Chapter 9

ROBUST STABILITY AND PERFORMANCE UNDER MIXED PERTURBATIONS

In this chapter we consider control systems containing parameter as well as unstructured uncertainty. Parametric uncertainty is modelled as usual by interval systems or linear interval systems. Two types of unstructured feedback perturbations are considered. First, we deal with unstructured uncertainty modelled as H_∞ norm bounded perturbations. A robust version of the Small Gain Theorem is developed for interval systems. An exact calculation of both the worst case parametric and the unstructured stability margins are given and the maximum values of various H_∞ norms over the parameter set are determined. It is shown that these occur on the same extremal subset which appear in the Generalized Kharitonov Theorem of Chapter 7. This solves the important problem of determining robust performance when it is specified in terms of H_∞ norms. Next, we deal with unstructured perturbations consisting of a family of nonlinear sector bounded feedback gains perturbing interval or linear interval systems. Extremal results for this robust version of the classical Absolute Stability problem are given. The constructive solution to this problem is also based on the extremal systems introduced in the Generalized Kharitonov Theorem (Chapter 7).

9.1 INTRODUCTION

Robustness of stability in the presence of unstructured uncertainty is an important and well developed subject in control system analysis. In the 1950's an important robustness problem called the *absolute stability problem* was formulated and studied. In this problem, also known as the Lur'e or Popov problem, a fixed linear system is subjected to perturbations consisting of all possible nonlinear feedback gains lying in a sector. In the 1980's a similar problem was studied by modelling the perturbations as H_∞ norm bounded perturbations of a fixed linear system.

In most practical systems it is important to consider at least two broad classes of uncertainties, namely *structured* and *unstructured* uncertainties. Unstructured uncertainties represent the effects of high frequency unmodeled dynamics, nonlinearities and the errors due to linearization, truncation errors, etc., and are usually modelled as a ball of norm-bounded operators. In the control literature it has been shown that certain types of robust performance problems, specified in terms of norms, can be posed as robust stability problems under unstructured perturbations. By structured uncertainty we mean parametric uncertainty representing lack of precise knowledge of the actual system parameters. Our goal in this chapter is to analyze the stability and performance of systems with uncertainties of mixed type with the objective of quantifying, as nonconservatively as possible, the amounts of the different kinds of perturbation that can be tolerated by the closed-loop system. We attempt to do this by considering unstructured perturbations of the above two types acting around an interval or linear interval system. We show that the extremal subsets introduced in the last two chapters again play a key role. These reduced sets in the plant parameter space are exactly where the worst case stability margins occur. Since these sets are one-parameter families of systems, the solution is constructive and computationally efficient. We begin in the next section with the case of H_∞ norm-bounded uncertainty.

9.2 SMALL GAIN THEOREM

In the approach to robust control using norms, uncertainty is usually modelled as norm bounded perturbations of the nominal transfer function in an appropriate normed algebra. The main tool for the analysis of stability of the closed-loop perturbed system under unstructured uncertainty is the Small Gain Theorem. In general, the Small Gain Theorem can be posed in any normed algebra, and it gives conditions under which a system of interconnected components is stable. In the algebra of stable transfer functions (H_∞), the Small Gain Theorem can be used to supply necessary and sufficient conditions for robust stability under stable (H_∞) perturbations.

In the following we will use the standard notation:

$$\mathbf{C}_+ := \{s \in C : \mathrm{Re}[s] > 0\},$$

and $H_\infty(\mathbf{C}_+)$ will represent the space of functions $f(s)$ that are bounded and analytic in $\mathbf{C}_+$ with the standard H_∞ norm,

$$\|f\|_\infty = \sup_{\omega \in \mathbf{R}} |\tilde{f}(j\omega)|,$$

where $\tilde{f}(j\omega)$ is the boundary function associated with $f(s)$.

We consider one version of the Small Gain Theorem. In this problem, a stable, linear time-invariant system is perturbed via feedback by a stable transfer function ΔP with bounded H_∞ norm, as illustrated by Figure 9.1. The question which

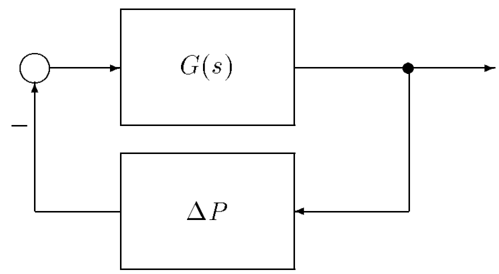

Figure 9.1. Stable system with H_∞ norm bounded feedback perturbation

is usually addressed in the robust stability literature is that of stability of the closed-loop system for all stable ΔP contained in an H_∞ ball of prescribed radius. Referring to Figure 9.1, we state the following well known result.

Theorem 9.1 (Small Gain Theorem)
If $G(s)$ is a stable transfer function, then the closed loop system remains stable for all perturbations ΔP satisfying $\|\Delta P\|_\infty < \alpha$, if and only if

$$\|G\|_\infty \le \frac{1}{\alpha}. \tag{9.1}$$

Notice that in this result ΔP can be any H_∞ function. However, it has been shown that if (9.1) is violated then it is always possible to find a destabilizing ΔP which is a real rational function. We remark that an identical result also holds for a matrix transfer function $G(s)$. This result provides a way to account for unstructured perturbations of the fixed linear system G. In the next section we extend this result to the case where $G(s)$ is subject to parameter perturbations as well.

9.3 SMALL GAIN THEOREM FOR INTERVAL SYSTEMS

We now turn our attention to the case where in addition to the unstructured feedback perturbations, the linear part of the configuration in Figure 9.1 is subject to parametric perturbations also. As in the previous two chapters we will model parametric uncertainty by considering interval or linear interval systems. We present the results first for the case of an interval system. Let

$$G(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ belongs to a family of real interval polynomials $\mathbf{N}(s)$ and $D(s)$ belongs to a family of real interval polynomials $\mathbf{D}(s)$. The interval family $\mathbf{G}(s)$ is written, following the notational convention of the last two chapters, as

$$\mathbf{G}(s) := \frac{\mathbf{N}(s)}{\mathbf{D}(s)}. \tag{9.2}$$

Let $K_N^i(s)$ and $K_D^i(s)$, $i = 1, 2, 3, 4$ denote the Kharitonov polynomials associated with $\mathbf{N}(s)$ and $\mathbf{D}(s)$, respectively. Introduce the Kharitonov systems

$$\mathbf{G}_{\mathrm{K}}(s) = \left\{ \frac{K_N^i(s)}{K_D^j(s)} \ : \ i, j \in \{1, 2, 3, 4\} \right\}. \tag{9.3}$$

The robust version of the Small Gain Theorem for interval systems can be stated as follows.

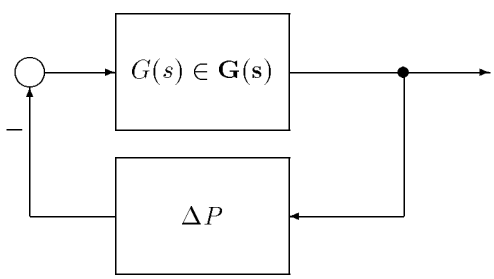

Figure 9.2. Closed loop system with H_∞ norm bounded function

Theorem 9.2 (Small Gain Theorem for Interval Systems)
Given the interval family $\mathbf{G}(s)$ of stable proper systems, the closed-loop in Figure 9.2 remains stable for all stable perturbations ΔP such that $\|\Delta P\|_\infty < \alpha$ if and only if

$$\alpha \leq \frac{1}{\max_{G \in \mathbf{G}_{\mathrm{K}}} \|G\|_\infty}. \tag{9.4}$$

The proof of this result follows from Theorem 9.1 (Small Gain Theorem) and the following lemma.

Lemma 9.1 $\|G\|_\infty < 1$ *for all* $G(s)$ *in* $\mathbf{G}(s)$ *if and only if* $\|G\|_\infty < 1$ *for the 16 elements of* $\mathbf{G}_{\mathrm{K}}(s)$.

The proof of this lemma is based on the following preliminary result, proved in Chapter 3, which characterizes proper rational functions $G(s)$ which are in $H_\infty(\mathbf{C}_+)$ and which satisfy $\|G\|_\infty < 1$.

Lemma 9.2 *Let*

$$G(s) = \frac{N(s)}{D(s)}$$

be a proper (real or complex) rational function in $H_\infty(\mathbf{C}_+)$, *with* $\text{degree}[D(s)] = q$ *and* n_q *and* d_q *denoting the leading coefficients of* $N(s)$ *and* $D(s)$, *respectively. Then* $\|G\|_\infty < 1$ *if and only if*

a1) $|n_q| < |d_q|$

b1) $D(s) + e^{j\theta}N(s)$ *is Hurwitz for all* θ *in* $[0, 2\pi)$.

The proof of this lemma is given in Chapter 3 (Lemma 3.1).

We can now prove Lemma 9.1.

Proof of Lemma 9.1 Necessity is obvious. For sufficiency we use the result of Lemma 9.2. First note that condition a1) of Lemma 9.2 is clearly true for all $G(s) \in \mathbf{G}(s)$ if it is true for the 16 Kharitonov systems.

Thus we prove that condition a2) is satisfied by any plant in $\mathbf{G}(s)$ if it is satisfied by the 16 plants of $\mathbf{G}_{\mathrm{K}}(s)$. We know from the bounding properties of Kharitonov polynomials that for an arbitrary element

$$G(s) = \frac{N(s)}{D(s)}$$

in $\mathbf{G}(s)$ and for any fixed $\omega \in \mathbb{R}$, there exists $i \in \{1, 2, 3, 4\}$ such that,

$$\left|\frac{N(j\omega)}{D(j\omega)}\right| \leq \left|\frac{K_N^i(j\omega)}{D(j\omega)}\right|.$$

Therefore, $\|G(s)\|_\infty < 1$, for all $G(s) \in \mathbf{G}(s)$, if and only if

$$\left\|\frac{K_N^i(s)}{D(s)}\right\|_\infty < 1, \qquad \text{for all } \ i \in \{1, 2, 3, 4\} \ \text{ and for all } \ D(s) \in \mathbf{D}(s).$$

Now for a fixed i, we have to check that, $D(s) + e^{j\theta}K_N^i(s)$ is Hurwitz, for all $D(s) \in \mathbf{D}(s)$, and for all $\theta \in [0, 2\pi)$. However, for a fixed $\theta \in [0, 2\pi)$, and a fixed $i \in \{1, 2, 3, 4\}$, the family of polynomials

$$\left\{D(s) + e^{j\theta}K_N^i(s) \ : \ D(s) \in \mathbf{D}(s)\right\}$$

is an interval family of complex polynomials (with constant imaginary part) and therefore by Kharitonov's theorem for complex polynomials (see Chapter 5) this family is Hurwitz if and only if

$$K_D^l(s) + e^{j\theta}K_N^i(s)$$

is Hurwitz for all $l \in \{1, 2, 3, 4\}$. The proof of the Lemma is now completed by using again Lemma 9.2. ♣

The proof of Theorem 9.2 is an immediate consequence of the Small Gain Theorem and Lemma 9.2. We remark that at a *particular* frequency ω the maximum magnitude of $||G(j\omega)||$ does *not* necessarily occur at a Kharitonov vertex and therefore the result established in Theorem 9.2 is quite nontrivial.

Before proceeding we present a generalization of Lemma 9.2 to the multivariable case. Even though we will not use it, we believe that this generalization is of interest in its own right. Let $H_\infty^{m\times p}(\mathbf{C}_+)$ be the space of matrix-valued functions $F(s)$ that are bounded and analytic in $\mathbf{C}_+$ with the norm,

$$||F||_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(\tilde{F}(j\omega))$$

where $\sigma_{\max}(\cdot)$ denotes the largest singular value of $(\cdot)$. For a constant complex matrix, $||A||$ will denote the following induced norm,

$$||A|| = \sup_{||v||_2\leq 1} ||Av||_2 = \sigma_{\max}(A) \ .$$

Lemma 9.3 *Let $G(s)$ be a proper rational transfer function matrix in $H_\infty^{m\times p}(\mathbf{C}_+)$. Assume without loss of generality that $p \geq m$, and let $G(s) = N(s)D^{-1}(s)$ be a right coprime description of $G(s)$ over the ring of polynomial matrices, with $D(s)$ column-reduced, then $||G||_\infty < 1$ if and only if*

a2) $||G(\infty)|| < 1$,

b2) $\det\left(D(s) + U\begin{bmatrix} I \\ 0 \end{bmatrix} N(s)\right)$ *is Hurwitz for all unitary matrices U in $C^{p\times p}$.*

Proof. *Sufficiency*: By contradiction if $||G||_\infty \geq 1$, then a2) implies that there exists $\omega_o \in \mathbb{R}$ such that $||G^*(j\omega_o)G(j\omega_o)|| = 1$. But it is well known that $||G(j\omega_o)||^2 = ||G^*(j\omega_o)G(j\omega_o)||$, and therefore for that same ω_o we have $||G(j\omega_o)|| = 1$. Thus there exists $v \in C^p$ such that,

$$||G(j\omega_o)v||_2 = ||v||_2 \ .$$

This implies that,

$$\left\|\begin{bmatrix} I \\ 0 \end{bmatrix} N(j\omega_o)D^{-1}(j\omega_o)v\right\|_2 = ||v||_2 \ ,$$

and therefore there exists a unitary matrix U such that,

$$U\begin{bmatrix} I \\ 0 \end{bmatrix} N(j\omega_o)D^{-1}(j\omega_o)v = v \ .$$

As a result,

$$\det\left(D(j\omega_o) - U\begin{bmatrix} I \\ 0 \end{bmatrix} N(j\omega_o)\right) = 0 \ ,$$

and this is a contradiction.

Necessity: It is clear that condition a2) is necessary and therefore we assume that it holds in the following. In order to establish b2), the first step is to prove that,

$$\det\left(D(s)+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N(s)\right)$$

is Hurwitz for all $\lambda \in [0,1]$. Consider the family of polynomials

$$\left\{P_\lambda(s)=\det\left(D(s)+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N(s)\right) : \lambda\in[0,1]\right\}.$$

Let us first show the following: a2) implies that this family has a constant degree. Indeed we can write,

$$D(s)=D_{hc}H_c(s)+D_{lc}(s),\;\; N(s)=N_{hc}H_c(s)+N_{lc}(s), \tag{9.5}$$

where $H_c(s)=\mathrm{diag}(s^{k_i}, i=1\cdots,p)$, k_i being the column-degree of the i^{th} column of $D(s)$, and $D_{lc}(s)$, $N_{lc}(s)$ contain the lower-degree terms.

