Chapter 5

INTERVAL POLYNOMIALS: KHARITONOV'S THEOREM

In this chapter we present Kharitonov's Theorem on robust Hurwitz stability of interval polynomials, dealing with both the real and complex cases. This elegant result forms the basis for many of the results, to be developed later in the book, on robustness under parametric uncertainty. We develop an important extremal property of the associated Kharitonov polynomials and give an application of this theorem to state feedback stabilization. An extension of Kharitonov's Theorem to nested families of interval polynomials is described. Robust Schur stability of interval polynomials is also discussed and it is shown that robust stability can be ascertained from that of the upper edges.

5.1 INTRODUCTION

We devote this chapter mainly to a result proved in 1978 by V. L. Kharitonov, regarding the Hurwitz stability of a family of *interval polynomials*. This result was so surprising and elegant that it has been the starting point of a renewed interest in robust control theory with an emphasis on *deterministic bounded parameter perturbations*. It is important therefore that control engineers thoroughly understand both result and proof, and this is why we considerably extend our discussion of this subject.

In the next section we first state and prove Kharitonov's Theorem for real polynomials. We emphasize how this theorem generalizes the Hermite-Biehler Interlacing Theorem which is valid for a single polynomial. This has an appealing frequency domain interpretation in terms of *interlacing of frequency bands*. We then give an interpretation of Kharitonov's Theorem based on the evolution of the complex plane image set of the interval polynomial family. Here again the Boundary Crossing Theorem and the monotonic phase increase property of Hurwitz polynomials (Chapter 1) are the key concepts that are needed to establish the Theorem. This proof gives useful geometric insight into the working of the theorem and shows how the result is related to the Vertex Lemma (Chapter 2). We then state the theorem for the case of an interval family of polynomials with complex coefficients. This proof follows quite naturally from the above interlacing point of view. Next, we develop an important *extremal property* of the Kharitonov polynomials. This property establishes that one of the four points represented by the Kharitonov polynomials is the closest to instability over the entire set of uncertain parameters. The latter result is independent of the norm used to measure the distance between polynomials in the coefficient space. We next give an application of the Kharitonov polynomials to robust state feedback stabilization. Following this we establish that Kharitonov's Theorem can be extended to nested families of interval polynomials which are neither interval or polytopic and in fact includes nonlinear dependence on uncertain parameters. In the last section we consider the robust Schur stability of an interval polynomial family. A stability test for this family is derived based on the upper edges which form a subset of all the exposed edges. We illustrate the application of these fundamental results to control systems with examples.

5.2 KHARITONOV'S THEOREM FOR REAL POLYNOMIALS

In this chapter stable will mean Hurwitz stable unless otherwise stated. Of course, we will say that a set of polynomials is stable if and only if each and every element of the set is a Hurwitz polynomial.

Consider now the set $\mathcal{I}(s)$ of real polynomials of degree n of the form

$$\delta(s) = \delta_o + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots + \delta_n s^n$$

where the coefficients lie within given ranges,

$$\delta_0 \in [x_0, y_0],\ \delta_1 \in [x_1, y_1],\ \cdots,\ \delta_n \in [x_n, y_n].$$

Write

$$\underline{\delta} := [\delta_0,\ \delta_1,\ \cdots,\ \delta_n]$$

and identify a polynomial $\delta(s)$ with its coefficient vector $\underline{\delta}$. Introduce the hyperrectangle or box of coefficients

$$\boldsymbol{\Delta} := \left\{ \underline{\delta} : \underline{\delta} \in \mathbb{R}^{n+1}, \quad x_i \le \delta_i \le y_i, \quad i = 0, 1, \cdots, n \right\}. \tag{5.1}$$

We assume that the degree remains invariant over the family, so that $0 \notin [x_n, y_n]$. Such a set of polynomials is called a real *interval* family and we loosely refer to $\mathcal{I}(s)$ as an interval polynomial. Kharitonov's Theorem provides a surprisingly simple necessary and sufficient condition for the Hurwitz stability of the entire family.

Theorem 5.1 (Kharitonov's Theorem)
Every polynomial in the family $\mathcal{I}(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$K^1(s) = x_o + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots,$$

$$
\begin{aligned}
K^2(s) &= x_o + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots, \\
K^3(s) &= y_o + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots, \\
K^4(s) &= y_o + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots.
\end{aligned}
\tag{5.2}
$$

The box $\mathbf{\Delta}$ and the vertices corresponding to the Kharitonov polynomials are shown in Figure 5.1. The proof that follows allows for the interpretation of Kharitonov's

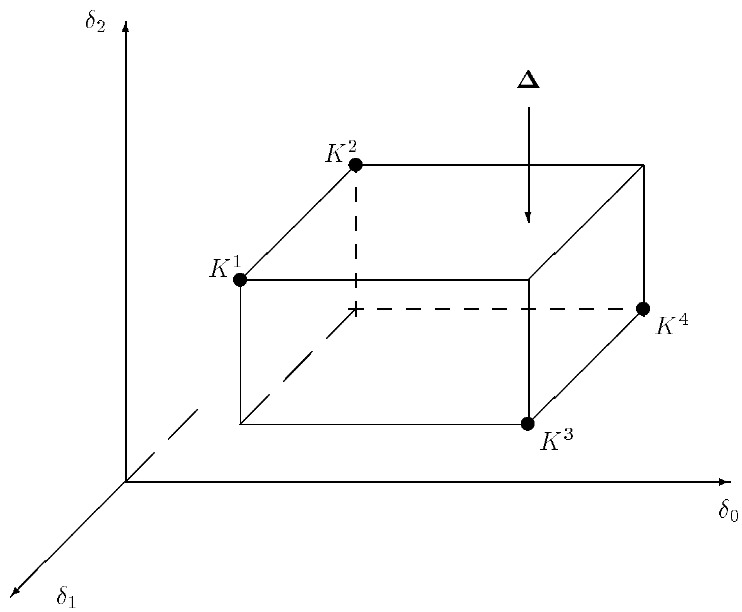

Figure 5.1. The box $\mathbf{\Delta}$ and the four Kharitonov vertices

Theorem as a generalization of the Hermite-Biehler Theorem for Hurwitz polynomials, proved in Theorem 1.7 of Chapter 1. We start by introducing two symmetric lemmas that will lead us naturally to the proof of the theorem.

Lemma 5.1 *Let*

$$
\begin{aligned}
P_1(s) &= P^{\text{even}}(s) + P_1^{\text{odd}}(s) \\
P_2(s) &= P^{\text{even}}(s) + P_2^{\text{odd}}(s)
\end{aligned}
$$

denote two stable polynomials of the same degree with the same even part $P^{\text{even}}(s)$ *and differing odd parts* $P_1^{\text{odd}}(s)$ *and* $P_2^{\text{odd}}(s)$ *satisfying*

$$
P_1^o(\omega) \le P_2^o(\omega), \qquad \text{for all } \omega \in [0, \infty]. \tag{5.3}
$$

Then,

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$$

is stable for every polynomial $P(s)$ *with odd part* $P^{\text{odd}}(s)$ *satisfying*

$$P_1^o(\omega) \le P^o(\omega) \le P_2^o(\omega), \qquad \text{for all } \omega \in [0, \infty]. \tag{5.4}$$

Proof. Since $P_1(s)$ and $P_2(s)$ are stable, $P_1^o(\omega)$ and $P_2^o(\omega)$ both satisfy the interlacing property with $P^e(\omega)$. In particular, $P_1^o(\omega)$ and $P_2^o(\omega)$ are not only of the same degree, but the sign of their highest coefficient is also the same since it is in fact the same as that of the highest coefficient of $P^e(\omega)$. Given this it is easy to see that $P^o(\omega)$ cannot satisfy (5.4) unless it also has this same degree and the same sign for its highest coefficient. Then, the condition in (5.4) forces the roots of $P^o(\omega)$ to interlace with those of $P^e(\omega)$. Therefore, according to the Hermite-Biehler Theorem (Theorem 1.7, Chapter 1), $P^{\text{even}}(s) + P^{\text{odd}}(s)$ is stable. ♣

We remark that Lemma 5.1 as well as its dual, Lemma 5.2 given below, are special cases of the Vertex Lemma, developed in Chapter 2 and follow immediately from it. We illustrate Lemma 5.1 in the example below (see Figure 5.2).

Example 5.1. Let

$$\begin{aligned} P_1(s) &= s^7 + 9s^6 + 31s^5 + 71s^4 + 111s^3 + 109s^2 + 76s + 12 \\ P_2(s) &= s^7 + 9s^6 + 34s^5 + 71s^4 + 111s^3 + 109s^2 + 83s + 12. \end{aligned}$$

Then

$$\begin{aligned} P^{\text{even}}(s) &= 9s^6 + 71s^4 + 109s^2 + 12 \\ P_1^{\text{odd}}(s) &= s^7 + 31s^5 + 111s^3 + 76s \\ P_2^{\text{odd}}(s) &= s^7 + 34s^5 + 111s^3 + 83s. \end{aligned}$$

Figure 5.2 shows that $P^e(\omega)$ and the tube bounded by $P_1^o(\omega)$ and $P_2^o(\omega)$ satisfy the interlacing property.

Thus, we conclude that every polynomial $P(s)$ with odd part $P^{\text{odd}}(s)$ satisfying

$$P_1^o(\omega) \le P^o(\omega) \le P_2^o(\omega), \qquad \text{for all } \omega \in [0, \infty]$$

is stable. For example, the dotted line shown inside the tube represents

$$P^{\text{odd}}(s) = s^7 + 32s^5 + 111s^3 + 79s.$$

The dual of Lemma 5.1 is:

Lemma 5.2 *Let*

$$\begin{aligned} P_1(s) &= P_1^{\text{even}}(s) + P^{\text{odd}}(s) \\ P_2(s) &= P_2^{\text{even}}(s) + P^{\text{odd}}(s) \end{aligned}$$

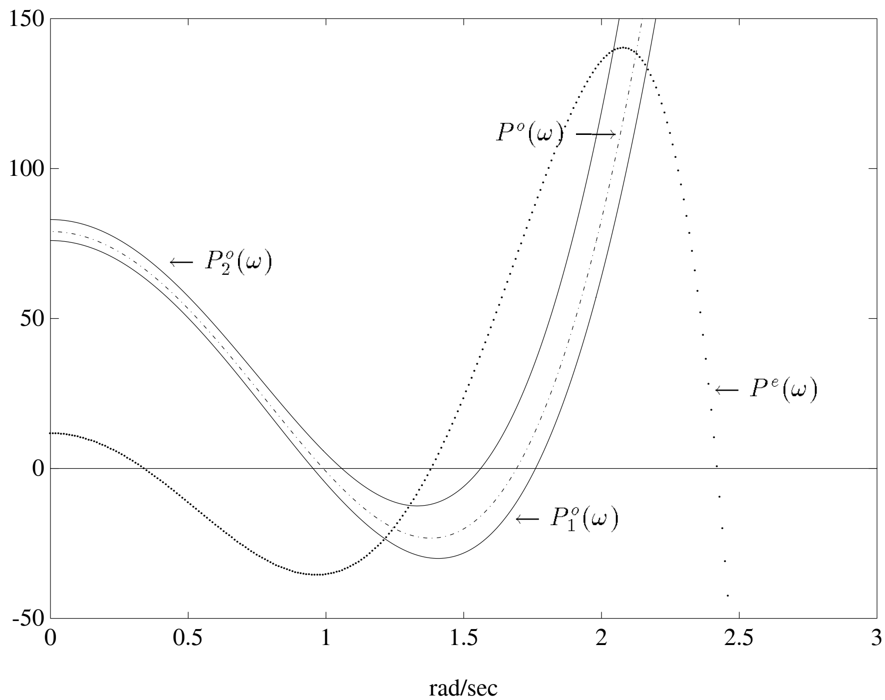

Figure 5.2. $P^e(\omega)$ and $(P_1^o(\omega), P_2^o(\omega))$ (Example 5.1)

denote two stable polynomials of the same degree with the same odd part $P^{\text{odd}}(s)$ *and differing even parts* $P_1^{\text{even}}(s)$ *and* $P_2^{\text{even}}(s)$ *satisfying*

$$P_1^e(\omega) \le P_2^e(\omega), \qquad \text{for all } \ \omega \in [0, \ \infty]. \tag{5.5}$$

Then,

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$$

is stable for every polynomial $P(s)$ *with even part* $P^{\text{even}}(s)$ *satisfying*

$$P_1^e(\omega) \le P^e(\omega) \le P_2^e(\omega), \qquad \text{for all } \ \omega \in [0, \ \infty]. \tag{5.6}$$

We are now ready to prove Kharitonov's Theorem.

Proof of Kharitonov's Theorem The Kharitonov polynomials repeated below, for convenience are four specific vertices of the box $\mathbf{\Delta}$:

$$K^1(s) = x_o + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots,$$
$$K^2(s) = x_o + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots,$$

$$
\begin{aligned}
K^3(s) &= y_o + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots, \\
K^4(s) &= y_o + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots.
\end{aligned} \tag{5.7}
$$

These polynomials are built from two different even parts $K_{\max}^{\text{even}}(s)$ and $K_{\min}^{\text{even}}(s)$ and two different odd parts $K_{\max}^{\text{odd}}(s)$ and $K_{\min}^{\text{odd}}(s)$ defined below:

$$
\begin{aligned}
K_{\max}^{\text{even}}(s) &:= y_o + x_2 s^2 + y_4 s^4 + x_6 s^6 + y_8 s^8 + \cdots, \\
K_{\min}^{\text{even}}(s) &:= x_o + y_2 s^2 + x_4 s^4 + y_6 s^6 + x_8 s^8 + \cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{\max}^{\text{odd}}(s) &:= y_1 s + x_3 s^3 + y_5 s^5 + x_7 s^7 + y_9 s^9 + \cdots, \\
K_{\min}^{\text{odd}}(s) &:= x_1 s + y_3 s^3 + x_5 s^5 + y_7 s^7 + x_9 s^9 + \cdots.
\end{aligned}
$$

The Kharitonov polynomials in (5.2) or (5.7) can be rewritten as:

$$
\begin{aligned}
K^1(s) &= K_{\min}^{\text{even}}(s) + K_{\min}^{\text{odd}}(s), \\
K^2(s) &= K_{\min}^{\text{even}}(s) + K_{\max}^{\text{odd}}(s), \\
K^3(s) &= K_{\max}^{\text{even}}(s) + K_{\min}^{\text{odd}}(s), \\
K^4(s) &= K_{\max}^{\text{even}}(s) + K_{\max}^{\text{odd}}(s).
\end{aligned} \tag{5.8}
$$

The motivation for the subscripts "max" and "min" is as follows. Let $\delta(s)$ be an arbitrary polynomial with its coefficients lying in the box $\mathbf{\Delta}$ and let $\delta^{\text{even}}(s)$ be its even part. Then

$$
\begin{aligned}
K_{\max}^e(\omega) &= y_0 - x_2\omega^2 + y_4\omega^4 - x_6\omega^6 + y_8\omega^8 + \cdots, \\
\delta^e(\omega) &= \delta_0 - \delta_2\omega^2 + \delta_4\omega^4 - \delta_6\omega^6 + \delta_8\omega^8 + \cdots, \\
K_{\min}^e(\omega) &= x_0 - y_2\omega^2 + x_4\omega^4 - y_6\omega^6 + x_8\omega^8 + \cdots,
\end{aligned}
$$

so that

$$
K_{\max}^e(\omega) - \delta^e(\omega) = (y_0 - \delta_0) + (\delta_2 - x_2)\omega^2 + (y_4 - \delta_4)w^4 + (\delta_6 - x_6)\omega^6 + \cdots,
$$

and

$$
\delta^e(\omega) - K_{\min}^e(\omega) = (\delta_0 - x_0) + (y_2 - \delta_2)\omega^2 + (\delta_4 - x_4)\omega^4 + (y_6 - \delta_6)\omega^6 + \cdots.
$$

Therefore,

$$
K_{\min}^e(\omega) \le \delta^e(\omega) \le K_{\max}^e(\omega), \qquad \text{for all } \ \omega \in [0, \infty]. \tag{5.9}
$$

Similarly, if $\delta^{\text{odd}}(s)$ denotes the odd part of $\delta(s)$, and $\delta^{\text{odd}}(j\omega) = j\omega\delta^o(\omega)$ it can be verified that

$$
K_{\min}^o(\omega) \le \delta^o(\omega) \le K_{max}^o(\omega), \qquad \text{for all } \ \omega \in [0, \infty]. \tag{5.10}
$$

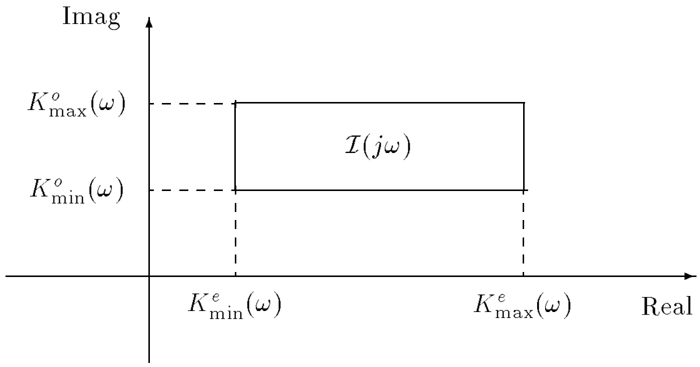

Figure 5.3. Axis parallel rectangle $\mathcal{I}(j\omega)$

Thus $\delta(j\omega)$ lies in an axis parallel rectangle $\mathcal{I}(j\omega)$ as shown in Figure 5.3.

