Chapter 7

THE GENERALIZED KHARITONOV THEOREM

In this chapter we study the Hurwitz stability of a family of polynomials which consists of a linear combination, with fixed polynomial coefficients, of interval polynomials. The Generalized Kharitonov Theorem given here provides a constructive solution to this problem by reducing it to the Hurwitz stability of a prescribed set of extremal line segments. The number of line segments in this test set is independent of the dimension of the parameter space. Under special conditions on the fixed polynomials this test set reduces to a set of vertex polynomials. This test set has many important extremal properties that are useful in control systems. These are developed in the subsequent chapters.

7.1 INTRODUCTION

In attempting to apply Kharitonov's Theorem directly to control systems we encounter a certain degree of conservatism. This is mainly due to the fact that the characteristic polynomial coefficients perturb interdependently, whereas a crucial assumption in Kharitonov's Theorem is that the coefficients of the characteristic polynomial vary *independently*. For example, in a typical situation, the closed loop characteristic polynomial coefficients may vary only through the perturbation of the plant parameters while the controller parameters remain fixed. We show by an example the precise nature of the conservativeness of Kharitonov's Theorem when faced by this problem.

Example 7.1. Consider the plant:

$$G(s) = \frac{n^p(s)}{d^p(s)} = \frac{s}{1 - s + \alpha s^2 + s^3}, \text{ where } \alpha \in [3.4, 5],$$

and has a nominal value

$$\alpha^0 = 4.$$

It is easy to check that the controller $C(s) = \dfrac{3}{s+1}$ stabilizes the nominal plant, yielding the nominal closed-loop characteristic polynomial,

$$\delta_4(s) = 1 + 3s + 3s^2 + 5s^3 + s^4.$$

To determine whether $C(s)$ also stabilizes the family of perturbed plants we observe that the characteristic polynomial of the system is

$$\delta_\alpha(s) = 1 + 3s + (\alpha - 1)s^2 + (\alpha + 1)s^3 + s^4.$$

In the space $(\delta_2,\ \delta_3)$, the coefficients of s^2 and s^3 describe the segment $[R_1, R_2]$ shown in Figure 7.1.

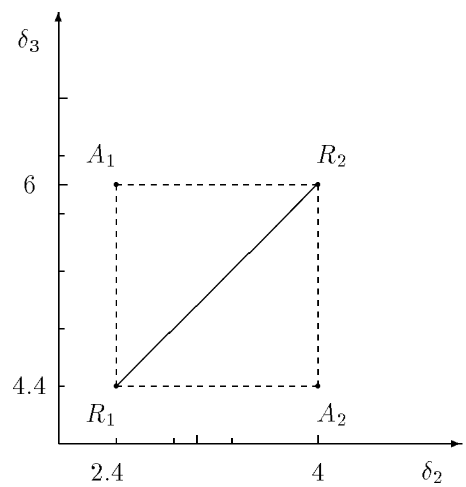

Figure 7.1. A box in parameter space is transformed into a segment in coefficient space (Example 7.1)

The only way to apply Kharitonov's theorem here is to enclose this segment in the box $\mathcal{B}$ defined by the two 'real' points R_1 and R_2 and two 'artificial' points A_1 and A_2 and to check the stability of the Kharitonov polynomials which correspond to the characteristic polynomial evaluated at the four corners of $\mathcal{B}$. But

$$\delta_{A_1}(s) = 1 + 3s + 2.4s^2 + 6s^3 + s^4,$$

is unstable because its third Hurwitz determinant H_3 is

$$H_3 = \begin{vmatrix} 6 & 3 & 0 \\ 1 & 2.4 & 1 \\ 0 & 6 & 3 \end{vmatrix} = -1.8 < 0.$$

Therefore, using Kharitonov's theorem here does not allow us to conclude the stability of the entire family of closed-loop systems. And yet, if one checks the values of the Hurwitz determinants along the segment $[R_1, R_2]$ one finds

$$H = \begin{vmatrix} 1+\alpha & 3 & 0 & 0 \\ 1 & \alpha-1 & 1 & 0 \\ 0 & 1+\alpha & 3 & 0 \\ 0 & 1 & \alpha-1 & 1 \end{vmatrix}$$

and

$$\begin{cases} H_1 &= 1+\alpha \\ H_2 &= \alpha^2-4 \\ H_3 &= 2\alpha^2-2\alpha-13 \\ H_4 &= H_3 \end{cases} \quad \text{all positive for } \alpha \in [3.4, 5].$$

This example demonstrates that Kharitonov's theorem provides only sufficient conditions which may sometimes be too conservative for control problems.

An alternative that we have in this type of situation is to apply the Edge Theorem of Chapter 6, since the parameters of the plant are within a box, which is, of course, a particular case of a polytope. However, we shall see that the solution given by the Edge Theorem, in general, requires us to carry out many redundant checks. Moreover, the Edge Theorem is not a generalization of Kharitonov's Theorem. An appropriate generalization of Kharitonov's Theorem would be expected to produce a test set that would enjoy the economy and optimality of the Kharitonov polynomials, without any unnecessary conservatism.

Motivated by these considerations, we formulate the problem of generalizing Kharitonov's Theorem in the next section. Before proceeding to the main results, we introduce some notation and notational conventions with a view towards streamlining the presentation.

7.2 PROBLEM FORMULATION AND NOTATION

We will be dealing with polynomials of the form

$$\delta(s) = F_1(s)P_1(s) + F_2(s)P_2(s) + \cdots + F_m(s)P_m(s). \tag{7.1}$$

Write

$$\underline{F}(s) := (F_1(s), F_2(s), \cdots, F_m(s)) \tag{7.2}$$
$$\underline{P}(s) := (P_1(s), P_2(s), \cdots, P_m(s)) \tag{7.3}$$

and introduce the notation

$$< \underline{F}(s), \underline{P}(s) > := F_1(s)P_1(s) + F_2(s)P_2(s) + \cdots + F_m(s)P_m(s). \tag{7.4}$$

We will say that $\underline{F}(s)$ *stabilizes* $\underline{P}(s)$ if $\delta(s) =< \underline{F}(s), \underline{P}(s) >$ is *Hurwitz stable*. Note that throughout this chapter, stable will mean Hurwitz stable, unless otherwise stated.

The polynomials $F_i(s)$ are assumed to be fixed real polynomials whereas $P_i(s)$ are real polynomials with coefficients varying independently in prescribed intervals. An extension of the results to the case where $F_i(s)$ are complex polynomials or quasipolynomials will also be given in a later section.

Let $d^o(P_i)$ be the degree of $P_i(s)$

$$P_i(s) := p_{i,0} + p_{i,1}s + \cdots + p_{i,d^o(P_i)}s^{d^o(P_i)}. \tag{7.5}$$

and

$$\mathbf{p}_i := [p_{i,0}, p_{i,1}, \cdots, p_{i,d^o(P_i)}]. \tag{7.6}$$

Let $\underline{n} = [1, 2, \cdots, n]$. Each $P_i(s)$ belongs to an interval family $\mathbf{P}_i(s)$ specified by the intervals

$$p_{i,j} \in [\alpha_{i,j}, \beta_{i,j}] \quad i \in \underline{m} \quad j = 0, \cdots, d^o(P_i). \tag{7.7}$$

The corresponding parameter box is

$$\mathbf{\Pi}_i := \{\mathbf{p}_i : \alpha_{i,j} \leq p_{i,j} \leq \beta_{i,j}, \quad j = 0, 1, \cdots, d^o(P_i)\}. \tag{7.8}$$

Write $\underline{P}(s) := [P_1(s), \cdots P_m(s)]$ and introduce the family of m-tuples

$$\mathbf{P}(s) := \mathbf{P}_1(s) \times \mathbf{P}_2(s) \times \cdots \times \mathbf{P}_m(s). \tag{7.9}$$

Let

$$\mathbf{p} := [\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_m] \tag{7.10}$$

denote the global parameter vector and let

$$\mathbf{\Pi} := \mathbf{\Pi}_1 \times \mathbf{\Pi}_2 \times \cdots \times \mathbf{\Pi}_m \tag{7.11}$$

denote the global parameter uncertainty set. Now let us consider the polynomial (7.1) and rewrite it as $\delta(s, \mathbf{p})$ or $\delta(s, \underline{P}(s))$ to emphasize its dependence on the parameter vector $\mathbf{p}$ or the m-tuple $\underline{P}(s)$. We are interested in determining the Hurwitz stability of the set of polynomials

$$\begin{aligned} \mathbf{\Delta}(s) &:= \{\delta(s, \mathbf{p}) : \mathbf{p} \in \mathbf{\Pi}\} \\ &= \{< \underline{F}(s), \underline{P}(s) >: \underline{P}(s) \in \mathbf{P}(s)\}. \end{aligned} \tag{7.12}$$

We call this a *linear interval polynomial* and adopt the convention

$$\mathbf{\Delta}(s) = F_1(s)\mathbf{P}_1(s) + F_2(s)\mathbf{P}_2(s) + \cdots + F_m(s)\mathbf{P}_m(s). \tag{7.13}$$

We shall make the following standing assumptions about this family.

Assumption 7.1.

a1) Elements of $\mathbf{p}$ perturb independently of each other. Equivalently, $\mathbf{\Pi}$ is an axis parallel rectangular box.

a2) Every polynomial in $\mathbf{\Delta}(s)$ is of the same degree.

The above assumptions will allow us to use the usual results such as the Boundary Crossing Theorem (Chapter 1) and the Edge Theorem (Chapter 6) to develop the solution. It is also justified from a control system viewpoint where loss of the degree of the characteristic polynomial also implies loss of bounded-input bounded-output stability. Henceforth we will say that $\mathbf{\Delta}(s)$ *is stable if every polynomial in* $\mathbf{\Delta}(s)$ *is Hurwitz stable.* An equivalent statement is that $\underline{F}(s)$ *stabilizes every* $\underline{P}(s) \in \mathbf{P}(s)$.

The solution developed below constructs an extremal set of line segments $\mathbf{\Delta}_{\mathrm{E}}(s) \subset \mathbf{\Delta}(s)$ with the property that the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ implies stability of $\mathbf{\Delta}(s)$. This solution is constructive because the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ can be checked, for instance by a set of root locus problems. The solution will be efficient since the number of elements of $\mathbf{\Delta}_{\mathrm{E}}(s)$ will be independent of the dimension of the parameter space $\mathbf{\Pi}$. The extremal subset $\mathbf{\Delta}_{\mathrm{E}}(s)$ will be generated by first constructing an extremal subset $\mathbf{P}_{\mathrm{E}}(s)$ of the m-tuple family $\mathbf{P}(s)$. The extremal subset $\mathbf{P}_{\mathrm{E}}(s)$ is constructed from the Kharitonov polynomials of $\mathbf{P}_i(s)$. We describe the construction of $\mathbf{\Delta}_{\mathrm{E}}(s)$ next.

Construction of the Extremal Subset

The Kharitonov polynomials corresponding to each $\mathbf{P}_i(s)$ are

$$
\begin{aligned}
K_i^1(s) &= \alpha_{i,0} + \alpha_{i,1}s + \beta_{i,2}s^2 + \beta_{i,3}s^3 + \cdots \\
K_i^2(s) &= \alpha_{i,0} + \beta_{i,1}s + \beta_{i,2}s^2 + \alpha_{i,3}s^3 + \cdots \\
K_i^3(s) &= \beta_{i,0} + \alpha_{i,1}s + \alpha_{i,2}s^2 + \beta_{i,3}s^3 + \cdots \\
K_i^4(s) &= \beta_{i,0} + \beta_{i,1}s + \alpha_{i,2}s^2 + \alpha_{i,3}s^3 + \cdots
\end{aligned}
$$

and we denote them as:

$$
\mathcal{K}_i(s) := \{K_i^1(s), K_i^2(s), K_i^3(s), K_i^4(s)\}. \tag{7.14}
$$

For each $\mathbf{P}_i(s)$ we introduce 4 line segments joining pairs of Kharitonov polynomials as defined below:

$$
\mathcal{S}_i(s) := \left\{[K_i^1(s), K_i^2(s)], [K_i^1(s), K_i^3(s)], [K_i^2(s), K_i^4(s)], [K_i^3(s), K_i^4(s)]\right\}. \tag{7.15}
$$

These four segments are called *Kharitonov segments.* They are illustrated in Figure 7.2 for the case of a polynomial of degree 2.