It is easy to see that $G(\infty)=N_{hc}D_{hc}^{-1}$. Using (9.5) we also get

$$D(s)+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N(s)=\left(D_{hc}+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N_{hc}\right)H_c(s)+D_{lc}(s)+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N_{lc}(s).$$

Now the fact that $D(s)$ is column-reduced implies that $\det(D_{hc})$ is nonzero. To prove our claim it is enough to show that

$$\det\left(D_{hc}+\lambda\begin{bmatrix} I \\ 0 \end{bmatrix}N_{hc}\right)\neq 0, \text{ for all } \lambda\in(0,1].$$

Suppose by contradiction that for some $\lambda_o \in (0,1]$,

$$\det\left(D_{hc}+\lambda_o\begin{bmatrix} I \\ 0 \end{bmatrix}N_{hc}\right)=0.$$

This implies immediately that

$$\det\left(I+\lambda_o\begin{bmatrix} I \\ 0 \end{bmatrix}N_{hc}D_{hc}^{-1}\right)=\det\left(I+\lambda_o\begin{bmatrix} I \\ 0 \end{bmatrix}c\,G(\infty)\right)=0.$$

Thus there would exist a vector $v\in C^p$ such that

$$\begin{bmatrix} I \\ 0 \end{bmatrix}G(\infty)v=\frac{1}{\lambda_o}v,$$

and therefore

$$\left\|\begin{bmatrix} I \\ 0 \end{bmatrix}G(\infty)v\right\|_2=\|G(\infty)v\|_2=\frac{1}{\lambda_o}\|v\|_2\geq\|v\|_2,$$

so that $\|G^*(\infty)G(\infty)\| = \|G(\infty)\|^2 \geq 1$, which is a contradiction.

With this claim we know that the family of polynomials $P_\lambda(s)$ contains one stable element, namely $P_0(s)$, and has a fixed degree. Using now the continuity of the roots of a polynomial with respect to its coefficients, we see that this family contains an unstable polynomial if and only if it also contains a polynomial with a root on the imaginary axis. However, if $P_{\lambda_o}(j\omega_o) = 0$ for some $\lambda_o \in (0, 1]$ and some $\omega_o \in \mathbb{R}$, then

$$\det\left(D(j\omega_o) + \lambda_o \begin{bmatrix} I \\ 0 \end{bmatrix} N(j\omega_o)\right) = 0,$$

and the same argument as above leads to the contradiction that

$$\|G^*(j\omega_o)G(j\omega_o)\| = \|G(j\omega_o)\|^2 \geq 1.$$

To complete the proof we need only apply the exact same reasoning as in the first step of the proof to the family of polynomials

$$\left\{P_U(s) = \det\left(D(s) + U \begin{bmatrix} I \\ 0 \end{bmatrix} N(s)\right) : U \text{ unitary matrix in } C^{p\times p}\right\}.$$

It can be proved first that this family has a constant degree, and the first part of the proof shows that it contains one stable polynomial (corresponding to $U = I$). Now note that the set of unitary matrices is pathwise connected. This is due to the fact that any unitary matrix U can be expressed as $U = e^{jF}$, where F is some Hermitian matrix, and therefore it is the image of a convex set under a continuous mapping. This implies in turn that the family of polynomials $P_U(s)$ is pathwise connected and thus allows us to use the continuity property of the roots of a polynomial with respect to its coefficients. ♣

9.3.1 Worst Case H_∞ Stability Margin

Theorem 9.2 states that in order to compute α the radius of the maximum allowable unstructured perturbation ball that does not destroy closed loop stability, it is sufficient to compute the maximum H_∞ norm of the Kharitonov systems. This is a tremendous reduction in computation since it replaces testing norms of an infinite family of functions to that of a finite set. In this context it is important to note that the boundary results of Chapter 8 already tell us that since $\partial\mathbf{G}(j\omega) \subset \partial\mathbf{G}_{\mathrm{E}}(j\omega)$, the maximum H_∞ norm over the parameter set occurs over the subset $\mathbf{G}_{\mathrm{E}}(s)$. The following example illustrates the use of these results.

Example 9.1. Consider the stable family $\mathbf{G}(s)$ of interval systems whose generic element is given by

$$G(s) = \frac{n_0 + n_1 s + n_2 s^2 + n_3 s^3}{d_0 + d_1 s + d_2 s^2 + d_3 s^3}$$

where

$$n_0 \in [1, 2], \quad n_1 \in [-3, 1], \quad n_2 \in [2, 4], \quad n_3 \in [1, 3],$$
$$d_0 \in [1, 3], \quad d_1 \in [2, 4], \quad d_2 \in [6, 7], \quad d_3 \in [1, 2].$$

$\mathbf{G}_{\mathrm{K}}(s)$ consists of the following 16 rational functions

$$
\begin{aligned}
G_1(s) &= \frac{1-3s+4s^2+3s^3}{1+2s+7s^2+2s^3}, & G_2(s) &= \frac{1-3s+4s^2+3s^3}{1+4s+7s^2+s^3} \\
G_3(s) &= \frac{1-3s+4s^2+3s^3}{3+2s+6s^2+2s^3}, & G_4(s) &= \frac{1-3s+4s^2+3s^3}{3+4s+6s^2+s^3}, \\
G_5(s) &= \frac{1+s+4s^2+s^3}{1+2s+7s^2+2s^3}, & G_6(s) &= \frac{1+s+4s^2+s^3}{1+4s+7s^2+s^3}, \\
G_7(s) &= \frac{1+s+4s^2+s^3}{3+2s+6s^2+2s^3}, & G_8(s) &= \frac{1+s+4s^2+s^3}{3+4s+6s^2+s^3}, \\
G_9(s) &= \frac{2-3s+2s^2+3s^3}{1+2s+7s^2+2s^3}, & G_{10}(s) &= \frac{2-3s+2s^2+3s^3}{1+4s+7s^2+s^3}, \\
G_{11}(s) &= \frac{2-3s+2s^2+3s^3}{3+2s+6s^2+2s^3}, & G_{12}(s) &= \frac{2-3s+2s^2+3s^3}{3+4s+6s^2+s^3}, \\
G_{13}(s) &= \frac{2+s+2s^2+s^3}{1+2s+7s^2+2s^3}, & G_{14}(s) &= \frac{2+s+2s^2+s^3}{1+4s+7s^2+s^3}, \\
G_{15}(s) &= \frac{2+s+2s^2+s^3}{3+2s+6s^2+2s^3}, & G_{16}(s) &= \frac{2+s+2s^2+s^3}{3+4s+6s^2+s^3}.
\end{aligned}
$$

The 16 corresponding H_∞ norms are given by,

$$
\begin{array}{llll}
\|G_1\|_\infty = 2.112, & \|G_2\|_\infty = 3.0, & \|G_3\|_\infty = 5.002, & \|G_4\|_\infty = 3.0, \\
\|G_5\|_\infty = 1.074, & \|G_6\|_\infty = 1.0, & \|G_7\|_\infty = 1.710, & \|G_8\|_\infty = 1.0, \\
\|G_9\|_\infty = 3.356, & \|G_{10}\|_\infty = 3.0, & \|G_{11}\|_\infty = 4.908, & \|G_{12}\|_\infty = 3.0, \\
\|G_{13}\|_\infty = 2.848, & \|G_{14}\|_\infty = 2.0, & \|G_{15}\|_\infty = 1.509, & \|G_{16}\|_\infty = 1.0\ .
\end{array}
$$

Therefore, the entire family of systems remains stable under any unstructured feedback perturbations of H_∞ norm less than

$$\alpha = \frac{1}{5.002} = 0.19992\ .$$

To compute the worst case H_∞ stability margin of this system, without using the result of Lemma 9.2, we would need to plot the frequency template $\mathbf{G}(j\omega)$. From the boundary properties developed in Chapter 8 it follows that the maximum H_∞ norm occurs on the subset $\mathbf{G}_{\mathrm{E}}(j\omega)$. Figure 9.3 shows this template from which we find the worst case H_∞ stability margin to be 0.2002. This compares favorably with the previous computation of the H_∞ norm at the Kharitonov vertices, which was based on Lemma 9.2.

9.3.2 Worst Case Parametric Stability Margin

The converse problem, where a bound is fixed on the level α (size of the ball) of unstructured H_∞ perturbations that are to be tolerated and the largest parametric

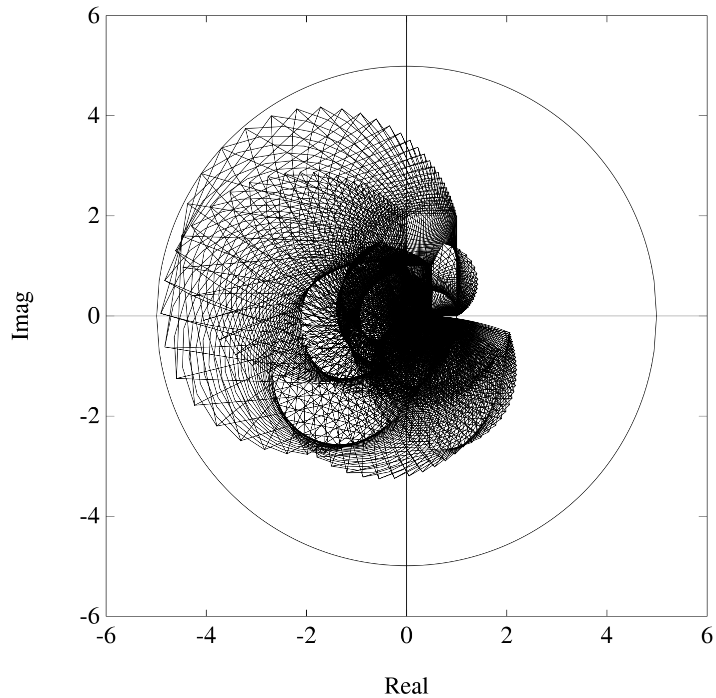

Figure 9.3. Frequency template of $\mathbf{G}_{\mathrm{E}}$ and H_∞ stability margin (Example 9.1)

stability ball is sought, is also important and will be considered in this subsection. In this case we start with a nominal stable system

$$G^o(s) = \frac{n_0^o + n_1^o s + \cdots + n_p^o s^p}{d_0^o + d_1^o s + \cdots + d_q^o s^q}$$

which satisfies $\|G^o\|_\infty = \alpha$. A bound $\frac{1}{\beta} < \frac{1}{\alpha}$ is then fixed on the desired level of unstructured perturbations. It is then possible to fix the structure of the parametric perturbations and to maximize a weighted l_∞ ball around the parameters of $G^o(s)$. More precisely, one can allow the parameters n_i, d_j of the plant to vary in intervals of the form

$$n_i \in [n_i^o - \epsilon\nu_i, n_i^o + \epsilon\nu_i], \quad d_j \in [d_j^o - \epsilon\mu_j, d_j^o + \epsilon\mu_j],$$

where the weights ν_i, μ_j are fixed and nonnegative. For each ϵ we get a family of interval systems $\mathbf{G}(s,\epsilon)$ and its associated set of Kharitonov systems $\mathbf{G}_{\mathrm{K}}(s,\epsilon)$. The structured stability margin is then given by the largest ϵ, say $\epsilon_{\max}$, for which every system $G(s)$ in the corresponding interval family $\mathbf{G}(s,\epsilon_{\max})$ satisfies $\|G\|_\infty \leq \beta$.

An upper bound ϵ_1 for $\epsilon_{\max}$ is easily found by letting ϵ_1 be the smallest number such that the interval family,

$$\left\{D(s) = d_0 + \cdots + d_q s^q : \ d_j \in [d_j^o - \epsilon_1 \mu_j, d_j^o + \epsilon_1 \mu_j]\right\}$$

contains an unstable polynomial. This upper bound is easily calculated using Kharitonov's Theorem. One way to compute $\epsilon_{\max}$ is then to use a bisection algorithm:

1. set LBOUND=0, UBOUND=ϵ_1.
2. Let $\epsilon = \dfrac{\text{UBOUND} + \text{LBOUND}}{2}$.
3. Update $\mathbf{G}_{\mathrm{K}}(s, \epsilon)$.
4. If the 16 systems in $\mathbf{G}_{\mathrm{K}}(s, \epsilon)$ have H_∞ norm $\leq \beta$ then set LBOUND $= \epsilon$, otherwise set UBOUND $= \epsilon$.
5. if $|\text{UBOUND} - \text{LBOUND}|$ is small enough then EXIT, otherwise GOTO 2.

This procedure requires, at each step, that we check the H_∞ norm of the 16 current Kharitonov systems. We illustrate the computation of the l_∞ structured margin in the following example.

Example 9.2. Let the nominal system be given by

$$G(s) = \frac{1-s}{1+3s+s^2}.$$

The H_∞ norm of $G(s)$ is equal to 1. Let us fix the bound on the unstructured margin to be equal to $\frac{1}{2}$. To simplify the notation we assume that the perturbed system is of the form

$$G_{a,b,c,d}(s) = \frac{1+a-(1+b)s}{1+c+(3+d)s+s^2}$$

where

$$|a| \leq \epsilon, \ |b| \leq \epsilon, \ |c| \leq \epsilon, \ |d| \leq \epsilon \ . \tag{9.6}$$

We seek the largest ϵ such that for all (a, b, c, d) satisfying (9.6)

$$||G_{a,b,c,d}||_\infty \leq 2.$$

For this simple example, $\epsilon_{\max}$ can be computed analytically and is readily found to be equal to 1/3. The extremal systems are,

$$\frac{\frac{4}{3} - \frac{2}{3}s}{\frac{2}{3} + \frac{10}{3}s + s^2}, \quad \frac{\frac{4}{3} - \frac{4}{3}s}{\frac{2}{3} + \frac{10}{3}s + s^2}, \quad \frac{\frac{4}{3} - \frac{2}{3}s}{\frac{2}{3} + \frac{8}{3}s + s^2}, \quad \frac{\frac{4}{3} - \frac{4}{3}s}{\frac{2}{3} + \frac{8}{3}s + s^2} \ .$$

In the next section we extend the results of this section to the case where a feedback controller is present and $\mathbf{G}(s)$ is a set of linear interval systems.

9.4 ROBUST SMALL GAIN THEOREM

We now extend the results of the last section to the case in which a fixed feedback controller is connected to a class of linear interval systems. As we have seen before, this case occurs when the parameters of interest enter affine linearly into the transfer function coefficients. Let

$$\mathbf{G}(s) := \frac{\mathbf{N}(s)}{\mathbf{D}(s)} \tag{9.7}$$

be a family of *strictly proper* linear interval systems. We refer the reader to Chapter 8 (Section 8.6, p. 358) for precise definition of this family of systems as well as the definition of the extremal subset $\mathbf{G}_{\mathrm{E}}(s)$.

Assume that we have found a stabilizing controller $C(s)$ for the entire family. We therefore have a family of stable closed-loop systems and we consider unstructured additive perturbations as shown in Figure 9.4. The family of perturbed plants under

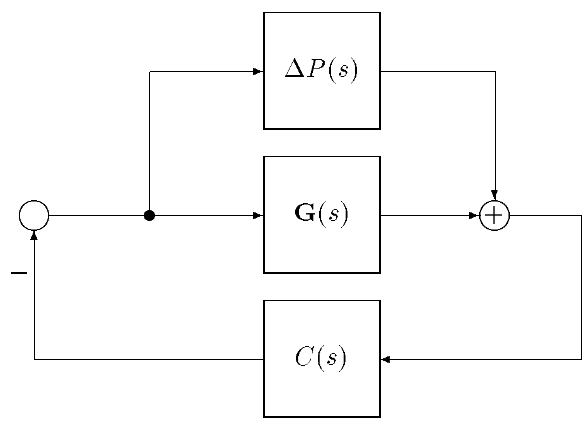

Figure 9.4. Closed loop system with additive norm bounded perturbations

consideration can be represented as

$$\{(G(s) + \Delta P) : G(s) \in \mathbf{G}(s), \quad \|\Delta P\|_\infty < \alpha\}.$$

In order to determine the amount of unstructured perturbations that can be tolerated by this family of additively perturbed interval systems we have to find the maximum of the H_∞ norm of the closed-loop transfer function

$$C(s)\,(1 + G(s)C(s))^{-1}$$

over all elements $G(s) \in \mathbf{G}(s)$.