To proceed with the proof of Kharitonov's Theorem we note that necessity of the condition is trivial since if all the polynomials with coefficients in the box $\mathbf{\Delta}$ are stable, it is clear that the Kharitonov polynomials must also be stable since their coefficients lie in $\mathbf{\Delta}$. For the converse, assume that the Kharitonov polynomials are stable, and let $\delta(s) = \delta^{\text{even}}(s) + \delta^{\text{odd}}(s)$ be an *arbitrary* polynomial belonging to the family $\mathcal{I}(s)$, with even part $\delta^{\text{even}}(s)$ and odd part $\delta^{\text{odd}}(s)$.

We conclude, from Lemma 5.1 applied to the stable polynomials $K^3(s)$ and $K^4(s)$ in (5.8), that

$$K^{\text{even}}_{\max}(s) + \delta^{\text{odd}}(s) \quad \text{is stable.} \tag{5.11}$$

Similarly, from Lemma 5.1 applied to the stable polynomials $K^1(s)$ and $K^2(s)$ in (5.8) we conclude that

$$K^{\text{even}}_{\min}(s) + \delta^{\text{odd}}(s) \quad \text{is stable.} \tag{5.12}$$

Now, since (5.9) holds, we can apply Lemma 5.2 to the two stable polynomials

$$K^{\text{even}}_{\max}(s) + \delta^{\text{odd}}(s) \quad \text{and} \quad K^{\text{even}}_{\min}(s) + \delta^{\text{odd}}(s)$$

to conclude that

$$\delta^{\text{even}}(s) + \delta^{\text{odd}}(s) = \delta(s) \quad \text{is stable.}$$

Since $\delta(s)$ was an arbitrary polynomial of $\mathcal{I}(s)$ we conclude that the entire family of polynomials $\mathcal{I}(s)$ is stable and this concludes the proof of the theorem. ♣

Remark 5.1. The Kharitonov polynomials can also be written with the highest order coefficient as the first term:

$$\hat{K}^1(s) = x_n s^n + y_{n-1}s^{n-1} + y_{n-2}s^{n-2} + x_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots,$$

$$\begin{aligned}\hat{K}^2(s) &= x_n s^n + x_{n-1}s^{n-1} + y_{n-2}s^{n-2} + y_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots,\\ \hat{K}^3(s) &= y_n s^n + x_{n-1}s^{n-1} + x_{n-2}s^{n-2} + y_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots,\\ \hat{K}^4(s) &= y_n s^n + y_{n-1}s^{n-1} + x_{n-2}s^{n-2} + x_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots.\end{aligned} \quad (5.13)$$

Remark 5.2. The assumption regarding invariant degree of the interval family can be relaxed. In this case some additional polynomials need to be tested for stability. This is dealt with in Exercise 5.13.

Remark 5.3. The assumption inherent in Kharitonov's Theorem that the coefficients perturb independently is crucial to the working of the theorem. In the examples below we have constructed some control problems where this assumption is satisfied. Obviously in many real world problems this assumption would fail to hold, since the characteristic polynomial coefficients would perturb interdependently through other primary parameters. However even in these cases Kharitonov's Theorem can give useful and computationally simple answers by overbounding the actual perturbations by an axis parallel box $\boldsymbol{\Delta}$ in coefficient space.

Remark 5.4. As remarked above Kharitonov's Theorem would give conservative results when the characteristic polynomial coefficients perturb interdependently. The Edge Theorem and the Generalized Kharitonov Theorem described in Chapters 6 and 7 respectively were developed precisely to deal nonconservatively with such dependencies.

Example 5.2. Consider the problem of checking the robust stability of the feedback system shown in Figure 5.4.

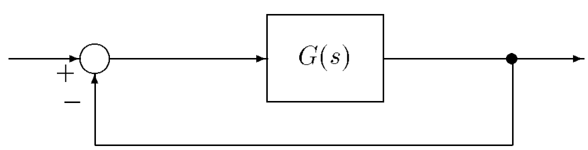

Figure 5.4. Feedback system (Example 5.2)

The plant transfer function is

$$G(s) = \frac{\delta_1 s + \delta_0}{s^2(\delta_4 s^2 + \delta_3 s^3 + \delta_2)}$$

with coefficients being bounded as

$$\delta_4 \in [x_4, y_4], \quad \delta_3 \in [x_3, y_3], \quad \delta_2 \in [x_2, y_2], \quad \delta_1 \in [x_1, y_1], \quad \delta_0 \in [x_0, y_0].$$

The characteristic polynomial of the family is written as

$$\delta(s) = \delta_4 s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0.$$

The associated even and odd polynomials for Kharitonov's test are as follows:

$$K^{\rm even}_{\rm min}(s) = x_0 + y_2 s^2 + x_4 s^4, \qquad K^{\rm even}_{\rm max}(s) = y_0 + x_2 s^2 + y_4 s^4,$$
$$K^{\rm odd}_{\rm min}(s) = x_1 s + y_3 s^3, \qquad K^{\rm odd}_{\rm max}(s) = y_1 s + x_3 s^3.$$

The Kharitonov polynomials are:

$$K^1(s) = x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4, \qquad K^2(s) = x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4,$$
$$K^3(s) = y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4, \qquad K^4(s) = y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4.$$

The problem of checking the Hurwitz stability of the family therefore is reduced to that of checking the Hurwitz stability of these four polynomials. This in turn reduces to checking that the coefficients have the same sign (positive, say; otherwise multiply $\delta(s)$ by -1) and that the following inequalities hold:

$$\begin{array}{lll}
K^1(s) & \text{Hurwitz}: y_2 y_3 > x_1 x_4, & x_1 y_2 y_3 > x_1^2 x_4 + y_3^2 x_0, \\
K^2(s) & \text{Hurwitz}: y_2 x_3 > y_1 x_4, & y_1 y_2 x_3 > y_1^2 x_4 + x_3^2 x_0, \\
K^3(s) & \text{Hurwitz}: x_2 y_3 > x_1 y_4, & x_1 x_2 y_3 > x_1^2 y_4 + y_3^2 y_0, \\
K^4(s) & \text{Hurwitz}: x_2 x_3 > y_1 y_4, & y_1 x_2 x_3 > y_1^2 y_4 + x_3^2 y_0.
\end{array}$$

Example 5.3. Consider the control system shown in Figure 5.5.

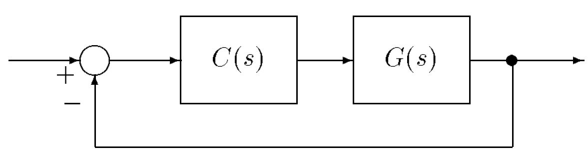

Figure 5.5. Feedback system with controller (Example 5.3)

The plant is described by the rational transfer function $G(s)$ with numerator and denominator coefficients varying independently in prescribed intervals. We refer to such a family of transfer functions $\mathbf{G}(s)$ as an *interval plant.* In the present example we take

$$\mathbf{G}(s) := \left\{ G(s) = \frac{n_2 s^2 + n_1 s + n_0}{s^3 + d_2 s^2 + d_1 s + d_0} : \begin{array}{l} n_0 \in [1, 2.5],\ n_1 \in [1, 6], \qquad n_2 \in [1, 7], \\ d_2 \in [-1, 1],\ d_1 \in [-0.5, 1.5],\ d_0 \in [1, 1.5] \end{array} \right\}.$$

The controller is a constant gain, $C(s) = k$ that is to be adjusted, if possible, to robustly stabilize the closed loop system. More precisely we are interested in determining the range of values of the gain $k \in [-\infty, +\infty]$ for which the closed loop system is robustly stable, i.e. stable for all $G(s) \in \mathbf{G}(s)$.

The characteristic polynomial of the closed loop system is:

$$\delta(k,s) = s^3 + \underbrace{(d_2 + kn_2)}_{\delta_2(k)} s^2 + \underbrace{(d_1 + kn_1)}_{\delta_1(k)} s + \underbrace{(d_0 + kn_0)}_{\delta_0(k)}.$$

Since the parameters d_i, n_j, $i = 0,1,2$, $j = 0,1,2$ vary independently it follows that for each fixed k, $\delta(k,s)$ is an interval polynomial. Using the bounds given to describe the family $\mathbf{G}(s)$ we get the following coefficient bounds for positive k:

$$\begin{aligned}\delta_2(k) &\in [-1+k, 1+7k],\\ \delta_1(k) &\in [-0.5+k, 1.5+6k],\\ \delta_0(k) &\in [-1+k, 1.5+2.5k].\end{aligned}$$

Since the leading coefficient is +1 the remaining coefficients must be all positive for the polynomial to be Hurwitz. This leads to the constraints:

(a) $-1 + k > 0, \quad -0.5 + k > 0, \quad 1 + k > 0.$

From Kharitonov's Theorem applied to third order interval polynomials it can be easily shown that to ascertain the Hurwitz stability of the entire family it suffices to check in addition to positivity of the coefficients, the Hurwitz stability of only the third Kharitonov polynomial $K_3(s)$. In this example we therefore have that the entire system is Hurwitz if and only if in addition to the above constraints (a) we have:

$$(-0.5+k)(-1+k) > 1.5 + 2.5k.$$

From this it follows that the closed loop system is robustly stabilized if and only if

$$k \in (2+\sqrt{5}, +\infty].$$

To complete our treatment of this important result we give Kharitonov's Theorem for polynomials with complex coefficients in the next section.

5.3 KHARITONOV'S THEOREM FOR COMPLEX POLYNOMIALS

Consider the set $\mathcal{I}^*(s)$ of all complex polynomials of the form,

$$\delta(s) = (\alpha_0 + j\beta_0) + (\alpha_1 + j\beta_1)s + \cdots + (\alpha_n + j\beta_n)s^n \tag{5.14}$$

with

$$\alpha_0 \in [x_0, y_0], \quad \alpha_1 \in [x_1, y_1], \quad \cdots, \quad \alpha_n \in [x_n, y_n] \tag{5.15}$$

and

$$\beta_0 \in [u_0, v_0], \quad \beta_1 \in [u_1, v_1], \quad \cdots, \quad \beta_n \in [u_n, v_n]. \tag{5.16}$$

This is a complex interval family of polynomials of degree n which includes the real interval family studied earlier as a special case. It is natural to consider the generalization of Kharitonov's Theorem for the real case to this family. The Hurwitz stability of complex interval families will also arise naturally in studying the extremal H_∞ norms of interval systems in Chapter 9. Complex polynomials also arise in the study of phase margins of control systems and in time-delay systems. Kharitonov extended his result for the real case to the above case of complex interval families. We assume as before that the degree remains invariant over the family. Introduce two sets of complex polynomials as follows:

$$\begin{aligned}
K_1^+(s) &:= (x_0 + ju_0) + (x_1 + jv_1)s + (y_2 + jv_2)s^2 + (y_3 + ju_3)s^3 \\
&\quad + (x_4 + ju_4)s^4 + (x_5 + jv_5)s^5 + \cdots, \\
K_2^+(s) &:= (x_0 + jv_0) + (y_1 + jv_1)s + (y_2 + ju_2)s^2 + (x_3 + ju_3)s^3 \\
&\quad + (x_4 + jv_4)s^4 + (y_5 + jv_5)s^5 + \cdots, \\
K_3^+(s) &:= (y_0 + ju_0) + (x_1 + ju_1)s + (x_2 + jv_2)s^2 + (y_3 + jv_3)s^3 \\
&\quad + (y_4 + ju_4)s^4 + (x_5 + u_5)s^5 + \cdots, \\
K_4^+(s) &:= (y_0 + jv_0) + (y_1 + ju_1)s + (x_2 + ju_2)s^2 + (x_3 + jv_3)s^3 \\
&\quad + (y_4 + jv_4)s^4 + (y_5 + ju_5)s^5 + \cdots,
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
K_1^-(s) &:= (x_0 + ju_0) + (y_1 + ju_1)s + (y_2 + jv_2)s^2 + (x_3 + jv_3)s^3 \\
&\quad + (x_4 + ju_4)s^4 + (y_5 + ju_5)s^5 + \cdots, \\
K_2^-(s) &:= (x_0 + jv_0) + (x_1 + ju_1)s + (y_2 + ju_2)s^2 + (y_3 + jv_3)s^3 \\
&\quad + (x_4 + jv_4)s^4 + (x_5 + ju_5)s^5 + \cdots, \\
K_3^-(s) &:= (y_0 + ju_0) + (y_1 + jv_1)s + (x_2 + jv_2)s^2 + (x_3 + ju_3)s^3 \\
&\quad + (y_4 + ju_4)s^4 + (y_5 + jv_5)s^5 + \cdots, \\
K_4^-(s) &:= (y_0 + jv_0) + (x_1 + jv_1)s + (x_2 + ju_2)s^2 + (y_3 + ju_3)s^3 \\
&\quad + (y_4 + jv_4)s^4 + (x_5 + jv_5)s^5 + \cdots.
\end{aligned} \tag{5.18}$$

Theorem 5.2 *The family of polynomials $\mathcal{I}^*(s)$ is Hurwitz if and only if the eight Kharitonov polynomials $K_1^+(s)$, $K_2^+(s)$, $K_3^+(s)$, $K_4^+(s)$, $K_1^-(s)$, $K_2^-(s)$, $K_3^-(s)$, $K_4^-(s)$ are all Hurwitz.*

Proof. The necessity of the condition is obvious because the eight Kharitonov polynomials are in $\mathcal{I}^*(s)$. The proof of sufficiency follows again from the Hermite-Biehler Theorem for complex polynomials (Theorem 1.8, Chapter 1).

