Chapter 4

THE PARAMETRIC STABILITY MARGIN

In this chapter we extend the stability ball calculation developed in Chapter 3 to accommodate *interdependent* perturbations among the polynomial coefficients. This is the usual situation that one encounters in control systems containing uncertain parameters. We calculate the radius of the largest stability ball in the space of real parameters under the assumption that these uncertain parameters enter the characteristic polynomial coefficients *linearly or affinely.* This radius serves as a quantitative measure of the real parametric stability margin for control systems. Both *ellipsoidal* and *polytopic* uncertainty regions are considered. Examples are included to illustrate the calculation of the parametric stability margin for continuous time and discrete time control systems and also those containing time delay. In the polytopic case we establish the stability testing property of the *exposed edges* and some *extremal properties* of edges and vertices. These are useful for calculating the *worst case stability margin* over an given uncertainty set which in turn is a measure of the *robust performance* of the system. The graphical Tsypkin-Polyak plot for stability margin calculation (parameter space version) and the theory of linear disc polynomials are described.

4.1 INTRODUCTION

In Chapter 3 we calculated the radius of the stability ball for a family of real polynomials in the space of the polynomial coefficients. The underlying assumption was that these coefficients could perturb *independently.* This assumption is rather restrictive. In control problems, the characteristic polynomial coefficients do not in general perturb independently. At the first level of detail every feedback system is composed of at least two subsystems, namely controller and plant connected in a feedback loop. The characteristic polynomial coefficients of such a system is a function of plant parameters and controller parameters. Although both parameters influence the coefficients, the nature of the two sets of parameters are quite different. The plant contains parameters that are subject to uncontrolled variations depending on the physical operating conditions, disturbances, modeling errors, etc. The controller parameters on the other hand, are often fixed during the operation of the system. However, at the design stage, they are also uncertain parameters to be chosen.

Consider the standard feedback system shown in Figure 4.1.

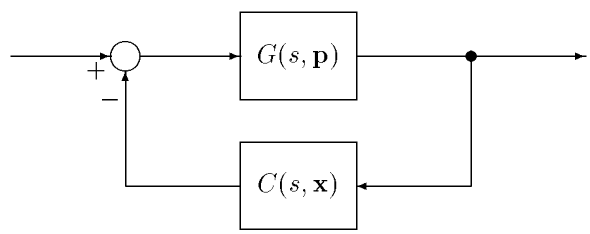

Figure 4.1. Standard feedback system with parameters

Suppose that the plant transfer function contains the real parameter vector $\mathbf{p}$ and the controller is characterized by the real vector $\mathbf{x}$. Let these transfer functions be respectively

$$G(s) = G(s, \mathbf{p}), \qquad C(s) = C(s, \mathbf{x}).$$

The parameter vector $\mathbf{p}$ specifies the plant completely and the choice of the vector $\mathbf{x}$ likewise uniquely determines the controller $C(s)$.

The Parametric Stability Margin

Suppose that $\mathbf{p}^0$ denotes the nominal value of the plant parameter vector $\mathbf{p}$. Consider a *fixed* controller $C^0(s)$, with parameter $\mathbf{x}^0$, which stabilizes the nominal plant $G(s, \mathbf{p}^0)$. Now let $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^0$ denote a perturbation of the plant parameter vector from the nominal $\mathbf{p}^0$. A natural question that occurs now is: How large can the perturbation $\boldsymbol{\Delta}\mathbf{p}$ be, without destroying closed loop stability? A bound on the size of $\boldsymbol{\Delta}\mathbf{p}$ for which closed loop stability is guaranteed is useful as it provides us with a ball in parameter space within which the parameters can freely vary without destroying closed loop stability. An even more useful quantity to know is the *maximal* size of such a stability ball. Besides providing us with a nonconservative evaluation of the size of the stability region, this would also also serve as a bona fide measure of the performance of the controller $C^0(s)$. Accordingly the *parametric stability margin* is defined as the length of the smallest perturbation $\Delta\mathbf{p}$ which destabilizes the closed loop. This margin serves as a quantitative measure of the robustness of the closed system with respect to parametric uncertainty evaluated at the nominal point $\mathbf{p}^0$. It is useful in controller *design* as a means of comparing the performance of proposed controllers.