In the case of multiplicative perturbations we consider the family of perturbed plants to be:

$$\{(I + \Delta P)G(s) : G(s) \in \mathbf{G}(s), \quad \|\Delta P\|_\infty < \alpha\}.$$

Here, the level of unstructured perturbations α that can be tolerated by the closed loop system without becoming unstable is determined by the maximum value of the H_∞ norm

$$\left\| G(s)C(s)\left(1+G(s)C(s)\right)^{-1} \right\|_\infty$$

as $G(s)$ ranges over $\mathbf{G}(s)$.

The following theorem shows us that the exact level of unstructured perturbations that can be tolerated by the *family* of closed-loop systems, can be computed, in both the additive and multiplicative cases, by replacing $\mathbf{G}(s)$ by the subset $\mathbf{G}_{\mathrm{E}}(s)$ in the corresponding block diagram.

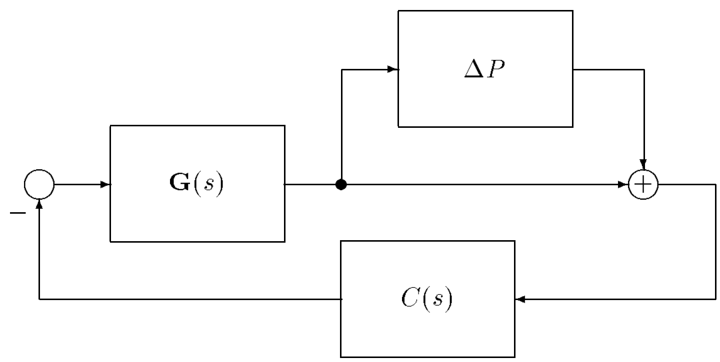

Figure 9.5. Closed loop system with multiplicative norm bounded perturbations

Theorem 9.3 (Robust Small Gain Theorem)
Given an interval family $\mathbf{G}(s)$ of strictly proper plants and a stabilizing controller $C(s)$ for $\mathbf{G}(s)$, the closed-loop in a) Figure 9.4 or b) Figure 9.5 remains stable for all stable perturbations ΔP such that $\|\Delta P\|_\infty < \alpha$ if and only if,

a) $\alpha \le \frac{1}{\sup_{G \in \mathbf{G}_{\mathrm{E}}} \left\| C(s)(1+G(s)C(s))^{-1} \right\|_\infty}$

b) $\alpha \le \frac{1}{\sup_{G \in \mathbf{G}_{\mathrm{E}}} \left\| G(s)C(s)(1+G(s)C(s))^{-1} \right\|_\infty}$.

Proof. For a) consider the family of transfer functions

$$\left\{ C(s)\left(1+G(s)C(s)\right)^{-1} \;:\; G(s) \in \mathbf{G}(s) \right\}$$

which we denote as

$$C(s)\left(1+\mathbf{G}(s)C(s)\right)^{-1}.$$

Recall from Theorem 8.4 (Chapter 8) that the boundary of the image of the above set at $s = j\omega$ is identical to the boundary of the image of the set

$$C(s)\left(1+\mathbf{G}_{\mathrm{E}}(s)C(s)\right)^{-1}.$$

In other words,

$$\partial\left(C(j\omega)\left(1+\mathbf{G}(j\omega)C(j\omega)\right)^{-1}\right) \subset C(j\omega)\left(1+\mathbf{G}_{\mathrm{E}}(j\omega)C(j\omega)\right)^{-1}$$

From this it follows that

$$\sup_{G\in\mathbf{G}}\left|C(j\omega)\left(1+G(j\omega)C(j\omega)\right)^{-1}\right| = \sup_{G\in\mathbf{G}_{\mathrm{E}}}\left|C(j\omega)\left(1+G(j\omega)C(j\omega)\right)^{-1}\right|$$

for each $\omega \geq 0$. Therefore,

$$\sup_{G\in\mathbf{G}}\left\|C(s)\left(1+G(s)C(s)\right)^{-1}\right\|_{\infty} = \sup_{G\in\mathbf{G}_{\mathrm{E}}}\left\|C(s)\left(1+G(s)C(s)\right)^{-1}\right\|_{\infty}.$$

This completes the proof of a). The proof of b) is similar. ♣

Remark 9.1. The theorems given above are stated for an interval plant $\mathbf{G}(s)$. From the boundary results Theorem 8.3 established in Chapter 8 we know that identical results hold for linear interval systems (p. 358) and also for linear fractional transformations of linear interval systems (p. 363).

Theorem 9.3 allows for the computation of the unstructured stability margin for a prescribed level of structured perturbations. This stability margin can also serve as a measure of the performance of the system. The computation of

$$\sup_{G\in\mathbf{G}_{\mathrm{E}}}\left\|C(s)\left(1+G(s)C(s)\right)^{-1}\right\|_{\infty}$$

is not very complicated as can be seen from the following example.

Example 9.3. (Computation of unstructured stability margin) Consider the interval plants,

$$\mathbf{G}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)} = \frac{\beta s}{1 - s + \gamma s^2 + s^3} \quad : \quad \beta \in [1,2],\ \gamma \in [3.4, 5].$$

Using the results of GKT (Chapter 7) one can easily check that the controller

$$C(s) = \frac{3}{s+1}$$

stabilizes the entire family. The transfer function family of interest is given by

$$C(s)\left(1+G_{\beta,\gamma}(s)C(s)\right)^{-1} = \frac{3(1-s+\gamma s^2+s^3)}{1+3\beta s+(\gamma-1)s^2+(\gamma+1)s^3+s^4}.$$

According to Theorem 9.3, to compute the H_∞ stability margin α, we have to find the maximum H_∞ norm of four one-parameter families of rational functions, namely

$$r_\lambda(s) = \frac{3(1-s+\lambda s^2+s^3)}{1+3s+(\lambda-1)s^2+(\lambda+1)s^3+s^4}, \quad \lambda \in [3.4, 5],$$

$$r_\mu(s) = \frac{3(1-s+\mu s^2+s^3)}{1+6s+(\mu-1)s^2+(\mu+1)s^3+s^4}, \quad \mu \in [3.4, 5],$$

$$r_\nu(s) = \frac{3(1-s+3.4s^2+s^3)}{1+3\nu s+2.4s^2+4.4s^3+s^4}, \quad \nu \in [1, 2],$$

$$r_\xi(s) = \frac{3(1-s+5s^2+s^3)}{1+3\xi s+4s^2+6s^3+s^4}, \quad \xi \in [1, 2].$$

Consider for example the case of $r_\lambda(s)$. We have

$$|r_\lambda(j\omega)|^2 = \frac{9\left((1-\lambda\omega^2)^2+\omega^2(1+\omega^2)^2\right)}{\left(1-(\lambda-1)\omega^2+\omega^4\right)^2+\omega^2\left(3-(\lambda+1)\omega^2\right)^2}.$$

Letting $t=\omega^2$ we have to find

$$\sup_{t\geq 0, \lambda\in[3.4,5]} f(t,\lambda) = \sup_{t\geq 0, \lambda\in[3.4,5]} \frac{9\left((1-\lambda t)^2+t(1+t)^2\right)}{\left(1-(\lambda-1)t+t^2\right)^2+t\left(3-(1+\lambda)t\right)^2}.$$

Differentiating with respect to λ we get a supremum at,

$$\lambda_1(t) = \frac{-2t+3+\sqrt{4t^3+12t^2+1}}{2t}$$

or

$$\lambda_2(t) = \frac{-2t+3-\sqrt{4t^3+12t^2+1}}{2t}.$$

It is then easy to see that $\lambda_1(t) \in [3.4, 5]$ if and only if $t \in [t_1, t_2] \cup [t_3, t_4]$ where,

$$t_1 \simeq 0.39796, \quad t_2 \simeq 0.64139, \quad t_3 \simeq 15.51766, \quad t_4 \simeq 32.44715\ ,$$

whereas, $\lambda_2(t) \in [3.4, 5]$ if and only if $t \in [t_5, t_6]$ where,

$$t_5 \simeq 0.15488, \quad t_6 \simeq 0.20095\ .$$

As a result, the maximum H_∞ norm for $r_\lambda(s)$ is given by,

$$\max\left(\|r_{3.4}\|_\infty, \|r_5\|_\infty, \sqrt{\sup_{t\in[t_1,t_2]\cup[t_3,t_4]} f(t,\lambda_1(t))}, \sqrt{\sup_{t\in[t_5,t_6]} f(t,\lambda_2(t))}\right)$$

where one can at once verify that,

$$f(t,\lambda_1(t)) = \frac{9\left(2t-1-\sqrt{4t^3+12t^2+1}\right)}{2t^2+7t-1-(t+1)\sqrt{4t^3+12t^2+1}}$$

and,

$$f(t,\lambda_2(t)) = \frac{9\left(2t-1+\sqrt{4t^3+12t^2+1}\right)}{2t^2+7t-1+(t+1)\sqrt{4t^3+12t^2+1}}.$$

This maximum is then easily found to be equal to,

$$\max(34.14944, 7.55235, 27.68284, 1.7028) = 34.14944\ .$$

Proceeding in the same way for $r_\mu(s)$, $r_\nu(s)$, and $r_\xi(s)$, we finally get

$$\max_{\beta\in[1,2],\gamma\in[3.4,5]} \left\| C(s)\left(1+G_{\beta,\gamma}(s)C(s)\right)^{-1} \right\|_\infty = 34.14944$$

where the maximum is in fact achieved for $\beta = 1$, $\gamma = 3.4$.

9.5 ROBUST PERFORMANCE

The result established in the last section that for the controller to tolerate a certain amount of unstructured uncertainty over the entire parametrized family $\mathbf{G}(s)$ it is necessary and sufficient that it achieves the same level of tolerance over the subset $\mathbf{G}_{\mathrm{E}}(s)$. In the H_∞ approach to robust control problems, system performance is measured by the size of the H_∞ norm of error, output and other transfer functions. When parameter uncertainty is present, it is appropriate to determine *robust performance* by determining the worst case performance over the parameter uncertainty set. This amounts to determining the maximum values of the H_∞ norm of various system transfer functions over the uncertainty set. For instance, in the control system shown in Figure 9.6 it may be desirable to minimize the H_∞ norm of the error transfer function

$$T^e(s) = \left(1 + C(s)G(s)\right)^{-1} \tag{9.8}$$

in order to minimize the worst case tracking error. The H_∞ norm of the output transfer function

$$T^y(s) = C(s)G(s)\left(1 + C(s)G(s)\right)^{-1} \tag{9.9}$$

known as *M - peak* in classical control is usually required to be small. To keep the control signal small the H_∞ norm of

$$T^u(s) = C(s)\left(1 + C(s)G(s)\right)^{-1} \tag{9.10}$$

should be small.

The following theorem shows us that the worst case performance measured in any of the above norms over the set of systems $\mathbf{G}(s)$ can in fact be determined by replacing $G(s)$ in the control system by elements of the one-parameter family of systems $\mathbf{G}_{\mathrm{E}}(s)$.

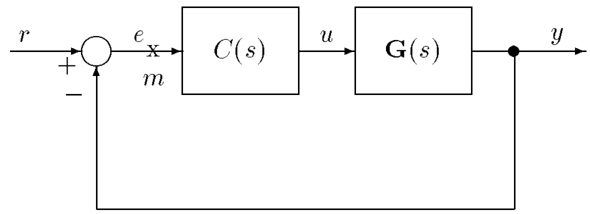

Figure 9.6. A standard unity feedback systems

Theorem 9.4 (Robust Performance)
The maximum value of the H_∞ norms of $T^e(s)$, $T^y(s)$ and $T^u(s)$ over $\mathbf{G}(s)$ is attained over $\mathbf{G}_{\mathrm{E}}(s)$:

$$\sup_{G\in\mathbf{G}} \left\| (1+C(s)G(s))^{-1} \right\|_\infty = \sup_{G\in\mathbf{G}_{\mathrm{E}}} \left\| (1+C(s)G(s))^{-1} \right\|_\infty$$

$$\sup_{G\in\mathbf{G}} \left\| C(s)G(s)\,(1+C(s)G(s))^{-1} \right\|_\infty = \sup_{G\in\mathbf{G}_{\mathrm{E}}} \left\| C(s)G(s)\,(1+C(s)G(s))^{-1} \right\|_\infty .$$

$$\sup_{G\in\mathbf{G}} \left\| C(s)\,(1+C(s)G(s))^{-1} \right\|_\infty = \sup_{G\in\mathbf{G}_{\mathrm{E}}} \left\| C(s)\,(1+C(s)G(s))^{-1} \right\|_\infty .$$

The proof of this theorem is identical to that of Theorem 9.3 and also follows from the boundary properties given in Chapter 8. Analogous results hold for systems where disturbances are present and also where the transfer functions under consideration are suitably weighted. These results precisely determine the role of the controller in robust stability and performance analysis and in the design of control systems containing parameter uncertainty.

Example 9.4. **(H_∞ Performance Example)** Let the plant and the stabilizing controller be

$$G(s) = \frac{\beta s}{s^3 + \alpha s^2 - s + 1}$$

and

$$C(s) = \frac{3}{s+1}$$

where

$$\alpha \in [3.4, 5], \quad \beta \in [1, 2].$$

A closed loop system transfer function is

$$T^u(s) = C(s)\left[1 + \mathbf{G}(s)C(s)\right]^{-1}.$$

To compute the worst case H_∞ stability margin of this closed loop system under additive perturbations, we only need to plot the frequency template

$$\mathbf{T}^u(j\omega) = C(j\omega)\left[1 + \mathbf{G}_{\mathrm{E}}(j\omega)C(j\omega)\right]^{-1}.$$

Figure 9.7 shows this template. From this we find that the worst case H_∞ stability margin is 0.0293.