Observe that the Kharitonov polynomials in (5.17) and (5.18) are composed of the following extremal polynomials:
For the "positive" Kharitonov polynomials define:

$$\begin{aligned} R^+_{\max}(s) &:= y_0 + ju_1 s + x_2 s^2 + jv_3 s^3 + y_4 s^4 + \cdots \\ R^+_{\min}(s) &:= x_0 + jv_1 s + y_2 s^2 + ju_3 s^3 + x_4 s^4 + \cdots \\ I^+_{\max}(s) &:= jv_0 + y_1 s + ju_2 s^2 + x_3 s^3 + jv_4 s^4 + \cdots \\ I^+_{\min}(s) &:= ju_0 + x_1 s + jv_2 s_2 + y_3 s^3 + ju_4 s^4 + \cdots \end{aligned}$$

so that

$$\begin{aligned} K_1^+(s) &= R^+_{\min}(s) + I^+_{\min}(s) \\ K_2^+(s) &= R^+_{\min}(s) + I^+_{\max}(s) \\ K_3^+(s) &= R^+_{\max}(s) + I^+_{\min}(s) \\ K_4^+(s) &= R^+_{\max}(s) + I^+_{\max}(s). \end{aligned}$$

For the "negative" Kharitonov polynomials we have

$$\begin{aligned} R^-_{\max}(s) &= y_0 + jv_1 s + x_2 s^2 + ju_3 s^3 + y_4 s^4 + \cdots \\ R^-_{\min}(s) &= x_0 + ju_1 s + y_2 s^2 + jv_3 s^3 + x_4 s^4 + \cdots \\ I^-_{\max}(s) &= jv_0 + x_1 s + ju_2 s^2 + y_3 s^3 + jv_4 s^4 + \cdots \\ I^-_{\min}(s) &= ju_0 + y_1 s + jv_2 s^2 + x_3 s^3 + ju_4 s^4 + \cdots \end{aligned}$$

and

$$\begin{aligned} K_1^-(s) &= R^-_{\min}(s) + I^-_{\min}(s) \\ K_2^-(s) &= R^-_{\min}(s) + I^-_{\max}(s) \\ K_3^-(s) &= R^-_{\max}(s) + I^-_{\min}(s) \\ K_4^-(s) &= R^-_{\max}(s) + I^-_{\max}(s). \end{aligned}$$

$R^\pm_{\max}(j\omega)$ and $R^\pm_{\min}(j\omega)$ are real and $I^\pm_{\max}(j\omega)$ and $I^\pm_{\min}(j\omega)$ are imaginary. Let $\mathrm{Re}[\delta(j\omega)] := \delta^r(\omega)$ and $\mathrm{Im}[\delta(j\omega)] := \delta^i(\omega)$ denote the real and imaginary parts of $\delta(s)$ evaluated at $s = j\omega$. Then we have:

$$\begin{aligned} \delta^r(\omega) &= \alpha_0 - \beta_1\omega - \alpha_2\omega^2 + \beta_3\omega^3 + \cdots , \\ \delta^i(\omega) &= \beta_0 + \alpha_1\omega - \beta_2\omega^2 - \alpha_3\omega^3 + \cdots . \end{aligned}$$

It is easy to verify that

$$\begin{cases} R^+_{\min}(j\omega) \le \delta^r(\omega) \le R^+_{\max}(j\omega), & \text{for all } \omega \in [0, \infty] \\ \dfrac{I^+_{\min}(j\omega)}{j} \le \delta^i(\omega) \le \dfrac{I^+_{\max}(j\omega)}{j}, & \text{for all } \omega \in [0, \infty] \end{cases} \tag{5.19}$$

$$\begin{cases} R^-_{\min}(j\omega) \le \delta^r(\omega) \le R^-_{\max}(j\omega), & \text{for all } \omega \in [0, -\infty] \\ \dfrac{I^-_{\min}(j\omega)}{j} \le \delta^i(\omega) \le \dfrac{I^-_{\max}(j\omega)}{j}, & \text{for all } \omega \in [0, -\infty] \end{cases} \tag{5.20}$$

The proof of the theorem is now completed as follows. The stability of the 4 positive Kharitonov polynomials guarantees interlacing of the "real tube" (bounded by $R^{+}_{\max}(j\omega)$ and $R^{+}_{\min}(j\omega)$) with the "imaginary tube" (bounded by $I^{+}_{\max}(j\omega)$ and $I^{+}_{\min}(j\omega)$) for $\omega \geq 0$. The relation in (5.19) then guarantees that the real and imaginary parts of an arbitrary polynomial in $\mathcal{I}^*(s)$ are forced to interlace for $\omega \geq 0$. Analogous arguments, using the bounds in (5.20) and the "negative" Kharitonov polynomials forces interlacing for $\omega \leq 0$. Thus by the Hermite-Biehler Theorem for complex polynomials δ(s) is Hurwitz. Since δ(s) was arbitrary, it follows that each and every polynomial in $\mathcal{I}^*(s)$ is Hurwitz. ♣

Remark 5.5. In the complex case the real and imaginary parts of $\delta(j\omega)$ are polynomials in ω and not ω^2, and therefore it is necessary to verify the interlacing of the roots on the entire imaginary axis and not only on its positive part. This is the reason why there are twice as many polynomials to check in the complex case.

5.4 INTERLACING AND IMAGE SET INTERPRETATION

In this section we interpret Kharitonov's Theorem in terms of the interlacing property or Hermite-Biehler Theorem and also in terms of the complex plane image of the set of polynomials $\mathcal{I}(s)$, evaluated at $s = j\omega$ for each $\omega \in [0, \infty]$. In Chapter 1 we have seen that the Hurwitz stability of a single polynomial $\delta(s) = \delta^{\text{even}}(s) + \delta^{\text{odd}}(s)$ is equivalent to the interlacing property of $\delta^e(\omega) = \delta^{\text{even}}(j\omega)$ and $\delta^o(\omega) = \dfrac{\delta^{\text{odd}}(j\omega)}{j\omega}$. In considering the Hurwitz stability of the interval family $\mathcal{I}(s)$ we see that the family is stable if and only if every element satisfies the interlacing property. In view of Kharitonov's Theorem it must therefore be true that verification of the interlacing property for the four Kharitonov polynomials guarantees the interlacing property of every member of the family. This point of view is expressed in the following version of Kharitonov's Theorem. Let $\omega_{e_i}^{\max}(\omega_{e_i}^{\min})$ denote the positive roots of $K^e_{\max}(\omega)(K^e_{\min}(\omega))$ and let $\omega_{o_i}^{\max}(\omega_{o_i}^{\min})$ denote the positive roots of $K^o_{\max}(\omega)(K^o_{\min}(\omega))$.

Theorem 5.3 (Interlacing Statement of Kharitonov's Theorem)
The family $\mathcal{I}(s)$ contains only stable polynomials if and only if

1) The polynomials $K^e_{\max}(\omega)$, $K^e_{\min}(\omega)$, $K^o_{\max}(\omega)$, $K^o_{\min}(\omega)$ have only real roots and the set of positive roots interlace as follows:

$$0 < \omega_{e_1}^{\min} < \omega_{e_1}^{\max} < \omega_{o_1}^{\min} < \omega_{o_1}^{\max} < \omega_{e_2}^{\min} < \omega_{e_2}^{\max} < \omega_{o_2}^{\min} < \omega_{o_2}^{\max} < \cdots,$$

2) $K^e_{\max}(0)$, $K^e_{\min}(0)$, $K^o_{\max}(0)$, $K^o_{\min}(0)$ are non zero and of the same sign.

This theorem is illustrated in Figure 5.6 which shows how the interlacing of the odd and even tubes implies the interlacing of the odd and even parts of each polynomial in the interval family. We illustrate this interlacing property of interval polynomials with an example.

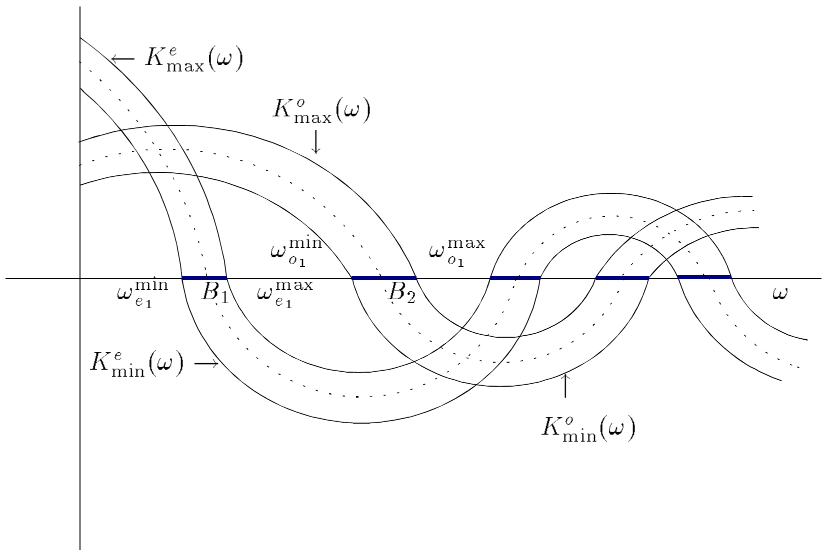

Figure 5.6. Interlacing odd and even tubes

Example 5.4. Consider the interval family

$$\delta(s) = s^7 + \delta_6 s^6 + \delta_5 s^5 + \delta_4 s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0$$

where

$$\begin{array}{lll} \delta_6 \in [9, 9.5], & \delta_5 \in [31, 31.5], & \delta_4 \in [71, 71.5], \\ \delta_3 \in [111, 111.5], & \delta_2 \in [109, 109.5], & \delta_1 \in [76, 76.5], \\ \delta_0 \in [12, 12.5] & & \end{array}$$

Then

$$\begin{aligned} K^e_{\max}(\omega) &= -9\omega^6 + 71.5\omega^4 - 109\omega^2 + 12.5 \\ K^e_{\min}(\omega) &= -9.5\omega^6 + 71\omega^4 - 109.5\omega^2 + 12 \\ K^o_{\max}(\omega) &= -\omega^6 + 31.5\omega^4 - 111\omega^2 + 76.5 \\ K^o_{\min}(\omega) &= -\omega^6 + 31\omega^4 - 111.5\omega^2 + 76. \end{aligned}$$

We can verify the interlacing property of these polynomials (see Figure 5.7).

The illustration in Figure 5.7 shows how all polynomials with even parts bounded by $K^e_{\max}(\omega)$ and $K^e_{\min}(\omega)$ and odd parts bounded by $K^o_{\max}(\omega)$ and $K^o_{\min}(\omega)$ on the imaginary axis satisfy the interlacing property when the Kharitonov polynomials

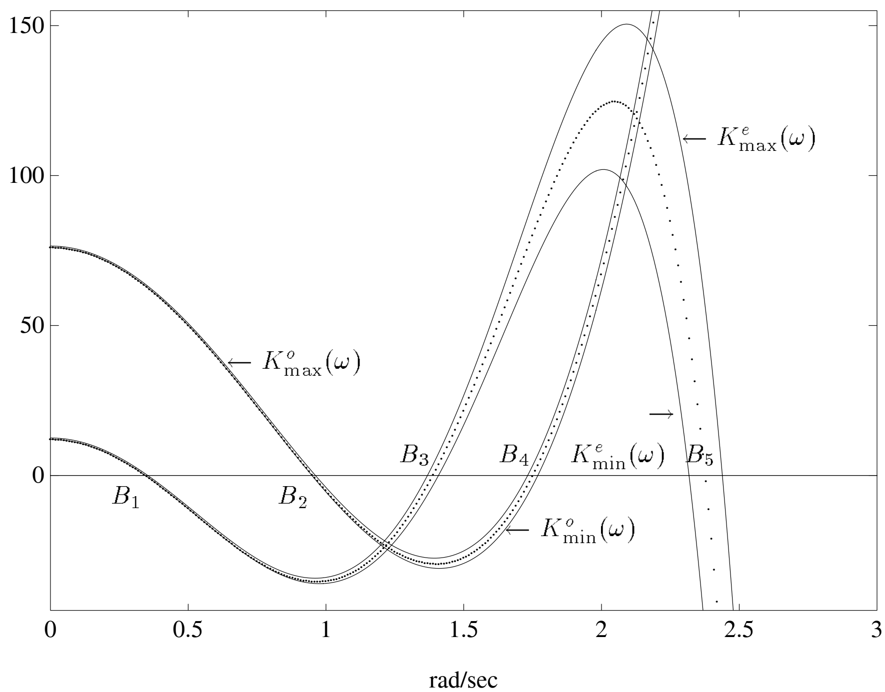

Figure 5.7. Interlacing property of an interval polynomial (Example 5.4)

are stable. Figure 5.7 also shows that the interlacing property for a *single* stable polynomial corresponding to a *point* δ in coefficient space generalizes to the *box* $\mathbf{\Delta}$ of stable polynomials as the requirement of "interlacing" of the odd and even "*tubes*." This interpretation is useful; for instance it can be used to show that for polynomials of order less than six, fewer than four Kharitonov polynomials need to be tested for robust stability (see Exercise 5.4).

Image Set Interpretation

It is instructive to interpret Kharitonov's Theorem in terms of the evolution of the complex plane image of $\mathcal{I}(s)$ evaluated along the imaginary axis. Let $\mathcal{I}(j\omega)$ denote the set of complex numbers $\delta(j\omega)$ obtained by letting the coefficients of $\delta(s)$ range over $\mathbf{\Delta}$:

$$\mathcal{I}(j\omega) := \{\delta(j\omega) : \underline{\delta} \in \mathbf{\Delta}\}.$$

Now it follows from the relations in (5.9) and (5.10) that $\mathcal{I}(j\omega)$ is a rectangle in the complex plane with the corners $K_1(j\omega)$, $K_2(j\omega)$, $K_3(j\omega)$, $K_4(j\omega)$ corresponding to the Kharitonov polynomials evaluated at $s = j\omega$. This is shown in Figure 5.8. As ω runs from 0 to ∞ the rectangle $\mathcal{I}(j\omega)$ varies in shape size and location but its sides always remain parallel to the real and imaginary axes of the complex plane. We illustrate this by using a numerical example.

Example 5.5. Consider the interval polynomial of Example 5.4. The image set of this family is calculated for various frequencies. These frequency dependent rectangles are shown in Figure 5.8.

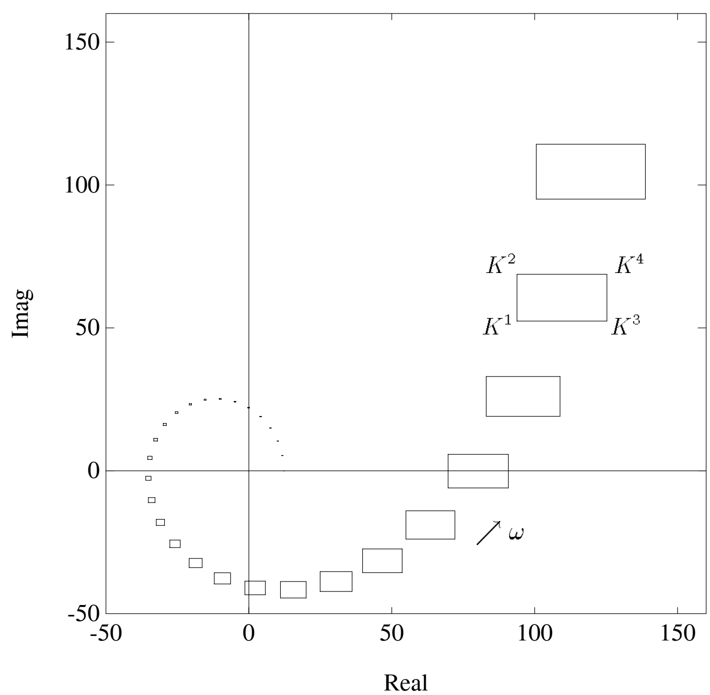

Figure 5.8. Image sets of interval polynomial (Kharitonov boxes) (Example 5.5)

5.4.1 Two Parameter Representation of Interval Polynomials

The observation that $\mathcal{I}(j\omega)$ is an axis parallel rectangle with the $K_i(j\omega)$, $i = 1, 2, 3, 4$ as corners motivates us to introduce a reduced family $\mathcal{I}_R(s) \subset \mathcal{I}(s)$ which generates the image set $\mathcal{I}(j\omega)$ at every ω. Let $\beta_0(s)$ denote the center polynomial of the family $\mathcal{I}(s)$:

$$\beta_0(s) := \frac{x_0 + y_0}{2} + \frac{x_1 + y_1}{2}s + \cdots + \frac{x_n + y_n}{2}s^n$$

and introduce the even and odd polynomials:

$$\beta_e(s) := K_{\max}^{\text{even}}(s) - K_{\min}^{\text{even}}(s) \tag{5.21}$$

$$\beta_o(s) := K_{\max}^{\text{odd}}(s) - K_{\min}^{\text{odd}}(s). \tag{5.22}$$

We define

$$\mathcal{I}_R(s) := \{\beta(s) = \beta_0(s) + \lambda_1\beta_e(s) + \lambda_2\beta_o(s) : |\lambda_i| \leq 1, i = 1, 2\}$$

It is easy to see that $\mathcal{I}_R(s) \subset \mathcal{I}(s)$ but

$$\mathcal{I}(j\omega) = \mathcal{I}_R(j\omega), \qquad \text{for all } \omega \geq 0.$$

This shows that the $n+1$-parameter interval polynomial family $\mathcal{I}(s)$ can *always* be replaced by the two-parameter testing family $\mathcal{I}_R(s)$ as far as any frequency evaluations are concerned since they both generate the same image at each frequency. We emphasize that this kind of parameter reduction based on properties of the image set holds in more general cases, i.e. even when the family under consideration is not interval and the stability region is not the left half plane. Of course in the interval Hurwitz case, Kharitonov's Theorem shows us a further reduction of the testing family to the four vertices $K_i(s)$. We show next how this can be deduced from the behaviour of the image set.