The determination of the parametric stability margin is the problem that one faces in robustness analysis. This problem can be solved with $\mathbf{p}$ fixed at the nominal point $\mathbf{p}^0$ or with $\mathbf{p}$ lying in a prescribed uncertainty set Ω. In the latter case it is desirable to find the *worst case stability margin* over the uncertainty set. This worst case margin is a measure of the worst case *performance* of the controller $\mathbf{x}^0$ over Ω. This is regarded also as a measure of the *robust performance* of the controller and can also be the basis of comparison of two proposed controllers.

In synthesis or design problems one is faced with the task of choosing the controller parameter $\mathbf{x}$ to *increase* or *maximize* these margins. In addition control design involves several other issues such as tolerance of unstructured uncertainty and nonlinearities, quality of transient response, integrity against loop failures, and boundedness of various signal levels in the closed loop system. Because of the complexity and interrelatedness of these issues it is probably tempting to treat the entire design exercise as an optimization problem with various mathematical constraints reflecting the physical requirements. In the control literature this approach has been developed in recent years, to a high level of sophistication, with the aid of the Youla parametrization of all stabilizing controllers. The latter parametrization allowed for the systematic search over the set of all stabilizing controllers to achieve performance objectives such as H_∞ norm or ℓ_1 norm optimization. However relatively little progress has been made with this approach in the sphere of control problems involving real parametric stability margins and also in problems where the controller has a fixed number of adjustable parameters.

In this chapter we deal only with the real parametric stability margin. Even in this apparently simple case the analysis and synthesis problems mentioned above are unsolved problems in general. In the special case where the parameters enter the characteristic polynomial coefficients in an affine *linear* or *multilinear* fashion the problem of calculating the real parametric stability margin now has neat solutions. Fortunately, both these cases fit nicely into the framework of control system problems. In the present chapter, we deal with the linear (including affine) case where an exact calculation of the parametric stability margin can be given. The multilinear case will be dealt with in Chapters 10 and 11. The calculation of stability margins for systems subject to both parametric and unstructured perturbations will be treated in Chapter 9.

The Linear Case

The characteristic polynomial of the standard closed loop system in Figure 4.1 can be expressed as

$$\delta(s, \mathbf{x}, \mathbf{p}) = \sum_{i=0}^{n} \delta_i(\mathbf{x}, \mathbf{p}) s^i.$$

As an example suppose that the plant and controller transfer functions, denoted respectively by $G(s, \mathbf{p})$ and $C(s, \mathbf{x})$ are:

$$G(s, \mathbf{p}) := \frac{p_3}{s^2 + p_1 s + p_2}, \qquad C(s, \mathbf{x}) := \frac{x_3}{x_1 s + x_2}.$$

Then the characteristic polynomial of the closed loop system is

$$\delta(s,\mathbf{x},\mathbf{p}) = \underbrace{x_1}_{\delta_3(\mathbf{x},\mathbf{p})} s^3 + \underbrace{(p_1x_1+x_2)}_{\delta_2(\mathbf{x},\mathbf{p})} s^2 + \underbrace{(p_2x_1+p_1x_2)}_{\delta_1(\mathbf{x},\mathbf{p})} s + \underbrace{(p_2x_2+p_3x_3)}_{\delta_0(\mathbf{x},\mathbf{p})}.$$

For fixed values of the plant parameter $\mathbf{p}$, we see that the coefficients of the characteristic polynomial are linear functions of the controller parameter $\mathbf{x}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ p_2 & p_1 & 0 \\ 0 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \delta_3(\mathbf{x}) \\ \delta_2(\mathbf{x}) \\ \delta_1(\mathbf{x}) \\ \delta_0(\mathbf{x}) \end{bmatrix}.$$