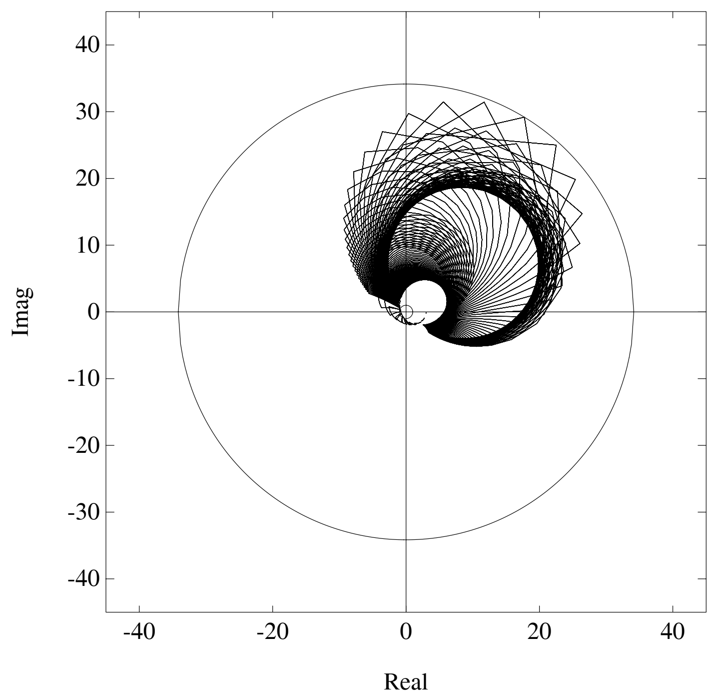

Figure 9.7. Frequency template of $\mathbf{T}^u(j\omega)$ and H_∞ stability margin (Example 9.4)

9.6 VERTEX RESULTS FOR EXTREMAL H_∞ NORMS

In this subsection we give some useful vertex results for the computation of the extremal H_∞ norm. We start by considering a control system containing an interval plant with multiplicative unstructured perturbations, connected to a feedback controller with a special structure.

Theorem 9.5 *Let a controller $C(s)$ be of the form*

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{A_1(s)E_1(s)}{s^t A_2(s)E_2(s)} \tag{9.11}$$

where t is a positive integer, $A_i(s)$ are anti-Hurwitz and $E_i(s)$ are even polynomials. $C(s)$ stabilizes the feedback system in Figure 9.5 for all ΔP such that $\|\Delta P\|_\infty < \alpha$ if and only if it stabilizes $\mathbf{G}_{\mathrm{K}}(s)$ and

$$\alpha \leq \frac{1}{\sup_{G \in \mathbf{G}_{\mathrm{K}}} \left\| G(s)C(s)\left(1 + G(s)C(s)\right)^{-1} \right\|_\infty}.$$

Proof. From Theorem 9.4 we already know that the extremal value of the H_∞ norm of $G(s)C(s)(1 + G(s)C(s))^{-1}$ over the family $\mathbf{G}(s)$ occurs over the subset $\mathbf{G}_{\mathrm{E}}(s)$. Thus, we need only show that the maximum H_∞ norm of the closed loop transfer function $G(s)C(s)(1 + G(s)C(s))^{-1}$ along an arbitrary segment of $\mathbf{G}_{\mathrm{E}}(s)$ is attained at one of the two vertex plants corresponding to its endpoints.

Now consider a specific extremal segment in $\mathbf{G}_{\mathrm{E}}(s)$ say,

$$G(s, \lambda) := \frac{K_N^1(s)}{K_D^3(s) + \lambda\left[K_D^2(s) - K_D^3(s)\right]}, \quad \lambda \in [0, 1].$$

Denote the associated vertex systems by

$$V_1(s) = \left(C(s)\frac{K_N^1(s)}{K_D^3(s)} \left(1 + \frac{K_N^1(s)}{K_D^3(s)}C(s)\right)^{-1} \right)$$
$$V_2(s) = \left(C(s)\frac{K_N^1(s)}{K_D^2(s)} \left(1 + \frac{K_N^1(s)}{K_D^2(s)}C(s)\right)^{-1} \right).$$

We need to establish the implication

$$\max\left(\|V_1(s)\|_\infty, \|V_2(s)\|_\infty\right) < \frac{1}{\alpha} \tag{9.12}$$

$$\Downarrow$$

$$\left\| G(s,\lambda)C(s)\left(1 + G(s,\lambda)C(s)\right)^{-1} \right\|_\infty < \frac{1}{\alpha}, \quad \text{for all} \quad \lambda \in [0, 1]. \tag{9.13}$$

By the hypotheses in (9.12) and Lemma 9.2, the following two polynomials are Hurwitz for any fixed real $\theta \in [0, 2\pi]$:

$$P_0(s) = K_N^1(s)N_c(s)(1 - \alpha e^{j\theta}) + D_c(s)K_D^3(s)$$
$$P_1(s) = K_N^1(s)N_c(s)(1 - \alpha e^{j\theta}) + D_c(s)K_D^2(s).$$

Consider now the complex segment of polynomials

$$P_\lambda(s) = P_0(s) + \lambda\left[P_1(s) - P_0(s)\right], \quad \lambda \in [0, 1].$$

We have

$$\delta_0(s) := P_1(s) - P_0(s) = D_c(s)[K_D^3(s) - K_D^2(s)] = s^t A_2(s)E_2(s)[K_D^3(s) - K_D^2(s)].$$

It is easy to see that under the assumptions made on $C(s)$

$$\frac{d}{d\omega}\arg\delta_0(j\omega) \le 0$$

since the anti-Hurwitz factor has nonpositive rate of change of phase while each of the other factors has a zero rate of change of phase. Thus $\delta_0(s)$ is a convex direction by Lemma 2.15 (Chapter 2) and therefore $P_\lambda(s)$ is Hurwitz for all $\lambda \in [0,1]$. This implies that

$$\begin{aligned} P_\lambda(j\omega) = {} & K_N^1(j\omega)N_c(j\omega)(1-\alpha e^{j\theta}) + D_c(j\omega)K_D^3(j\omega) \\ & +\lambda D_c(j\omega)\left[K_D^2(j\omega) - K_D^3(j\omega)\right] \neq 0, \\ & \text{for all } \ \omega, \quad \text{for all } \ \lambda \in [0,1]. \end{aligned} \tag{9.14}$$

Since (9.14) holds for any $\theta \in [0, 2\pi]$, the conditions in Lemma 9.2 are satisfied for the transfer function

$$\alpha G(s,\lambda)C(s)\left(1 + G(s,\lambda)C(s)\right)^{-1}, \quad \text{for all } \ \lambda \in [0,1].$$

Thus, it follows that (9.13) holds. This type of argument can be applied to each segment in $\mathbf{G}_E(s)$ to complete the proof. ♣

The extremal H_∞ norm calculation of the sensitivity and complementary sensitivity functions of a unity feedback system containing an interval plant $\mathbf{G}(s)$ also enjoy the vertex property. The sensitivity function is

$$S(s) = \frac{1}{1+G(s)} \tag{9.15}$$

and the complementary sensitivity function is

$$T(s) = 1 - S(s). \tag{9.16}$$

Lemma 9.4

$$\begin{aligned} \sup_{G\in\mathbf{G}} \|S(s)\|_\infty &= \sup_{G\in\mathbf{G}_{\mathrm{K}}} \|S(s)\|_\infty \\ \sup_{G\in\mathbf{G}} \|T(s)\|_\infty &= \sup_{G\in\mathbf{G}_{\mathrm{K}}} \|T(s)\|_\infty . \end{aligned}$$

We leave the proof to the reader as it is identical to the proof of the previous result, and based on Lemma 9.2 and the Complex Convex Direction Lemma (Lemma 2.15, Chapter 2).

To conclude this section we present without proof two special vertex results that do not follow from the Convex Direction Lemma. The first result holds for interval plants multiplied by a special type of weighting factor. As usual, let $\mathbf{G}(s)$ denote an interval transfer function family.

Lemma 9.5 *Let $p(s)$ be an arbitrary polynomial and β a positive real number such that $\frac{p(s)}{s+\beta}G(s)$ is proper and stable for every $G(s) \in \mathbf{G}(s)$. Then*

$$\sup_{G \in \mathbf{G}} \left\| \frac{p(s)}{s+\beta} G(s) \right\|_\infty = \sup_{G \in \mathbf{G}_{\mathrm{K}}} \left\| \frac{p(s)}{s+\beta} G(s) \right\|_\infty$$

The next vertex result holds for a limited class of weighted sensitivity and complementary sensitivity functions.

Lemma 9.6 *Let α and β be positive numbers with $\alpha \neq \beta$, $K_S > 1$ and $K_T < \frac{\beta}{\alpha}$. If the transfer functions*

$$\frac{s+\alpha}{s+\beta} S(s) \quad \text{and} \quad \frac{s+\alpha}{s+\beta} T(s)$$

are proper and stable for all $G(s) \in \mathbf{G}(s)$, then

$$\begin{aligned} \sup_{G \in \mathbf{G}} \left\| K_S \frac{s+\alpha}{s+\beta} S(s) \right\|_\infty &= \sup_{G \in \mathbf{G}_{\mathrm{K}}} \left\| K_S \frac{s+\alpha}{s+\beta} S(s) \right\|_\infty \\ \sup_{G \in \mathbf{G}} \left\| K_T \frac{s+\alpha}{s+\beta} T(s) \right\|_\infty &= \sup_{G \in \mathbf{G}_{\mathrm{K}}} \left\| K_T \frac{s+\alpha}{s+\beta} T(s) \right\|_\infty . \end{aligned}$$

In the following section we turn to another model of unstructured perturbation, namely nonlinear feedback gains lying in a sector.

9.7 THE ABSOLUTE STABILITY PROBLEM

The classical Lur'e or Popov problem considers the stability of a fixed linear time invariant dynamic system perturbed by a family of nonlinear feedback gains. This problem is also known as the *absolute stability* problem. This framework is a device to account for unstructured perturbations of the fixed linear system. Consider the configuration in Figure 9.8 where a stable, linear time-invariant system is connected by feedback to a memoryless, time-varying nonlinearity.

The Absolute Stability problem is the following: Under what conditions is the closed-loop system in the configuration above globally, uniformly asymptotically stable for all nonlinearities in a prescribed class? We first consider the allowable nonlinearities to be time-varying and described by sector bounded functions. Specifically, the nonlinearity $\phi(t, \sigma)$ is assumed to be single-valued and satisfying (see Figure 9.9)

$$\begin{aligned} \phi(t, 0) = 0, \quad &\text{for all } t \geq 0, \\ 0 \leq \sigma\phi(t, \sigma) \leq k\sigma^2&. \end{aligned} \tag{9.17}$$

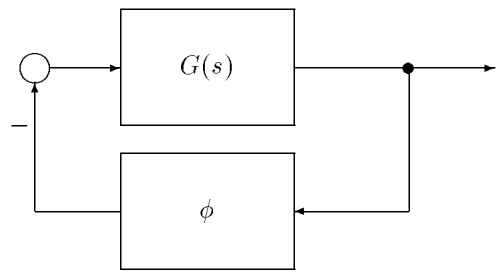

Figure 9.8. Absolute stability problem

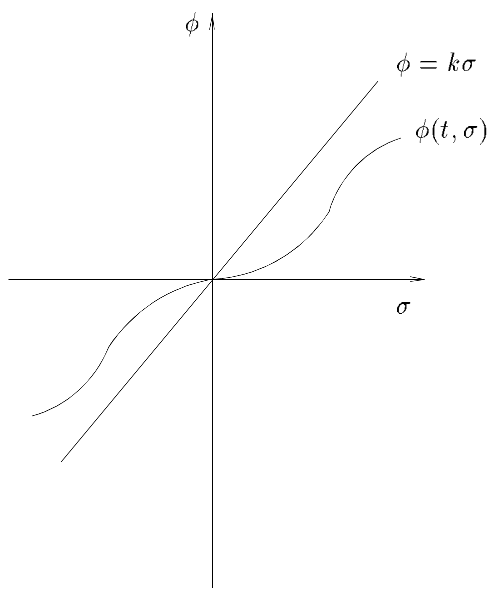

Figure 9.9. Sector bounded nonlinear function

Assumption (9.17) implies that $\phi(t, \sigma)$ is bounded by the lines $\phi = 0$ and $\phi = k\sigma$. Such nonlinearities are said to belong to a sector $[0, k]$.

An important condition that arises in the solution of this absolute stability problem is the property of strict positive realness (SPR) of a transfer function. This property is closely related to passivity and robustness of the system. The SPR property is defined as follows:

Definition 9.1. A proper transfer function $G(s)$ is said to be strictly positive real (SPR) if

1) $G(s)$ has no poles in the closed right half plane, and

2) $\mathrm{Re}[G(j\omega)] > 0, \quad \omega \in (-\infty, +\infty)$.

Referring to Figure 9.8, we first state a well-known result on absolute stability.

Theorem 9.6 (Lur'e Criterion)
If $G(s)$ is a stable transfer function, and ϕ belongs to the sector $[0, k_L]$, then a sufficient condition for absolute stability is that

$$\frac{1}{k_L} + \mathrm{Re}\,[G(j\omega)] > 0, \qquad \textit{for all } \omega \in \mathbb{R}. \tag{9.18}$$

We illustrate this with an example.

Example 9.5. Let us consider the following stable transfer function

$$G(s) = \frac{3.1s + 3.2}{s^4 + 1.1s^3 + 24.5s^2 + 2.5s + 3.5}$$

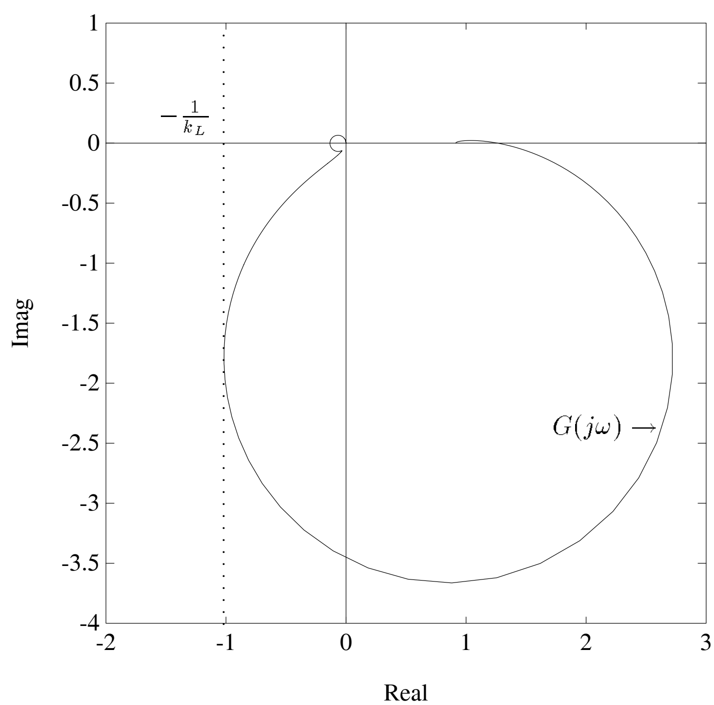

Figure 9.10. $G(j\omega)$: Lur'e problem (Example 9.5)

Figure 9.10 shows that the Lur'e gain k_L is obtained from the minimum real value of $G(j\omega)$. From Figure 9.10 we obtain $k_L = 0.98396$. Using the Lur'e gain, Figure 9.11 shows that

$$\frac{1}{k_L} + \mathrm{Re}\,[G(s)]$$

is SPR.