5.4.2 Image Set Based Proof of Kharitonov's Theorem

We give an alternative proof of Kharitonov's Theorem based on analysis of the image set. Suppose that the family $\mathcal{I}(s)$ is of degree n and contains at least one stable polynomial.Then stability of the family $\mathcal{I}(s)$ can be ascertained by verifying that no polynomial in the family has a root on the imaginary axis. This follows immediately from the Boundary Crossing Theorem of Chapter 1. Indeed if some element of $\mathcal{I}(s)$ has an unstable root then there must also exist a frequency ω^* and a polynomial with a root at $s = j\omega^*$. The case $\omega^* = 0$ is ruled out since this would contradict the requirement that $K_{\max}^e(0)$ and $K_{\min}^e(0)$ are of the same sign. Thus it is only necessary to check that the rectangle $\mathcal{I}(j\omega^*)$ excludes the origin of the complex plane for every $\omega^* > 0$. Suppose that the Kharitonov polynomials are stable. By the monotonic phase property of Hurwitz polynomials it follows that the corners $K^1(j\omega)$, $K^2(j\omega)$, $K^3(j\omega)$, $K^4(j\omega)$ of $\mathcal{I}(j\omega)$ start on the positive real axis (say), turn strictly counterclockwise around the origin and do not pass through it as ω runs from 0 to ∞. Now suppose by contradiction that $0 \in \mathcal{I}(j\omega^*)$ for some $\omega^* > 0$. Since $\mathcal{I}(j\omega)$ moves continuously with respect to ω and the origin lies outside of $\mathcal{I}(0)$ it follows that there exists $\omega_0 \leq \omega^*$ for which the origin just begins to enter the set $\mathcal{I}(j\omega_0)$. We now consider this limiting situation in which the origin lies on the boundary of $\mathcal{I}(j\omega_0)$ and is just about to enter this set as ω increases from ω_0. This is depicted in Figure 5.9. The origin can lie on one of the four sides of the image set rectangle, say AB. The reader can easily verify that in each of these cases the entry of the origin implies that the phase angle (argument) of one of the corners, A or B on the side through which the entry takes place, *decreases* with increasing ω at $\omega = \omega_0$. Since the corners correspond to Kharitonov polynomials which are Hurwitz stable, we have a contradiction with the monotonic phase increase property of Hurwitz polynomials.

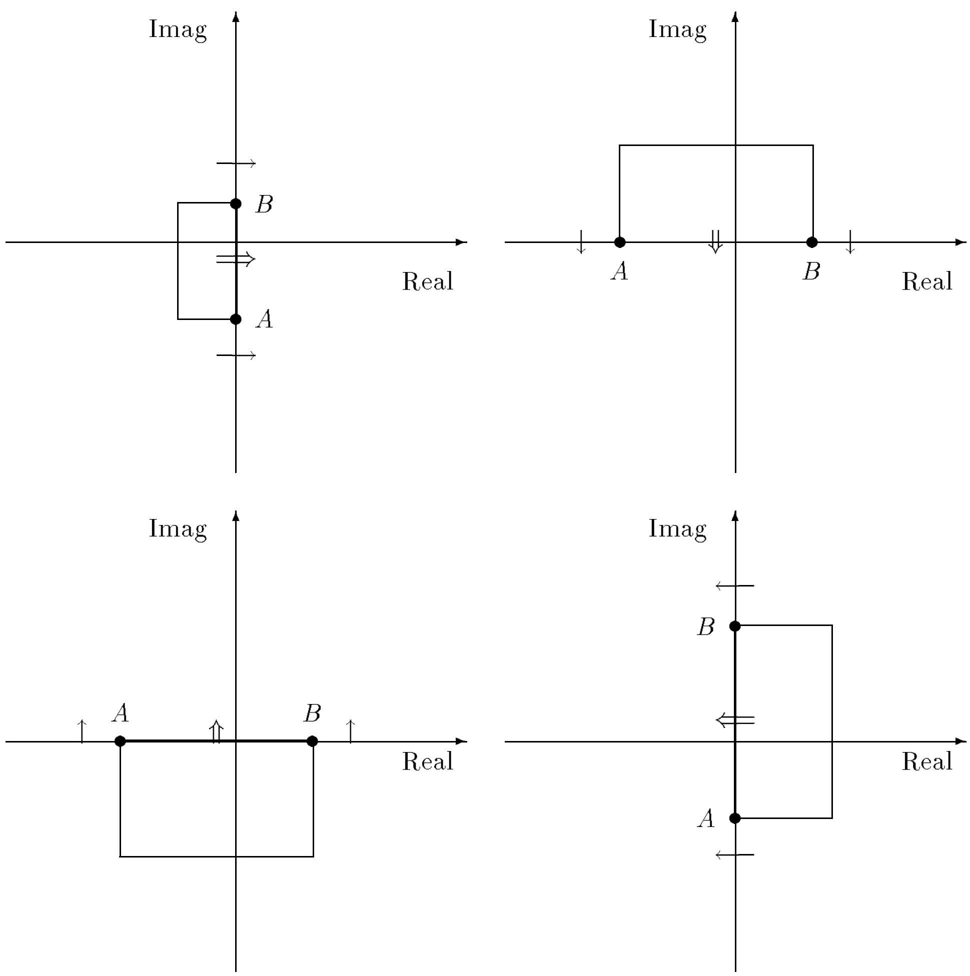

Figure 5.9. Alternative proof of Kharitonov's Theorem

5.4.3 Image Set Edge Generators and Exposed Edges

The interval family $\mathcal{I}(s)$, or equivalently, the coefficient set $\mathbf{\Delta}$, is a polytope and therefore, as discussed in Chapter 4, its stability is equivalent to that of its exposed edges. There are in general $(n+1)2^{n+1}$ such exposed edges. However from the image set arguments given above, it is clear that stability of the family $\mathcal{I}(s)$ is, in fact, also equivalent to that of the *four* polynomial segments $[K_1(s), K_2(s)]$, $[K_1(s), K_3(s)]$, $[K_2(s), K_4(s)]$ and $[K_3(s), K_4(s)]$. This follows from the previous continuity arguments and the fact that these polynomial segments generate the boundary of the image set $\mathcal{I}(j\omega)$ for *each* ω. We now observe that each of the differences $K_2(s)-K_1(s)$,$K_3(s)-K_1(s)$,$K_4(s)-K_2(s)$ and $K_4(s)-K_3(s)$ is either an *even* or an *odd* polynomial. It follows then from the Vertex Lemma of Chapter 2 that these segments are Hurwitz stable if and only if the endpoints $K_1(s)$, $K_2(s)$, $K_3(s)$ and $K_4(s)$ are. These arguments serve as yet another proof of Kharitonov's Theorem. They serve to highlight 1) the important fact that it is only necessary to check stability of that subset of polynomials which generate the boundary of the image set and 2) the role of the Vertex Lemma in reducing the stability tests to that of fixed polynomials.

5.5 EXTREMAL PROPERTIES OF THE KHARITONOV POLYNOMIALS

In this section we derive some useful extremal properties of the Kharitonov polynomials. Suppose that we have proved the stability of the family of polynomials

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n,$$

with coefficients in the box

$$\mathbf{\Delta} = [x_0, y_0] \times [x_1, y_1] \times \cdots \times [x_n, y_n].$$

Each polynomial in the family is stable. A natural question that arises now is the following: What point in $\mathbf{\Delta}$ is closest to instability? The stability margin of this point is in a sense the worst case stability margin of the interval system. It turns out that a precise answer to this question can be given in terms of the parametric stability margin as well as in terms of the gain margins of an associated interval system. We first deal with the extremal parametric stability margin problem.

5.5.1 Extremal Parametric Stability Margin Property

We consider a stable interval polynomial family. It is therefore possible to associate with each polynomial of the family the largest stability ball centered around it. Write

$$\underline{\delta} = [\delta_0, \delta_1, \cdots, \delta_n],$$

and regard $\underline{\delta}$ as a point in $\mathbb{R}^{n+1}$. Let $\|\underline{\delta}\|_p$ denote the p norm in $\mathbb{R}^{n+1}$ and let this be associated with $\delta(s)$. The set of polynomials which are unstable of degree n or of degree less than n is denoted by $\mathcal{U}$. Then the radius of the stability ball centered at δ is

$$\rho(\delta) = \inf_{u \in \mathcal{U}} \|\underline{\delta} - \mathbf{u}\|_p.$$

We thus define a mapping from $\mathbf{\Delta}$ to the set of all positive real numbers:

$$\begin{aligned} \mathbf{\Delta} &\xrightarrow{\rho} \mathcal{R}^+ \backslash \{0\} \\ \delta(s) &\longrightarrow \rho(\delta). \end{aligned}$$

A natural question to ask is the following: Is there a point in $\mathbf{\Delta}$ which is the nearest to instability? Or stated in terms of functions: Has the function ρ a minimum and is there a precise point in $\mathbf{\Delta}$ where it is reached? The answer to that question is given in the following theorem. In the discussion to follow we drop the subscript p from the norm since the result holds for any norm chosen.

Theorem 5.4 (Extremal property of the Kharitonov polynomials)
The function

$$\begin{aligned} \mathbf{\Delta} &\xrightarrow{\rho} \mathcal{R}^+ \backslash \{0\} \\ \delta(s) &\longrightarrow \rho(\delta) \end{aligned}$$

has a minimum which is reached at one of the four Kharitonov polynomials associated with $\mathbf{\Delta}$.

Proof. Let $K^i(s)$, $i = 1, 2, 3, 4$ denote the four Kharitonov polynomials . Consider the four radii associated with these four extreme polynomials, and let us assume for example that

$$\rho(K^1) = \min\left[\rho(K^1), \rho(K^2), \rho(K^3), \rho(K^4)\right]. \tag{5.23}$$

Let us now suppose, by way of contradiction, that some polynomial $\gamma(s)$ in the box is such that

$$\rho(\gamma) < \rho(K^1). \tag{5.24}$$

For convenience we will denote $\rho(\gamma)$ by ρ_γ, and $\rho(K^1)$ by ρ_1.

By definition, there is at least one polynomial situated on the hypersphere $S(\gamma(s), \rho_\gamma)$ which is unstable or of degree less than n. Let $\beta(s)$ be such a polynomial. Since $\beta(s)$ is on $S(\gamma(s), \rho_\gamma)$, there exists $\underline{\alpha} = [\alpha_0, \alpha_1, \cdots, \alpha_n]$ with $\|\underline{\alpha}\| = 1$ such that

$$\beta(s) = \gamma_0 + \alpha_0\rho_\gamma + (\gamma_1 + \alpha_1\rho_\gamma)s + \cdots + (\gamma_n + \alpha_n\rho_\gamma)s^n, \tag{5.25}$$

(α_0, α_1, $\cdots$, α_n can be positive or non positive here.)

But by (5.23), ρ_1 is the smallest of the four extreme radii and by (5.24) ρ_γ is less than ρ_1. As a consequence, the four new polynomials

$$\begin{aligned}
\delta_n^1(s) &= (x_0 - |\alpha_0|\rho_\gamma) + (x_1 - |\alpha_1|\rho_\gamma)s + (y_2 + |\alpha_2|\rho_\gamma)s^2 + (y_3 + |\alpha_3|\rho_\gamma)s^3 + \cdots \\
\delta_n^2(s) &= (x_0 - |\alpha_0|\rho_\gamma) + (y_1 + |\alpha_1|\rho_\gamma)s + (y_2 + |\alpha_2|\rho_\gamma)s^2 + (x_3 - |\alpha_3|\rho_\gamma)s^3 + \cdots \\
\delta_n^3(s) &= (y_0 + |\alpha_0|\rho_\gamma) + (x_1 - |\alpha_1|\rho_\gamma)s + (x_2 - |\alpha_2|\rho_\gamma)s^2 + (y_3 + |\alpha_3|\rho_\gamma)s^3 + \cdots \\
\delta_n^4(s) &= (y_0 + |\alpha_0|\rho_\gamma) + (y_1 + |\alpha_1|\rho_\gamma)s + (x_2 - |\alpha_2|\rho_\gamma)s^2 + (x_3 - |\alpha_3|\rho_\gamma)s^3 + \cdots
\end{aligned}$$

are all stable because

$$\|\delta_n^i - K^i\| = \rho_\gamma < \rho_i, \qquad i = 1, 2, 3, 4.$$

By applying Kharitonov's Theorem, we thus conclude that the new box

$$\mathbf{\Delta}_n = [x_0 - |\alpha_0|\rho_\gamma,\ y_0 + |\alpha_0|\rho_\gamma] \times \cdots \times [x_n - |\alpha_n|\rho_\gamma,\ y_n + |\alpha_n|\rho_\gamma] \tag{5.26}$$

contains only stable polynomials of degree n. The contradiction now clearly follows from the fact that $\beta(s)$ in (5.25) certainly belongs to $\mathbf{\Delta}_n$, and yet it is unstable or of degree less than n, and this proves the theorem. ♣

The above result tells us that, over the entire box, one is closest to instability at one of the Kharitonov corners, say $K^i(s)$. It is clear that if we take the box $\mathbf{\Delta}_n$, constructed in the above proof, (5.26) and replace ρ_γ by $\rho(K^i)$, the resulting box is larger than the original box $\mathbf{\Delta}$. This fact can be used to develop an algorithm that enlarges the stability box to its maximum limit. We leave the details to the reader but give an illustrative example.

Example 5.6. Consider the system given in Example 5.2:

$$G(s) = \frac{\delta_2 s^2 + \delta_1 s + \delta_0}{s^3(\delta_6 s^3 + \delta_5 s^2 + \delta_4 s + \delta_3)}$$

with the coefficients being bounded as follows:

$$\begin{array}{llll} \delta_0 \in [300, 400], & \delta_1 \in [600, 700], & \delta_2 \in [450, 500], & \\ \delta_3 \in [240, 300], & \delta_4 \in [70, 80], & \delta_5 \in [12, 14], & \delta_6 \in [1, 1]. \end{array}$$

We wish to verify if the system is robustly stable, and if it is we would like to calculate the smallest value of the stability radius in the space of $\delta = [\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6]$ as these coefficients range over the uncertainty box.

The characteristic polynomial of the closed loop system is

$$\delta(s) = \delta_6 s^6 + \delta_5 s^5 + \delta_4 s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0.$$

Since all coefficients of the polynomial are perturbing independently, we can apply Kharitonov's Theorem. This gives us the following four polynomials to check:

$$\begin{aligned} K^1(s) &= 300 + 600s + 500s^2 + 300s^3 + 70s^4 + 12s^5 + s^6 \\ K^2(s) &= 300 + 700s + 500s^2 + 240s^3 + 70s^4 + 14s^5 + s^6 \\ K^3(s) &= 400 + 600s + 450s^2 + 300s^3 + 80s^4 + 12s^5 + s^6 \\ K^4(s) &= 400 + 700s + 450s^2 + 240s^3 + 80s^4 + 14s^5 + s^6. \end{aligned}$$

Since all four Kharitonov polynomials are Hurwitz, we proceed to calculate the worst case stability margin.

From the result established above,Theorem 5.4 we know that this occurs at one of the Kharitonov vertices and thus it suffices to determine the stability radius at these four vertices. The stability radius will be determined using a weighted ℓ_p norm. In other words we want to find the largest value of ρ_p such that the closed loop system remains stable for all δ satisfying

$$\left\{ [\delta_0, \cdots, \delta_6] : \left[\sum_{k=0}^{6} \left| \frac{\delta_k - \delta_k^0}{\alpha_k} \right|^p \right]^{\frac{1}{p}} \leq \rho_p(\delta) \right\}$$

where δ_k^0 represent the coefficients corresponding to the center (a Kharitonov vertex). It is assumed that $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6] = [43,\ 33,\ 25,\ 15,\ 5,\ 1.5,\ 1]$. We can compute the stability radius by simply applying the techniques of Chapter 3 (Tsypkin-Polyak Locus, for example) to each of these fixed Kharitonov polynomials and taking the minimum value of the stability margin. We illustrate this calculation using two different norm measures, corresponding to $p = 2$ and $p = \infty$.