Similarly, for fixed values of the controller parameter $\mathbf{x}$, the coefficients of the characteristic polynomial are linear functions of the plant parameter $\mathbf{p}$ as follows:

$$\begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 0 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \delta_3(\mathbf{p}) \\ \delta_2(\mathbf{p}) \\ \delta_1(\mathbf{p}) \\ \delta_0(\mathbf{p}) \end{bmatrix}.$$

The characteristic polynomial can also be written in the following forms:

$$\begin{aligned} \delta(s,\mathbf{p}) &= \underbrace{(x_1s^2+x_2s)}_{a_1(s)} p_1 + \underbrace{(x_1s+x_2)}_{a_2(s)} p_2 + \underbrace{x_3}_{a_3(s)} p_3 + \underbrace{(x_1s^3+x_2s^2)}_{b(s)} \\ &= \underbrace{(x_1s+x_2)}_{F_1(s)} \underbrace{(s^2+p_1s+p_2)}_{P_1(s)} + \underbrace{x_3}_{F_2(s)} \underbrace{p_3}_{P_2(s)}. \end{aligned}$$

Motivated by this we will consider in this chapter the case where the controller $\mathbf{x}$ is fixed and therefore $\delta_i(\mathbf{x},\mathbf{p})$ are linear functions of the parameters $\mathbf{p}$. We will sometimes refer to this simply as the linear case. Therefore, we will consider the characteristic polynomial of the form

$$\delta(s,\mathbf{p}) = a_1(s)p_1 + a_2(s)p_2 + \cdots + a_l(s)p_l + b(s) \tag{4.1}$$

where $a_i(s)$, $i = 1, 2, \cdots, l$ and $b(s)$ are fixed polynomials and p_1, p_2, $\cdots$, p_l are the uncertain parameters. We will also develop results for the following form of the characteristic polynomial:

$$\delta(s,\mathbf{p}) = F_1(s)P_1(s) + F_2(s)P_2(s) + \cdots + F_m(s)P_m(s) \tag{4.2}$$

wherein the $F_i(s)$ are fixed and the coefficients of $P_i(s)$ are the uncertain parameters. The uncertain parameters will be allowed to lie in a set Ω which could be ellipsoidal or box-like.

The problems of interest will be: Are all characteristic polynomials corresponding to the parameter set Ω, stable? How large can the set Ω be without losing stability? What is the smallest value (worst case) of the stability margin over a given uncertainty set Ω? How do we determine the stability margin if time delays are also present in the system of Figure 4.1? We shall provide constructive solutions to these questions in this chapter. We start in the next section with a characterization of the stability ball in parameter space based on the Boundary Crossing Theorem.

4.2 THE STABILITY BALL IN PARAMETER SPACE

In this section we give a useful characterization of the parametric stability margin in the general case. This can be done by finding the largest stability ball in parameter space, centered at a "stable" nominal parameter value $\mathbf{p}^0$. Let $\mathcal{S} \subset \mathcal{C}$ denote as usual an open set which is symmetric with respect to the real axis. $\mathcal{S}$ denotes the stability region of interest. In continuous-time systems $\mathcal{S}$ may be the open left half plane or subsets thereof. For discrete-time systems $\mathcal{S}$ is the open unit circle or a subset of it. Now let $\mathbf{p}$ be a vector of real parameters,

$$\mathbf{p} = [p_1, p_2, \cdots, p_l]^T.$$

The characteristic polynomial of the system is denoted by

$$\delta(s, \mathbf{p}) = \delta_n(\mathbf{p})s^n + \delta_{n-1}(\mathbf{p})s^{n-1} + \cdots + \delta_0(\mathbf{p}).$$

The polynomial $\delta(s, \mathbf{p})$ is a real polynomial with coefficients that depend continuously on the real parameter vector $\mathbf{p}$. We suppose that for the *nominal parameter* $\mathbf{p} = \mathbf{p}^0$, $\delta(s, \mathbf{p}^0) := \delta^0(s)$ is stable with respect to $\mathcal{S}$ (has its roots in $\mathcal{S}$). Write

$$\triangle\mathbf{p} := \mathbf{p} - \mathbf{p}^0 = [p_1 - p_1^0, p_2 - p_2^0, \cdots, p_l - p_l^0]$$

to denote the perturbation in the parameter $\mathbf{p}$ from its nominal value $\mathbf{p}^0$. Now let us introduce a norm $\|\cdot\|$ in the space of the parameters $\mathbf{p}$ and introduce the open ball of radius ρ

$$\mathcal{B}(\rho, \mathbf{p}^0) = \{\mathbf{p} : \|\mathbf{p} - \mathbf{p}^0\| < \rho\}. \tag{4.3}$$