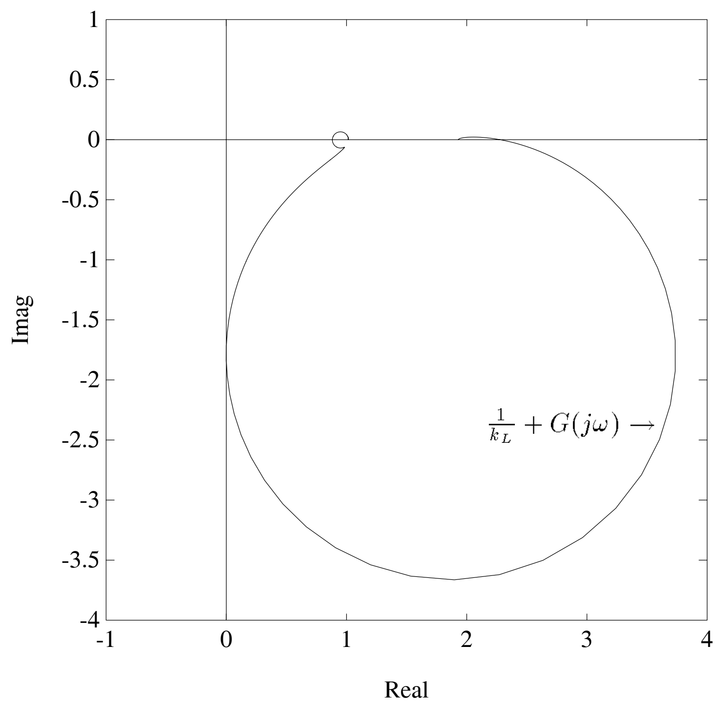

Figure 9.11. SPR property of $\frac{1}{k_L} + G(s)$ (Example 9.5)

We now impose the further restriction that the nonlinearity ϕ is time-invariant and enunciate the Popov criterion.

Theorem 9.7 (Popov Criterion)
If $G(s)$ is a stable transfer function, and ϕ is a time-invariant nonlinearity which belongs to the sector $[0, k_P]$, then a sufficient condition for absolute stability is that there exist a real number q such that

$$\frac{1}{k_P} + \mathrm{Re}\,[(1 + qj\omega)G(j\omega)] > 0, \qquad \textit{for all } \omega \in \mathbb{R}. \tag{9.19}$$

This theorem has a graphical interpretation which is illustrated in the next example.

Example 9.6. Consider the transfer function used in Example 9.5. To illustrate the Popov criterion, we need the *Popov plot*

$$\tilde{G}(j\omega) = \mathrm{Re}\,[G(j\omega)] + j\omega \mathrm{Im}\,[G(j\omega)]\,.$$

As shown in Figure 9.12, the limiting value of the Popov gain k_P is obtained by selecting a straight line in the Popov plane such that the Popov plot of $\tilde{G}(j\omega)$ lies below this line. From Figure 9.12 we obtain $k_P = 2.5$ We remark that the Lur'e gain corresponds to the case $q = 0$ in the Popov plot.

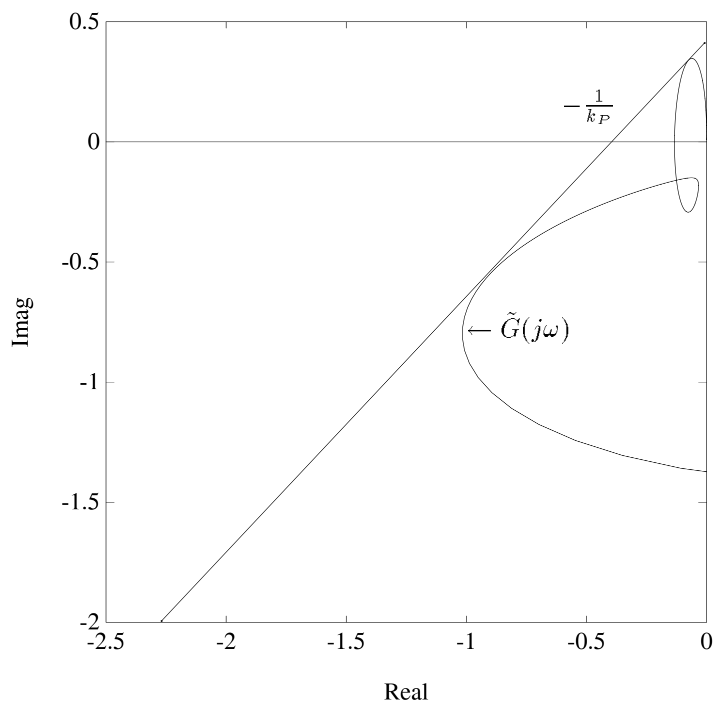

Figure 9.12. Popov criterion (Example 9.6)

In addition to the Lur'e and Popov criteria, there is another useful result in robust stability of nonlinear systems known as the *Circle Criterion.* It is assumed that the nonlinearity is time-invariant and lies in the sector $[k_1, k_2]$:

$$0 \leq k_1 \leq \phi(\sigma) \leq k_2. \tag{9.20}$$

Introduce the complex plane circle $\mathcal{C}$ centered on the negative real axis and cutting it at the points $-\dfrac{1}{k_1}$ and $-\dfrac{1}{k_2}$.

Theorem 9.8 (Circle Criterion)
If $G(s)$ is a stable transfer function and ϕ is a time-invariant nonlinearity which belongs to the sector $[k_1, k_2]$, then a sufficient condition for absolute stability is that the Nyquist plot $G(j\omega)$ stays out of the circle $\mathcal{C}$.

We illustrate this theorem with an example.

Example 9.7. Consider again the transfer function $G(s)$ used in Example 9.5. Figure 9.13 shows the plot of $G(j\omega)$ for $0 \leq \omega < \infty$. From Figure 9.13 we see that the smallest circle centered at -1 touches the $G(j\omega)$ locus and cuts the negative real axis at $-\dfrac{1}{k_2} = -0.3$ and $-\dfrac{1}{k_1} = -1.7$. This gives the absolute stability sector $[0.59, 3.33]$.

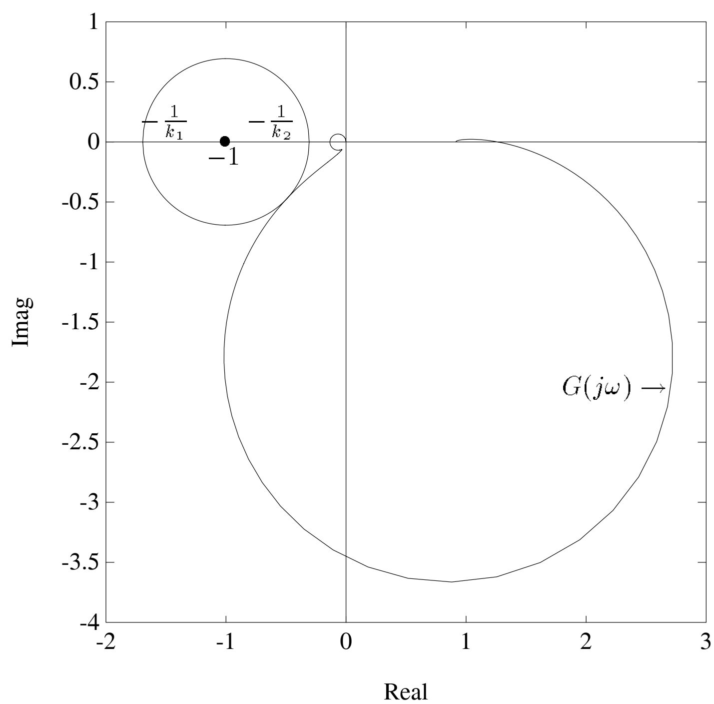

Figure 9.13. Circle criterion (Example 9.7)

In a later section we will consider the robust version of the absolute stability problem by letting the transfer function $G(s)$ vary over a family $\mathbf{G}(s)$. In this case it will be necessary to determine the infimum value of the stability sectors as $G(s)$

ranges over the family $\mathbf{G}(s)$. We will see that in each case these stability sectors can be found from the extremal systems $\mathbf{G}_{\mathrm{E}}(s)$ in a constructive manner. We develop some preliminary results on the SPR property that will aid in this calculation.

9.8 CHARACTERIZATION OF THE SPR PROPERTY

The importance of the SPR property in robustness can be seen from the fact that a unity feedback system containing a forward transfer function which is SPR has infinite gain margin and at least 90° phase margin. Our first result is a stability characterization of proper stable real transfer functions satisfying the SPR property. More precisely, let

$$G(s) = \frac{N(s)}{D(s)} \tag{9.21}$$

be a real proper transfer function with no poles in the closed right-half plane.

Theorem 9.9 *$G(s)$ is SPR if and only if the following three conditions are satisfied:*

a) $\mathrm{Re}[G(0)] > 0$,

b) $N(s)$ is Hurwitz stable,

c) $D(s) + j\alpha N(s)$ is Hurwitz stable for all α in $\mathbb{R}$.

Proof. Let us first assume that $G(s)$ is SPR and let us show that conditions b) and c) are satisfied since condition a) is clearly true in that case. Consider the family of polynomials:

$$\mathcal{P} := \{P_\alpha(s) = D(s) + j\alpha N(s) : \ \alpha \in \mathbb{R}\}.$$

Every polynomial in this family has the same degree as that of $D(s)$. Since this family contains a stable element, namely $P_0(s) = D(s)$, it follows from the continuity of the roots of a polynomial (Boundary Crossing Theorem of Chapter 1) that $\mathcal{P}$ will contain an unstable polynomial if and only if it also contains an element with a root at $j\omega$ for some $\omega \in \mathbb{R}$. Assume that for some α_o and ω in $\mathbb{R}$ we had that:

$$P_{\alpha_o}(j\omega) = D(j\omega) + j\alpha_o D(j\omega) = 0.$$

We write

$$D(j\omega) = D^e(\omega) + j\omega D^o(\omega) \tag{9.22}$$

with similar notation for $N(j\omega)$. Separating the real and imaginary parts of $P_{\alpha_o}(j\omega)$, we deduce that

$$D^e(\omega) - \alpha_o \omega N^o(\omega) = 0 \quad \text{and} \quad \omega D^o(\omega) + \alpha_o N^e(\omega) = 0.$$

But this implies necessarily that

$$N^e(\omega)D^e(\omega) + \omega^2 N^o(\omega) D^o(\omega) = 0$$

that is

$$\mathrm{Re}\,[G(j\omega)] = 0$$

and this contradicts the fact that $G(s)$ is SPR. Thus c) is also true.
Since c) is true it now implies that,

$$N(s) + j\beta D(s) \ \text{ is Hurwitz stable for all } \ \beta \neq 0.$$

Therefore, letting β tend to 0 we see that $N(s)$ is a limit of Hurwitz polynomials of bounded degree. Rouché's theorem immediately implies that the unstable roots of $N(s)$, if any, can only be on the $j\omega-$axis. However, if $N(s)$ has a root on the $j\omega-$axis then $\mathrm{Re}\,[G(j\omega)] = 0$ at this root and again this contradicts the fact that $G(s)$ is SPR.

To prove the converse, we use the fact that a) and b) hold and reason by contradiction. Since a) holds it follows by continuity that $G(s)$ is not SPR if and only if for some $\omega \in \mathbb{R}$, $0 \neq \omega$, we have that $\mathrm{Re}\,[G(j\omega)] = 0$, or equivalently

$$N^e(\omega)D^e(\omega) + \omega^2 N^o(\omega)D^o(\omega) = 0. \tag{9.23}$$

Now, assume that at this particular ω, $N(s)$ satisfies

$$N^e(\omega) \neq 0, \quad \text{and} \quad N^o(\omega) \neq 0.$$

From (9.23) we then conclude that

$$\frac{D^e(\omega)}{\omega N^o(\omega)} = \frac{-\omega D^o(\omega)}{N^e(\omega)} = \alpha_o \tag{9.24}$$

and therefore

$$D^e(\omega) - \alpha_o \omega N^o(\omega) = 0, \quad \text{and} \quad \omega D^o(\omega) + \alpha_o N^e(\omega) = 0 \tag{9.25}$$

so that

$$[D + j\alpha_o N]\,(j\omega) = 0,$$

contradicting c).

On the other hand assume for example that $N^e(\omega) = 0$. Since $N(s)$ is stable, we deduce that $N^o(\omega) \neq 0$, and from (9.23) we also have that $D^o(\omega) = 0$. Therefore (9.24) is still true with

$$\alpha_o = \frac{D^e(\omega)}{\omega N^o(\omega)}.$$

♣

9.8.1 SPR Conditions for Interval Systems

Now consider the following family $\mathbf{G}(s)$ of transfer functions

$$G(s) = \frac{A(s)}{B(s)},$$

where $A(s)$ belongs to a family of real interval polynomials $\mathbf{A}(s)$, and $B(s)$ belongs to a family of real interval polynomials $\mathbf{B}(s)$, defined as follows:

$$\mathbf{A}(s) = \{A(s) \ : \ A(s) = a_0 + a_1 s + \cdots + a_p s^p, \ \text{and } a_i \in [\alpha_i, \beta_i], \ \text{for all } \ i = 0, \cdots, p\}$$
$$\mathbf{B}(s) = \{B(s) \ : \ B(s) = b_0 + b_1 s + \cdots + b_n s^n, \ \text{and } b_j \in [\gamma_j, \delta_j], \ \text{for all } \ j = 0, \cdots, n\}.$$

Let $K_A^i(s)$, $i = 1, 2, 3, 4$ and $K_B^i(s)$, $i = 1, 2, 3, 4$ denote the Kharitonov polynomials associated with $\mathbf{A}(s)$ and $\mathbf{B}(s)$ respectively. We call $\mathbf{G}(s)$ a family of interval plants and the *Kharitonov systems* associated with $\mathbf{G}(s)$ are naturally defined to be the 16 plants of the following set,

$$\mathbf{G}_{\mathrm{K}}(s) := \left\{ \frac{K_A^i(s)}{K_B^j(s)} \ : \ i, j \in \{1, 2, 3, 4\} \right\}.$$

We assume that the interval family $\mathbf{G}(s)$ is stable. Let γ be any given real number. We want to find necessary and sufficient conditions under which it is true that for all $G(s)$ in $\mathbf{G}(s)$:

$$\mathrm{Re}\,[G(j\omega)] + \gamma > 0, \quad \text{for all } \ \omega \in \mathbb{R}. \tag{9.26}$$

In other words we ask the question: Under what conditions is $G(s) + \gamma$ SPR for all $G(s)$ in $\mathbf{G}(s)$? The answer to this question is given in the following lemma.