For each $l \in \{1, \cdots, m\}$ let us define

$$
\mathbf{P}_{\mathrm{E}}^l(s) := \mathcal{K}_1(s) \times \cdots \times \mathcal{K}_{l-1}(s) \times \mathcal{S}_l(s) \times \mathcal{K}_{l+1}(s) \times \cdots \times \mathcal{K}_m(s). \tag{7.16}
$$

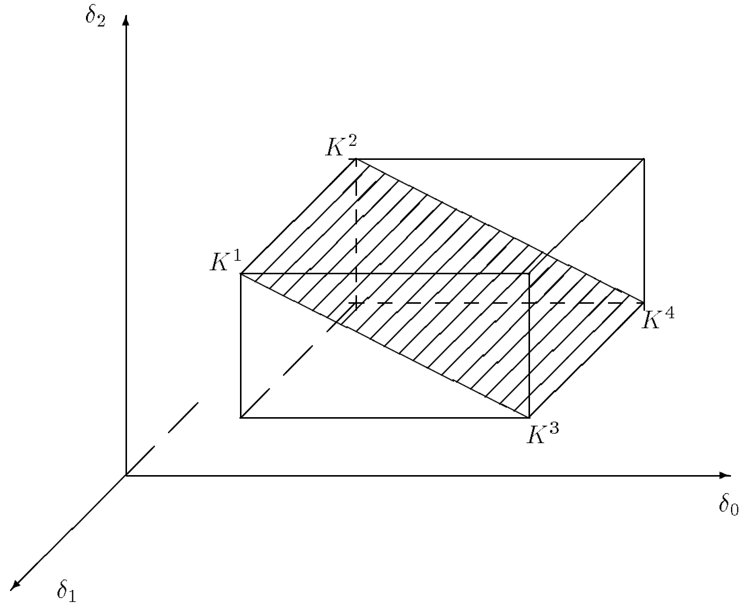

Figure 7.2. The four Kharitonov segments

A typical element of $\mathbf{P}^l_{\mathrm{E}}(s)$ is

$$\left(K_1^{j_1}(s), K_2^{j_2}(s), \cdots, K_{l-1}^{j_{l-1}}(s), (1-\lambda)K_l^1(s) + \lambda K_l^2(s), K_{l+1}^{j_{l+1}}(s), \cdots, K_m^{j_m}(s)\right) \tag{7.17}$$

with $\lambda \in [0,1]$. This can be rewritten as

$$\begin{aligned}(1-\lambda)&\left(K_1^{j_1}(s), K_2^{j_2}(s), \cdots, K_{l-1}^{j_{l-1}}(s), K_l^1(s), K_{l+1}^{j_{l+1}}(s), \cdots, K_m^{j_m}(s)\right)\\ +\lambda&\left(K_1^{j_1}(s), K_2^{j_2}(s), \cdots, K_{l-1}^{j_{l-1}}(s), K_l^2(s), K_{l+1}^{j_{l+1}}(s), \cdots, K_m^{j_m}(s)\right).\end{aligned} \tag{7.18}$$

Corresponding to the m-tuple $\mathbf{P}^l_{\mathrm{E}}(s)$, introduce the polynomial family

$$\mathbf{\Delta}^l_{\mathrm{E}}(s) := \{< \underline{F}(s), \underline{P}(s) >: \underline{P}(s) \in \mathbf{P}^l_{\mathrm{E}}(s)\}. \tag{7.19}$$

The set $\mathbf{\Delta}^l_{\mathrm{E}}(s)$ is also described as

$$\begin{aligned}\mathbf{\Delta}^l_{\mathrm{E}}(s) = {}& F_1(s)\mathcal{K}_1(s) + \cdots + F_{l-1}(s)\mathcal{K}_{l-1}(s) + F_l(s)\mathcal{S}_l(s) + F_{l+1}(s)\mathcal{K}_{l+1}(s) + \\ & \cdots + F_m(s)\mathcal{K}_m(s).\end{aligned} \tag{7.20}$$

A typical element of $\mathbf{\Delta}^l_{\mathrm{E}}(s)$ is the line segment of polynomials

$$F_1(s)K_1^{j_1}(s)+F_2(s)K_2^{j_2}(s)+\cdots+F_{l-1}(s)K_{l-1}^{j_{l-1}}(s)+F_l(s)\left[(1-\lambda)K_l^1(s)+\lambda K_l^2(s)\right]$$
$$+F_{l+1}(s)K_{l+1}^{j_{l+1}}(s)+\cdots+F_m(s)K_m^{j_m}(s) \tag{7.21}$$

with $\lambda \in [0,1]$.

The *extremal subset* of $\mathbf{P}(s)$ is defined by

$$\mathbf{P}_{\mathrm{E}}(s) := \cup_{l=1}^{m} \mathbf{P}_{\mathrm{E}}^{l}(s). \tag{7.22}$$

The corresponding *generalized Kharitonov segment* polynomials are

$$\begin{aligned}\mathbf{\Delta}_{\mathrm{E}}(s) &:= \cup_{l=1}^{m} \mathbf{\Delta}_{\mathrm{E}}^{l}(s)\\ &= \{< \underline{F}(s), \underline{P}(s) > : \ \underline{P}(s) \in \mathbf{P}_{\mathrm{E}}(s)\}.\end{aligned} \tag{7.23}$$

The set of m-tuples of Kharitonov polynomials are denoted $\mathbf{P}_{\mathrm{K}}(s)$ and referred to as the *Kharitonov vertices* of $\mathbf{P}(s)$:

$$\mathbf{P}_{\mathrm{K}}(s) := \mathcal{K}_1(s)\mathrm{x}\mathcal{K}_2(s)\mathrm{x}\cdots\mathrm{x}\mathcal{K}_m(s) \subset \mathbf{P}_{\mathrm{E}}(s). \tag{7.24}$$

The corresponding set of *Kharitonov vertex* polynomials is

$$\mathbf{\Delta}_{\mathrm{K}}(s) := \{< \underline{F}(s), \underline{P}(s) > : \ \underline{P}(s) \in \mathbf{P}_{\mathrm{K}}(s)\}. \tag{7.25}$$

A typical element of $\mathbf{\Delta}_{\mathrm{K}}(s)$ is

$$F_1(s)K_1^{j_1}(s) + F_2(s)K_2^{j_2}(s) + \cdots + F_m(s)K_m^{j_m}(s). \tag{7.26}$$

The set $\mathbf{P}_{\mathrm{E}}(s)$ is made up of one parameter families of polynomial vectors. It is easy to see that there are $m4^m$ such segments in the most general case where there are four distinct Kharitonov polynomials for each $\mathbf{P}_i(s)$. The parameter space subsets corresponding to $\mathbf{P}_{\mathrm{E}}^{l}(s)$ and $\mathbf{P}_{\mathrm{E}}(s)$ are denoted by $\mathbf{\Pi}_l$ and

$$\mathbf{\Pi}_{\mathrm{E}} := \cup_{l=1}^{m} \mathbf{\Pi}_l, \tag{7.27}$$

respectively. Similarly, let $\mathbf{\Pi}_{\mathrm{K}}$ denote the vertices of $\mathbf{\Pi}$ corresponding to the Kharitonov polynomials. Then, we also have

$$\mathbf{\Delta}_{\mathrm{E}}(s) := \{\delta(s,\mathbf{p}) : \mathbf{p} \in \mathbf{\Pi}_{\mathrm{E}}\} \tag{7.28}$$
$$\mathbf{\Delta}_{\mathrm{K}}(s) := \{\delta(s,\mathbf{p}) : \mathbf{p} \in \mathbf{\Pi}_{\mathrm{K}}\} \tag{7.29}$$

The set $\mathbf{P}_{\mathrm{K}}(s)$ in general has 4^m distinct elements when each $\mathbf{P}_i(s)$ has four distinct Kharitonov polynomials. Thus $\mathbf{\Delta}_{\mathrm{K}}(s)$ is a discrete set of polynomials, $\mathbf{\Delta}_{\mathrm{E}}(s)$ is a set of line segments of polynomials, $\mathbf{\Delta}(s)$ is a polytope of polynomials, and

$$\mathbf{\Delta}_{\mathrm{K}}(s) \subset \mathbf{\Delta}_{\mathrm{E}}(s) \subset \mathbf{\Delta}(s). \tag{7.30}$$

With these preliminaries, we are ready to state the Generalized Kharitonov Theorem (GKT).

7.3 THE GENERALIZED KHARITONOV THEOREM

Let us say that $\underline{F}(s)$ stabilizes a set of m-tuples if it stabilizes every element in the set. We can now enunciate the Generalized Kharitonov Theorem.

Theorem 7.1 (Generalized Kharitonov Theorem (GKT))
For a given m-tuple $\underline{F}(s) = (F_1(s), \cdots, F_m(s))$ of real polynomials:

I) $\underline{F}(s)$ stabilizes the entire family $\mathbf{P}(s)$ of m-tuples if and only if $\underline{F}$ stabilizes every m-tuple segment in $\mathbf{P}_{\mathrm{E}}(s)$. Equivalently, $\mathbf{\Delta}(s)$ is stable if and only if $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable.

II) Moreover, if the polynomials $F_i(s)$ are of the form

$$F_i(s) = s^{t_i}(a_i s + b_i)U_i(s)Q_i(s)$$

where $t_i \geq 0$ is an arbitrary integer, a_i and b_i are arbitrary real numbers, $U_i(s)$ is an anti-Hurwitz polynomial, and $Q_i(s)$ is an even or odd polynomial, then it is enough that $\underline{F}(s)$ stabilizes the finite set of m-tuples $\mathbf{P}_{\mathrm{K}}(s)$, or equivalently, that the set of Kharitonov vertex polynomials $\mathbf{\Delta}_{\mathrm{K}}(s)$ are stable.

III) Finally, stabilizing the finite set $\mathbf{P}_{\mathrm{K}}(s)$ is not sufficient to stabilize $\mathbf{P}(s)$ when the polynomials $F_i(s)$ do not satisfy the conditions in II). Equivalently, stability of $\mathbf{\Delta}_{\mathrm{K}}(s)$ does not imply stability of $\mathbf{\Delta}(s)$ when $F_i(s)$ do not satisfy the conditions in II).

The strategy of the proof is to construct an intermediate polytope $\mathbf{\Delta}_{\mathrm{I}}(s)$ of dimension $2m$ such that

$$\mathbf{\Delta}_{\mathrm{E}}(s) \subset \mathbf{\Delta}_{\mathrm{I}}(s) \subset \mathbf{\Delta}(s). \tag{7.31}$$

In the first lemma we shall show that the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ implies the stability of $\mathbf{\Delta}_{\mathrm{I}}(s)$. The next two lemmas will be used recursively to show further that the stability of $\mathbf{\Delta}_{\mathrm{I}}(s)$ implies the stability of $\mathbf{\Delta}(s)$.

Recall that Kharitonov polynomials are built from even and odd parts as follows:

$$\begin{aligned} K_i^1(s) &= K_i^{\mathrm{even,min}}(s) + K_i^{\mathrm{odd,min}}(s) \\ K_i^2(s) &= K_i^{\mathrm{even,min}}(s) + K_i^{\mathrm{odd,max}}(s) \\ K_i^3(s) &= K_i^{\mathrm{even,max}}(s) + K_i^{\mathrm{odd,min}}(s) \\ K_i^4(s) &= K_i^{\mathrm{even,max}}(s) + K_i^{\mathrm{odd,max}}(s), \end{aligned} \tag{7.32}$$

where

$$\begin{aligned} K_i^{\mathrm{even,min}}(s) &= \alpha_{i,0} + \beta_{i,2}s^2 + \alpha_{i,4}s^4 + \cdots \\ K_i^{\mathrm{even,max}}(s) &= \beta_{i,0} + \alpha_{i,2}s^2 + \beta_{i,4}s^4 + \cdots \\ K_i^{\mathrm{odd,min}}(s) &= \alpha_{i,1}s + \beta_{i,3}s^3 + \alpha_{i,5}s^5 + \cdots \\ K_i^{\mathrm{odd,max}}(s) &= \beta_{i,1}s + \alpha_{i,3}s^3 + \beta_{i,5}s^5 + \cdots. \end{aligned} \tag{7.33}$$

Now introduce the polytope $\mathbf{\Delta}_{\mathrm{I}}(s)$:

$$\mathbf{\Delta}_{\mathrm{I}}(s) := \left\{ \sum_{i=1}^{m} F_i(s) \Big((1-\lambda_i) K_i^{\text{even,min}}(s) + \lambda_i K_i^{\text{even,max}}(s) \right.$$
$$+ (1-\mu_i) K_i^{\text{odd,min}}(s) + \mu_i K_i^{\text{odd,max}}(s) \Big) :$$
$$\left. (\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots, \lambda_m, \mu_m), \quad \lambda_i \in [0,1], \mu_i \in [0,1] \right\}. \tag{7.34}$$

Lemma 7.1 *$\mathbf{\Delta}_{\mathrm{I}}(s)$ is stable if and only if $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable.*

Proof. It is clear that the stability of $\mathbf{\Delta}_{\mathrm{I}}(s)$ implies the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ since $\mathbf{\Delta}_{\mathrm{E}}(s) \subset \mathbf{\Delta}_{\mathrm{I}}(s)$. To prove the converse, we note that the degree of all polynomials in $\mathbf{\Delta}_{\mathrm{I}}(s)$ is the same (see Assumption a2). Moreover, the exposed edges of $\mathbf{\Delta}_{\mathrm{I}}(s)$ are obtained by setting $2m-1$ coordinates of the set

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots, \lambda_m, \mu_m)$$

to 0 or 1 and letting the remaining one range over $[0,1]$. It is easy to see that this family of line segments is nothing but $\mathbf{\Delta}_{\mathrm{E}}(s)$. Therefore, by the Edge Theorem it follows that the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ implies the stability of $\mathbf{\Delta}_{\mathrm{I}}(s)$. ♣

We shall also need the following two symmetric lemmas.

Lemma 7.2 *Let $\mathcal{B}^e(s)$ be the family of real even polynomials*

$$B(s) = b_0 + b_2 s^2 + b_4 s^4 + \cdots + b_{2p} s^{2p},$$

$$\textit{where:} \qquad b_0 \in [x_0, y_0], b_2 \in [x_2, y_2], \cdots, b_{2p} \in [x_{2p}, y_{2p}],$$

and define,

$$K_1(s) = x_0 + y_2 s^2 + x_4 s^4 + \cdots$$
$$K_2(s) = y_0 + x_2 s^2 + y_4 s^4 + \cdots.$$

Let also $A(s)$ and $C(s)$ be two arbitrary but fixed real polynomials. Then,

A) $A(s) + C(s)B(s)$ is stable for every polynomial $B(s)$ in $\mathcal{B}^e(s)$ if and only if the segment $[A(s) + C(s)K_1(s), A(s) + C(s)K_2(s)]$ is stable.

