrarily large* by choosing small values of B. This suggests that complexifying real parameters could lead to highly conservative results. On the other hand it is also necessary to keep in mind that as B tends to 0 and μ_R tends to ∞, the poles of $G(s)$ approach the imaginary axis, indicating heightened sensitivity to unstructured perturbations.

12.6 TWO SPECIAL CLASSES OF INTERVAL MATRICES

In this section, we consider two special classes of interval matrices namely nonnegative matrices and the so-called Metzlerian matrices for which stronger results can be stated.

12.6.1 Robust Schur Stability of Nonnegative Matrices

Definition 12.2. A *nonnegative matrix* is a real matrix whose entries are nonnegative. Similarly, the matrix is said to be a *nonnegative interval matrix* if every element is nonnegative throughout their respective intervals:

$$\mathbf{A} := \left\{ A \in \mathbb{R}^{n\times n} : 0 \leq a_{ij}^- \leq a_{ij} \leq a_{ij}^+, \text{ for all } i, j \right\}. \tag{12.55}$$

Using the lower and upper bound matrices

$$A^- := \begin{bmatrix} a_{11}^- & \cdots & a_{1n}^- \\ \vdots & & \vdots \\ a_{n1}^- & \cdots & a_{nn}^- \end{bmatrix} \qquad A^+ := \begin{bmatrix} a_{11}^+ & \cdots & a_{1n}^+ \\ \vdots & & \vdots \\ a_{n1}^+ & \cdots & a_{nn}^+ \end{bmatrix},$$

we can represent a typical element of $\mathbf{A}$ by the notation:

$$0 \leq A^- \leq A \leq A^+. \tag{12.56}$$

The following definition and properties will play an important role throughout the section.

Definition 12.3. A real square matrix P is called a nonsingular *M-Matrix* if the following two conditions are satisfied:

a) $p_{ii} > 0$ for all i and $p_{ij} \leq 0$ for all $i \neq j$.

b) $P^{-1} \geq 0$ (P^{-1} is nonnegative).

Property 12.1.

A) Let $Q \in \mathbb{R}^{n\times n}$ with $Q \geq 0$. The matrix $\lambda I - Q$ is a nonsingular M-matrix if and only if $\rho(Q) < \lambda$ where $\rho(\cdot)$ is the spectral radius of the matrix $(\cdot)$.

B) The matrix P with $p_{ii} > 0$ for all i and $p_{ij} \leq 0$ for all $i \neq j$ is a nonsingular M-matrix if and only if all the leading principal minors of P are positive.

C) The characteristic polynomial of an M-matrix is antiHurwitz. Equivalently, all the eigenvalues of an M-matrix have positive real parts.

D) For $X, Y \in \mathbb{R}^{n\times n}$, if $0 \leq X \leq Y$, then $\rho(X) \leq \rho(Y)$.

The proofs of these properties may be found in the matrix theory literature referred to in the notes and references section of this chapter.

Let $A(\alpha)$ denote the α dimensional leading principal submatrix of A which consists of the first α rows and columns of A.

Theorem 12.5 *A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable for all $A \in [A^-, A^+]$ if and only if $\rho(A^+) < 1$. Equivalently, all leading principal minors of $I - A^+$, $\det[I - A^+(\alpha)]$, are positive for $\alpha = 1, \cdots, n$.*

Proof. Necessity follows trivially since $A^+ \in [A^-, A^+]$ and Schur stability of A is equivalent to $\rho(A) < 1$. To prove sufficiency, we use Property 12.1. Since $A^- \leq A \leq A^+$, we know that $\rho(A) \leq \rho(A^+)$ from Property 12.1.D. Therefore, the nonnegative matrix A is Schur stable for all $A \in [A^-, A^+]$ if and only if $\rho(A^+) < 1$. Moreover, from Property 12.1.A we also know that $I - A^+$ is a nonsingular M-matrix since $\rho(A^+) < 1$. From Property 12.1.B, this is also equivalent to all the leading principal minors of $I - A^+$, $\det[I - A^+(\alpha)]$, being positive. ♣

12.6.2 Robust Hurwitz Stability of Metzlerian Matrices

An extremal result also holds for the case of Hurwitz stability of Metzlerian matrices defined below:

Definition 12.4. A matrix A is called a *Metzlerian matrix* if $a_{ii} < 0$ for all i and $a_{ij} \geq 0$ for all $i \neq j$. Similarly, an interval matrix $\mathbf{A}$ is said to be a *Metzlerian interval matrix* if every matrix $A \in \mathbf{A}$ is a Metzlerian matrix.