Lemma 9.7 *Equation (9.26) is satisfied by every element in $\mathbf{G}(s)$ if and only if it is satisfied for the 16 Kharitonov systems in $\mathbf{G}_{\mathrm{K}}(s)$.*

Proof. For an arbitrary $A(s)$ in $\mathbf{A}(s)$ and an arbitrary $B(s)$ in $\mathbf{B}(s)$ we can write:

$$\mathrm{Re}\left[\frac{A(j\omega)}{B(j\omega)}\right] + \gamma > 0$$
$$\Longleftrightarrow (A^e(\omega) + \gamma B^e(\omega))\, B^e(\omega) + \omega^2 \,(A^o(\omega) + \gamma B^o(\omega))\, B^o(\omega) > 0. \tag{9.27}$$

The right hand side of this last inequality is linear in $A^e(\omega)$ and $A^o(\omega)$ and thus from the facts (see Chapter 5)

$$K_A^{\mathrm{e,min}}(\omega) := K_A^{\mathrm{even,min}}(j\omega) \le A^e(\omega) \le K_A^{\mathrm{e,max}}(\omega) := K_A^{\mathrm{even,max}}(j\omega)$$
$$K_A^{\mathrm{o,min}}(\omega) := \frac{K_A^{\mathrm{odd,min}}(j\omega)}{j\omega} \le A^o(\omega) \le K_A^{\mathrm{o,max}}(\omega) := \frac{K_A^{\mathrm{odd,max}}(j\omega)}{j\omega},$$

it is clear that it is enough to check (9.27) when $A(s)$ is fixed and equal to one of the 4 Kharitonov polynomials associated with $\mathbf{A}(s)$. To further explain this point, let $A(s)$ and $B(s)$ be arbitrary polynomials in $\mathbf{A}(s)$ and $\mathbf{B}(s)$ respectively and suppose arbitrarily that at a given ω we have $B^e(\omega) > 0$ and $B^o(\omega) > 0$. Then the expression in (9.27) is obviously bounded below by

$$\left(K_A^{\mathrm{e,min}}(\omega) + \gamma B^e(\omega)\right) B^e(\omega) + \omega^2 \left(K_A^{\mathrm{o,min}}(\omega) + \gamma B^o(\omega)\right) B^o(\omega),$$

which corresponds to $A(s) = K_A^1(s)$.

Now, since $A(s)$ is a fixed polynomial, we deduce from Theorem 9.9 that the following is true:

$$\mathrm{Re}\left[\frac{A(j\omega)}{B(j\omega)}\right] + \gamma > 0, \quad \text{for all } \omega \in \mathbb{R}, \text{ and for all } B(s) \in \mathbf{B}(s),$$

if and only if the following three conditions are satisfied

1) $\mathrm{Re}\left[\dfrac{A(0)}{B(0)}\right] + \gamma > 0$, for all $B(s) \in \mathbf{B}(s)$,
2) $A(s) + \gamma B(s)$ is Hurwitz stable for all $B(s) \in \mathbf{B}(s)$,
3) $B(s) + \dfrac{j\alpha}{1 + j\alpha\gamma} A(s)$ is Hurwitz stable for all $\alpha \in \mathbb{R}$ and all $B(s) \in \mathbf{B}(s)$.

Note that in condition 3) above we have used the fact that

$$\mathrm{Re}\left[\frac{A(j\omega)}{B(j\omega)}\right] + \gamma > 0 \Longleftrightarrow \mathrm{Re}\left[\frac{A(j\omega) + \gamma B(j\omega)}{B(j\omega)}\right] > 0,$$

and therefore condition c) of Theorem 9.9 can be written as

$$B(s) + j\alpha\left(A(s) + \gamma B(s)\right) \text{ stable for all } \alpha \in \mathbb{R}$$

which is of course equivalent to

$$B(s) + \frac{j\alpha}{1 + j\alpha\gamma} A(s) \quad \text{is Hurwitz stable for all } \alpha \in \mathbb{R}.$$

The family of polynomials defined by condition 2) is a real interval family so that by using Kharitonov's theorem for real polynomials, we deduce that condition 2) is equivalent to:

2′)

$$\begin{array}{ll} A(s) + \gamma K_B^1(s), & A(s) + \gamma K_B^2(s), \\ A(s) + \gamma K_B^3(s), & A(s) + \gamma K_B^4(s) \end{array}$$

stable.

The polynomials defined in 3) is a *complex interval family* for every α and thus Kharitonov's theorem for complex polynomials applies and 3) is equivalent to:

3′)

$$K_B^1(s) + \frac{j\alpha}{1+j\alpha\gamma}A(s), \qquad K_B^2(s) + \frac{j\alpha}{1+j\alpha\gamma}A(s),$$
$$K_B^3(s) + \frac{j\alpha}{1+j\alpha\gamma}A(s), \qquad K_B^4(s) + \frac{j\alpha}{1+j\alpha\gamma}A(s)$$

stable for all $\alpha \in \mathbb{R}$. Also 1) is equivalent to

1′)

$$\mathrm{Re}\left[\frac{A(0)}{K_B^1(0)}\right] + \gamma > 0 \text{ and } \mathrm{Re}\left[\frac{A(0)}{K_B^3(0)}\right] + \gamma > 0.$$

Thus by using Theorem 9.9 in the other direction, we conclude that when $A(s)$ is fixed,

$$\mathrm{Re}\left[\frac{A(j\omega)}{B(j\omega)}\right] + \gamma > 0, \text{ for all } \omega \in \mathbb{R}, \text{ and for all } B(s) \in \mathbf{B}(s)$$

if and only if

$$\mathrm{Re}\left[\frac{A(j\omega)}{K_B^k(j\omega)}\right] + \gamma > 0, \text{ for all } \omega \in \mathbb{R}, \text{ and for all } k \in \{1,2,3,4\},$$

and this concludes the proof of the lemma. ♣

As a consequence of Lemma 9.7 we have the following result.

Theorem 9.10 *Given a proper stable family $\mathbf{G}(s)$ of interval plants, the minimum of $\mathrm{Re}(G(j\omega))$ over all ω and over all $G(s)$ in $\mathbf{G}(s)$ is achieved at one of the 16 Kharitonov systems in $\mathbf{G}_{\mathrm{K}}(s)$*

Proof. First, since $\mathbf{G}(s)$ is proper it is clear that this overall minimum is finite. Assume for the sake of argument that the minimum of $\mathrm{Re}\,[G(j\omega)]$ over all ω and over the 16 Kharitonov systems is γ_0, but that some plant $G^*(s)$ in $\mathbf{G}(s)$ satisfies

$$\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[G^*(j\omega)] = \gamma_1 < \gamma_0. \tag{9.28}$$

Take any γ satisfying $\gamma_1 < \gamma < \gamma_0$. By assumption we have that

$$\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[G(j\omega)] - \gamma > 0, \tag{9.29}$$

whenever $G(s)$ is one of the 16 Kharitonov systems. By Lemma 9.7 this implies that (9.29) is true for all $G(s)$ in $\mathbf{G}(s)$, and this obviously contradicts (9.28). ♣

We now look more carefully at the situation where one only needs to check that every plant $G(s)$ in $\mathbf{G}(s)$ has the SPR property. In other words, we are interested in the special case in which $\gamma = 0$. A line of reasoning similar to that of Theorem 9.10 would show that here again it is enough to check the 16 Kharitonov systems. However a more careful analysis shows that it is enough to check only 8 systems and we have the following result.

Theorem 9.11 *Every plant $G(s)$ in $\mathbf{G}(s)$ is SPR if and only if it is the case for the 8 following plants:*

$$G_1(s) = \frac{K_A^2(s)}{K_B^1(s)}, \quad G_2(s) = \frac{K_A^3(s)}{K_B^1(s)}, \quad G_3(s) = \frac{K_A^1(s)}{K_B^2(s)}, \quad G_4(s) = \frac{K_A^4(s)}{K_B^2(s)},$$

$$G_5(s) = \frac{K_A^1(s)}{K_B^3(s)}, \quad G_6(s) = \frac{K_A^4(s)}{K_B^3(s)}, \quad G_7(s) = \frac{K_A^2(s)}{K_B^4(s)}, \quad G_8(s) = \frac{K_A^3(s)}{K_B^4(s)}.$$

Proof. Using Definition 9.1 and Theorem 9.9, it is easy to see that every transfer function

$$G(s) = \frac{A(s)}{B(s)}$$

in the family is SPR if and only if the following three conditions are satisfied:

1) $A(0)B(0) > 0$ for all $A(s) \in \mathbf{A}(s)$ and all $B(s) \in \mathbf{B}(s)$,

2) $A(s)$ is Hurwitz stable for all $A(s) \in \mathbf{A}(s)$,

3) $B(s) + j\alpha A(s)$ is stable for all $A(s) \in \mathbf{A}(s)$, all $B(s) \in \mathbf{B}(s)$, and all $\alpha \in \mathbb{R}$.

By Kharitonov's theorem for real polynomials, it is clear that condition 2) is equivalent to:

2′) $K_A^1(s)$, $K_A^2(s)$, $K_A^3(s)$, $K_A^4(s)$ are Hurwitz stable.

Now, the simplification over Theorem 9.10 stems from the fact that here in condition 3), even if $A(s)$ is not a fixed polynomial, we still have to deal with a *complex interval family* since $\gamma = 0$. Hence, using Kharitonov's theorem for complex polynomials (see Chapter 5), we conclude that 3) is satisfied if and only if:

3′)

$$\begin{array}{ll} K_B^1(s) + j\alpha K_A^2(s), & K_B^1(s) + j\alpha K_A^3(s), \\ K_B^2(s) + j\alpha K_A^1(s), & K_B^2(s) + j\alpha K_A^4(s), \\ K_B^3(s) + j\alpha K_A^1(s), & K_B^3(s) + j\alpha K_A^4(s), \\ K_B^4(s) + j\alpha K_A^2(s), & K_B^4(s) + j\alpha K_A^3(s) \end{array}$$

are Hurwitz stable for all α in $\mathbb{R}$.

Note that you only have to check these eight polynomials whether α is positive or negative. As for condition 1) it is clear that it is equivalent to:

1$'$) $K_A^2(0)K_B^1(0) > 0,\ K_A^3(0)K_B^1(0) > 0,\ K_A^1(0)K_B^3(0) > 0,\ K_A^4(0)K_B^3(0) > 0.$

Once again using Theorem 9.9 in the other direction we can see that conditions 1$'$), 2$'$) and 3$'$) are precisely equivalent to the fact that the eight transfer functions specified in Theorem 9.11 satisfy the SPR property. ♣

As a final remark on the SPR property we see that when the entire family is SPR as in Theorem 9.11, there are two cases. On one hand, if the family is strictly proper then the overall minimum is 0. On the other hand, when the family is proper but not strictly proper, then the overall minimum is achieved at one of the 16 Kharitonov systems even though one only has to check eight plants to verify the SPR property for the entire family. In fact, the minimum need not be achieved at one of these eight plants as the following example shows.

Example 9.8. Consider the following stable family $\mathbf{G}(s)$ of interval systems whose generic element is given by

$$G(s) = \frac{1 + \alpha s + \beta s^2 + s^3}{\gamma + \delta s + \epsilon s^2 + s^3}$$

where

$$\alpha \in [1, 2],\ \beta \in [3, 4],\ \gamma \in [1, 2],\ \delta \in [5, 6],\ \epsilon \in [3, 4].$$

$\mathbf{G}_{\mathrm{K}}(s)$ consists of the following 16 rational functions,

$$\begin{aligned}
r_1(s) &= \frac{1+s+3s^2+s^3}{1+5s+4s^2+s^3}, & r_2(s) &= \frac{1+s+3s^2+s^3}{1+6s+4s^2+s^3},\\
r_3(s) &= \frac{1+s+3s^2+s^3}{2+5s+3s^2+s^3}, & r_4(s) &= \frac{1+s+3s^2+s^3}{2+6s+3s^2+s^3},\\
r_5(s) &= \frac{1+s+4s^2+s^3}{1+5s+4s^2+s^3}, & r_6(s) &= \frac{1+s+4s^2+s^3}{1+6s+4s^2+s^3},\\
r_7(s) &= \frac{1+s+4s^2+s^3}{2+5s+3s^2+s^3}, & r_8(s) &= \frac{1+s+4s^2+s^3}{2+6s+3s^2+s^3},\\
r_9(s) &= \frac{1+2s+3s^2+s^3}{1+5s+4s^2+s^3}, & r_{10}(s) &= \frac{1+2s+3s^2+s^3}{1+6s+4s^2+s^3},\\
r_{11}(s) &= \frac{1+2s+3s^2+s^3}{2+5s+3s^2+s^3}, & r_{12}(s) &= \frac{1+2s+3s^2+s^3}{2+6s+3s^2+s^3},\\
r_{13}(s) &= \frac{1+2s+4s^2+s^3}{1+5s+4s^2+s^3}, & r_{14}(s) &= \frac{1+2s+4s^2+s^3}{1+6s+4s^2+s^3},\\
r_{15}(s) &= \frac{1+2s+4s^2+s^3}{2+5s+3s^2+s^3}, & r_{16}(s) &= \frac{1+2s+4s^2+s^3}{2+6s+3s^2+s^3}.
\end{aligned}$$

The corresponding minima of their respective real parts along the imaginary axis are given by,

$$\begin{aligned}
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_1(j\omega)] &= 0.1385416, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_2(j\omega)] &= 0.1134093,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_3(j\omega)] &= 0.0764526, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_4(j\omega)] &= 0.0621581,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_5(j\omega)] &= 0.1540306, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_6(j\omega)] &= 0.1262789,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_7(j\omega)] &= 0.0602399, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_8(j\omega)] &= 0.0563546,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_9(j\omega)] &= 0.3467740, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{10}(j\omega)] &= 0.2862616,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{11}(j\omega)] &= 0.3011472, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{12}(j\omega)] &= 0.2495148,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{13}(j\omega)] &= 0.3655230, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{14}(j\omega)] &= 0.3010231,\\
\inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{15}(j\omega)] &= 0.2706398, & \inf_{\omega\in\mathbf{R}} \mathrm{Re}\,[r_{16}(j\omega)] &= 0.2345989\ .
\end{aligned}$$

Therefore the entire family is SPR and the minimum is achieved at $r_8(s)$. However $r_8(s)$ corresponds to

$$\frac{K_A^1(s)}{K_B^4(s)}$$

which is not among the eight rational functions of Theorem 9.11.

Complex Rational Functions

It is possible to extend the above results to the case of complex rational functions. In the following we give the corresponding results and sketch the small differences in the proofs. The SPR property for a complex rational function is again given by Definition 9.1. Thus a proper complex rational function

$$G(s) = \frac{N(s)}{D(s)}$$

is SPR if

1) $G(s)$ has no poles in the closed right half plane,

2) $\mathrm{Re}\,[G(j\omega)] > 0$ for all $\omega \in \mathbb{R}$, or equivalently,

$$\mathrm{Re}\,[N(j\omega)]\ \ \mathrm{Re}\,[D(j\omega)] + \mathrm{Im}\,[N(j\omega)]\ \ \mathrm{Im}\,[D(j\omega)] > 0, \ \text{for all}\ \omega \in \mathbb{R}$$

As with real rational functions, the characterization given by Theorem 9.9 is true and we state this below.