B) Moreover if $C(s) = s^t(as+b)U(s)R(s)$ where $t \geq 0$, a and b are arbitrary real numbers, $U(s)$ is an anti-Hurwitz polynomial, and $R(s)$ is an even or odd polynomial, then $A(s) + C(s)B(s)$ is Hurwitz stable for every polynomial $B(s)$ in $\mathcal{B}^e(s)$ if and only if

$$A(s) + C(s)K_1(s), \quad \textit{and} \quad A(s) + C(s)K_2(s) \quad \textit{are Hurwitz stable.}$$

Proof. Let us assume that $A(s)+C(s)B(s)$ is stable for every polynomial $B(s)$ in $[K_1(s),\, K_2(s)]$, that is for every polynomial of the form,

$$B(s)=(1-\lambda)K_1(s)+\lambda K_2(s), \quad \lambda\in[0,1].$$

Let us now assume by way of contradiction, that $A(s)+C(s)P(s)$ was unstable for some polynomial $P(s)$ in $\mathcal{B}^e(s)$. Then we know that there must also exist a polynomial $Q(s)$ in $\mathcal{B}^e(s)$ such that

$$A(s)+C(s)Q(s)$$

has a root at the origin or a pure imaginary root. Let us at once discard the case of a polynomial $Q(s)$ in the box $\mathcal{B}^e(s)$, being such that

$$A(0)+C(0)Q(0)=0. \tag{7.35}$$

Indeed, since $Q(0)=q_0$ belongs to $[x_0,y_0]$, it can be written

$$q_0=(1-\lambda)x_0+\lambda y_0, \quad \text{for some } \lambda \text{ in } [0,1].$$

Then (7.35) would imply

$$\begin{aligned} A(0)+C(0)\left((1-\lambda)x_0+\lambda y_0\right) &= A(0)+C(0)\left((1-\lambda)K_1(0)+\lambda K_2(0)\right)\\ &=0, \end{aligned}$$

which would contradict our assumption that $A(s)+C(s)B(s)$ is stable. Suppose now that $A(s)+C(s)Q(s)$ has a pure imaginary root $j\omega$, for some $\omega>0$. If this is true then we have

$$\left\{\begin{array}{l} A^e(\omega)+C^e(\omega)Q(\omega)=0\\ A^o(\omega)+C^o(\omega)Q(\omega)=0. \end{array}\right. \tag{7.36}$$

Notice here that since $Q(s)$ is an even polynomial, we simply have

$$Q(j\omega)=Q^e(\omega)=Q_{even}(j\omega):=Q(\omega).$$

Now, (7.36) implies that for this particular value of ω we have

$$A^e(\omega)C^o(\omega)-A^o(\omega)C^e(\omega)=0. \tag{7.37}$$

On the other hand, consider the two polynomials

$$B_1(s)=A(s)+C(s)K_1(s), \text{ and } B_2(s)=A(s)+C(s)K_2(s).$$

We can write for $i=1,2$

$$B_i^e(\omega)=(A+CK_i)^e(\omega)=A^e(\omega)+C^e(\omega)K_i(\omega)$$

and

$$B_i^o(\omega)=(A+CK_i)^o(\omega)=A^o(\omega)+C^o(\omega)K_i(\omega).$$

Thinking then of using the Segment Lemma (Chapter 2) we compute

$$B_1^e(\omega)B_2^o(\omega) - B_2^e(\omega)B_1^o(\omega) = \Big(A^e(\omega) + C^e(\omega)K_1(\omega)\Big)\Big(A^o(\omega) + C^o(\omega)K_2(\omega)\Big) - \Big(A^e(\omega) + C^e(\omega)K_2(\omega)\Big)\Big(A^o(\omega) + C^o(\omega)K_1(\omega)\Big),$$

which can be written as

$$B_1^e(\omega)B_2^o(\omega) - B_2^e(\omega)B_1^o(\omega) = (K_2(\omega) - K_1(\omega))(A^e(\omega)C^o(\omega) - A^o(\omega)C^e(\omega)),$$

and therefore because of (7.37),

$$B_1^e(\omega)B_2^o(\omega) - B_2^e(\omega)B_1^o(\omega) = 0. \tag{7.38}$$

Moreover, assume without loss of generality that

$$C^e(\omega) \geq 0, \text{ and } C^o(\omega) \leq 0, \tag{7.39}$$

and remember that due to the special form of $K_1(s)$ and $K_2(s)$ we have

$$K_1(\omega) \leq Q(\omega) \leq K_2(\omega), \qquad \text{for all } \omega \in [0, +\infty).$$

Then we conclude from (7.36) and (7.39) that

$$\begin{aligned} B_1^e(\omega) &= A^e(\omega) + C^e(\omega)K_1(\omega) \leq 0 \leq B_2^e(\omega) = A^e(\omega) + C^e(\omega)K_2(\omega), \\ B_2^o(\omega) &= A^o(\omega) + C^o(\omega)K_2(\omega) \leq 0 \leq B_1^o(\omega) = A^o(\omega) + C^o(\omega)K_1(\omega). \end{aligned} \tag{7.40}$$

But if we put together equations (7.38) and (7.40) we see that

$$\begin{cases} B_1^e(\omega)B_2^o(\omega) - B_2^e(\omega)B_1^o(\omega) = 0 \\ B_1^e(\omega)B_2^e(\omega) \leq 0 \\ B_1^o(\omega)B_2^o(\omega) \leq 0. \end{cases}$$

We see therefore from the Segment Lemma (Chapter 2) that some polynomial on the segment $[B_1(s), B_2(s)]$ has the same $j\omega$ root, which here again is a contradiction of our original assumption that $A(s)+C(s)B(s)$ is stable. This concludes the proof of part A.

To prove part B, let us assume that $C(s)$ is of the form specified and that

$$B_1(s) = A(s) + C(s)K_1(s), \text{ and } B_2(s) = A(s) + C(s)K_2(s),$$

are both Hurwitz stable. Then

$$B_1(s) - B_2(s) = s^t(as+b)U(s)R(s)\left(K_1(s) - K_2(s)\right). \tag{7.41}$$

Since $K_1(s) - K_2(s)$ is even we conclude from the Vertex Lemma (Chapter 2) that the segment $[B_1(s), B_2(s)]$ is Hurwitz stable. This proves part B. ♣

The dual lemma is stated without proof.

Lemma 7.3 *Let $\mathcal{B}^o(s)$ be the family of real odd polynomials*

$$B(s) = b_1 s + b_3 s^3 + b_5 s^5 + \cdots + b_{2p+1} s^{2p+1},$$

$$\textit{where:} \qquad b_1 \in [x_1, y_1], \quad b_3 \in [x_3, y_3], \quad \cdots, \quad b_{2p+1} \in [x_{2p+1}, y_{2p+1}],$$

and define,

$$K_1(s) = x_1 s + y_3 s^3 + x_5 s^5 + \cdots$$
$$K_2(s) = y_1 s + x_3 s^3 + y_5 s^5 + \cdots.$$

Let also $D(s)$ and $E(s)$ be two arbitrary but fixed real polynomials. Then

a) $D(s) + E(s)B(s)$ is stable for every polynomial $B(s)$ in $\mathcal{B}^o(s)$ if and only if the segment $[D(s) + E(s)K_1(s), D(s) + E(s)K_2(s)]$ is Hurwitz stable.

b) Moreover if $E(s) = s^t(as + b)U(s)R(s)$ where $t \geq 0$, a and b are arbitrary real numbers, $U(s)$ is an anti-Hurwitz polynomial, and $R(s)$ is an even or odd polynomial, then $D(s) + E(s)B(s)$ is stable for every polynomial $B(s)$ in $\mathcal{B}^o(s)$ if and only if

$$D(s) + E(s)K_1(s), \ \textit{and} \ D(s) + E(s)K_2(s) \ \textit{are Hurwitz stable.}$$

Proof of GKT (Theorem 7.1) Since $\mathbf{\Delta}_{\mathrm{E}}(s) \subset \mathbf{\Delta}(s)$, it is only necessary to prove that the stability of $\mathbf{\Delta}_{\mathrm{E}}(s)$ implies that of $\mathbf{\Delta}(s)$. Therefore, let us assume that $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable, or equivalently that $\underline{F}(s)$ stabilizes $\mathbf{P}_{\mathrm{E}}(s)$. Now consider an arbitrary m-tuple of polynomials in $\mathbf{P}(s)$

$$\underline{P}(s) = (P_1(s), \cdots, P_m(s)).$$

Our task is to prove that $\underline{F}(s)$ stabilizes this $\underline{P}(s)$. For the sake of convenience we divide the proof into four steps.

Step 1 Write as usual

$$P_i(s) = P_{i,\mathrm{even}}(s) + P_{i,\mathrm{odd}}(s), \quad i = 1, \cdots, m.$$

Since $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable, it follows from Lemma 7.1 that $\mathbf{\Delta}_{\mathrm{I}}(s)$ is stable. In other words,

$$\sum_{i=1}^{m} F_i(s) \Big((1-\lambda_i) K_i^{\mathrm{even,min}}(s) + \lambda_i K_i^{\mathrm{even,max}}(s) + (1-\mu_i) K_i^{\mathrm{odd,min}}(s) + \mu_i K_i^{\mathrm{odd,max}}(s) \Big), \tag{7.42}$$

is Hurwitz stable for all possible

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots, \lambda_m, \mu_m), \qquad \text{all in } [0,1].$$

Step 2 In this step we show that stability of $\mathbf{\Delta}_{\rm I}(s)$ implies the stability of $\mathbf{\Delta}(s)$. In equation (7.42) set

$$D(s) = \sum_{i=1}^{m-1} F_i(s)\left((1-\lambda_i)K_i^{\rm even,min}(s) + \lambda_i K_i^{\rm even,max}(s) + (1-\mu_i)K_i^{\rm odd,min}(s) + \mu_i K_i^{\rm odd,max}(s)\right) + F_m(s)\left((1-\lambda_m)K_m^{\rm even,min}(s) + \lambda_m K_m^{\rm even,max}(s)\right),$$

and

$$E(s) = F_m(s).$$

We know from (7.42) that

$$D(s) + E(s)\left((1-\mu_m)K_m^{\rm odd,min}(s) + \mu_m K_m^{\rm odd,max}(s)\right)$$

is stable for all μ_m in $[0,1]$. But $K_m^{\rm odd,min}(s)$ and $K_m^{\rm odd,max}(s)$ play exactly the role of $K_1(s)$ and $K_2(s)$ in Lemma 7.3, and therefore we conclude that $D(s)+E(s)P_{m,\rm odd}(s)$ is stable. In other words

$$\sum_{i=1}^{m-1} F_i(s)\left((1-\lambda_i)K_i^{\rm even,min}(s) + \lambda_i K_i^{\rm even,max}(s) + (1-\mu_i)K_i^{\rm odd,min}(s) + \mu_i K_i^{\rm odd,max}(s)\right) + F_m(s)\left((1-\lambda_m)K_m^{\rm even,min}(s) + \lambda_m K_m^{\rm even,max}(s)\right) + F_m(s)P_{m,\rm odd}(s), \tag{7.43}$$

is stable, and remains stable for all possible values

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots, \lambda_m), \quad \text{all in } [0,1],$$

since we fixed them arbitrarily. Proceeding, we can now set

$$A(s) = \sum_{i=1}^{m-1} F_i(s)\left((1-\lambda_i)K_i^{\rm even,min}(s) + \lambda_i K_i^{\rm even,max}(s) + (1-\mu_i)K_i^{\rm odd,min}(s) + \mu_i K_i^{\rm odd,max}(s)\right) + F_m(s)P_{m,\rm odd}(s)$$

and

$$C(s) = F_m(s).$$

Then we know by (7.43) that

$$A(s) + C(s)\left((1-\lambda_m)K_m^{\rm even,min}(s) + \lambda_m K_m^{\rm even,max}(s)\right)$$

is stable for all λ_m in $[0,1]$. But, here again, $K_m^{\rm even,min}(s)$ and $K_m^{\rm even,max}(s)$ play exactly the role of $K_1(s)$ and $K_2(s)$ in Lemma 7.2, and hence we conclude that $A(s) + C(s)P_{m,\rm even}(s)$ is stable. That is

$$\sum_{i=1}^{m-1} F_i(s)\left((1-\lambda_i)K_i^{\rm even,min}(s) + \lambda_i K_i^{\rm even,max}(s) + (1-\mu_i)K_i^{\rm odd,min}(s) + \mu_i K_i^{\rm odd,max}(s)\right) + F_m(s)P_{m,\rm odd}(s) + F_m(s)P_{m,\rm even}(s)$$

or finally

$$\sum_{i=1}^{m-1} F_i(s)\Big((1-\lambda_i)K_i^{\text{even,min}}(s)+\lambda_i K_i^{\text{even,max}}(s)+(1-\mu_i)K_i^{\text{odd,min}}(s) + \mu_i K_i^{\text{odd,max}}(s)\Big)+F_m(s)P_m(s)$$

is stable, and this is true for all possible values

$$(\lambda_1,\mu_1,\lambda_2,\mu_2,\cdots,\lambda_{m-1},\mu_{m-1}), \quad \text{in } [0,1].$$

The same reasoning can be carried out by induction until one reaches the point where

$$F_1(s)P_1(s)+F_2(s)P_2(s)+\cdots+F_m(s)P_m(s),$$

is found to be stable. Since

$$\underline{P}(s)=(P_1(s),\cdots,P_m(s)),$$

was an arbitrary element of $\mathbf{P}(s)$, this proves that $\underline{F}(s)$ stabilizes $\mathbf{P}(s)$. Equivalently, $\mathbf{\Delta}(s)$ is stable. This concludes the proof of part I of the Theorem.