We have the following theorem.

Theorem 12.6 *A Metzlerian matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz stable for all $A \in [A^-, A^+]$ if and only if A^+ is Hurwitz stable. An equivalent condition is that all leading principal minors of $-A^+$ are positive, i.e.* $\det[-A^+(\alpha)] > 0$ *for* $\alpha = 1, \cdots, n$.

Proof. All we have to prove is that $A \in [A^-, A^+]$ is Hurwitz stable if and only if $-A^+$ is an M-matrix. The necessity of this condition is obvious because the stability of $A \in [A^-, A^+]$ implies that of A^+. Therefore $-A^+$ is antiHurwitz and consequently, $-A^+$ is an M-matrix. From Property 12.1.B, all the leading principal minors of $-A^+$ must be positive.

To prove sufficiency assume that $-A^+$ is an M-matrix. From the structure of the matrix $-A$, one can always find $\lambda > 0$ such that $\lambda I - (-A) \geq 0$ for all $A \in [A^-, A^+]$. We have

$$\lambda I - \left(\lambda I - (-A^+)\right) = -A^+$$

and since $-A^+$ is a nonsingular M-matrix it follows from Property 12.1.A that

$$\rho(\lambda I + A^+) < \lambda.$$

We also know from Property 12.1.D that

$$0 \leq \lambda I + A \leq \lambda I + A^+$$

implies that

$$\rho(\lambda I + A) \leq \rho(\lambda I + A^+)$$

Thus

$$\rho(\lambda I + A) < \lambda$$

and therefore from Property 12.1.A we conclude that $-A$ is an M-matrix. Therefore by Property 12.1.C A is Hurwitz stable. ♣

In the next subsection, we show that a much simpler solution can be obtained for the real stability radius problem if the interval matrix falls into one of these two classes.

12.6.3 Real Stability Radius

From the previous subsection, we know that the real stability radius of a matrix is obtained by first solving a minimization problem involving one variable and next performing a frequency sweep. However, for the case of both nonnegative and Metzlerian matrices, a direct formula can be given for their respective real stability radii.

Consider the real nonnegative system matrix A with $A \geq 0$ and with $\rho(A) < 1$ and subject to structured perturbations. The real matrix stability radius with respect to the unit circle is defined by

$$\mu_R = \inf \{ \|\Delta\| : \Lambda(A + D\Delta E) \cap \mathcal{U} \neq \emptyset, \quad A + D\Delta E \geq 0 \} \tag{12.57}$$

where the condition $A + D\Delta E \geq 0$ is imposed in order to ensure nonnegativity of the entire set. The calculation will depend on the well known Perron-Frobenius Theorem of matrix theory.

Theorem 12.7 (Perron-Frobenuis Theorem)
If the matrix $A \in \mathbb{R}^{n \times n}$ is nonnegative, then

a) A has a positive eigenvalue, r, equal to the spectral radius of A

b) There is a positive (right) eigenvector associated with the eigenvalue r;

c) The eigenvalue r has algebraic multiplicity 1.

The eigenvalue r will be called the Perron-Frobenius eigenvalue.

Theorem 12.8 *Let A be Schur stable. Then the real stability radius of the nonnegative system is given by*

$$\mu_R = \frac{1}{\bar{\sigma}\left(E(I-A)^{-1}D\right)}$$

Proof. $A + D\Delta E$ becomes unstable if and only if $\rho(A + D\Delta E) \geq 1$. Hence from the definition of the real stability radius, we have

$$\begin{aligned}\mu_R &= \inf\left\{\|\Delta\| : \rho(A + D\Delta E) \geq 1, \quad A + D\Delta E \geq 0\right\} \\ &= \inf\left\{\|\Delta\| : \rho(A + D\Delta E) = 1, \quad A + D\Delta E \geq 0\right\}.\end{aligned}$$

From Theorem 12.7 it follows that $A + D\Delta E \geq 0$ is unstable if and only if its Perron-Frobenius eigenvalue (spectral radius) $r \geq 1$. Therefore we have

$$\begin{aligned}\mu_R &= \inf\left\{\bar{\sigma}(\Delta) : \lambda_i(A + D\Delta E) = 1 \text{ for some } i\right\} \\ &= \inf\left\{\bar{\sigma}(\Delta) : \det[I - A - D\Delta C] = 0\right\} \\ &= \inf\left\{\bar{\sigma}(\Delta) : \det\left[I - \Delta E(I-A)^{-1}D\right] = 0\right\} \\ &= \frac{1}{\bar{\sigma}\left(E(I-A)^{-1}D\right)}.\end{aligned}$$

♣

A similar result holds for Hurwitz stability (see Exercise 12.1).