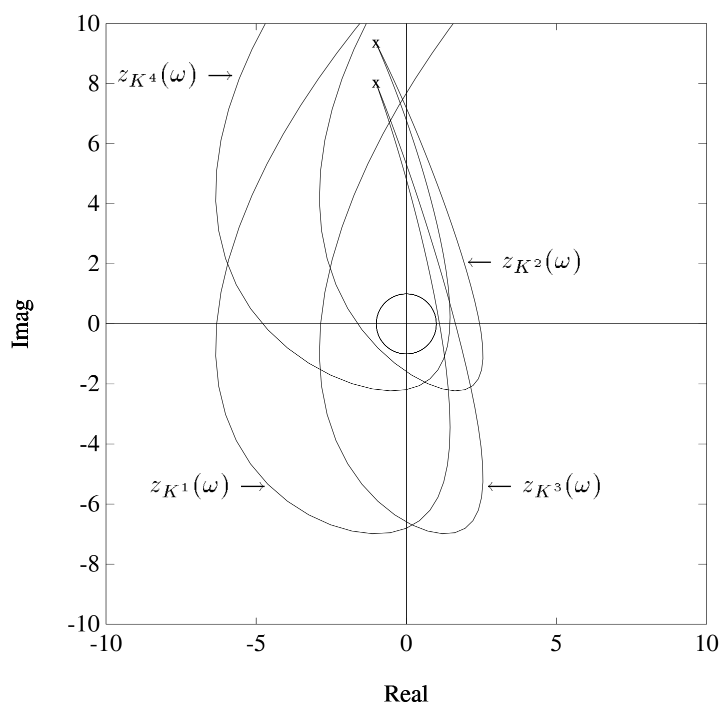

Figure 5.10. ℓ_2 stability margin (Example 5.6)

Figure 5.10 shows the four Tsypkin-Polyak loci corresponding to the four Kharitonov polynomials. The radius of the inscribed circle indicates the minimum weighted ℓ_2 stability margin in coefficient space. Figure 5.11 shows the extremal weighted ℓ_∞ stability margin. From these figures, we have

$$\rho_2(\delta) = 1 \quad \text{and} \quad \rho_\infty(\delta) = 0.4953.$$

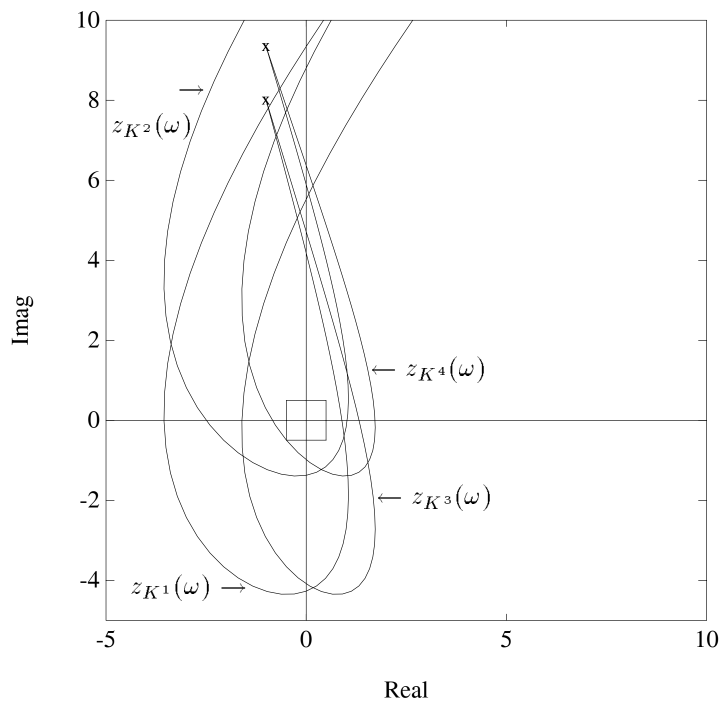

Figure 5.11. ℓ_∞ stability margin (Example 5.6)

5.5.2 Extremal Gain Margin for Interval Systems

Let us now consider the standard unity feedback control system shown below in Figure 5.12. We assume that the system represented by $G(s)$ contains parameter uncertainty. In particular let us assume that $G(s)$ is a proper transfer function which is a ratio of polynomials $n(s)$ and $d(s)$ the coefficients of which vary in independent intervals. Thus the polynomials $n(s)$ and $d(s)$ vary in respective independent interval polynomial families $\mathcal{N}(s)$ and $\mathcal{D}(s)$ respectively. We refer to such a family of

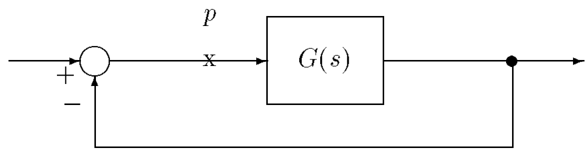

Figure 5.12. A feedback system

systems as an *interval system*. Let

$$\mathbf{G}(s) := \left\{ G(s) = \frac{n(s)}{d(s)} : n(s) \in \mathcal{N}(s), \quad d(s) \in \mathcal{D}(s) \right\}.$$

represent the interval family of systems in which the open loop transfer function lies.

We assume that the closed loop system containing the interval family $\mathbf{G}(s)$ is robustly stable. In other words we assume that the characteristic polynomial of the closed loop system given by

$$\Pi(s) = d(s) + n(s)$$

is of invariant degree n and is Hurwitz for all $(n(s),d(s)) \in (\mathcal{N}(s) \times \mathcal{D}(s))$. Let

$$\mathbf{\Pi}(s) = \{\Pi(s) = n(s) + d(s) : n(s) \in \mathcal{N}(s), \quad d(s) \in \mathcal{D}(s)\}.$$

Then robust stability means that every polynomial in $\mathbf{\Pi}(s)$ is Hurwitz and of degree n. This can in fact be verified constructively. Let $K_N^i(s), i = 1,2,3,4$ and $K_D^j(s), j = 1,2,3,4$ denote the Kharitonov polynomials associated with $\mathcal{N}(s)$ and $\mathcal{D}(s)$ respectively. Now introduce the positive set of *Kharitonov systems* $\mathbf{G}_{\mathrm{K}}^{+}(s)$ associated with the interval family $\mathbf{G}(s)$ as follows:

$$\mathbf{G}_{\mathrm{K}}^{+}(s) := \left\{ \frac{K_N^i(s)}{K_D^i(s)} : i = 1,2,3,4 \right\}.$$

Theorem 5.5 *The closed loop system of Figure 5.12 containing the interval plant* $\mathbf{G}(s)$ *is robustly stable if and only if each of the positive Kharitonov systems in* $\mathbf{G}_{K}^{+}(s)$ *is stable.*

Proof. We need to verify that $\Pi(s)$ remains of degree n and Hurwitz for all $(n(s), d(s)) \in \mathcal{N}(s) \times \mathcal{D}(s)$. It follows from the assumption of independence of the families $\mathcal{N}(s)$ and $\mathcal{D}(s)$ that $\mathbf{\Pi}(s)$ is itself an interval polynomial family. It is easy to check that the Kharitonov polynomials of $\mathbf{\Pi}(s)$ are the four $K_N^i(s) + K_D^i(s), \quad i = 1,2,3,4$. Thus the family $\Pi(s)$ is stable if and only if the subset $K_N^i(s) + K_D^i(s), \quad i = 1,2,3,4$ are all stable and of degree n. The latter in turn is equivalent to the stability of the feedback systems obtained by replacing $G(s)$ by each element of $\mathbf{G}_{\mathrm{K}}^{+}(s)$. ♣

In classical control *gain margin* is commonly regarded as a useful measure of robust stability. Suppose the closed system is stable. The gain margin at the loop breaking point p is defined to be the largest value k^* of $k \geq 1$ for which closed loop stability is preserved with $G(s)$ replaced by $kG(s)$ for all $k \in [1, k^*)$. Suppose now that we have verified the robust stability of the interval family of systems $\mathbf{G}(s)$. The next question that is of interest is: What is the gain margin of the system at the loop breaking point p? To be more precise we need to ask: What is the *worst case* gain margin of the system at the point p as $G(s)$ ranges over $\mathbf{G}(s)$? An exact answer to this question can be given as follows.

Theorem 5.6 *The worst case gain margin of the system at the point p over the family $\mathbf{G}(s)$ is the minimum gain margin corresponding to the positive Kharitonov systems $\mathbf{G}_{\mathrm{K}}^{+}(s)$.*

Proof. Consider the characteristic polynomial $\Pi(s) = d(s) + kn(s)$ corresponding to the open loop system $kG(s)$. For each fixed value of k this is an interval family. For positive k the Kharitonov polynomials of this family are $K_D^i(s) + kK_N^i(s), i = 1, 2, 3, 4$. Therefore the minimum value of the gain margin over the set $\mathbf{G}(s)$ is in fact attained over the subset $\mathbf{G}_{\mathrm{K}}^{+}(s)$. ♣

Remark 5.6. A similar result can be stated for the case of a positive feedback system by introducing the set of negative Kharitonov systems (see Exercise 5.9).

5.6 ROBUST STATE-FEEDBACK STABILIZATION

In this section we give an application of Kharitonov's Theorem to robust stabilization. We consider the following problem: Suppose that you are given a set of n nominal parameters

$$\left\{a_0^0, a_1^0, \cdots, a_{n-1}^0\right\},$$

together with a set of prescribed uncertainty ranges: Δa_0, Δa_1, $\cdots$, Δa_{n-1}, and that you consider the family $\mathcal{I}_{\underline{0}}(s)$ of monic polynomials,

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_{n-1} s^{n-1} + s^n,$$

where

$$\delta_0 \in \left[a_0^0 - \frac{\Delta a_0}{2}, a_0^0 + \frac{\Delta a_0}{2}\right], \cdots, \delta_{n-1} \in \left[a_{n-1}^0 - \frac{\Delta a_{n-1}}{2}, a_{n-1}^0 + \frac{\Delta a_{n-1}}{2}\right].$$

To avoid trivial cases assume that the family $\mathcal{I}_{\underline{0}}(s)$ contains at least one unstable polynomial.

Suppose now that you can use a vector of n free parameters

$$\underline{k} = (k_0, k_1, \cdots, k_{n-1}),$$

to transform the family $\mathcal{I}_{\underline{0}}(s)$ into the family $\mathcal{I}_{\underline{k}}(s)$ described by:

$$\delta(s) = (\delta_0 + k_0) + (\delta_1 + k_1)s + (\delta_2 + k_2)s^2 + \cdots + (\delta_{n-1} + k_{n-1})s^{n-1} + s^n.$$

The problem of interest, then, is the following: Given Δa_0, Δa_1, $\cdots$, Δa_{n-1} the perturbation ranges fixed a priori, find, if possible, a vector $\underline{k}$ so that the new family of polynomials $\mathcal{I}_{\underline{k}}(s)$ is entirely Hurwitz stable. This problem arises, for example, when one applies a state-feedback control to a single input system where the matrices A, b are in controllable companion form and the coefficients of the characteristic polynomial of A are subject to bounded perturbations. The answer to this problem is always affirmative and is precisely given in Theorem 5.7. Before stating it, however, we need to prove the following lemma.

Lemma 5.3 *Let n be a positive integer and let $P(s)$ be a stable polynomial of degree $n-1$:*

$$P(s) = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1}, \qquad \textit{with all} \;\; p_i > 0.$$

Then there exists $\alpha > 0$ such that:

$$Q(s) = P(s) + p_n s^n = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1} + p_n s^n,$$

is stable if and only if: $\quad p_n \in [0, \alpha)$.

Proof. To be absolutely rigorous there should be four different proofs depending on whether n is of the form $4r$ or $4r+1$ or $4r+2$ or $4r+3$. We will give the proof of this lemma when n is of the form $4r$ and one can check that only slight changes are needed if n is of the form $4r+j$, $\quad j = 1, 2, 3$.

If $n = 4r$, $r > 0$, we can write

$$P(s) = p_0 + p_1 s + \cdots + p_{4r-1}s^{4r-1},$$

and the even and odd parts of $P(s)$ are given by:

$$\begin{aligned} P_{\text{even}}(s) &= p_0 + p_2 s^2 + \cdots + p_{4r-2}s^{4r-2}, \\ P_{\text{odd}}(s) &= p_1 s + p_3 s^3 + \cdots + p_{4r-1}s^{4r-1}. \end{aligned}$$

Let us also define

$$\begin{aligned} P^e(\omega) &:= P_{\text{even}}(j\omega) = p_0 - p_2\omega^2 + p_4\omega^4 - \cdots - p_{4r-2}\omega^{4r-2}, \\ P^o(\omega) &:= \frac{P_{\text{odd}}(j\omega)}{j\omega} = p_1 - p_3\omega^2 + p_5\omega^4 - \cdots - p_{4r-1}\omega^{4r-2}. \end{aligned}$$

$P(s)$ being stable, we know by the Hermite-Biehler Theorem that $P^e(\omega)$ has precisely $2r-1$ positive roots $\omega_{e,1}$, $\omega_{e,2}$, $\cdots$, $\omega_{e,2r-1}$, that $P^o(\omega)$ has also $2r-1$ positive roots $\omega_{o,1}$, $\omega_{o,2}$, $\cdots$, $\omega_{o,2r-1}$, and that these roots interlace in the following manner:

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \omega_{o,2} < \cdots < \omega_{e,2r-1} < \omega_{o,2r-1}.$$

It can be also checked that,

$$P^e(\omega_{o,j}) < 0 \text{ if and only if } j \text{ is odd, and } P^e(\omega_{o,j}) > 0 \text{ if and only if } j \text{ is even,}$$

that is,

$$P^e(\omega_{o,1}) < 0,\ P^e(\omega_{o,2}) > 0,\ \cdots,\ P^e(\omega_{o,2r-2}) > 0,\ P^e(\omega_{o,2r-1}) < 0. \tag{5.27}$$

Let us denote

$$\alpha = \min_{j \text{ odd}} \left\{ \frac{-P^e(\omega_{o,j})}{(\omega_{o,j})^{4r}} \right\}. \tag{5.28}$$

By (5.27) we know that α is positive. We can now prove the following:

$$Q(s) = P(s) + p_{4r}s^{4r} \text{ is stable if and only if } \quad p_{4r} \in [0, \alpha).$$

$Q(s)$ is certainly stable when $p_{4r} = 0$. Let us now suppose that

$$0 < p_{4r} < \alpha. \tag{5.29}$$

$Q^o(\omega)$ and $Q^e(\omega)$ are given by

$$Q^o(\omega) = P^o(\omega) = p_1 - p_3\omega^2 + p_5\omega^4 - \cdots - p_{4r-1}\omega^{4r-1},$$
$$Q^e(\omega) = P^e(\omega) + p_{4r}\omega^{4r} = p_0 - p_2\omega^2 + p_4\omega^4 - \cdots - p_{4r-2}\omega^{4r-2} + p_{4r}\omega^{4r}.$$

We are going to show that $Q^e(\omega)$ and $Q^o(\omega)$ satisfy the Hermite-Biehler Theorem provided that p_{4r} remains within the bounds defined by (5.29).
First we know the roots of $Q^o(\omega) = P^o(\omega)$. Then we have that $Q^e(0) = p_0 > 0$, and also

$$Q^e(\omega_{o,1}) = P^e(\omega_{o,1}) + p_{4r}(\omega_{o,1})^{4r}.$$

But, by (5.28) and (5.29) we have

$$Q^e(\omega_{o,1}) < \underbrace{P^e(\omega_{o,1}) - \frac{P^e(\omega_{o,1})}{(\omega_{o,1})^{4r}}(\omega_{o,1})^{4r}}_{=0}.$$

Thus $Q^e(\omega_{o,1}) < 0$. Then we have

$$Q^e(\omega_{o,2}) = P^e(\omega_{o,2}) + p_{4r}(\omega_{o,2})^{4r}.$$

But by (5.27), we know that $P^e(\omega_{o,2}) > 0$, and therefore we also have

$$Q^e(\omega_{o,2}) > 0.$$

Pursuing the same reasoning we could prove in exactly the same way that the following inequalities hold

$$Q^e(0) > 0,\ Q^e(\omega_{o,1}) < 0,\ Q^e(\omega_{o,2}) > 0,\ \cdots, Q^e(\omega_{o,2r-2}) > 0,\ Q^e(\omega_{o,2r-1}) < 0. \tag{5.30}$$

From this we conclude that $Q^e(\omega)$ has precisely $2r-1$ roots in the open interval $(0, \omega_{o,2r-1})$, namely

$$\omega'_{e,1},\ \omega'_{e,2},\ \cdots,\ \omega'_{e,2r-1},$$

and that these roots interlace with the roots of $Q^o(\omega)$,

$$0 < \omega'_{e,1} < \omega_{o,1} < \omega'_{e,2} < \omega_{o,2} < \cdots < \omega'_{e,2r-1} < \omega_{o,2r-1}. \tag{5.31}$$

Moreover, we see in (5.30) that

$$Q^e(\omega_{o,2r-1}) < 0,$$

and since $p_{4r} > 0$, we also obviously have

$$Q^e(+\infty) > 0.$$

Therefore $Q^e(\omega)$ has a final positive root $\omega'_{e,2r}$ which satisfies

$$\omega_{o,2r-1} < \omega'_{e,2r}. \tag{5.32}$$

From (5.31) and (5.32) we conclude that $Q^o(\omega)$ and $Q^e(\omega)$ satisfy the Hermite-Biehler Theorem and therefore $Q(s)$ is stable.