Step 3 To prove part II observe that a typical segment of $\mathbf{\Delta}_{\mathrm{E}}(s)$ is

$$\delta_\lambda(s):=F_1(s)K_1^{j_1}(s)+\cdots+F_l(s)[\lambda K_l^{j_l}(s)+(1-\lambda)K_l^{i_l}(s)]+\cdots+F_m(s)K_m^{j_m}(s).$$

The endpoints of this segment are

$$\begin{aligned}\delta_1(s)&=F_1(s)K_1^{j_1}(s)+\cdots+F_l(s)K_l^{j_l}(s)+\cdots+F_m(s)K_m^{j_m}(s)\\ \delta_2(s)&=F_1(s)K_1^{j_1}(s)+\cdots+F_l(s)K_l^{i_l}(s)+\cdots+F_m(s)K_m^{j_m}(s).\end{aligned}$$

The difference between the end points of this segment is

$$\begin{aligned}\delta_0(s)&:=\delta_1(s)-\delta_2(s)\\ &=F_l(s)[K_l^{j_l}(s)-K_l^{i_l}(s)].\end{aligned}$$

If $F_l(s)$ is of the form $s^t(as+b)U(s)R(s)$ where $t\geq 0$, a and b are arbitrary real numbers, $U(s)$ are anti-Hurwitz, and $R(s)$ is even or odd, then so is $\delta_0(s)$ since $K_l^{j_l}(s)-K_l^{i_l}(s)$ is either even or odd. Therefore, by the Vertex Lemma of Chapter 2, stability of the segment $[\delta_1(s),\delta_2(s)]$ for $\lambda\in[0,1]$ is implied by the stability of the vertices. We complete the proof of part II by applying this reasoning to every segment in $\mathbf{\Delta}_{\mathrm{E}}(s)$.

Step 4 We prove part III by giving a counter example. Consider

$$\underline{P}(s)=(1.5-s-s^2,\,2+3s+\gamma s^2), \text{ where } \gamma\in[2,16],$$

and

$$\underline{F}(s) = (1, 1 + s + s^2).$$

Then

$$\delta_\gamma(s) := F_1(s)P_1(s) + F_2(s)P_2(s) = 3.5 + 4s + (4+\gamma)s^2 + (3+\gamma)s^3 + \gamma s^4.$$

Here $\mathbf{P}_{\rm K}(s)$ consists of the two 2-tuples

$$P_1(s) = 1.5 - s - s^2, \quad P_2'(s) = 2 + 3s + 2s^2$$

and

$$P_1(s) = 1.5 - s - s^2, \quad P_2''(s) = 2 + 3s + 16s^2.$$

The corresponding polynomials of $\mathbf{\Delta}_{\rm K}(s)$ are

$$\begin{aligned} \delta_2(s) &= 3.5 + 4s + 6s^2 + 5s^3 + 2s^4, \\ \delta_{16}(s) &= 3.5 + 4s + 20s^2 + 19s^3 + 16s^4. \end{aligned}$$

The Hurwitz matrix for δ_γ is

$$H = \begin{vmatrix} 3+\gamma & 4 & 0 & 0 \\ \gamma & 4+\gamma & 3.5 & 0 \\ 0 & 3+\gamma & 4 & 0 \\ 0 & \gamma & 4+\gamma & 3.5 \end{vmatrix}$$

and the Hurwitz determinants are

$$\left\{ \begin{array}{l} H_1 = 3+\gamma \\ H_2 = \gamma^2 + 3\gamma + 12 \\ H_3 = 0.5\gamma^2 - 9\gamma + 16.5 \\ H_4 = 3.5H_3. \end{array} \right.$$

Now one can see that H_1, H_2 are positive for all values of γ in $[2, 16]$. However H_3 and H_4 are positive for $\gamma = 2$, or $\gamma = 16$, but negative when, for example, $\gamma = 10$. Therefore it is not enough to check the stability of the extreme polynomials $\delta_\gamma(s)$ corresponding to the couples of polynomials in $\mathbf{P}_{\rm K}$ and one must check the stability of the entire segment

$$\left(P_1(s), (\lambda P_2'(s) + (1-\lambda)P_2''(s))\right), \quad \lambda \in [0, 1],$$

which is the only element in $\mathbf{P}_{\rm E}$ for this particular example. This completes the proof. ♣

An alternative way to prove step 2 is to show that if $\mathbf{\Delta}(s)$ contains an unstable polynomial then the polytope $\mathbf{\Delta}_{\rm I}(s)$ contains a polynomial with a $j\omega$ root. This contradicts the conclusion reached in step 1. This approach to the proof is sketched below.

Alternative Proof of Step 2 of GKT (Theorem 7.1)

If $\underline{F}(s)$ stabilizes every m-tuple segment in $\mathbf{P}_{\mathrm{E}}(s)$, we conclude from Step 1 that every polynomial of the form

$$\beta(s) = \sum_{i=1}^{m} F_i(s)\Big((1-\lambda_i)K_i^{\text{even,min}}(s) + \lambda_i K_i^{\text{even,max}}(s) \\ +(1-\mu_i)K_i^{\text{odd,min}}(s) + \mu_i K_i^{\text{odd,max}}(s)\Big) \quad (7.44)$$

is stable for all possible

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots, \lambda_m, \mu_m), \qquad \lambda_i \in [0,1], \quad \mu_i \in [0,1].$$

To complete the proof of part I we have to prove that the stability of these polynomials implies the stability of every polynomial in $\mathbf{\Delta}(s)$.

If every polynomial in (7.44) is stable, $\underline{F}(s)$ will not stabilize the entire family $\mathbf{P}(s)$ if and only if for at least one m-tuple

$$\underline{R}(s) := (R_1(s), R_2(s), \cdots, R_m(s))$$

in $\mathbf{P}(s)$ the corresponding polynomial

$$\delta(s) = F_1(s)R_1(s) + F_2(s)R_2(s) + \cdots + F_m(s)R_m(s)$$

has a root at $j\omega^*$ for some $\omega^* \geq 0$. This last statement is a consequence of the Boundary Crossing Theorem (Chapter 2). In other words, for this ω^* we would have

$$\delta(j\omega^*) = F_1(j\omega^*)R_1(j\omega^*) + F_2(j\omega^*)R_2(j\omega^*) + \cdots + F_m(j\omega^*)R_m(j\omega^*) = 0. \quad (7.45)$$

Consider now one of the polynomials $R_i(s)$. We can decompose $R_i(s)$ into its odd and even part

$$R_i(s) = R_i^{\text{even}}(s) + R_i^{\text{odd}}(s)$$

and we know that on the imaginary axis, $R_i^{\text{even}}(j\omega)$ and $\frac{1}{j}R_i^{\text{odd}}(j\omega)$, are, respectively, the real and imaginary parts of $R_i(j\omega)$. Then the associated extremal polynomials

$$K_i^{\text{even,min}}(s), \quad K_i^{\text{even,max}}(s), \quad K_i^{\text{odd,min}}(s), \quad K_i^{\text{odd,max}}(s)$$

satisfy the inequalities

$$K_i^{\text{even,min}}(j\omega) \leq R_i^{\text{even}}(j\omega) \leq K_i^{\text{even,max}}(j\omega), \qquad \text{for all } \omega \in [0,\infty)$$

and

$$\frac{1}{j}K_i^{\text{odd,min}}(j\omega) \leq \frac{1}{j}R_i^{\text{odd}}(j\omega) \leq \frac{1}{j}K_i^{\text{odd,max}}(j\omega), \qquad \text{for all } \omega \in [0,\infty). \quad (7.46)$$

Using (7.46) we conclude that we can find $\lambda_i \in [0,1]$ and $\mu_i \in [0,1]$ such that

$$R_i^{\text{even}}(j\omega^*) = (1-\lambda_i)K_i^{\text{even,min}}(j\omega^*) + \lambda_i K_i^{\text{even,max}}(j\omega^*)$$
$$\frac{1}{j}R_i^{\text{odd}}(j\omega^*) = (1-\mu_i)\frac{1}{j}K_i^{\text{odd,min}}(j\omega^*) + \mu\frac{1}{j}K_i^{\text{odd,max}}(j\omega^*). \tag{7.47}$$

From (7.47) we deduce that we can write

$$\begin{aligned} R_i(j\omega^*) = &(1-\lambda_i)K_i^{\text{even,min}}(j\omega^*) + \lambda_i K_i^{\text{even,max}}(j\omega^*) \\ &+(1-\mu_i)K_i^{\text{odd,min}}(j\omega^*) + \mu_i K_i^{\text{odd,max}}(j\omega^*). \end{aligned} \tag{7.48}$$

However, substituting (7.48) for every $i = 1, \cdots, m$ into (7.45), we eventually get

$$\begin{aligned} \sum_{i=1}^{m} F_i(j\omega^*) \Big(&(1-\lambda_i)K_i^{\text{even,min}}(j\omega^*) + \lambda_i K_i^{\text{even,max}}(j\omega^*) \\ &+(1-\mu_i)K_i^{\text{odd,min}}(j\omega^*) + \mu_i K_i^{\text{odd,max}}(j\omega^*)\Big) = 0 \end{aligned}$$

but this is a contradiction with the fact that every polynomial $\beta(s)$ in (7.44) is stable as proved in Step 1.

Remark 7.1. One can immediately see that in the particular case $m = 1$ and $F_1(s) = 1$, the GKT (Theorem 7.1) reduces to Kharitonov's Theorem because $F_1(s) = 1$ is even and thus part II of the theorem applies.

Comparison with the Edge Theorem

The problem addressed in the Generalized Kharitonov Theorem (GKT) deals with a polytope and therefore it can also be solved by using the Edge Theorem. This would require us to check the stability of the exposed edges of the polytope of polynomials $\mathbf{\Delta}(s)$. GKT on the other hand requires us to check the stability of the segments $\mathbf{\Delta}_{\text{E}}(s)$. In general these two sets are quite different. Consider the simplest case of an interval polynomial containing three variable parameters. The 12 exposed edges and 4 extremal segments are shown in Figure 7.2. While two of the extremal segments are also exposed edges, the other two extremal segments lie on the exposed faces and are not edges at all. More importantly, the number of exposed edges depends exponentially on the number of the uncertain parameters (dimension of $\mathbf{p} \in \mathbf{\Pi}$). The number of extremal segments, on the other hand, depends only on m (the number of uncertain polynomials). To compare these numbers, consider for instance that each uncertain polynomial $P_i(s)$ has q uncertain parameters. Table 7.1 shows the number of exposed edges and number of segments $\mathbf{P}_{\text{E}}(s)$ for various values of m and q. We can see that the number of exposed edges grows exponentially with the number of parameters whereas the number of extremal segments remains constant for a fixed m.

Table 7.1. Number of exposed edges vs. number of extremal segments

m	q	Exposed Edges	Extremal Segments
2	2	32	32
2	3	80	32
2	4	192	32
.	.		
.	.		
3	4	24,576	192
.	.		
.	.		

Remark 7.2. In some situations, not all the coefficients of the polynomials are necessarily going to vary. In such cases, the number of extremal segments to be checked would be smaller than the maximum theoretical number, $m4^m$. With regard to the vertex result given in part II, it can happen that *some* $F_i(s)$ satisfy the conditions given in part II whereas other $F_i(s)$ do not. Suppose $F_l(s)$ satisfies the vertex conditions in part II. Then we can replace the stability check of the segments corresponding to $\mathbf{P}^l_{\mathrm{E}}(s)$ by the stability check of the corresponding vertices.