Theorem 9.12 *The complex rational function $G(s)$ is SPR if and only if the conditions a),b) and c) of Theorem 9.9 hold.*

Proof. The proof is similar to that for Theorem 9.9. However there is a slight difference in proving that the SPR property implies part c) which is,

$$D(s) + j\alpha N(s) \quad \text{is Hurwitz for all} \quad \alpha \in \mathbb{R}. \tag{9.30}$$

To do so in the real case, we consider the family of polynomials:

$$\mathcal{P} := \{P_\alpha(s) = D(s) + j\alpha N(s) \ : \ \alpha \in \mathbb{R}\}$$

and we start by arguing that this family of polynomials has constant degree. This may not be true in the complex case when the rational function is proper but not strictly proper. To prove that (9.30) is nevertheless correct we first observe that the same proof carries over in the strictly proper case. Let us suppose now that $N(s)$ and $D(s)$ have the same degree p and their leading coefficients are

$$n_p = n_p^r + jn_p^i, \qquad d_p = d_p^r + jd_p^i.$$

Then it is easy to see that the family $\mathcal{P}$ does not have constant degree if and only if

$$d_p^r n_p^r + d_p^i n_p^i = 0. \tag{9.31}$$

Thus if $G(s)$ is SPR and (9.31) is not satisfied then again the same proof works and (9.30) is true.

Now, let us assume that $G(s)$ is SPR, proper but not strictly proper, and that

$$G_\gamma(s) = G(s) + \gamma = \frac{N'(s)}{D(s)}, \quad \text{where } N'(s) = N(s) + \gamma D(s).$$

It is clear that $G_\gamma(s)$ is still SPR, and it can be checked that it is always proper and not strictly proper. Moreover, (9.31) cannot hold for $N'(s)$ and $D(s)$ since in that case,

$$d_p^r {n'}_p^r + d_p^i {n'}_p^i = d_p^r(n_p^r + \gamma d_p^r) + d_p^i(n_p^i + \gamma d_p^i) = \gamma((d_p^r)^2 + (d_p^i)^2) > 0.$$

Thus we conclude that for all $\alpha \in \mathbb{R}$,

$$D(s) + j\alpha(N(s) + \gamma D(s)) \quad \text{is Hurwitz stable.} \tag{9.32}$$

Now letting γ go to 0, we see that $D(s) + j\alpha N(s)$ is a limit of Hurwitz polynomials of bounded degree and therefore Rouché's theorem implies that the unstable roots of $D(s) + j\alpha N(s)$, if any, can only be on the $j\omega-$axis. However since $G(s)$ is SPR this cannot happen since,

$$d(j\omega) + j\alpha n(j\omega) = 0 \implies \begin{cases} \mathrm{Re}\,[D(j\omega)] - \alpha \mathrm{Im}\,[N(j\omega)] = 0 \\ \mathrm{Im}\,[D(j\omega)] + \alpha \mathrm{Re}\,[N(j\omega)] = 0, \end{cases}$$

and these two equations in turn imply that

$$\mathrm{Re}\,[N(j\omega)] \ \mathrm{Re}\,[D(j\omega)] + \mathrm{Im}\,[N(j\omega)] \ \mathrm{Im}\,[D(j\omega)] = 0,$$

a contradiction. ♣

Now consider a family $\mathbf{G}(s)$ of proper complex interval rational functions

$$G(s) = \frac{A(s)}{B(s)}$$

where $A(s)$ belongs to a family of complex interval polynomials $\mathbf{A}(s)$, and $B(s)$ belongs to a family of complex interval polynomials $\mathbf{B}(s)$. The Kharitonov polynomials for such a family are 8 extreme polynomials. We refer the reader to Chapter 5 for the definition of these polynomials. The Kharitonov systems associated with $\mathbf{G}(s)$ are the 64 rational functions in the set

$$\mathbf{G}_{\mathrm{K}}(s) = \left\{ \frac{K_A^i(s)}{K_B^j(s)} \;:\; i,j \in \{1,2,3,4,5,6,7,8\} \right\}.$$

Similar to the real case we have the following theorem.

Theorem 9.13 *Given a proper stable family $\mathbf{G}(s)$ of complex interval rational functions, the minimum of $\mathrm{Re}\,[G(j\omega)]$ over all ω and over all $G(s)$ in $\mathbf{G}(s)$ is achieved at one of the 64 Kharitonov systems.*

The proof is identical to that for the real case and is omitted.

One may also consider the problem of only checking that the entire family is SPR, and here again a stronger result holds in that case.

Theorem 9.14 *Every rational function $G(s)$ in $\mathbf{G}(s)$ is SPR if and only if it is the case for the 16 following rational functions:*

$$G_1(s) = \frac{K_A^2(s)}{K_B^1(s)},\quad G_2(s) = \frac{K_A^3(s)}{K_B^1(s)},\quad G_3(s) = \frac{K_A^1(s)}{K_B^2(s)},\quad G_4(s) = \frac{K_A^4(s)}{K_B^2(s)},$$

$$G_5(s) = \frac{K_A^1(s)}{K_B^3(s)},\quad G_6(s) = \frac{K_A^4(s)}{K_B^3(s)},\quad G_7(s) = \frac{K_A^2(s)}{K_B^4(s)},\quad G_8(s) = \frac{K_A^3(s)}{K_B^4(s)}.$$

$$G_9(s) = \frac{K_A^6(s)}{K_B^5(s)},\quad G_{10}(s) = \frac{K_A^7(s)}{K_B^5(s)},\quad G_{11}(s) = \frac{K_A^5(s)}{K_B^6(s)},\quad G_{12}(s) = \frac{K_A^8(s)}{K_B^6(s)},$$

$$G_{13}(s) = \frac{K_A^5(s)}{K_B^7(s)},\quad G_{14}(s) = \frac{K_A^8(s)}{K_B^7(s)},\quad G_{15}(s) = \frac{K_A^6(s)}{K_B^8(s)},\quad G_{16}(s) = \frac{K_A^7(s)}{K_B^8(s)}.$$

The proof is the same as for Theorem 9.11 and is omitted.

9.9 THE ROBUST ABSOLUTE STABILITY PROBLEM

We now extend the classical absolute stability problem by allowing the linear system $G(s)$ to lie in a family of systems $\mathbf{G}(s)$ containing parametric uncertainty. Thus, we are dealing with a robustness problem where parametric uncertainty as well as sector bounded nonlinear feedback gains are simultaneously present. For a given class of nonlinearities lying in a prescribed sector the closed loop system will be said to be *robustly absolutely stable* if it is absolutely stable for every $G(s) \in \mathbf{G}(s)$. In this section we will give a constructive procedure to calculate the size of the stability sector using the Lur'e, Popov or Circle Criterion when $\mathbf{G}(s)$ is an interval system or a linear interval system. In each case we shall see that an appropriate sector can be determined by replacing the family $\mathbf{G}(s)$ by the extremal set $\mathbf{G}_{\mathrm{E}}(s)$. Specifically we deal with the Lur'e problem. However, it will be obvious from the boundary generating properties of the set $\mathbf{G}_{\mathrm{E}}(j\omega)$ that identical results will hold for the Popov sector and the Circle Criterion with time-invariant nonlinearities.

First consider the Robust Lur'e problem where the forward loop element $G(s)$ shown in Figure 9.14 lies in an interval family $\mathbf{G}(s)$ and the feedback loop contains as before a time-varying sector bounded nonlinearity ϕ lying in the sector $[0, k]$. As usual let $\mathbf{G}_{\mathrm{K}}(s)$ denote the transfer functions of the Kharitonov systems associated with the family $\mathbf{G}(s)$.

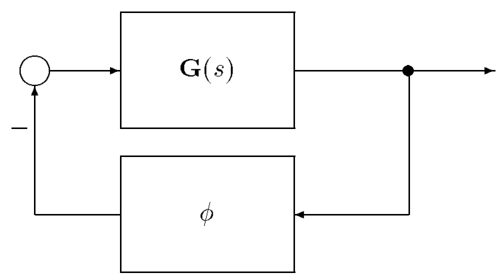

Figure 9.14. Robust absolute stability problem

Theorem 9.15 (Absolute Stability for Interval Systems)
The feedback system in Figure 9.14 is absolutely stable for every $G(s)$ in the the interval family $\mathbf{G}(s)$ of stable proper systems, if the time-varying nonlinearity ϕ belongs to the sector $[0, k]$ where

$$k = \infty, \ \textit{if} \ \inf_{\mathbf{G}_{\mathrm{K}}} \inf_{\omega \in \mathbf{R}} \mathrm{Re}\,[G(j\omega)] \geq 0,$$

otherwise

$$k < -\frac{1}{\inf_{\mathbf{G}_{\mathrm{K}}} \inf_{\omega \in \mathbf{R}} \mathrm{Re}\,[G(j\omega)]},$$

where $\mathbf{G}_{\mathrm{K}}(s)$ is the set of sixteen Kharitonov systems corresponding to $\mathbf{G}(s)$.

Proof. Let $G(s)$ be any member of the interval family. The following inequality holds because of Lemma 9.7.

$$\mathrm{Re}\left[\frac{1}{k} + G(j\omega)\right] \geq \inf_{G \in \mathbf{G}} \inf_{\omega} \mathrm{Re}\left[\frac{1}{k} + G(j\omega)\right]$$

$$= \inf_{G \in \mathbf{G}_{\mathrm{K}}} \inf_{\omega} \mathrm{Re}\left[\frac{1}{k} + G(j\omega)\right] > 0$$

By Theorem 9.6, the *absolute stability* of the closed loop system follows. ♣

We can extend this absolute stability result to feedback systems. Consider a feedback system in which a fixed controller $C(s)$ stabilizes each plant $G(s)$ belonging to a family of linear interval systems $\mathbf{G}(s)$. Let $\mathbf{G}_{\mathrm{E}}(s)$ denote the extremal set for this family $\mathbf{G}(s)$. Now suppose that the closed loop system is subject to nonlinear sector bounded feedback perturbations. Refer to Figure 9.15. Our task is to determine the size of the sector for which absolute stability is preserved for each $G(s) \in \mathbf{G}(s)$. The solution to this problem is given below.

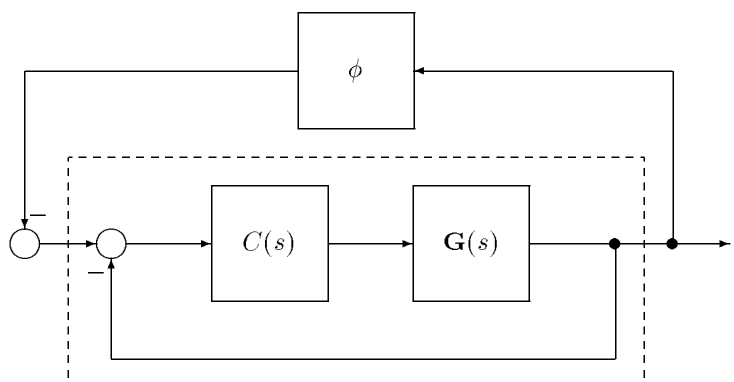

Figure 9.15. Closed loop system with nonlinear feedback perturbations

Theorem 9.16 (Absolute Stability of Interval Control Systems)
Given the feedback system in Figure 9.15 where $G(s)$ belongs to a linear interval family $\mathbf{G}(s)$, the corresponding nonlinear system is absolutely stable if the nonlinearity ϕ belongs to the sector $[0, k]$, where $k > 0$ must satisfy:

$$k = \infty, \ \textit{if} \ \inf_{\mathbf{G}_{\mathrm{E}}} \inf_{\omega \in \mathbf{R}} \mathrm{Re}\left[C(j\omega)G(j\omega)\left(1 + C(j\omega)G(j\omega)\right)^{-1}\right] \geq 0,$$

otherwise,

$$k < -\frac{1}{\inf_{\mathbf{G}_{\mathrm{E}}} \inf_{\omega \in \mathbf{R}} \mathrm{Re}\left[C(j\omega)G(j\omega)\left(1 + C(j\omega)G(j\omega)\right)^{-1}\right]}.$$

Proof. The stability of the system in Figure 9.15 is equivalent, to that of the system in Figure 9.16. The proof now follows from the fact that the boundary of

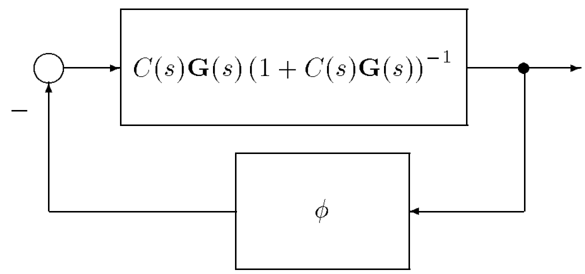

Figure 9.16. Transformed system

the set

$$\left\{C(j\omega)G(j\omega)\left(1+C(j\omega)G(j\omega)\right)^{-1} \ : \ G(s)\in\mathbf{G}(s)\right\}$$

is identical with that of

$$\left\{C(j\omega)G(j\omega)\left(1+C(j\omega)G(j\omega)\right)^{-1} \ : \ G(s)\in\mathbf{G}_{\mathrm{E}}(s)\right\}.$$

This boundary result was proved in Chapter 8. ♣

Remark 9.2. The result given above is stated in terms of an interval plant $\mathbf{G}(s)$. However it is straightforward to show, from the extremal and boundary results developed in Chapter 8, that identical results hold for linear interval systems and for linear fractional transformations of interval systems. This allows us to handle a much more general class of parametrized models.

We illustrate these results with some examples.

Example 9.9. (Robust Lur'e Problem) Consider the feedback configuration in Figure 9.15 and let the plant be given as

$$G(s)=\frac{\alpha_1 s+\alpha_0}{\beta_2 s^2+\beta_1 s+\beta_0}$$

where

$$\begin{array}{lll}\alpha_0\in[0.9,1.1], & \alpha_1\in[0.1,0.2] & \\ \beta_0\in[1.9,2.1], & \beta_1\in[1.8,2], & \beta_2\in[0.9,1].\end{array}$$

A stabilizing controller is

$$C(s)=\frac{s^2+2s+1}{s^4+2s^3+2s^2+s}.$$

We suppose that the closed loop system is perturbed by sector bounded time-varying nonlinear gains. Then the closed loop transfer function family of interest is

$$\left\{ C(s)G(s)\left[1 + C(s)G(s)\right]^{-1} \ : \ G(s) \in \mathbf{G}(s) \right\}.$$

To compute an appropriate robust Lur'e stability sector for this closed loop system, we only need to plot the frequency template

$$\mathbf{T}(j\omega) = \{C(j\omega)G(j\omega)\left[1 + C(j\omega)G(j\omega)\right]^{-1} : \ G \ \in \ \mathbf{G}_{\mathrm{E}}\}$$

for each frequency. Figure 9.17 shows this set of templates.