To complete the proof of this lemma, notice that $Q(s)$ is obviously unstable if $p_{4r} < 0$ since we have assumed that all the p_i are positive. Moreover it can be shown that for $p_{4r} = \alpha$, the polynomial $P(s) + \alpha s^{4r}$ has a pure imaginary root and therefore is unstable. Now, it is impossible that $P(s) + p_{4r}s^{4r}$ be stable for some $p_{4r} > \alpha$, because otherwise we could use Kharitonov's Theorem and say,

$$P(s) + \frac{\alpha}{2}s^{4r} \text{ and } P(s) + p_{4r}s^{4r} \text{ both stable } \Longrightarrow P(s) + \alpha s^{4r} \text{ stable },$$

which would be a contradiction. This completes the proof of the theorem when $n = 4r$.

For the sake of completeness, let us just make precise that in general we have,

$$\begin{aligned}
\text{if } n = 4r, &\qquad \alpha = \min_{j \text{ odd}} \left\{ \frac{-P^e(\omega_{o,j})}{(\omega_{o,j})^{4r}} \right\}, \\
\text{if } n = 4r+1, &\qquad \alpha = \min_{j \text{ even}} \left\{ \frac{-P^o(\omega_{e,j})}{(\omega_{o,j})^{4r+1}} \right\}, \\
\text{if } n = 4r+2, &\qquad \alpha = \min_{j \text{ even}} \left\{ \frac{P^e(\omega_{o,j})}{(\omega_{o,j})^{4r+2}} \right\}, \\
\text{if } n = 4r+3, &\qquad \alpha = \min_{j \text{ odd}} \left\{ \frac{P^o(\omega_{e,j})}{(\omega_{o,j})^{4r+3}} \right\}.
\end{aligned}$$

The details of the proof for the other cases are omitted. ♣

We can now enunciate the following theorem to answer the question raised at the beginning of this section.

Theorem 5.7 *For any set of nominal parameters* $\{a_0, a_1, \cdots, a_{n-1}\}$*, and for any set of positive numbers* Δa_0*,* Δa_1*,* $\cdots$*,* Δa_{n-1}*, it is possible to find a vector* $\underline{k}$ *such that the entire family* $\mathcal{I}_{\underline{k}}(s)$ *is stable.*

Proof. The proof is constructive.

Step 1: Take any stable polynomial $R(s)$ of degree $n-1$. Let $\rho(R(\cdot))$ be the radius of the largest stability hypersphere around $R(s)$. It can be checked from the formulas of Chapter 3, that for any positive real number λ, we have

$$\rho(\lambda R(\cdot)) = \lambda\rho(R(\cdot)).$$

Thus it is possible to find a positive real λ such that if $P(s) = \lambda R(s)$,

$$\rho(P(\cdot)) > \sqrt{\frac{\Delta a_0{}^2}{4} + \frac{\Delta a_1{}^2}{4} + \cdots + \frac{\Delta a_{n-1}{}^2}{4}}. \tag{5.33}$$

Denote

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_{n-1}s^{n-1},$$

and consider the four following Kharitonov polynomials of degree $n-1$:

$$\begin{aligned}
P^1(s) &= \left(p_0 - \frac{\Delta a_0}{2}\right) + \left(p_1 - \frac{\Delta a_1}{2}\right)s + \left(p_2 + \frac{\Delta a_2}{2}\right)s^2 + \cdots,\\
P^2(s) &= \left(p_0 - \frac{\Delta a_0}{2}\right) + \left(p_1 + \frac{\Delta a_1}{2}\right)s + \left(p_2 + \frac{\Delta a_2}{2}\right)s^2 + \cdots,\\
P^3(s) &= \left(p_0 + \frac{\Delta a_0}{2}\right) + \left(p_1 - \frac{\Delta a_1}{2}\right)s + \left(p_2 - \frac{\Delta a_2}{2}\right)s^2 + \cdots,\\
P^4(s) &= \left(p_0 + \frac{\Delta a_0}{2}\right) + \left(p_1 + \frac{\Delta a_1}{2}\right)s + \left(p_2 - \frac{\Delta a_2}{2}\right)s^2 + \cdots.
\end{aligned} \tag{5.34}$$

We conclude that these four polynomials are stable since $\|P^i(s) - P(s)\| < \rho(P(s))$.

Step 2: Now, applying Lemma 5.3, we know that we can find four positive numbers α_1, α_2, α_3, α_4, such that

$$P^j(s) + p_n s^n \text{ is stable for } 0 \le p_n < \alpha_j, \quad j = 1, 2, 3, 4.$$

Let us select a single positive number α such that the polynomials,

$$P^j(s) + \alpha s^n \tag{5.35}$$

are all stable. If α can be chosen to be equal to 1 (that is if the four α_j are greater than 1) then we do choose $\alpha = 1$; otherwise we multiply everything by $\frac{1}{\alpha}$ which is greater than 1 and we know from (5.35) that the four polynomials

$$K^j(s) = \frac{1}{\alpha}P^j(s) + s^n,$$

are stable. But the four polynomials $K^j(s)$ are nothing but the four Kharitonov polynomials associated with the family of polynomials

$$\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_{n-1}s^{n-1} + s^n,$$

where

$$\delta_0 \in \left[\frac{1}{\alpha}p_0 - \frac{1}{\alpha}\frac{\Delta a_0}{2},\ \frac{1}{\alpha}p_0 + \frac{1}{\alpha}\frac{\Delta a_0}{2}\right], \cdots$$

$$\cdots, \delta_{n-1} \in \left[\frac{1}{\alpha}p_{n-1} - \frac{1}{\alpha}\frac{\Delta a_{n-1}}{2},\ \frac{1}{\alpha}p_{n-1} + \frac{1}{\alpha}\frac{\Delta a_{n-1}}{2}\right],$$

and therefore this family is entirely stable.

Step 3: It suffices now to chose the vector $\underline{k}$ such that

$$k_i + a_i^0 = \frac{1}{\alpha}p_i, \quad \text{for } i = 1, \cdots, n-1.$$

♣

Remark 5.7. It is clear that in step 1 one can determine the largest box around $R(\cdot)$ with sides proportional to Δa_i. The dimensions of such a box are also enlarged by the factor λ when $R(\cdot)$ is replaced by $\lambda R(\cdot)$. This change does not affect the remaining steps of the proof.

Example 5.7. Suppose that our nominal polynomial is

$$s^6 - s^5 + 2s^4 - 3s^3 + 2s^2 + s + 1,$$

that is

$$(a_0^0, a_1^0, a_2^0, a_3^0, a_4^0, a_5^0) = (1, 1, 2, -3, 2, -1).$$

And suppose that we want to handle the following set of uncertainty ranges:

$$\Delta a_0 = 3, \quad \Delta a_1 = 5, \quad \Delta a_2 = 2, \quad \Delta a_3 = 1, \quad \Delta a_4 = 7, \quad \Delta a_5 = 5.$$

Step 1: Consider the following stable polynomial of degree 5

$$R(s) = (s+1)^5 = 1 + 5s + 10s^2 + 10s^3 + 5s^4 + s^5.$$

The calculation of $\rho(R(\cdot))$ gives: $\rho(R(\cdot)) = 1$. On the other hand we have

$$\sqrt{\frac{\Delta a_0{}^2}{4} + \frac{\Delta a_1{}^2}{4} + \cdots + \frac{\Delta a_{n-1}{}^2}{4}} = 5.31.$$

Taking therefore $\lambda = 6$, we have that

$$P(s) = 6 + 30s + 60s^2 + 60s^3 + 30s^4 + 6s^5,$$

has a radius $\rho(P(\cdot)) = 6$ that is greater than 5.31. The four polynomials $P^j(s)$ are given by

$$\begin{aligned}
P^1(s) &= 4.5 + 27.5s + 61s^2 + 60.5s^3 + 26.5s^4 + 3.5s^5, \\
P^2(s) &= 4.5 + 32.5s + 61s^2 + 59.5s^3 + 26.5s^4 + 8.5s^5, \\
P^3(s) &= 7.5 + 27.5s + 59s^2 + 60.5s^3 + 33.5s^4 + 3.5s^5, \\
P^4(s) &= 7.5 + 32.5s + 59s^2 + 59.5s^3 + 33.5s^4 + 8.5s^5.
\end{aligned}$$

Step 2: The application of Lemma 5.3 gives the following values

$$\alpha_1 \simeq 1.360, \quad \alpha_2 \simeq 2.667, \quad \alpha_3 \simeq 1.784, \quad \alpha_4 \simeq 3.821,$$

and therefore we can chose $\alpha = 1$, so that the four polynomials

$$\begin{aligned}
K^1(s) &= 4.5 + 27.5s + 61s^2 + 60.5s^3 + 26.5s^4 + 3.5s^5 + s^6, \\
K^2(s) &= 4.5 + 32.5s + 61s^2 + 59.5s^3 + 26.5s^4 + 8.5s^5 + s^6, \\
K^3(s) &= 7.5 + 27.5s + 59s^2 + 60.5s^3 + 33.5s^4 + 3.5s^5 + s^6, \\
K^4(s) &= 7.5 + 32.5s + 59s^2 + 59.5s^3 + 33.5s^4 + 8.5s^5 + s^6,
\end{aligned}$$

are stable.

Step 3: We just have to take

$$k_0 = p_0 - a_0 = 5, \quad k_1 = p_1 - a_1 = 29, \quad k_2 = p_2 - a_2 = 58,$$

$$k_3 = p_3 - a_3 = 63, \quad k_4 = p_4 - a_4 = 28, \quad k_5 = p_5 - a_5 = 7.$$

5.7 POLYNOMIAL FUNCTIONS OF INTERVAL POLYNOMIALS

In this section we extend the family of polynomials to which Kharitonov's Theorem applies. Consider the real interval polynomial

$$\mathcal{I}(s) = \left\{a(s) : a_j \in [a_j^-, a_j^+], \quad j = 0, 1, 2, \cdots, n\right\}, \tag{5.36}$$

and a given polynomial

$$\varphi(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m. \tag{5.37}$$

We generate the polynomial family

$$\varphi(\mathcal{I}(s)) = \{\varphi(a(s)) : a(s) \in \mathcal{I}(s)\} \tag{5.38}$$

which consists of polynomials of the form

$$\alpha_0(s) + \alpha_1 a(s) + \alpha_2 a^2(s) + \cdots + \alpha_m a^m(s) \tag{5.39}$$

where $a(s) \in \mathcal{I}(s)$. In the following we answer the question: Under what conditions is the family $\varphi(\mathcal{I}(s))$ Hurwitz stable?

We note that the family $\varphi(\mathcal{I}(s))$ is in general neither interval nor polytopic. Moreover, even though the image set $\mathcal{I}(j\omega)$ is a rectangle the image set of $\varphi(\mathcal{I}(j\omega))$ is, in general a very complicated set. Therefore it is not expected that a vertex testing set or an edge testing will be available. However the result given below shows that once again it suffices to test the stability of four polynomials.

Let

$$K^1(s), K^2(s), K^3(s), K^4(s)$$

denote the Kharitonov polynomials associated with the family $\mathcal{I}(s)$. We begin with a preliminary observation.

Lemma 5.4 *Given the real interval polynomial $\mathcal{I}(s)$ and a complex number z the Hurwitz stability of the family*

$$\mathcal{I}(s) - z = \{a(s) - z : a(s) \in \mathcal{I}(s)\}$$

is equivalent to the Hurwitz stability of the polynomials $K^j(s) - z$, $j = 1, 2, 3, 4$.

The proof of this lemma follows easily from analysis of the image set at $s = j\omega$ of $\mathcal{I}(s) - z$ and is omitted.

Stability Domains

For a given domain Γ of the complex plane let us say that a polynomial is Γ-stable if all its roots lie in the domain Γ.

Consider a polynomial $a_k(s)$ of degree n. The plot $z = a_k(j\omega)$, $\omega \in (-\infty, +\infty)$, partitions the complex plane into a finite number of open disjoint domains. With each such domain Λ we associate the integer n_Λ, the number of the roots of the polynomial $a_k(s) - z$ in the left half plane, where z is taken from Λ. This number is independent of the particular choice of z in the domain and there is at most one domain Λ_k for which $n_\Lambda = n$. Let us associate this domain Λ_k with $a_k(s)$ and call it the *stability domain*. If it so happens that there is no domain for which $n_\Lambda = n$ we set $\Lambda_k = \emptyset$.

Define for each polynomial $K^j(s)$, the associated stability domain Λ_j, $j = 1, 2, 3, 4$ and let

$$\Lambda_0 := \cap_{k=1}^4 \Lambda_k.$$

Theorem 5.8 *Suppose that*

$$\Lambda_0 \neq \emptyset.$$

The polynomial family $\varphi(\mathcal{I}(s))$ is Hurwitz stable if and only if $\varphi(z)$ is Λ_0-stable, that is, all the roots of $\varphi(z)$ lie in the domain Λ_0.

Proof. Let $z_1, z_2, \cdots, z_m$ be the roots of (5.37). Then

$$\varphi(z) = \alpha_m(z - z_1)(z - z_2)\cdots(z - z_m)$$

and

$$\varphi(a(s)) = \alpha_m (a(s) - z_1)(a(s) - z_2)\cdots(a(s) - z_m).$$

The Hurwitz stability of $\varphi(a(s))$ is equivalent to Hurwitz stability of the factors $a(s) - z_j$. This shows that Hurwitz stability of $\varphi(\mathcal{I}(s))$ is equivalent to Hurwitz stability of the interval polynomials $\mathcal{I}(s) - z_j$, $j = 1, 2, \cdots, m$. By Lemma 5.4 the family $\mathcal{I}(s) - z_j$ is Hurwitz stable if and only if $z_j \in \Lambda_0$. As a result the Hurwitz stability of $\varphi(\mathcal{I}(s))$ is equivalent to Λ_0-stability of $\varphi(z)$. ♣

This leads to the following useful result.

Theorem 5.9 *The polynomial family $\varphi(\mathcal{I}(s))$ is Hurwitz stable if and only if the four polynomials $\varphi\left(K^j(s)\right)$, $j = 1, 2, 3, 4$ are Hurwitz stable.*

Proof. Necessity is obvious because these four polynomials are members of $\varphi(\mathcal{I}(s))$. For sufficiency, we know that Hurwitz stability of $\varphi\left(K^j(s)\right)$ implies the Λ_j-stability of $\varphi(z)$. This means that $\varphi(z)$ is Λ_0-stable and hence, by the previous Theorem 5.8, $\varphi(\mathcal{I}(s))$ is Hurwitz stable. ♣

The above theorems describe two different methods of checking the stability of (5.38). In Theorem 5.8 one first has to construct the domains Λ_k and then check the Λ_0-stability of (5.37). In Theorem 5.9, on the other hand, one can directly check the Hurwitz stability of the four fixed polynomials $\varphi\left(K^j(s)\right)$.

Thus far we have considered the polynomial $\varphi(z)$ to be fixed. Suppose now that $\varphi(z)$ is an uncertain polynomial, and in particular belongs to a polytope

$$\Phi(z) = \{\varphi(z) : (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbf{\Delta}\} \tag{5.40}$$

where $\mathbf{\Delta}$ is a convex polytope. We ask the question: Given an interval family (5.36) and a polytopic family (5.40) determine under what conditions is the polynomial family

$$\Phi(\mathcal{I}(s)) = \{\varphi(a(s)) : a(s) \in \mathcal{I}(s), \quad \varphi \in \Phi\} \tag{5.41}$$

Hurwitz stable?

We assume all polynomials in $\mathcal{I}$ have the same degree n and all polynomials in Φ have the same degree m.

Theorem 5.10 *Let $\Lambda_0 \neq \emptyset$. Then the family $\Phi(\mathcal{I}(s))$ is stable if and only if $\Phi(z)$ is Λ_0-stable.*

Proof. The family $\Phi(\mathcal{I}(s))$ is of the form

$$\Phi(\mathcal{I}(s)) = \{\varphi(\mathcal{I}(s)) : \varphi \in \Phi(z)\}.$$

By Theorem 5.8, the family $\varphi(\mathcal{I}(s))$ is Hurwitz stable if and only if $\varphi(z)$ is Λ_0-stable. ♣

From Theorem 5.9 applied to the above we have the following result.

Theorem 5.11 *The family $\Phi(\mathcal{I}(s))$ is Hurwitz stable if and only if the four families*

$$\Phi\left(K^j(s)\right) = \left\{\varphi\left(K^j(s)\right) : \varphi \in \Phi\right\}, \qquad j = 1, 2, 3, 4 \tag{5.42}$$

are Hurwitz stable.