7.4 EXAMPLES

Example 7.2. Consider the plant

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{s^3 + \alpha s^2 - 2s + \beta}{s^4 + 2s^3 - s^2 + \gamma s + 1}$$

where

$$\alpha \in [-1, -2], \quad \beta \in [0.5, 1], \quad \gamma \in [0, 1].$$

Let

$$C(s) = \frac{F_1(s)}{F_2(s)}$$

denote the compensator. To determine if $C(s)$ robustly stabilizes the set of plants given we must verify the Hurwitz stability of the family of characteristic polynomials $\mathbf{\Delta}(s)$ defined as

$$F_1(s)(s^3 + \alpha s^2 - 2s + \beta) + F_2(s)(s^4 + 2s^3 - s^2 + \gamma s + 1)$$

with $\alpha \in [-1, -2]$, $\beta \in [0.5, 1]$, $\gamma \in [0, 1]$. To construct the generalized Kharitonov segments, we start with the Kharitonov polynomials. There are two Kharitonov polynomials associated with $P_1(s)$

$$K_1^1(s) = K_1^2(s) = 0.5 - 2s - s^2 + s^3$$
$$K_1^3(s) = K_1^4(s) = 1 - 2s - 2s^2 + s^3$$

and also two Kharitonov polynomials associated with $P_2(s)$

$$K_2^1(s) = K_2^3(s) = 1 - s^2 + 2s^3 + s^4$$
$$K_2^2(s) = K_2^4(s) = 1 + s - s^2 + 2s^3 + s^4.$$

The set $\mathbf{P}_{\mathrm{E}}^1(s)$ therefore consists of the 2 plant segments

$$\frac{\lambda_1 K_1^1(s) + (1-\lambda_1)K_1^3(s)}{K_2^1(s)} : \lambda_1 \in [0,1]$$
$$\frac{\lambda_2 K_1^1(s) + (1-\lambda_2)K_1^3(s)}{K_2^2(s)} : \lambda_2 \in [0,1].$$

The set $\mathbf{P}_{\mathrm{E}}^2(s)$ consists of the 2 plant segments

$$\frac{K_1^1(s)}{\lambda_3 K_2^1(s) + (1-\lambda_3)K_2^2(s)} : \lambda_3 \in [0,1]$$
$$\frac{K_1^3(s)}{\lambda_4 K_2^1(s) + (1-\lambda_4)K_2^2(s)} : \lambda_4 \in [0,1].$$

Thus, the extremal set $\mathbf{P}_{\mathrm{E}}(s)$ consists of the following four plant segments.

$$\frac{0.5(1+\lambda_1) - 2s - (1+\lambda_1)s^2 + s^3}{1 - s^2 + 2s^3 + s^4} : \lambda_1 \in [0,1]$$
$$\frac{0.5(1+\lambda_2) - 2s - (1+\lambda_2)s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} : \lambda_2 \in [0,1]$$

$$\frac{0.5 - 2s - s^2 + s^3}{1 + \lambda_3 s - s^2 + 2s^3 + s^4} : \lambda_3 \in [0,1], \qquad \frac{1 - 2s - 2s^2 + s^3}{1 + \lambda_4 s - s^2 + 2s^3 + s^4} : \lambda_4 \in [0,1].$$

Therefore, we can verify robust stability by checking the Hurwitz stability of the set $\mathbf{\Delta}_{\mathrm{E}}(s)$ which consists of the following four polynomial segments.

$$\begin{aligned}
&F_1(s)\left(0.5(1+\lambda_1) - 2s - (1+\lambda_1)s^2 + s^3\right) + F_2(s)\left(1 - s^2 + 2s^3 + s^4\right)\\
&F_1(s)\left(0.5(1+\lambda_2) - 2s - (1+\lambda_2)s^2 + s^3\right) + F_2(s)\left(1 + s - s^2 + 2s^3 + s^4\right)\\
&F_1(s)\left(0.5 - 2s - s^2 + s^3\right) + F_2(s)\left(1 + \lambda_3 s - s^2 + 2s^3 + s^4\right)\\
&F_1(s)\left(1 - 2s - 2s^2 + s^3\right) + F_2(s)\left(1 + \lambda_4 s - s^2 + 2s^3 + s^4\right)\\
&\qquad\qquad \lambda_i \in [0,1] : i = 1,2,3,4.
\end{aligned}$$

In other words, any compensator that stabilizes the family of plants $\mathbf{P}(s)$ must stabilize the 4 one-parameter family of extremal plants $\mathbf{P}_{\mathrm{E}}(s)$. If we had used the

Edge Theorem it would have been necessary to check the 12 line segments of plants corresponding to the exposed edges of $\mathbf{\Delta}(s)$.

If the compensator polynomials $F_i(s)$ satisfy the "vertex conditions" in part II of GKT, it is enough to check that they stabilize the plants corresponding to the four Kharitonov vertices. This corresponds to checking the Hurwitz stability of the four *fixed* polynomials

$$\begin{array}{l} F_1(s)\left(1-2s-2s^2+s^3\right)+F_2(s)\left(1-s^2+2s^3+s^4\right) \\ F_1(s)\left(1-2s-2s^2+s^3\right)+F_2(s)\left(1+s-s^2+2s^3+s^4\right) \\ F_1(s)\left(0.5-2s-s^2+s^3\right)+F_2(s)\left(1+s-s^2+2s^3+s^4\right) \\ F_1(s)\left(0.5-2s-s^2+s^3\right)+F_2(s)\left(1-s^2+2s^3+s^4\right) \end{array}$$

Example 7.3. (Stable Example) Consider the interval plant and controller pair

$$G(s)=\frac{P_1(s)}{P_2(s)}=\frac{a_1s+a_0}{b_2s^2+b_1s+b_0} \quad \text{and} \quad C(s)=\frac{F_1(s)}{F_2(s)}=\frac{s^2+2s+1}{s^4+2s^3+2s^2+s}$$

where the plant parameters vary as follows:

$$a_1\in[0.1,0.2],\quad a_0\in[0.9,1],\quad b_2\in[0.9,1.0],\quad b_1\in[1.8,2.0],\quad b_0\in[1.9,2.1].$$

The characteristic polynomial of the closed loop system is

$$\delta(s)=F_1(s)P_1(s)+F_2(s)P_2(s).$$

From GKT, the robust stability of the closed loop system is equivalent to that of the set of 32 generalized Kharitonov segments. To construct these segments, we begin with the Kharitonov polynomials of the interval polynomials $P_1(s)$ and $P_2(s)$, respectively:

$$K_1^1(s)=0.9+0.1s,\quad K_1^2(s)=0.9+0.2s,\quad K_1^3(s)=1+0.1s,\quad K_1^4(s)=1+0.2s$$

and

$$\begin{array}{ll} K_2^1(s)=1.9+1.8s+s^2, & K_2^2(s)=1.9+2s+s^2, \\ K_2^3(s)=2.1+1.8s+0.9s^2, & K_2^4(s)=2.1+2s+0.9s^2. \end{array}$$

Then the corresponding generalized Kharitonov segments are

$$F_1(s)K_1^i(s)+F_2(s)\left(\lambda K_2^j(s)+(1-\lambda)K_2^k(s)\right)$$

and

$$F_1(s)\left(\lambda K_1^i(s)+(1-\lambda)K_1^j(s)\right)+F_2(s)K_2^k(s)$$

where $i, j, k \in \underline{4} \times \underline{4} \times \underline{4}$. For example, two such segments are

$$F_1(s)K_1^1(s) + F_2(s)\left(\lambda K_2^1(s) + (1-\lambda)K_2^2(s)\right) = (s^2 + 2s + 1)(0.9 + 0.1s)$$
$$+(s^4 + 2s^3 + 2s^2 + s)\left(\lambda(1.9 + 1.8s + s^2) + (1-\lambda)(1.9 + 2s + s^2)\right)$$

and

$$F_1(s)\left(\lambda K_1^1(s) + (1-\lambda)K_1^2(s)\right) + F_2(s)K_2^1 = (s^2 + 2s + 1)$$
$$(\lambda(0.9 + 0.1s) + (1-\lambda)(0.9 + 0.2s)) + (s^4 + 2s^3 + 2s^2 + s)(1.9 + 1.8s + s^2)$$

for $\lambda \in [0, 1]$. The stability of these 32 segments can be checked in a number of ways such as the Segment Lemma (Chapter 2), Bounded Phase Conditions (Chapter 4) or the Zero Exclusion Theorem (Chapter 1). In Figure 7.3 we show the evolution of the image sets with frequency.

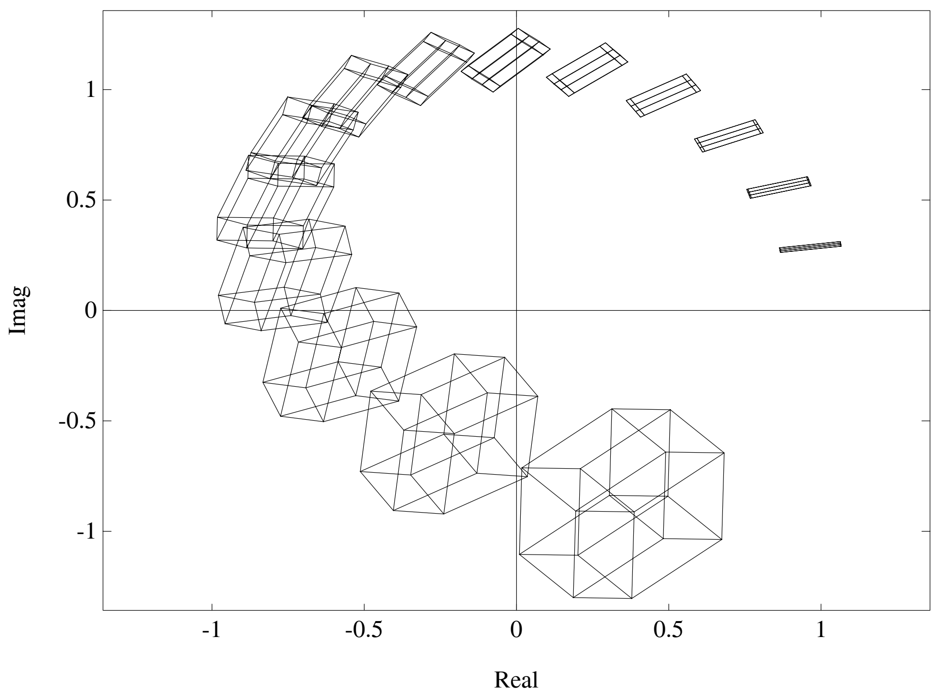

Figure 7.3. Image set of generalized Kharitonov segments (Example 7.3)

We see that the origin is excluded from the image sets for all frequency. In addition, since at least one element (a vertex polynomial) in the family is Hurwitz, the entire family is Hurwitz. Thus, we conclude that the controller $C(s)$ robustly stabilizes the interval plant.

Example 7.4. **(Unstable Example)** Consider the interval plant and controller pair

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0} \quad \text{and} \quad C(s) = \frac{F_1(s)}{F_2(s)} = \frac{s^2 + 2s + 1}{s^4 + 2s^3 + 2s^2 + s}$$

where the plant parameters vary in intervals as follows:

$$a_1 \in [0.1, 0.2], \quad a_0 \in [0.9, 1.5], \quad b_2 \in [0.9, 1.0], \quad b_1 \in [1.8, 2.0], \quad b_0 \in [1.9, 2.1].$$

The characteristic polynomial of the closed loop system is

$$\delta(s) = F_1(s)P_1(s) + F_2(s)P_2(s).$$

From GKT, the robust stability of the closed loop system is equivalent to that of the set of 32 generalized Kharitonov segments. We construct this set as in the previous example. The image set of these segments are displayed as a function of frequency in Figure 7.4.

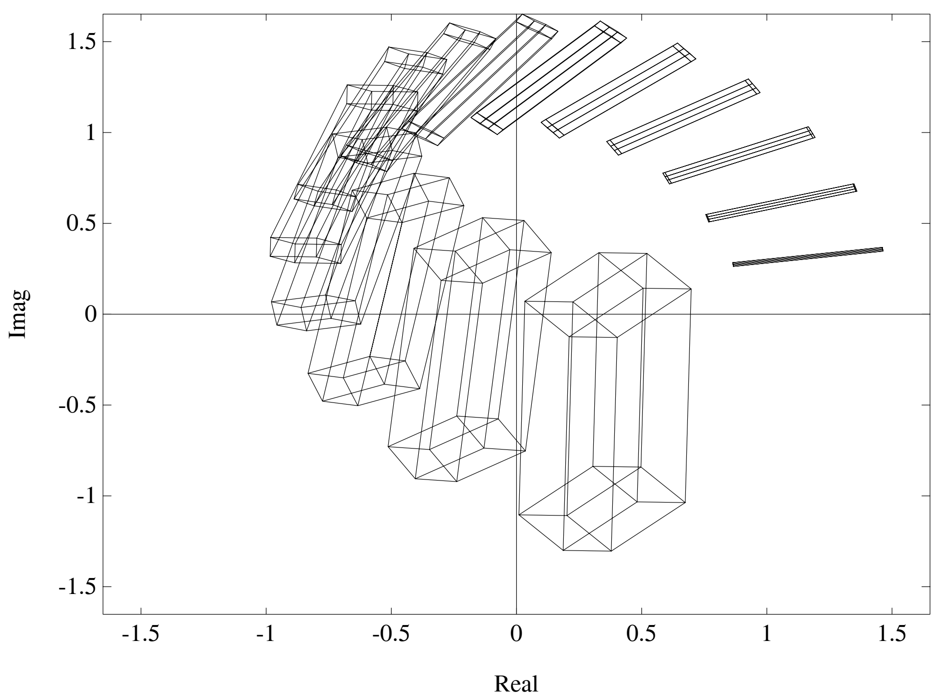

Figure 7.4. Image set of generalized Kharitonov segments (Example 7.4)

From this figure, we see that the origin is *included* in the image set at some frequencies. Thus we conclude that the controller $C(s)$ does not robustly stabilize the given family of plants $G(s)$.