12.6.4 Robust Stabilization

Using the results obtained for nonnegative and Metzlerian matrices above, here we consider the problem of designing a robust state feedback controller for an interval state space system.

Let us first consider the discrete time system

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k)\end{aligned} \tag{12.58}$$

where $A \in [A^-, A^+]$ while the matrices B,C are fixed. Note that in this problem neither A^- nor A^+ is necessarily nonnegative.

Let

$$u(k) = v + Kx(k) \tag{12.59}$$

so that the closed loop system is given by

$$x(k+1) = \underbrace{(A + BK)}_{A_{cl}} x(k) + Bv(k). \tag{12.60}$$

The following theorem provides a solution to this problem.

Theorem 12.9 *The feedback control law (12.59) robustly stabilizes the system (12.60) and A_{cl} remains nonnegative for all $A \in [A^-, A^+]$ if and only if*

$$A^- + BK \geq 0 \quad \text{and} \quad A^+ + BK \geq 0$$

and $A^+ + BK$ is Schur stable. Equivalently all the leading principal minors of the matrix $I - (A^+ + BK)$ are positive.

The proof of this theorem is a direct consequence of Theorem 12.5 and is omitted here.

Example 12.6. Consider the unstable interval discrete system

$$x(k+1) = \begin{bmatrix} [0.5, 0.6] & 0 & 0.5 \\ 1 & 0.5 & 1 \\ 0.5 & 0 & [-0.2, -0.1] \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k).$$

Suppose that we want to robustly stabilize the system by state feedback

$$u(k) = v(k) + Kx(k)$$

where $K = [k_1 \ k_2 \ k_3]$. Then the requirements $A^- + BK \geq 0$ and $A^+ + BK \geq 0$ lead to

$$\begin{aligned} 0.5 + k_1 &\geq 0 \\ k_2 &\geq 0 \\ -0.2 + k_3 &\geq 0 \end{aligned}$$

and the requirement that all the leading principal minors of $I - (A^+ + BK)$ be positive leads to

$$-0.25k_1 - 0.9k_2 - 0.2k_3 + 0.095 > 0.$$

Therefore, we select

$$K = [0 \ 0 \ 0.3].$$

A result similar to the previous theorem can be stated for the Hurwitz case and is given without proof:

Theorem 12.10 *The feedback control law $u(t) = Fx(t)$ robustly stabilizes the system*

$$x(t) = Ax(t) + Bu(t)$$

and the closed loop matrix $A + BF$ is a Metzlerian matrix for all $A \in [A^-, A^+]$ if and only if

$$A^- + BF \quad \text{and} \quad A^+ + BF$$

are Metzlerian and $A^+ + BF$ is Hurwitz. Equivalently all the leading principal minors of $-(A^+ + BF)$ are positive.

12.7 EXERCISES

12.1 Using the result in Theorem 12.8, derive the real stability radius formula for Metzlerian matrices with respect to the Hurwitz stability region.
Answer: The term $(I - A)$ in Theorem 12.8 is replaced by $-A$.

12.2 Consider the matrix

$$A(\mathbf{p}) = \begin{bmatrix} -2 + p_1 & -3 + p_2 \\ 2 + p_3 & 0 \end{bmatrix}.$$

Estimate the stability radius in the space of the parameters p_i by using the result in Theorem 12.1. Assume that the nominal values of the parameters p_i are zero. (You may choose $Q = I$ for a first estimate then optimize over Q as in Section 12.4.1.)

12.3

$$A = \begin{bmatrix} 0.5 & 1 + p_1 & 0 \\ 0 & 0.5 & p_2 \\ p_3 & 0 & 0.25 \end{bmatrix}$$

Let p_i vary in the interval $[0, \epsilon]$. Find the maximum value of ϵ such that the interval matrix A remains Schur stable.

12.4

$$A = \begin{bmatrix} -1 & 1 + p_1 & p_4 \\ 0 & -2 & p_5 \\ p_2 & p_3 & -3 \end{bmatrix}$$

Find the maximum value of ϵ such that the interval matrix A remains Hurwitz stable for all $p_i \in [0, \epsilon]$.

12.5 With

$$A_0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

let

$$A := A_0 + D\Delta E.$$

1) Find the real matrix stability radius.

2) Find the complex matrix stability radius.