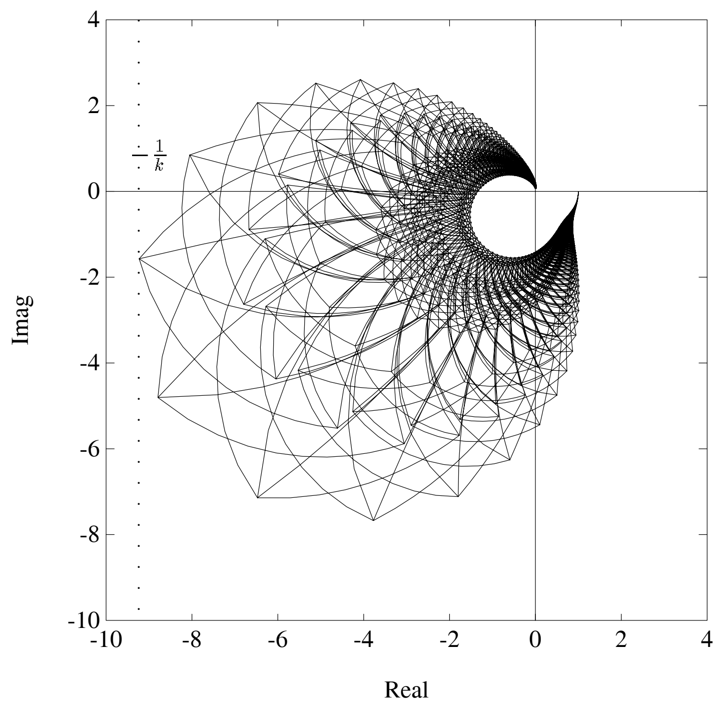

Figure 9.17. Frequency template of closed loop system $\mathbf{T}(j\omega)$ (Example 9.9)

We find the robust Lur'e gain from this as the largest value of k for which $\frac{1}{k} + \mathbf{T}(j\omega)$ becomes SPR. We get

$$\frac{1}{k} = 9.2294$$

and Figure 9.18 shows that adding this gain makes the entire family SPR.

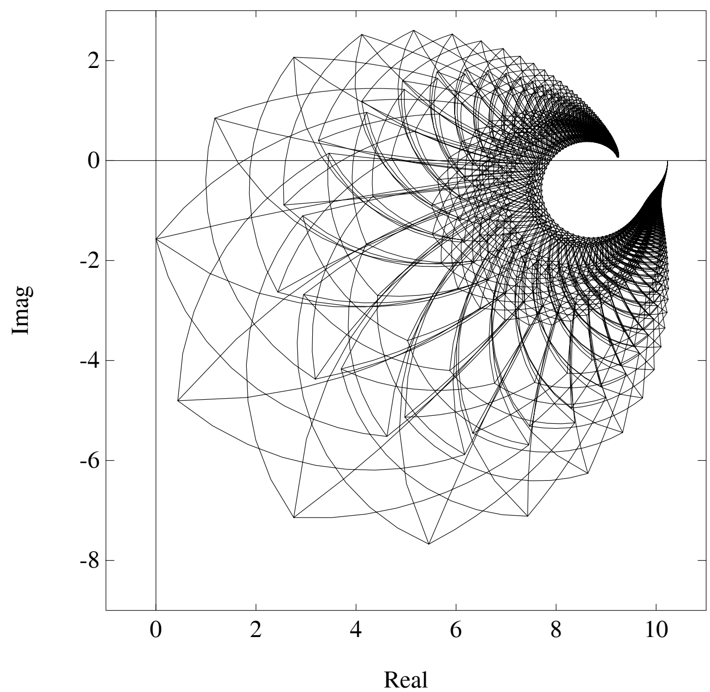

Figure 9.18. Frequency template of SPR system $\frac{1}{k} + \mathbf{T}(j\omega)$(Example 9.9)

Example 9.10. (Robust Popov Criterion) Consider the stable interval plant

$$G(s) = \frac{\alpha_1 s + \alpha_0}{s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}$$

where

$$\alpha_0 \in [3, 3.3], \quad \alpha_1 \in [3, 3.2]$$
$$\beta_0 \in [3, 4], \quad \beta_1 \in [2, 3], \quad \beta_2 \in [24, 25], \quad \beta_3 \in [1, 1.2].$$

To obtain the robust Popov gain we need to plot

$$\mathrm{Re}[G(j\omega)] + j\omega \mathrm{Im}[G(j\omega)] : G(j\omega) \in \mathbf{G}_(j\omega) \qquad \text{for } 0 \le \omega < \infty$$

We can instead generate the boundary of the robust Popov plot by determining:

$$\mathrm{Re}[G(j\omega)] + j\omega \mathrm{Im}[G(j\omega)] : G(j\omega) \in \mathbf{G}_{\mathrm{E}}(j\omega). \qquad \text{for } 0 \le \omega < \infty$$

This plot is shown in Figure 9.19. The Popov line is also shown in this figure from which we find the limiting value of the robust Popov gain to be

$$k_P \approx 0.4.$$

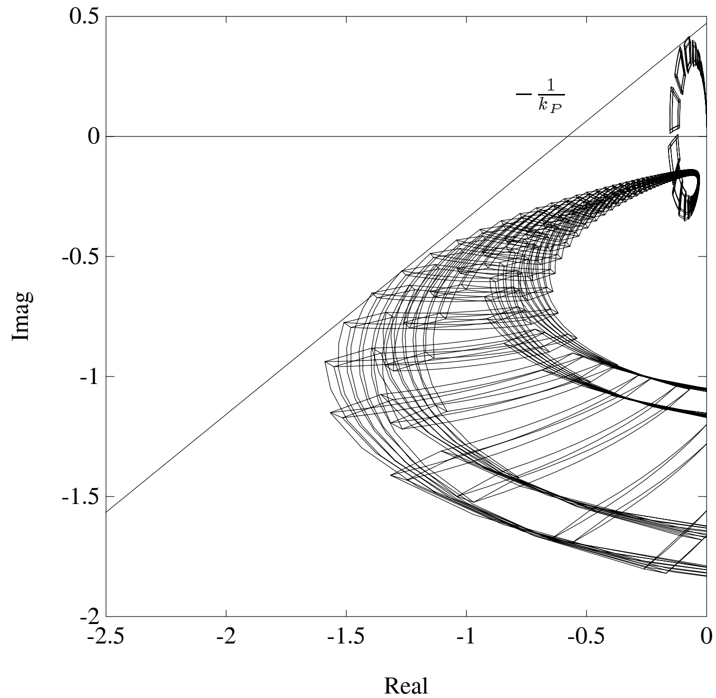

Figure 9.19. Popov criterion (Example 9.10)

Example 9.11. (Robust Circle Criterion) For the system in the previous example we can also use the Circle Criterion to determine a robust absolute stability sector for the family from the Nyquist envelope of $\mathbf{G}_{\mathrm{E}}(j\omega)$ for $0 \le \omega < \infty$. Figure 9.20 shows that the radius of the smallest real axis centered circle (centered at -1) that touches the envelope is found to be $k_C = 0.6138$. From this the values of the robust absolute stability sector $[k_1, k_2]$ can be found as shown in Figure 9.20.

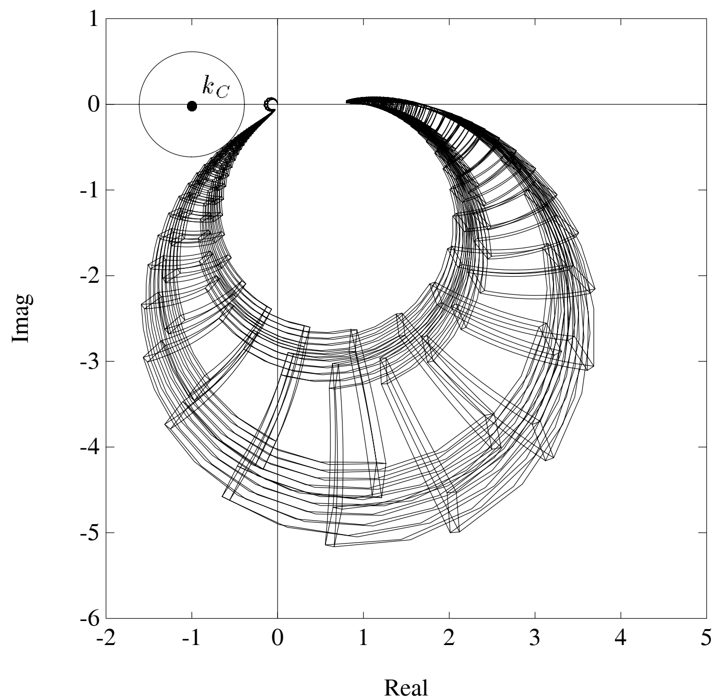

Figure 9.20. Circle criterion (Example 9.11)

9.10 EXERCISES

9.1 Consider the interval system

$$G(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0},$$

where

$$a_0 \in [1,2], \quad a_1 \in [2,3], \quad b_0 \in [1,2], \quad b_1 \in [2,3].$$

Determine the H_∞ stability margin of this system using the Robust Small Gain Theorem.

9.2 For the interval system of Exercise 9.1 determine the size of a Popov and a Lur'e sector for which Robust Absolute Stability can be guaranteed.

9.3 Consider the feedback system consisting of the interval system given in Exercise 9.1 with a controller $C(s) = 5$. Assuming that the plant is perturbed by unstructured additive perturbations, determine the H_∞ stability margin of the closed loop system.

9.4 Repeat Exercise 9.3 for the case of multiplicative uncertainty.

9.5 For the interval plant of Exercise 9.1 and the controller $C(s) = 5$ determine the size of a sector such that the closed loop system is absolutely robustly stable for all feedback gains perturbing the plant and lying in the prescribed sector.

9.6 In Exercises 9.3 - 9.5 let $C(s) = \alpha$ a variable gain lying in the range $\alpha \in [1, 100]$. Determine in each case the optimum value of α to maximize

a) the additive H_∞ stability margin

b) the multiplicative H_∞ margin

c) the size of the feedback sector guaranteeing robust absolute stability.

9.7 Consider the feedback system shown in Figure 9.21.

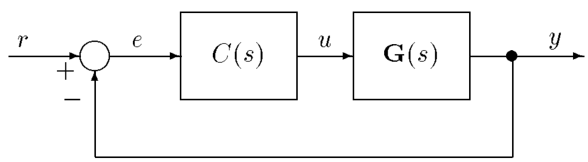

Figure 9.21. Feedback control system

Let

$$C(s) = \frac{2s^2 + 4s + 3}{s^2 + 3s + 4} \quad \text{and} \quad G(s) = \frac{s^2 + a_1 s + a_0}{s(s^2 + b_1 s + b_0)}$$

with the nominal values of the parameters being

$$a_1^0 = -2, \quad a_0^0 = 1, \quad b_0^0 = 2, \quad b_1^0 = 1.$$

Suppose that each parameter perturbs within the range ± 0.0875 Determine the worst case H_∞ stability margin assuming additive uncertainty around the interval plant $\mathbf{G}(s)$.
Answer: 0.3168

9.8 Consider again the interval transfer function $\mathbf{G}(s)$ and the control system in Exercise 9.7. Determine the worst case performance over the parameter set of the system measured in the H_∞ norm of

a) the sensitivity function.

b) the complementary sensitivity function.

9.9 In the block diagram shown in Figure 9.21

$$C(s) = \frac{20 + 30s + 12s^2}{-4 - 6s + s^2} \quad \text{and} \quad G(s) = \frac{a_0 + a_1 s + a2s^2}{s(b_0 + b_1 s + s^2)}$$

and the nominal plant transfer function is

$$G^0(s) = \frac{1 + s + s^2}{s(1 + 2s + s^2)}.$$

Assume that the parameters in $G(s)$ vary as

$$\begin{aligned} a_i &\in [a_i^0 - \epsilon, a_i^0 + \epsilon], \qquad & i = 0, 1, 2 \\ b_j &\in [b_j^0 - \epsilon, b_j^0 + \epsilon], \qquad & j = 0, 1, 2 \end{aligned}$$

where $\epsilon = 0.2275$. Sketch the Bode magnitude and phase envelopes of the closed loop transfer function. Find the worst case H_∞ norm of this transfer function.
Answer: Worst case (maximum) H_∞ norm $= 4.807$.

9.10 Consider the block diagram in Figure 9.21 and let

$$C(s) = \frac{s + 3}{4s + 7} \quad \text{and} \quad G(s) = \frac{s^2 + a_1 s + a_0}{s(s^2 + b_1 s + b_0)}$$

with the nominal values of the parameters being

$$a_1^0 = 5, \quad a_0^0 = 8, \quad b_1^0 = 6, \quad b_0^0 = 15.$$

If the parameters vary ± 2.8737 centered around their respective nominal values,

a) Determine the Bode magnitude and phase envelopes of the transfer function $\frac{y(s)}{r(s)}$. Find the maximum H_∞ norm of the transfer function over the parameters, i.e. the worst case M_p.

Answer: $M_p = 2.1212$

b) Find the worst case H_∞ additive stability margin (i.e. the unstructured block is an additive perturbation around $\mathbf{G}(s)$).

Answer: 2.1330

9.11 Suppose that the closed loop system given in Exercise 9.10 is perturbed by the nonlinear gain ϕ as shown in Figure 9.15.

a) Find the sector $[0, k]$ such that the corresponding nonlinear system is absolutely stable for all $G(s) \in \mathbf{G}(s)$ by plotting the Nyquist plot of an appropriate transfer function family.

b) Verify that the obtained sector guarantees the robust absolute stability of the system.

9.11 NOTES AND REFERENCES

The Small Gain Theorem (Theorem 9.1) is credited to Zames [244] and can be found in the book of Vidyasagar [232]. The Robust Small Gain Theorem for interval systems (Theorem 9.2), the worst case stability margin computations, and the Robust Small Gain Theorem for interval control systems (Theorem 9.3) were developed by Chapellat, Dahleh, and Bhattacharyya [63]. Theorem 9.2 was given by Mori and Barnett [182]. The vertex results given in Lemmas 9.4, 9.5 and 9.6 are due to Hollot and Tempo [116] who also showed by examples that the vertex results given in Lemma 9.6 do not hold if the weights are more complex than 1st order. The vertex result in Theorem 9.5 is due to Dahleh, Vicino and Tesi [72].

The SPR problem for interval transfer function families was first treated in Dasgupta [74], Dasgupta and Bhagwat [76] and Bose and Delansky [48]. The SPR problem for continuous and discrete time polytopic families has been considered by Šiljak [213]. The vertex results for the SPR problem for interval systems were first obtained in Chapellat, Dahleh, and Bhattacharyya [64].

The absolute stability problem was formulated in the 1950's by the Russian school of control theorists and has been studied as a practical and effective approach to the Robust Control problem for systems containing nonlinearities. For an account of this theory, see the books of Lur'e [165] and Aizerman and Gantmacher [9]. The Robust Absolute Stability problem was formulated and solved for interval systems in Chapellat, Dahleh, and Bhattacharyya [64] where the Lur'e problem was treated. These results were extended by Tesi and Vicino [222, 223] to the case where a controller is present. Marquez and Diduch [176] have shown that the SPR property for such systems can be verified by checking fewer segments than that specified by GKT. In Dahleh, Tesi and Vicino [71] the Popov version of the problem was considered. The crucial facts that make each of these results possible are the boundary results on transfer functions based on the GKT and established in Chapter 8. The Robust Lur'e problem with multiple nonlinearities has been studied by Grujič and Petkovski [104]. In [227] Tsypkin and Polyak treated the absolute stability problem in the presence of H_∞ uncertainty.