Example 7.5. **(Vertex Example)** Let us consider the plant and controller

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0} \quad \text{and} \quad C(s) = \frac{F_1(s)}{F_2(s)} = \frac{(3s+5)(s^2+1)}{s^2(s-5)}$$

where the plant parameters vary as $a_2 = -67$ and

$$a_1 \in [248, 250], \quad a_0 \in [623, 626], \quad b_2 \in [202, 203], \quad b_1 \in [624, 626], \quad b_0 \in [457, 458].$$

Then the characteristic polynomial of the closed loop system is

$$\delta(s) = F_1(s)P_1(s) + F_2(s)P_2(s).$$

In this particular problem, we observe that

$$F_1(s) : \text{(1st order)(even)} \quad \text{and} \quad F_2(s) : s^t(\text{anti} - \text{Hurwitz}).$$

This satisfies the vertex condition of GKT. Thus, the stability of the 16 vertex

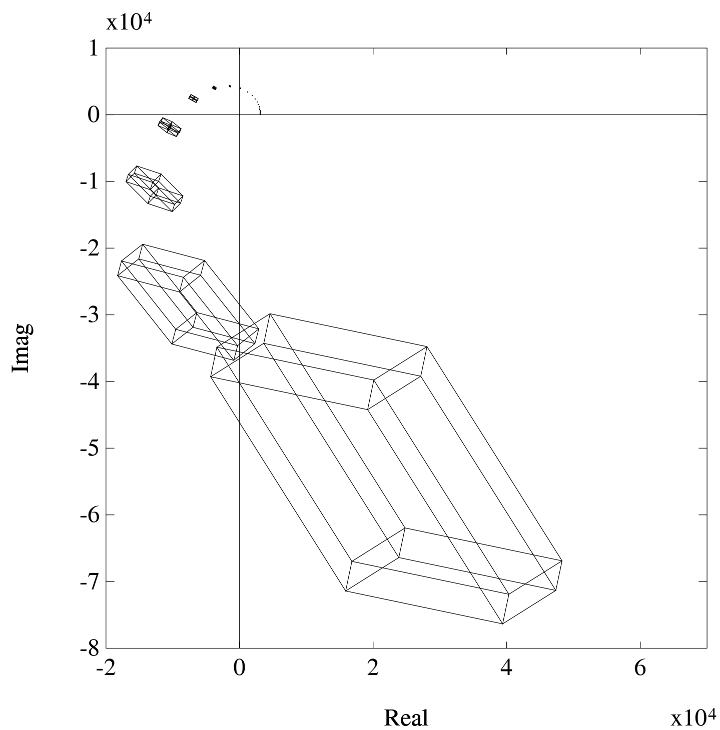

Figure 7.5. Image set of generalized Kharitonov segments (Example 7.5)

polynomials is equivalent to that of the closed loop system. Since all the roots of the 16 vertex polynomials

$$F_1(s)K_1^i(s) + F_2(s)K_2^j(s), \quad i = 1, 2, 3, 4; \quad j = 1, 2, 3, 4$$

lie in the left half of the complex plane, we conclude that the closed loop system is robustly stable. Figure 7.5 confirms this fact.

7.5 IMAGE SET INTERPRETATION

The Generalized Kharitonov Theorem has an appealing geometric interpretation in terms of the image set $\mathbf{\Delta}(j\omega)$ of $\mathbf{\Delta}(s)$. Recall that in Step 1 of the proof, the stability of $\mathbf{\Delta}(s)$ was reduced to that of the $2m$ parameter polytope $\mathbf{\Delta}_I(s)$. It is easy to see that even though $\mathbf{\Delta}_I(s)$ is in general a proper subset of $\mathbf{\Delta}(s)$, the image sets are in fact equal:

$$\mathbf{\Delta}(j\omega) = \mathbf{\Delta}_I(j\omega).$$

This follows from the fact, established is Chapter 5, that each of the m interval polynomials $\mathbf{P}_i(s)$ in $\mathbf{\Delta}(s)$ can be replaced by a 2-parameter family as far as its $j\omega$ evaluation is concerned. This proves that regardless of the dimension of the parameter space $\mathbf{\Pi}$, a linear interval problem with m terms can always be replaced by a $2m$ parameter problem. Of course in the rest of the Theorem we show that this $2m$ parameter problem can be further reduced to a set of one-parameter problems.

In fact, $\mathbf{\Delta}(j\omega)$ is a convex polygon in the complex plane and it may be described in terms of its vertices or exposed edges. Let $\partial\mathbf{\Delta}(j\omega)$ denote the exposed edges of $\mathbf{\Delta}(j\omega)$ and $\mathbf{\Delta}_{\mathrm{V}}(j\omega)$ denote its vertices. Then it is easy to establish the following.

Lemma 7.4

$$1)\ \ \partial\mathbf{\Delta}(j\omega) \subset \mathbf{\Delta}_{\mathrm{E}}(j\omega) \qquad\qquad 2)\ \ \mathbf{\Delta}_{\mathrm{V}}(j\omega) \subset \mathbf{\Delta}_{\mathrm{K}}(j\omega)$$

Proof. Observe that $\mathbf{\Delta}(j\omega)$ is the sum of complex plane sets as follows:

$$\mathbf{\Delta}(j\omega) = F_1(j\omega)\mathbf{P}_1(j\omega) + F_2(j\omega)\mathbf{P}_2(j\omega) + \cdots + F_m(j\omega)\mathbf{P}_m(j\omega).$$

Each polygon $F_i(j\omega)\mathbf{P}_i(j\omega)$ is a rectangle with vertex set $F_i(j\omega)\mathcal{K}_i(j\omega)$ and edge set $F_i(j\omega)\mathcal{S}_i(j\omega)$. Since the vertices of $\mathbf{\Delta}(j\omega)$ can only be generated by the vertices of $F_i(j\omega)\mathbf{P}_i(j\omega)$, we immediately have 2). To establish 1) we note that the boundary of the sum of two complex plane polygons can be generated by summing over all vertex-edge pairs with the vertices belonging to one and the edges belonging to the other. This fact used recursively to add m polygons shows that one has to sum vertices from $m-1$ of the sets to edges of the remaining set and repeat this over all possibilities. This leads to 1). ♣

The vertex property in 2) allows us to check robust stability of the family $\mathbf{\Delta}(s)$ by using the phase conditions for a polytopic family described in Chapter 4. More precisely, define

$$\phi_\delta(\lambda) := \arg\left(\frac{\delta(\lambda)}{\delta_0(\lambda)}\right). \tag{7.49}$$

and with $\delta_0(j\omega) \in \mathbf{\Delta}(j\omega)$,

$$\phi^+(j\omega) := \sup_{\delta_i(j\omega)\in\mathbf{\Delta}_{\mathrm{K}}(j\omega)} \phi_{\delta_i}(j\omega), \qquad 0 \le \phi^+ \le \pi$$
$$\phi^-(j\omega) := \inf_{\delta_i(j\omega)\in\mathbf{\Delta}_{\mathrm{K}}(j\omega)} \phi_{\delta_i}(j\omega), \qquad -\pi < \phi^- \le 0$$

and

$$\Phi_{\mathbf{\Delta}_{\mathrm{K}}}(j\omega) := \phi^+(j\omega) - \phi^-(j\omega). \tag{7.50}$$

Theorem 7.2 *Assume that $\mathbf{\Delta}(s)$ has at least one polynomial which is stable, then the entire family is stable if and only if $\Phi_{\mathbf{\Delta}_{\mathrm{K}}}(j\omega) < \pi$ for all ω.*

Example 7.6. Consider the plant controller pair of Example 7.3. We first check the stability of an arbitrary point in the family, say, one of the Kharitonov vertices. Next for each ω, we evaluate the maximum phase difference over the following 16 Kharitonov vertices

$$F_1(s)K_1^i(s) + F_2(s)K_2^j(s), \quad i = 1,2,3,4; \quad j = 1,2,3,4$$

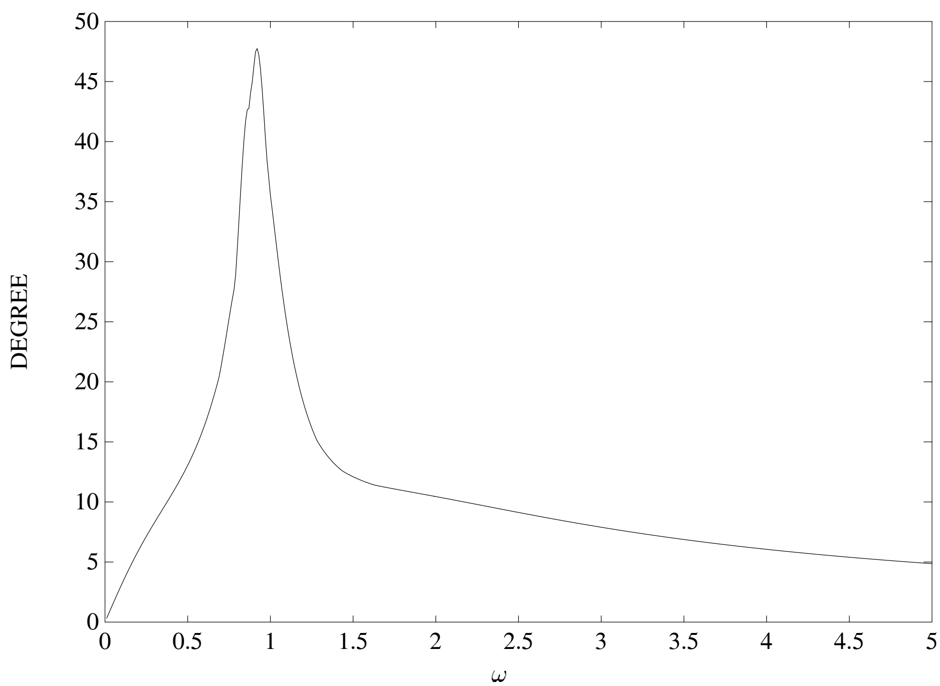

Figure 7.6. Maximum phase difference of Kharitonov vertices (Example 7.6)

The result is plotted in Figure 7.6. It shows that the maximum phase difference over these vertices never reaches 180° at any frequency. Thus we conclude that the system is robustly stable which agrees with the conclusion reached using the image set plot shown in Figure 7.3.

Example 7.7. For the plant controller pair of Example 7.4, we evaluate the maximum phase difference over the Kharitonov vertices at each frequency. The plot is shown in Figure 7.7.

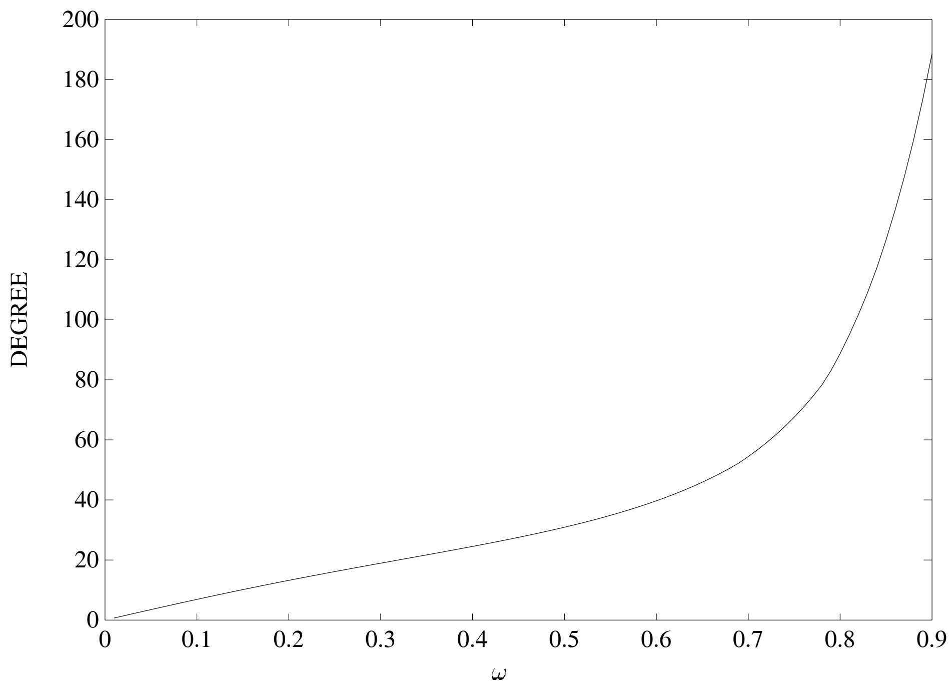

Figure 7.7. Maximum phase difference of Kharitonov vertices (Example 7.7)

This graph shows that the maximal phase difference reaches 180° showing that the family is *not* stable. This again agrees with the analysis using the image sets given in Figure 7.4.

7.6 EXTENSION TO COMPLEX QUASIPOLYNOMIALS

The main statement, namely part I, of the Generalized Kharitonov Theorem also holds when the $F_i(s)$ are complex polynomials and also when $F_i(s)$ are quasipolynomials. We state these results formally below but omit the detailed proof which, in general, follows from the fact that the image set of the family $\mathbf{\Delta}(j\omega)$ is still generated by the same extremal segments even in these cases. Thus all that is needed is to ensure that the Boundary Crossing Theorem can be applied.

Consider the family of quasipolynomials

$$\mathbf{\Delta}(s) = F_1(s)\mathbf{P}_1(s) + F_2(s)\mathbf{P}_2(s) + \cdots + F_m(s)\mathbf{P}_m(s). \tag{7.51}$$

where $\mathbf{P}_i(s)$ are real independent interval polynomials as before, but now

$$\underline{F}(s) = (F_1(s), \cdots, F_m(s)),$$

is a fixed m-tuple of complex quasipolynomials, of the form

$$F_i(s) = F_i^0(s) + e^{-sT_i^1}F_i^1(s) + e^{-sT_i^2}F_i^2(s) + \cdots$$

with the $F_i^j(s)$ being complex polynomials satisfying the condition

$$\text{degree}\left[F_i^0(s)\right] > \text{degree}\left[F_i^j(s)\right], \quad j \neq 0. \tag{7.52}$$

We assume that every polynomial in the family

$$\mathbf{\Delta}^0(s) := F_1^0(s)\mathbf{P}_1(s) + F_2^0(s)\mathbf{P}_2(s) + \cdots + F_m^0(s)\mathbf{P}_m(s) \tag{7.53}$$

is of the same degree. Let $\mathbf{P}_{\mathrm{E}}(s)$ and $\mathbf{\Delta}_{\mathrm{E}}(s)$ be as defined in Section 2. The above degree conditions guarantee that the Boundary Crossing Theorem can be applied to the family $\mathbf{\Delta}(s)$. We therefore have the following extension of GKT to the case of complex quasipolynomials.