12.6 Consider Exercise 12.5, find the real stability radius subject to the restriction that the matrix A remains Metzlerian.

12.7 Consider the discrete time system:

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1 & p_2 & p_3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

where

$$p_1 \in [-2, 2], \quad p_2 \in [1, 3], \quad p_3 \in [-3, -1].$$

Design a state feedback control law that robustly stabilizes the system.
Hint: Constrain the closed loop matrix to remain nonnegative.

12.8 Consider the following transfer function matrix:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s+3} & \dfrac{1}{s+4} \end{bmatrix}$$

Suppose that this plant is perturbed by feedback perturbations as shown in Figure 12.4:

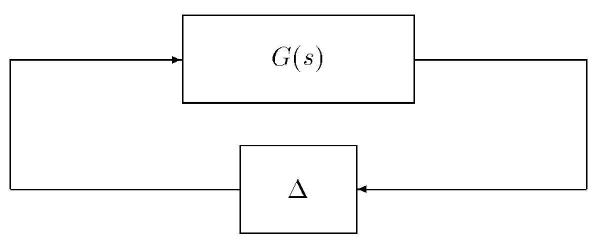

Figure 12.4. Feedback system

Compute the complex matrix stability radius with respect to Δ.

12.9 In Exercise 12.8, suppose that all the entries of Δ perturb within the interval $[-\epsilon, +\epsilon]$. Compute the real matrix stability radius $\epsilon_{\max}$ such that the closed loop system remains stable.
Hint: The characteristic equation of the closed loop system is multilinear in the parameters. One can apply the Mapping Theoerm based technique to this characteristic polynomial.

12.10 Consider the following discrete time system:

$$x(k+1) = \begin{bmatrix} 1 & 0 & p_1 \\ p_2 & -1 & 1 \\ 1 & p_3 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

where the parameters vary as

$$p_1 \in [1-\epsilon, 1+\epsilon], \quad p_2 \in [2-\epsilon, 2+\epsilon], \quad p_3 \in [1-\epsilon, 1+\epsilon].$$

1) Find the state feedback control law such that the closed loop system poles are assigned at 0.7 and $-0.5 \pm j0.3$.

2) With this control compute the maximum value $\epsilon_{\max}$ such that the closed loop system remains Schur stable.

12.8 NOTES AND REFERENCES

In 1980, Patel and Toda [185] gave a sufficient condition for the robust stability of interval matrices using unstructured perturbations. Numerous results have followed since then. Most of these follow-up results treated the case of structured perturbations because of its practical importance over the unstructured perturbation case. Some of these works are found in Yedavalli [238], Yedavalli and Liang [239], Martin [177], Zhou and Khargonekar [247], Keel, Bhattacharyya and Howze [139], Sezer and Šiljak [204], Leal and Gibson [159], Foo and Soh, [95], and Tesi and Vicino [219]. Theorem 12.1 and the formula for the gradients given in Section 12.4.1 are proved in Keel, Bhattacharyya and Howze [139]. Most of the cited works employed either the Lyapunov equation or norm inequalities and provided sufficient conditions with various degrees of inherent conservatism. Using robust eigenstructure assignment techniques, Keel, Lim and Juang [141] developed a method to robustify a controller such that the closed loop eigenvalues remain inside circular regions centered at the nominal eigenvalues while it allows the maximum parameter perturbation. The alforithm for determining the stability of an interval matrix is reported in Keel and Bhattacharyya [138]. Hinrichsen and Pritchard [113] have given a good survey of the matrix stability radius problem. The formula for the real matrix stability radius is due to Qiu, Bernhardsson, Rantzer, Davison, Young, and Doyle [193] to which we refer the reader for the proof. The results given in Section 12.5 are taken from Hinrichsen and Pritchard [114] and [193]. The example in Section 12.5.3 is taken from [113].

Some of the difficulties of dealing with general forms of interval matrices along with various results are discussed in Mansour [168]. Theorem 12.5 is due to Shafai, Perev, Cowley and Chehab [208]. A different class of matrices called Morishima matrices is treated in Sezer and Šiljak [205]. The proofs of Property 12.1 may be found in the book by Berman and Plemmons [27] and [208]. The proof of Theorem 12.7 (Perron-Frobenius Theorem) is found in the book of Lancaster and

Tismenetsky [158, 27]. The real stability radius problem for nonnegative and Metzlerian matrices is credited to Shafai, Kothandaraman and Chen [207]. The robust stabilization problem described in Section 12.6.4 is due to Shafai and Hollot [206].