Each of the families (5.42) has a polytopic structure and we can test for stability by testing the exposed edges. Let $\mathcal{E}_\Phi(z)$ denote the exposed edges of $\Phi(z)$. Each exposed edge is a one parameter family of the form

$$(1-\mu)\varphi_1(z) + \mu\varphi_2(z), \qquad \mu \in [0, 1]. \tag{5.43}$$

It generates the corresponding family

$$(1-\mu)\varphi_1\left(K^j(s)\right) + \mu\varphi_2\left(K^j(s)\right), \qquad \mu \in [0, 1] \tag{5.44}$$

which is an element of the exposed edge of $\Phi\left(K^j(s)\right)$. Collecting all such families (5.44) corresponding to the exposed edges of $\mathbf{\Delta}$, we have the following result.

Theorem 5.12 *The family $\Phi(\mathcal{I}(s))$ is stable if and only if the following four collections of one parameter families*

$$\mathcal{E}_{\Phi(K^j)} = \left\{\varphi\left(K^j(s)\right) : \varphi \in \mathcal{E}_\Phi\right\}, \qquad j = 1, 2, 3, 4$$

are stable.

This result is constructive as it reduces the test for robust stability to a set of one-parameter problems. We remark here that the uncertain parameters occurring in this problem appear both linearly (those from $\mathbf{\Delta}$) as well as *nonlinearly* (those from $\mathcal{I}(s)$).

5.8 SCHUR STABILITY OF INTERVAL POLYNOMIALS

We emphasize that Kharitonov's Theorem holds for Hurwitz stability, but in general does not apply to arbitrary regions. The following examples illustrates this fact for the case of Schur stability.

Example 5.8. The interval polynomial

$$\delta(z, p) = z^4 + pz^3 + \frac{3}{2}z^2 - \frac{1}{3}, \qquad p \in \left[-\frac{17}{8}, +\frac{17}{8}\right]$$

has the endpoints $\delta(z, -\frac{17}{8})$ and $\delta(z, \frac{17}{8})$ Schur stable but the midpoint $\delta(z, 0)$ is not Schur stable.

Example 5.9. Consider the interval polynomial

$$\delta(z) := z^4 + \delta_3 z^3 + \delta_2 z^2 + \delta_1 z - \frac{1}{2}$$

with

$$\delta_3 \in [-1, 0], \quad \delta_2 \in \left[\frac{109}{289}, \frac{109}{287}\right], \quad \delta_1 \in \left[\frac{49}{100}, \frac{51}{100}\right].$$

The four Kharitonov polynomials associated with $\delta(z)$

$$\begin{aligned}
K^1(z) &= -\frac{1}{2} + \frac{49}{100}z + \frac{109}{287}z^2 + z^4 \\
K^2(z) &= -\frac{1}{2} + \frac{51}{100}z + \frac{109}{287}z^2 - z^3 + z^4 \\
K^3(z) &= -\frac{1}{2} + \frac{49}{100}z + \frac{109}{289}z^2 + z^4 \\
K^4(z) &= -\frac{1}{2} + \frac{51}{100}z + \frac{109}{289}z^2 - z^3 + z^4
\end{aligned}$$

are all Schur stable. Furthermore, the rest of the vertex polynomials

$$\begin{aligned}
\hat{K}^1(z) &= -\frac{1}{2} + \frac{49}{100}z + \frac{109}{289}z^2 - z^3 + z^4 \\
\hat{K}^2(z) &= -\frac{1}{2} + \frac{49}{100}z + \frac{109}{287}z^2 - z^3 + z^4 \\
\hat{K}^3(z) &= -\frac{1}{2} + \frac{51}{100}z + \frac{109}{289}z^2 + z^4 \\
\hat{K}^4(z) &= -\frac{1}{2} + \frac{51}{100}z + \frac{109}{287}z^2 + z^4
\end{aligned}$$

are also all Schur stable. However, the polynomial

$$\hat{\delta}(z) = -\frac{1}{2} + \frac{1}{2}z + \frac{109}{288}z^2 - \frac{1}{4}z^3 + z^4$$

in the family has two roots at $0.25 \pm j0.9694$ which are not inside the unit circle. This shows that neither the stability of the Kharitonov polynomials nor even the stability of all the vertex polynomials guarantees the Schur stability of the entire family.

Schur stability of a polynomial is equivalent to the interlacing of the symmetric and antisymmetric parts of the polynomial evaluated along the unit circle. It is clear from the above example that interlacing of the vertex polynomials cannot guarantee the interlacing of the symmetric and antisymmetric parts of every member of the interval family. In the case of Hurwitz stability, the interlacing of the odd and even parts of the four Kharitonov polynomials in fact guarantees the interlacing along the $j\omega$ axis of every polynomial in the family.

In view of the above facts let us see what we *can* say about the Schur stability of an interval family of real polynomials. Let $\mathcal{I}(z)$ be the family of polynomials of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with coefficients belonging to a box $\mathbf{A}$:

$$\mathbf{A} := \{\underline{a} := (a_0, \cdots, a_n) \mid a_i \in [a_i^-, a_i^+], \quad i = 0, \cdots, n\}. \tag{5.45}$$

Introduce the vertices $\mathbf{V}$ and edges $\mathbf{E}$ of the box $\mathbf{A}$:

$$\mathbf{V} := \{(a_n, \cdots, a_0): \ a_i = a_i^- \text{ or } a_i^+, \quad i = 0, \cdots, n\}. \tag{5.46}$$

and

$$\begin{aligned} \mathbf{E}_k := \{(a_n, \cdots, a_0): \quad & a_i = a_i^- \text{ or } a_i^+, \quad i = 0, \cdots, n, \quad i \neq k \\ & a_k \in [a_k^-, a_k^+]\} \end{aligned} \tag{5.47}$$

and

$$\mathbf{E} = \cup_{k=0}^{n} \mathbf{E}_k. \tag{5.48}$$

The corresponding family of vertex and edge polynomials are defined by

$$\mathcal{I}_{\mathrm{V}}(z) := \{P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 : (a_n, \cdots, a_0) \in \mathbf{V}\} \tag{5.49}$$

$$\mathcal{I}_{\mathrm{E}}(z) := \{P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 : (a_n, \cdots, a_0) \in \mathbf{E}\}. \tag{5.50}$$

It is obvious that the interval family is a polytopic family and therefore stability of the family can be determined by that of the exposed edges. We state this preliminary result below.

Theorem 5.13 *Assume that the family of polynomials $\mathcal{I}(z)$ has constant degree. Then $\mathcal{I}(z)$ is Schur stable if and only if $\mathcal{I}_{\mathrm{E}}(z)$ is Schur stable.*

The proof is omitted as it follows from the image set arguments given in Chapter 4. The above result in fact holds for any stability region. It turns out that when we specifically deal with Schur stability, the number of edges to be tested for stability can be reduced and this is the result we present next.

In the rest of this section, *stable* will mean *Schur stable.* The first lemma given establishes a vertex result for an interval family with fixed upper order coefficients.

Lemma 5.5 *Let $n > 1$ and assume that in the family $\mathcal{I}(z)$ we have fixed upper order coefficients, namely that $a_i^- = a_i^+$ for $i = \frac{n}{2} + 1, \cdots, n$ if n is even, and $i = \frac{(n+1)}{2} + 1, \cdots, n$ if n is odd. Then the entire family $\mathcal{I}(z)$ is stable if and only if the family of vertex polynomials $\mathcal{I}_{\mathrm{V}}(z)$ is stable.*

Proof. If the entire family is stable it is obviously necessary that the vertex polynomials must be stable. Therefore we proceed to prove sufficiency of the condition. We know that the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

is Schur stable if and only if the polynomial

$$F(s) := (s-1)^n P\left(\frac{s+1}{s-1}\right)$$

is Hurwitz stable. Now

$$\begin{aligned} F(s) := {} & a_0(s-1)^n + a_1(s-1)^{n-1}(s+1) + a_2(s-1)^{n-2}(s+1)^2 \\ & + \cdots + a_{\frac{n}{2}}(s-1)^{\frac{n}{2}}(s+1)^{\frac{n}{2}} + a_{\frac{n}{2}+1}(s-1)^{\frac{n}{2}-1}(s+1)^{\frac{n}{2}+1} \\ & + \cdots + a_n(s+1)^n. \end{aligned} \quad (5.51)$$

Let $\mathcal{F}(s)$ be the family of polynomials of the form (5.51) with the parameter vector $\underline{a}$ ranging over $\mathbf{A}$:

$$\mathcal{F}(s) := \left\{ F(s) : a_i \in [a_i^-, a_i^+], \quad i \in (0, 1, \cdots, n) \right\}.$$

Since $\mathcal{F}(s)$ is a polytopic family, by the Edge Theorem, it is stable if and only if its exposed edges are. These edges correspond to the edges of $\mathbf{A}$, that is, to letting each a_i vary at a time. We give the detailed proof for the case in which n is even. Consider the set of *lower edges* obtained by letting a_k for some $k \in \left\{0, 1, \cdots, \frac{n}{2}\right\}$ vary in $[a_k^-, a_k^+]$ and fixing a_i, $i \in \{0, 1, \cdots, n\}$, $i \neq k$ at a vertex. With a view towards applying the Vertex Lemma (Chapter 2, Lemma 2.18) we determine the difference of the endpoints of these edge polynomials. These differences are of the form

$$\left[a_0^+ - a_0^-\right](s-1)^n, \left[a_1^+ - a_1^-\right](s-1)^{n-1}(s+1), \cdots, \left[a_{\frac{n}{2}}^+ - a_{\frac{n}{2}}^-\right](s-1)^{\frac{n}{2}}(s+1)^{\frac{n}{2}}.$$

From the Vertex Lemma we know that whenever these differences are of the form $A(s)s^t(as+b)Q(s)$ with $A(s)$ antiHurwitz and $Q(s)$ even or odd, edge (segment) stability is guaranteed by that of the vertices. We see that the difference polynomials above are precisely of this form. Therefore stability of the vertex polynomials suffices to establish the stability of the entire family. An identical proof works for the case n odd with the only difference being that the lower edges are defined with $k \in \left\{0, 1, \cdots, \frac{n}{2}+1\right\}$. ♣

In the above result, we considered a special interval family where the upper order coefficients were fixed. Now let us consider an arbitrary interval family where all the coefficients are allowed to vary. For convenience let n_u be defined as follows:

$$\text{for n even} : n_u = \left\{\frac{n}{2}+1, \frac{n}{2}+2, \cdots, n\right\}$$
$$\text{for n odd} : n_u = \left\{\frac{n+1}{2}+1, \frac{n+1}{2}+2, \cdots, n\right\}.$$

We refer to the coefficients a_k, $k \in n_u$ of the polynomial $P(z)$ as the upper coefficients. Also introduce a subset $\mathbf{E}^*$ of the edges $\mathbf{E}$ which we call *upper edges*. These edges are obtained by letting only one upper coefficient *at a time* vary within its interval bounds while all other coefficients are fixed at their upper or lower limits. The corresponding family of polynomials denoted $\mathcal{I}_{\mathrm{E}^*}(z)$ is

$$\mathcal{I}_{\mathrm{E}^*}(z) := \{P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 : (a_n, \cdots, a_0) \in \mathbf{E}^*\} \qquad (5.52)$$

A typical upper edge in $\mathcal{I}_{\mathrm{E}^*}(z)$ is defined by:

$$a_n z^n + \cdots + (\lambda a_k^- + (1-\lambda) a_k^+) z^k + \cdots + a_0 \qquad (5.53)$$

$$k \in n_u, \quad a_i = a_i^- \text{ or } a_i^+, \quad i = 0, \cdots, n, \quad i \neq k.$$

There are

$$\binom{\frac{n}{2}} 2^n, \ n \text{ even} \qquad (5.54)$$

$$\left(\frac{n+1}{2}\right) 2^n, \ n \text{ odd} \qquad (5.55)$$

such upper edges.

Example 5.10. Consider a second order polynomial

$$P(z) = a_2 z^2 + a_1 z + a_0$$

where

$$a_0 \in [a_0^-, a_0^+], \quad a_1 \in [a_1^-, a_1^+], \quad a_2 \in [a_2^-, a_2^+].$$

There are 4 upper edges given by:

$$\begin{array}{l}
(\lambda a_2^- + (1-\lambda) a_2^+) z^2 + a_1^- z + a_0^- \\
(\lambda a_2^- + (1-\lambda) a_2^+) z^2 + a_1^- z + a_0^+ \\
(\lambda a_2^- + (1-\lambda) a_2^+) z^2 + a_1^+ z + a_0^- \\
(\lambda a_2^- + (1-\lambda) a_2^+) z^2 + a_1^+ z + a_0^+
\end{array}$$

In this case, the set of all exposed edges is 12.

We now have the following main result on the Schur stability of interval polynomials. As usual we assume that the degree of all polynomials in $\mathcal{I}(z)$ is n.

Theorem 5.14 *The family $\mathcal{I}(z)$ is stable if and only if the family of edge polynomials $\mathcal{I}_{\mathrm{E}^*}(z)$ is stable.*

Proof. By Lemma 5.5, the stability of the polytope $\mathcal{I}(z)$ is equivalent to that of the subsets obtained by fixing the lower coefficients at their vertices and letting the upper coefficients vary over intervals. The stability of each of these subpolytopes in turn can be obtained from their exposed edges. These edges precisely generate the family $\mathcal{I}_{\mathrm{E}^*}(z)$. This completes the proof. ♣

It is important to note here that even though this result is not Kharitonov like in the sense that the number of segments to be checked increases with the order of the polynomial, it still yields significant computational advantages over checking *all* the exposed edges. Indeed for an interval polynomial of degree n, we would have $(n+1)2^n$ exposed edges to check whereas the present result requires us to check (in the case n even) only $\left(\frac{n}{2}\right)2^n$. As an example, for a second order interval polynomial there are 12 exposed edges, but only 4 upper edges. For a sixth order interval polynomial there are 448 exposed edges but only 192 upper edges.

Example 5.11. Consider the following second order polynomial

$$P(z) = a_2 z^2 + a_1 z + a_0$$

with coefficients varying in independent intervals

$$a_2 \in [2.5, 4], \qquad a_1 \in [-0.1, 0.25], \qquad a_0 \in [0.2, 0.8]$$

The corresponding exposed edges are obtained by letting one of the coefficients vary in an interval and fixing the other coefficients at a vertex. We obtain :

$$\begin{aligned}
E^1(\lambda, z) &= 2.5z^2 - 0.1z + 0.8 - 0.6\lambda \\
E^2(\lambda, z) &= 2.5z^2 + 0.25z + 0.8 - 0.6\lambda \\
E^3(\lambda, z) &= 4z^2 - 0.1z + 0.8 - 0.6\lambda \\
E^4(\lambda, z) &= 4z^2 + 0.25z + 0.8 - 0.6\lambda \\
E^5(\lambda, z) &= 2.5z^2 + (0.25 - 0.35\lambda)z + 0.2 \\
E^6(\lambda, z) &= 2.5z^2 + (0.25 - 0.35\lambda)z + 0.8 \\
E^7(\lambda, z) &= 4z^2 + (0.25 - 0.35\lambda)z + 0.2 \\
E^8(\lambda, z) &= 4z^2 + (0.25 - 0.35\lambda)z + 0.8 \\
E^9(\lambda, z) &= (4 - 1.5\lambda)z^2 - 0.1z + 0.2 \\
E^{10}(\lambda, z) &= (4 - 1.5\lambda)z^2 - 0.1z + 0.8 \\
E^{11}(\lambda, z) &= (4 - 1.5\lambda)z^2 + 0.25z + 0.2 \\
E^{12}(\lambda, z) &= (4 - 1.5\lambda)z^2 + 0.25z + 0.8
\end{aligned}$$

The upper edges associated with $P(z)$ are

$$\begin{aligned}
U^1(\lambda, z) &= (4 - 1.5\lambda)z^2 - 0.1z + 0.2 \\
U^2(\lambda, z) &= (4 - 1.5\lambda)z^2 - 0.1z + 0.8 \\
U^3(\lambda, z) &= (4 - 1.5\lambda)z^2 + 0.25z + 0.2 \\
U^4(\lambda, z) &= (4 - 1.5\lambda)z^2 + 0.25z + 0.8
\end{aligned}$$

One can see that the upper edges are a subset of the exposed edges. The evolution of the image set, evaluated along the unit circle, of the exposed edges and upper edges is shown in Figures 5.13 and 5.14 respectively.