Theorem 7.3 (Generalized Kharitonov Theorem: Complex Polynomials and Quasipolynomials)
Let $\underline{F} = (F_1(s), \cdots, F_m(s))$, be a given m-tuple of complex quasipolynomials satisfying the conditions (7.52) above, and $\mathbf{P}_i(s), 1 = 1, 2...m$ be independent real interval polynomials satisfying the invariant degree assumption for the family (7.53). Then $\underline{F}$ stabilizes the entire family $\mathbf{P}(s)$ of m-tuples if and only if $\underline{F}$ stabilizes every m-tuple segment in $\mathbf{P}_{\mathrm{E}}(s)$. Equivalently, $\mathbf{\Delta}(s)$ is stable if and only if $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable.

The complex *polynomial* case is a special case of the above result obtained by setting $F_i^j(s) = 0$, $i = 1, 2, \cdots, m$; $j \neq 0$. We note that the vertex result, part II of GKT, that holds in the real case does not, in general, carry over to this complex quasipolynomial case. However in the case of complex polynomials the following vertex result holds.

Corollary 7.1 (Theorem 7.3)
Under the conditions of Theorem 7.3, suppose that

$$\underline{F}(s) = (F_1(s), \cdots, F_m(s)),$$

is an m-tuple of complex polynomials such that

$$\frac{d}{d\omega}\arg F_i(j\omega) \le 0, \qquad i = 1, 2, \cdots, m.$$

Then $\underline{F}$ stabilizes the entire family $\mathbf{P}(s)$ of m-tuples if and only if $\underline{F}$ stabilizes every m-tuple in $\mathbf{P}_{\mathrm{K}}(s)$. Equivalently, $\mathbf{\Delta}(s)$ is stable if and only if the Kharitonov vertex set $\mathbf{\Delta}_{\mathrm{K}}(s)$ is stable.

Proof. The proof follows from the fact that the difference of the endpoints of a typical segment in $\mathbf{\Delta}_E(s)$ is of the form

$$\delta_0(s) = F_l(s)[K_l^{j_l}(s) - K_l^{i_l}(s)].$$

Since $[K_l^{j_l}(s) - K_l^{i_l}(s)]$ is real and odd or even, it follows that

$$\frac{d}{d\omega}\arg \delta_0(j\omega) = \frac{d}{d\omega}\arg F_i(j\omega) \le 0, \quad i = 1, 2, \cdots, m.$$

Therefore, by the Convex Direction Lemma for the complex case (Lemma 2.15, Chapter 2), such segments are stable if and only if the endpoints are. These endpoints constitute the set $\mathbf{\Delta}_{\mathrm{K}}(s)$ and this completes the proof. ♣

The above results can be used to determine the robust stability of systems containing time delay. The theorem can be further extended to the case where $P_i(s)$ are complex interval polynomials by using Kharitonov's Theorem for the complex case given in Chapter 5. The detailed development of these ideas is left to the reader.

Example 7.8. (Time Delay Example) Let us consider the plant and controller

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$$

and

$$C(s) = \frac{F_1(s)}{F_2(s)} = \frac{s^2 + 2s + 1}{s^4 + 2s^3 + 2s^2 + s}$$

with a time delay of 0.1 sec where the plant parameters vary as

$$a_1 \in [0.1, 0.2], \quad a_0 \in [0.9, 1],$$
$$b_2 \in [0.9, 1.0], \quad b_1 \in [1.8, 2.0], \quad b_0 \in [1.9, 2.1].$$

Then the characteristic equation of the closed loop system is

$$\delta(s) = F_2(s)P_2(s) + e^{-sT} F_1(s)P_1(s)$$

where $T = 0.1$.

The robust stability of this time delay system can also be tested by GKT. Figure 7.8 shows that the image set excludes the origin. Therefore, the closed loop system is robustly stable. This fact is also verified by the Bounded Phase Conditions as shown in Figure 7.9.

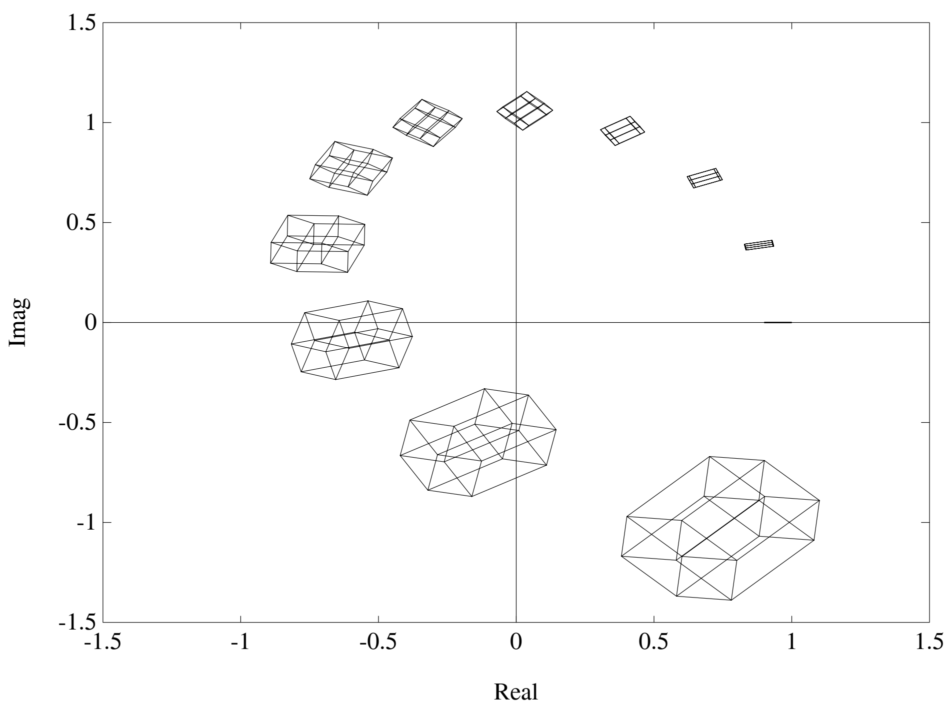

Figure 7.8. Image set of generalized Kharitonov segments (Example 7.8)

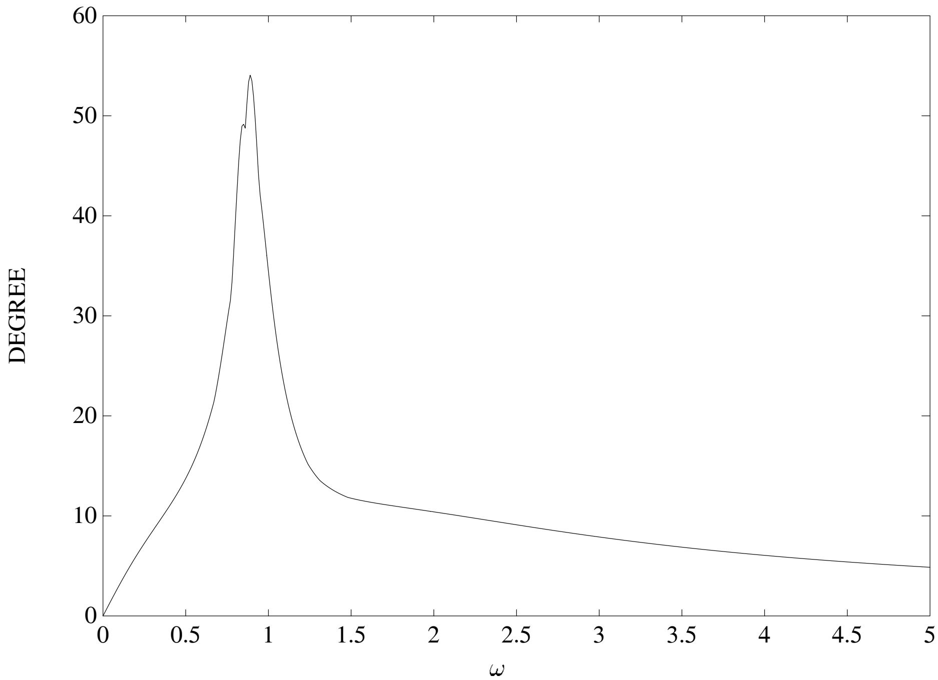

Figure 7.9. Maximum phase difference of Kharitonov vertices (Example 7.8)

We conclude this chapter by giving an application of the GKT to interval polynomials.

7.7 σ AND θ HURWITZ STABILITY OF INTERVAL POLYNOMIALS

Let

$$h(s) = h_0 + h_1 s + h_2 s^2 + \cdots + h_n s^n,$$

and consider the real interval polynomial family of degree n:

$$\mathcal{I}(s) := \left\{ h(s) : h_i^- \leq h_i \leq h_i^+, \quad i = 0, 1, \cdots, n \right\}$$

with the assumption $0 \notin [h_n^-, h_n^+]$. We consider the stability of the family $\mathcal{I}(s)$ with respect to two special stability regions.

First consider the *shifted Hurwitz* stability region (see Figure 7.10) defined for for a fixed real number $\sigma > 0$ by

$$\mathcal{S}_\sigma := \left\{ s : s \in \mathbb{C}^-, \quad \text{Re}[s] < -\sigma \right\}.$$

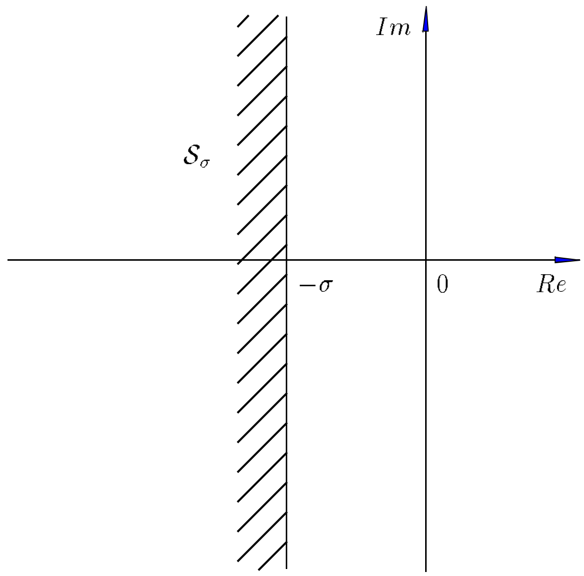

Figure 7.10. Shifted Hurwitz stability region

We shall say that $\mathcal{I}(s)$ is *σ-Hurwitz stable* if each polynomial in $\mathcal{I}(s)$ has all its roots in $\mathcal{S}_\sigma$. Introduce the set of *vertex polynomials* of $\mathcal{I}(s)$:

$$\mathbf{V}_I(s) := \left\{ h(s) : h_i = h_i^+ \text{ or } h_i = h_i^-, \ i = 0, 1, 2, \cdots, n \right\}.$$

We have the following result.

Theorem 7.4 *The family $\mathcal{I}(s)$ is σ-Hurwitz stable if and only if the vertex polynomials $\mathbf{V}_I(s)$ are σ-Hurwitz stable.*

Proof. We set $s = -\sigma + p$ and write

$$h(s) = g(p) = h_0 + h_1(p - \sigma) + h_2(p - \sigma)^2 + \cdots + h_n(p - \sigma)^n.$$

The σ-Hurwitz stability of the family $\mathcal{I}(s)$ is equivalent to the Hurwitz stability of the *real linear interval family*

$$\mathcal{G}(p) := \left\{ g(p) = h_0 + h_1(p - \sigma) + h_2(p - \sigma)^2 + \cdots + h_n(p - \sigma)^n : \right.$$
$$\left. h_i \in [h_i^-, h_i^+], \quad i = 0, 1, 2, \cdots, n \right\}$$

Let $\mathbf{V}_G(p)$ denote the vertex set associated with the family $\mathcal{G}(p)$. We now apply GKT to this family $\mathcal{G}(p)$ with

$$F_i = (p - \sigma)^{i-1} \text{ and } P_i = h_{i-1}, \quad i = 0, 1, 2, \cdots, n + 1$$

Since $F_0 = 1$ is even and $F_1, ...F_{n+1}$ are anti-Hurwitz it follows from part II of GKT that it suffices to check the Hurwitz stability of the vertex set $\mathbf{V}_G(p)$. This is equivalent to checking the σ-Hurwitz stability of the $\mathbf{V}_I(s)$. ♣

Remark 7.3. The above lemma can also be used to find the largest value of σ for which the interval family $\mathcal{I}(s)$ is σ-Hurwitz stable. This value is sometimes called the *stability degree* of the family and indicates the distance of the root set of the family from the imaginary axis.

Using the above result we can derive a useful vertex result on σ-Hurwitz stability for the family of polynomials

$$\mathbf{\Delta}(s) := F_1(s)\mathbf{P}_1(s) + F_2(s)\mathbf{P}_2(s) + \cdots + F_m(s)\mathbf{P}_m(s). \tag{7.54}$$

Corollary 7.2 *The family $\mathbf{\Delta}(s)$, with $F_i(s)$ satisfying*

$$F_i(s) = s^{t_i} A_i(s)(a_i s + b_i), \qquad i = 1, \cdots, m$$

with $A_i(s)$ antiHurwitz is σ-Hurwitz stable for $\sigma \geq 0$ if and only if the Kharitonov vertex set of polynomials $\mathbf{\Delta}_{\mathrm{K}}(s)$ is σ-Hurwitz stable.

The proof of this result follows from Part II of GKT and is omitted.

Next let us consider a rotation of the complex plane obtained by setting $s = s' e^{j\theta}$. We define the *rotated Hurwitz* region (see Figure 7.11) by

$$\mathcal{S}_\theta := \left\{ s : \ s \in \mathbb{C}, \quad \mathrm{Re}\left[s e^{-j\theta} \right] < 0 \right\}.$$

We say that the family $\mathcal{I}(s)$ is *θ-Hurwitz stable* if each polynomial in $\mathcal{I}(s)$ has all its roots in $\mathcal{S}_\theta$. Note that if the family is Hurwitz stable and is also θ-Hurwitz stable then its root set lies in the shaded region shown in Figure 7.11. The following lemma establishes that $\mathcal{S}_\theta$ stability of $\mathcal{I}(s)$ can also be determined from the vertex set.

Theorem 7.5 *The family $\mathcal{I}(s)$ is θ-Hurwitz stable if and only if the vertex polynomials $\mathbf{V}_I(s)$ are θ-Hurwitz stable.*

Proof. The result can be readily derived by first substituting $s' = s e^{-j\theta}$ in $h(s)$ and then applying GKT to test the Hurwitz stability of the resulting *complex* linear interval family

$$1 \cdot h_0 + e^{j\theta} h_1 s' + e^{j2\theta} h_2 (s')^2 + \cdots + e^{jn\theta} h_n (s')^n, \quad h_i \in [h_i^-, h_i^+], \quad i = 0, 1, \cdots, n.$$

To apply GKT we set

$$F_i(s') = (e^{j\theta})^{i-1}, \quad P_i(s') = h_i (s')^i, \quad i = 1, 2, \cdots, n+1.$$

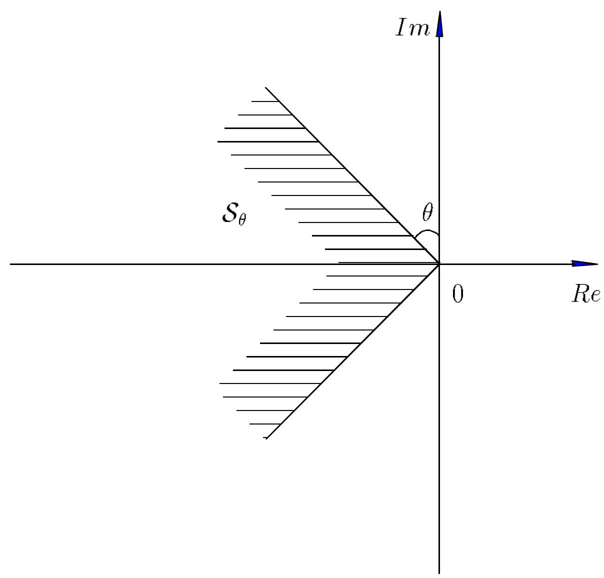

Figure 7.11. Rotated Hurwitz stability region

and test the Hurwitz stability of the family in the s' plane. This shows that the set is Hurwitz stable if and only if the corresponding segments with *only one* coefficient h_i varying at a time, with the rest of the $h_j, j \neq i$ set to the vertices, are Hurwitz stable in the s' plane. Consider a typical such complex segment. The difference between the endpoints of such a segment is a polynomial of the form

$$\delta_0(s') = (h_i^+ - h_i^-)(s')^i(e^{j\theta})^i.$$

Now by applying the Convex Direction Lemma for the complex case (Chapter 2, Lemma 2.15) we can see that

$$\frac{d}{d\omega} \arg \delta_0(j\omega) = 0.$$

Therefore such complex segments are stable if and only if the endpoints are. But this is equivalent to the θ-Hurwitz stability of the vertex set $\mathbf{V}_I(s)$. ♣

The maximum value θ^* of θ for which θ-Hurwitz stability is preserved for the family can be found using this result. The above two results can then be used to estimate the root space boundary of an interval polynomial family without using the excessive computations associated with the Edge Theorem.

7.8 EXERCISES

7.1 In a unity feedback system the plant transfer function is:

$$G(s) = \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}.$$

The nominal values of the plant parameters are

$$\begin{array}{lll} \alpha_2^0 = 1, & \alpha_1^0 = 5, & \alpha_0^0 = -2 \\ \beta_2^0 = -3, & \beta_1^0 = -4, & \beta_0^0 = 6. \end{array}$$

Determine a feedback controller of second order that places the 5 closed loop poles at -1, -2, -3, $-2+2j$, $-2-2j$. Suppose that the parameters of $G(s)$ are subject to perturbation as follows:

$$\alpha_1 \in [3, 7], \quad \alpha_0 \in [-3, -1], \quad \beta_1 \in [-6, -2], \quad \beta_0 \in [5, 7].$$

Determine if the closed loop is robustly stable with the controller designed as above for the nominal system.

7.2 Consider the two masses connected together by a spring and a damper as shown in Figure 7.12:

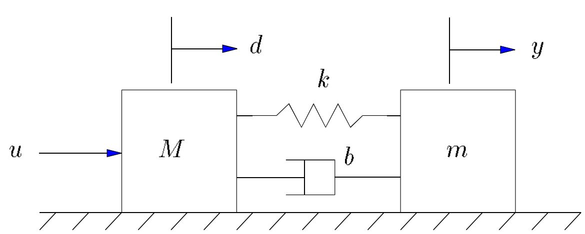

Figure 7.12. Mass-spring-damper system

Assuming that there is no friction between the masses and the ground, then we have the following dynamic equations:

$$\begin{aligned} M\ddot{d} + b(\dot{d} - \dot{y}) + k(d - y) &= u \\ m\ddot{y} + b(\dot{y} - \dot{d}) + k(y - d) &= 0. \end{aligned}$$

The transfer functions of the system are as follows:

$$\frac{y(s)}{u(s)} = \frac{\left(\frac{b}{m}s + \frac{k}{m}\right)}{Ms^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]}$$

$$\frac{d(s)}{u(s)} = \frac{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)}{Ms^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]}.$$

The feedback controller

$$C(s) = \frac{n_{c_1}(s)}{d_c(s)}y(s) + \frac{n_{c_2}(s)}{d_c(s)}d(s)$$

with $d_c(s) = s^2 + \beta_1 s + \beta_0$, $n_{c_1}(s) = \delta_1 s + \delta_0$, $n_{c_2}(s) = \gamma_1 s + \gamma_0$, is to be designed so that the closed loop is stabilized. With nominal parameters $m = 1$, $M = 2$, $b = 2$, $k = 3$ determine the controller parameters so that the closed loop poles for the nominal system are all at -1. Determine if the closed loop remains stable when the parameters b and k suffer perturbations in their magnitudes of 50%. Determine the largest l^∞ box centered at the above nominal parameter in the b, k parameter space for which closed loop stability is preserved with the controller designed. Also, use the Boundary Crossing Theorem of Chapter 1 to plot the entire stability region in this parameter space.

7.3 In a unity feedback system

$$G(s) = \frac{s - z_0}{3s^3 - p_2 s^2 + s + p_0} \quad \text{and} \quad C(s) = \frac{\alpha_0 + \alpha_1 s + \alpha_2 s^2}{s^2 + \beta_1 s + \beta_0}$$

represent the transfer functions of the plant and controller respectively. The nominal values of the parameters $[z_0, p_0, p_2]$ are given by $[z_0^0, p_0^0, p_2^0] = [1, 1, 2]$. Find the controller parameters so that the closed loop poles are placed at $[-1, -2, -3, -2 - j, -2 + j]$. Determine if the closed loop system that results remains stable if the parameters $[z_0, p_0, p_2]$ are subject to a 50% variation in their numerical values centered about the nominal, i.e. $z_0 \in [0.5, 1.5]$, $p_0 \in [0.5, 1.5]$, $p_2 \in [1, 3]$.

7.4 In the previous problem let the plant parameters have their nominal values and assume that the nominal controller has been designed to place the closed loop poles as specified in the problem. Determine the largest l^∞ ball in the controller parameter space $x = [\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1]$, centered at the nominal value calculated above, for which the closed loop system with the nominal plant remains stable.

7.5 In a unity feedback system with

$$G(s) = \frac{s+z_0}{s^2+p_1s+p_0} \quad \text{and} \quad C(s) = \frac{\alpha_2 s^2+\alpha_1 s+\alpha_0}{s(s+\beta)}.$$

Assume that the nominal values of the plant parameters are

$$(z_0^0, p_0^0, p_1^0) := (1,1,1).$$

Choose the controller parameters $(\alpha_0, \alpha_1, \alpha_2, \beta)$ so that the closed loop system is stabilized at the nominal plant parameters. Check if your controller robustly stabilizes the family of the closed loop systems under plant parameter perturbations of 20%.

7.6 Consider the interval polynomial $s^3 + a_2 s^2 + a_1 s + a_0$ with

$$a_2 \in [15-\epsilon, 15+\epsilon], \qquad a_1 \in [10-\epsilon, 10+\epsilon], \qquad a_0 \in [7-\epsilon, 7+\epsilon].$$

For each value of $\epsilon = 1,2,3,4,5,$ and 6 determine the maximum value $\sigma^*(\epsilon)$ for which the family is σ^* Hurwitz stable. Plot a graph of ϵ vs. $\sigma^*(\epsilon)$ to display the result.

7.7 Repeat Exercise 7.6 this time computing the maximal value $\theta^*(\epsilon)$ for which the family is θ^*-Hurwitz stable.

7.8 Consider the unity feedback configuration with the plant and controller being

$$C(s) = \frac{s+1}{s+2} \quad \text{and} \quad G(s) = \frac{s+b_0}{s^2+a_1s+a_0}$$

where

$$a_0 \in [2,4], \quad a_1 \in [2,4], \quad b_0 \in [1,3].$$

Is this closed loop system robustly stable?

7.9 Consider the unity feedback system shown in Figure 7.13 where

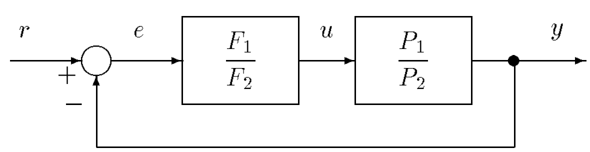

Figure 7.13. Feedback control system

$$\frac{F_1(s)}{F_2(s)} = \frac{2s^2+4s+3}{s^2+3s+4} \quad \text{and} \quad \frac{P_1(s)}{P_2(s)} = \frac{s^2+a_1s+a_0}{s(s^2+b_1s+b_0)}$$

with

$$a_1^0 = -2, \quad a_0^0 = 1, \quad b_0^0 = 2, \quad b_1^0 = 1.$$

Now let

$$\begin{aligned} a_0 &\in [1-\epsilon, 1+\epsilon], \quad b_0 \in [2-\epsilon, 2+\epsilon] \\ a_1 &\in [-2-\epsilon, -2+\epsilon], \quad b_1 \in [1-\epsilon, 1+\epsilon] \end{aligned}$$

Find $\epsilon_{\max}$ for which the system is robustly stable.
Answer: $\epsilon_{\max} = 0.175$

7.10 Referring to the system given in Figure 7.13 with

$$\frac{F_1(s)}{F_2(s)} = \frac{2s^2+4s+3}{s^2+3s+4} \quad \text{and} \quad \frac{P_1(s)}{P_2(s)} = \frac{s^2+a_1s+a_0}{s(s^2+b_1s+b_0)}.$$

Let the nominal system be

$$\frac{P_1^0(s)}{P_2^0(s)} = \frac{s^2-2s+7}{s(s^2+8s-0.25)}.$$

Suppose that the parameters vary within intervals:

$$\begin{aligned} a_1 &\in [-2-\epsilon, -2+\epsilon], \qquad a_0 \in [7-\epsilon, 7+\epsilon] \\ b_1 &\in [8-\epsilon, 8+\epsilon], \qquad b_0 \in [-0.25-\epsilon, -0.25+\epsilon]. \end{aligned}$$

Find the maximum value of ϵ for robust stability of the family using GKT.
Answer: $\epsilon_{\max} = 0.23$

7.11 For the same configuration in Figure 7.13 with

$$\frac{F_1(s)}{F_2(s)} = \frac{26+27s}{-17+2s} \quad \text{and} \quad \frac{P_1(s)}{P_2(s)} = \frac{s+a_0}{s^2+b_1s+b_0}$$

with

$$a_0 \in [-1.5, -0.5], \quad b_1 \in [-2.5, -1.5], \quad b_0 \in [-1.5, -0.5]$$

Show that the family of closed loop systems is unstable using GKT.

7.9 NOTES AND REFERENCES

The Generalized Kharitonov Theorem was proved by Chapellat and Bhattacharyya in [58], where it was called the Box Theorem. The vertex condition that was given in [58] dealt only with the case where the $F_i(s)$ were even or odd. The more general vertex condition given here in part II of Theorem 7.1 is based on the Vertex Lemma (Chapter 2). In Bhattacharyya [30] and Bhattacharyya and Keel [33] a comprehensive survey of the applications of this theorem were given. In the latter papers this result was referred to as the CB Theorem. A special case of the vertex results of part II of GKT is reported by Barmish, Hollot, Kraus, and Tempo [15]. A discrete time counterpart of GKT was developed by Katbab and Jury [130] but in this case the stability test set that results consists of manifolds rather than line segments. The results on σ and θ-Hurwitz stability are due to Datta and Bhattacharyya [78] and Theorem 7.4 was applied in Datta and Bhattacharyya [77] for quantitative estimates of robustness in Adaptive Control. Kharitonov and Zhabko [147] have used GKT to develop robust stability results for time-delay systems.