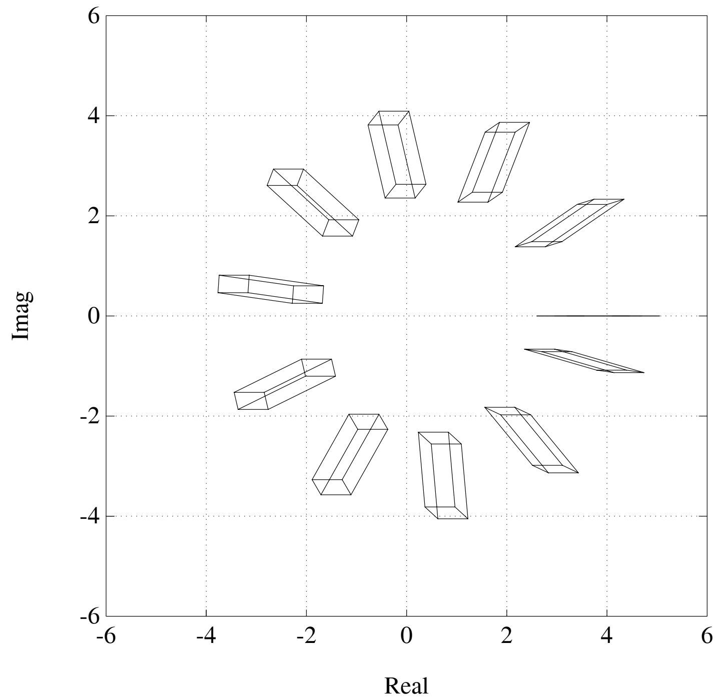

Figure 5.13. Evolution of the image set of the exposed edges (Example 5.11)

The image set excludes the origin, which shows that the entire family of polynomials is stable. One can note that the image set of the upper edges is a reduced subset of the image set of the exposed edges.

As a final note we again point out that robust Schur stability of an interval family can be ascertained by determining phase differences of the vertex polynomials. In the previous example this would have required the computation of the phases of eight vertex polynomials along with the stability check of a single polynomial in the family.

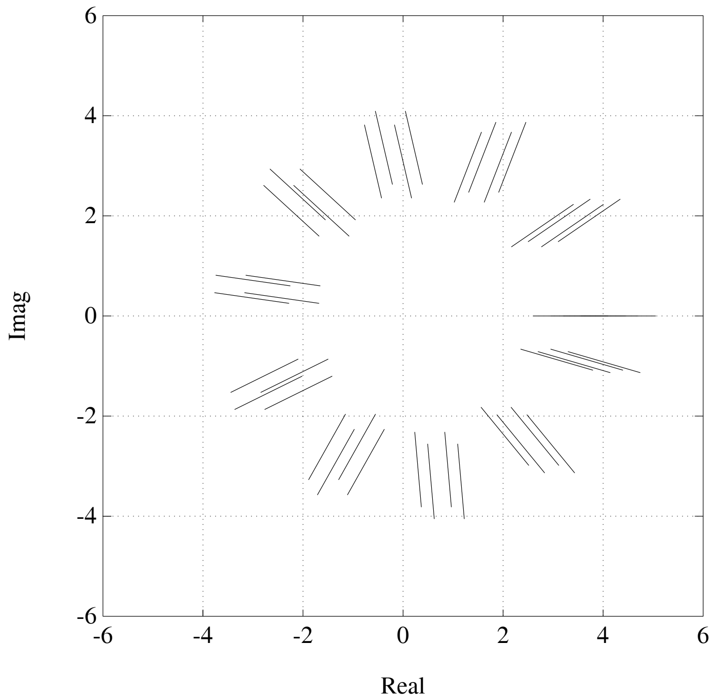

Figure 5.14. Evaluation of the image set of the upper edges (Example 5.11)

5.9 EXERCISES

5.1 Consider the control system shown in Figure 5.15. The parameters α_i, β_j vary

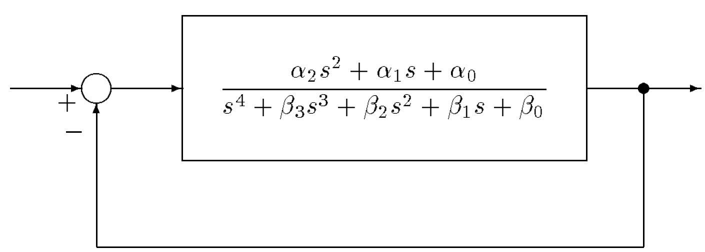

Figure 5.15. A feedback system (Exercise 5.1)

in the following ranges:

$$\alpha_0 \in [2,6], \quad \alpha_1 \in [0,2], \quad \alpha_2 \in [-1,3],$$

and

$$\beta_0 \in [4, 8], \quad \beta_1 \in [0.5, 1.5], \quad \beta_2 \in [2, 6], \quad \beta_3 \in [6, 14].$$

Determine if the closed loop system is Hurwitz stable or not for this class of perturbations.

5.2 For the system in Exercise 5.1 determine the largest box with the same center and shape (i.e. the ratios of the lengths of the sides are prespecified) as prescribed in the problem, for which the closed loop remains stable.

5.3 Show by an example that Kharitonov's Theorem does not hold when the stability region is the shifted half plane $\mathrm{Re}[s] \leq -\alpha, \ \alpha > 0$.

5.4 Show that for interval polynomials of degree less than six it suffices to test fewer than four polynomials in applying Kharitonov's test. Determine for each degree the number of polynomials to be checked, in addition to the condition that the signs of the coefficients are the same.
Hint: Consider the interlacing tubes corresponding to the interval family for each degree.

5.5 Consider the Hurwitz stability of an interval family where the only coefficient subject to perturbation is δ_k, the coefficient of s^k for an arbitrary $k \in [0, 1, 2, .., n]$. Show that the δ_k axis is partitioned into at most one stable segment and one or two unstable segments.

5.6 Apply the result of Exercise 5.5 to the Hurwitz polynomial

$$\delta(s) = s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0$$

with nominal parameters

$$\delta_3^0 = 4, \quad \delta_2^0 = 10, \quad \delta_1^0 = 12, \quad \delta_0^0 = 5.$$

Suppose that all coefficients except δ_3 remain fixed and δ_3 varies as follows:

$$\delta_3^- \leq \delta_3 \leq \delta_3^+.$$

Determine the largest interval (δ_3^-, δ_3^+) for which $\delta(s)$ remains Hurwitz. Repeat this for each of the coefficients δ_2, δ_1, and δ_0.

5.7 Consider the Hurwitz polynomial

$$\delta(s) = s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0$$

with nominal parameters $\underline{\delta}^0 = [\delta_3^0, \delta_2^0, \delta_1^0, \delta_0^0]$ given by:

$$\delta_3^0 = 4, \quad \delta_2^0 = 10, \quad \delta_1^0 = 12, \quad \delta_0^0 = 5.$$

Suppose that the coefficients vary independently within a weighted l^∞ box of size ρ given by:

$$\mathbf{\Delta}_\rho := \left\{\underline{\delta} : \delta_i^0 - \rho w_i \leq \delta_i \leq \delta_i^0 + \rho w_i, \quad i = 0,1,2,3\right\}$$

with weights $w_i \geq 0$. Find the maximal value of ρ for which stability is preserved assuming that $w_i = \delta_i^0$.

5.8 Consider an interval family and show that if the Kharitonov polynomials are completely unstable (i.e. their roots are all in the closed right half plane) then the entire family is completely unstable.

5.9 Consider the *positive feedback* control system shown in Figure 5.16 below:

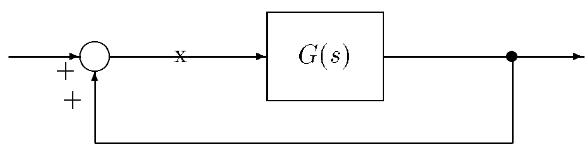

Figure 5.16. A feedback system (Exercise 5.9)

Let $G(s)$ belong to a family of interval systems $\mathbf{G}(s)$ as in Section 5.2. Show that the closed loop system is robustly stable (stable for all $G(s) \in \mathbf{G}(s)$) if and only if it is stable for each system in the set of negative Kharitonov systems

$$\mathbf{G}_{\mathrm{K}}^{-}(s) = \left\{\frac{K_N^{5-i}(s)}{K_D^i(s)}, \quad i = 1,2,3,4\right\}$$

Prove that if the system is robustly stable, the minimum gain margin over the interval family $\mathbf{G}(s)$ is attained over the subset $\mathbf{G}_{\mathrm{K}}^{-}(s)$.

5.10 Let the state space representation of the system (A, b) be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_0 & a_1 & a_2 & a_3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where

$$a_0 \in [0, 1.5], \quad a_1 \in [-1.5, 2], \quad a_2 \in [0, 1], \quad a_3 \in [1, 2].$$

Find the state feedback control law that robustly stabilizes the closed loop system.

5.11 Consider the Hurwitz stable interval polynomial family

$$\delta(s) = \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0$$

$$\delta_3 \in [1.5, 2.5], \quad \delta_2 \in [2, 6], \quad \delta_1 \in [4, 8], \quad \delta_0 \in [0.5, 1.5].$$

Determine the worst case parametric stability margin $\rho(\delta)$ over the parameter box in the ℓ_2 and ℓ_∞ norms.

5.12 For the interval polynomial

$$\delta(z) = \delta_3 z^3 + \delta_2 z^2 + \delta_1 z + \delta_0$$

$$\delta_3 \in [1-\epsilon, 1+\epsilon], \;\; \delta_2 \in \left[-\frac{1}{4} - \epsilon, -\frac{1}{4} + \epsilon\right],$$
$$\delta_1 \in \left[-\frac{3}{4} - \epsilon, -\frac{3}{4} + \epsilon\right], \;\; \delta_0 \in \left[\frac{3}{16} - \epsilon, \frac{3}{16} + \epsilon\right]$$

determine the maximal value of ϵ for which the family is Schur stable using Theorem 5.13.

5.13 Consider the interval family $\mathcal{I}(s)$ of real polynomials

$$\delta(s) = \delta_o + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots + \delta_n s^n$$

where the coefficients lie within given ranges,

$$\delta_0 \in [x_0, y_0], \;\; \delta_1 \in [x_1, y_1], \;\; \cdots, \;\; \delta_n \in [x_n, y_n].$$

Suppose now that $x_n = 0$ and that $x_i > 0, \;\; i = 0, 1, 2, \cdots, n-1$. Show that the Hurwitz stability of the family can be determined by checking, in addition to the usual four Kharitonov polynomials

$$\begin{aligned}
\hat{K}^1(s) &= x_n s^n + y_{n-1}s^{n-1} + y_{n-2}s^{n-2} + x_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots, \\
\hat{K}^2(s) &= x_n s^n + x_{n-1}s^{n-1} + y_{n-2}s^{n-2} + y_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots, \\
\hat{K}^3(s) &= y_n s^n + x_{n-1}s^{n-1} + x_{n-2}s^{n-2} + y_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots, \\
\hat{K}^4(s) &= y_n s^n + y_{n-1}s^{n-1} + x_{n-2}s^{n-2} + x_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots.
\end{aligned}$$

the following two additional polynomials

$$\begin{aligned}
\hat{K}_5(s) &= x_{n-1}s^{n-1} + x_{n-2}s^{n-2} + y_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots \\
\hat{K}_6(s) &= y_{n-1}s^{n-1} + x_{n-2}s^{n-2} + x_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots
\end{aligned}$$

Hint: Note that $\hat{K}_5(s)$ and $\hat{K}_6(s)$ can be obtained from $\hat{K}_3(s)$ and $\hat{K}_4(s)$ respectively by setting $y_n = 0$. Now use the argument that for a given polynomial $q(s)$ of degree $n-1$, the family $s^n + \mu q(s)$, $\mu \in \left[\frac{1}{y_n}, \infty\right)$ is Hurwitz stable if and only if $q(s)$ and $y_n s^n + q(s)$ are Hurwitz stable.)

5.14 Prove Lemma 5.3 for the case $n = 4r + j, \quad j = 1, 2, 3$.

5.15 Let $\mathcal{I}_\rho(s)$ denote the interval polynomial family

$$a(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

where

$$a_3 \in [1-\rho, 1+\rho], \quad a_2 \in [4-\rho, 4+\rho], \quad a_1 \in [6-\rho, 6+\rho], \quad a_0 \in [1-\rho, 1+\rho]$$

and let

$$\varphi(z) = \alpha_2 z^2 + \alpha_1 z + \alpha_0$$

with $\alpha_2 = 1$, $\alpha_1 = 3$, $\alpha_0 = 4$. Determine the maximum value of ρ for which the family $\varphi(\mathcal{I}_\rho(s))$ is Hurwitz stable.

5.16 In Exercise 5.15, suppose that the polynomial $\varphi(z)$ varies in a polytope

$$\Phi(z) = \{\varphi(z) = \alpha_2 z^2 + \alpha_1 z + \alpha_0 : \alpha_2 = 1, \alpha_1 \in [2, 4], \alpha_0 \in [3, 5]\}.$$

Determine the maximum value of ρ for which the family $\Phi(\mathcal{I}_\rho(s))$ is Hurwitz stable.

5.10 NOTES AND REFERENCES

The interval polynomial problem was originally posed by Faedo [92] who attempted to solve it using the Routh-Hurwitz conditions. Some necessary and some sufficient conditions were obtained by Faedo and the problem remained open until Kharitonov gave a complete solution. Kharitonov first published his theorem for real polynomials in 1978 [143], and then extended it to the complex case in [144]. The papers of Bialas [38] and Barmish [11] are credited with introducing this result to the Western literature. Several treatments of this theorem are available in the literature. Among them we can mention Bose [44], Yeung and Wang [240] and Minnichelli, Anagnost and Desoer [181] and Chapellat and Bhattacharyya [57]. A system-theoretic proof of Kharitonov's Theorem for the complex case was given by Bose and Shi [50] using complex reactance functions. That the set $\mathcal{I}(j\omega)$ is a rectangle was first pointed out by Dasgupta [75] and hence it came to be known as Dasgupta's rectangle. The proof in Minnichelli et. al. is based on the image set analysis given in Section 5.4. The proof in Chapellat and Bhattacharyya [57] is based on the Segment Lemma (Chapter 2). Mansour and Anderson [171] have proved Kharitonov's Theorem using the second method of Lyapunov. The computational approach to enlarging the ℓ_∞ box described in Exercise 5.7 was first reported in Barmish [11]. The extremal property of the Kharitonov polynomials, Theorem 5.4 was first proved by Chapellat and Bhattacharyya [56] and the robust stabilization result of Theorem 5.7 is adapted from Chapellat and Bhattacharyya [59]. Mansour, Kraus and Anderson [174] and Kraus, Anderson and Mansour [152] have given several results on robust Schur stability and strong Kharitonov theorems for Schur interval systems. Lemma 5.5 and

Theorem 5.14 were proved by Pérez, Abdallah and Docampo [188] and extend similar results due to Hollot and Bartlett [115], and Kraus, Mansour, and Jury [155]. In Kraus and Mansour [153] the minimal number of edges to be checked for Schur stability of an interval polynomial is derived. This number of course depends on n unlike the Hurwitz case where it is always 4. Rantzer [194] studied the problem of characterizing stability regions in the complex plane for which it is true that stability of *all* the vertices of an interval family guarantee that of the entire family. He showed that such regions $\mathcal{D}$, called Kharitonov regions, are characterized by the condition that $\mathcal{D}$ as well as $1/\mathcal{D}$ are both convex. Meressi, Chen and Paden [179] have applied Kharitonov's Theorem to mechanical systems. Mori and Kokame [183] dealt with the modifications required to extend Kharitonov's Theorem to the case where the degree can drop, i.e. $x_n = 0$ (see Exercise 5.12). Kharitonov's Theorem has been generalized by Chapellat and Bhattacharyya [58] for the control problem. This is described in Chapter 7. Various other generalizations of the Kharitonov's Theorem have been reported. In [45] Bose generalized Kharitonov's Theorem in another direction and showed that the scattering Hurwitz property of a set of bivariate interval polynomials could be established by checking only a finite number of extreme bivariate polynomials. Multidimensional interval polynomials were also studied by Basu [23]. Barmish [13] has reported a generalization of the so-called four polynomial concept of Kharitonov. The extension of Kharitonov's Theorem to polynomial functions of interval polynomials described in Section 5.7 are due to Kharitonov [146].