The hypersphere of radius ρ is defined by

$$\mathcal{S}(\rho, \mathbf{p}^0) = \{\mathbf{p} : \|\mathbf{p} - \mathbf{p}^0\| = \rho\}. \tag{4.4}$$

With the ball $\mathcal{B}(\rho, \mathbf{p}^0)$ we associate the family of uncertain polynomials:

$$\mathbf{\Delta}_\rho(s) := \{\delta(s, \mathbf{p}^0 + \Delta\mathbf{p}) : \|\Delta\mathbf{p}\| < \rho\}. \tag{4.5}$$

Definition 4.1. The *real parametric stability margin* in parameter space is defined as the radius, denoted $\rho^*(\mathbf{p}^0)$, of the *largest* ball centered at $\mathbf{p}^0$ for which $\delta(s, \mathbf{p})$ remains stable whenever $\mathbf{p} \in \mathcal{B}(\rho^*(\mathbf{p}^0), \mathbf{p}^0)$.

This stability margin then tells us how much we can perturb the original parameter $\mathbf{p}^0$ and yet remain stable. Our first result is a characterization of this maximal stability ball. To simplify notation we write ρ^* instead of $\rho^*(\mathbf{p}^0)$.

Theorem 4.1 *With the assumptions as above the parametric stability margin ρ^* is characterized by:*

a) There exists a largest stability ball $\mathcal{B}(\rho^*, \mathbf{p}^0)$ centered at $\mathbf{p}^0$, with the property that:

a1) For every $\mathbf{p}'$ within the ball, the characteristic polynomial $\delta(s, \mathbf{p}')$ is stable and of degree n.

a2) At least one point $\mathbf{p}''$ on the hypersphere $\mathcal{S}(\rho^*, \mathbf{p}^0)$ itself is such that $\delta(s, \mathbf{p}'')$ is unstable or of degree less than n.

b) Moreover if $\mathbf{p}''$ is any point on the hypersphere $\mathcal{S}(\rho^*, \mathbf{p}^0)$ such that $\delta(s, \mathbf{p}'')$ is unstable, then the unstable roots of $\delta(s, \mathbf{p}'')$ can only be on the stability boundary.

The proof of this theorem is identical to that of Theorem 3.1 of Chapter 3 and is omitted. It is based on continuity of the roots on the parameter $\mathbf{p}$. This theorem gives the first simplification for the calculation of the parametric stability margin ρ^*. It states that to determine ρ^* it suffices to calculate the minimum "distance" of $\mathbf{p}^0$ from the set of those points $\mathbf{p}$ which endow the characteristic polynomial with a root on the stability boundary, or which cause loss of degree. This last calculation can be carried out using the complex plane image of the family of polynomials $\mathbf{\Delta}_\rho(s)$ evaluated along the stability boundary. We will describe this in the next section.

The parametric stability margin or distance to instability is measured in the norm $\|\cdot\|$, and therefore the numerical value of ρ^* will depend on the specific norm chosen. We will consider, in particular, the weighted ℓ_p norms. These are defined as follows: Let $w = [w_1, w_2, \cdots, w_l]$ with $w_i > 0$ be a set of positive *weights*.

$$
\begin{aligned}
\ell_1 \text{ norm} &: \quad \|\Delta\mathbf{p}\|_1^w := \sum_{i=1}^{l} w_i |\Delta p_i| \\
\ell_2 \text{ norm} &: \quad \|\Delta\mathbf{p}\|_2^w := \sqrt{\sum_{i=1}^{l} w_i^2 \Delta p_i^2} \\
\ell_p \text{ norm} &: \quad \|\Delta\mathbf{p}\|_p^w := \left[\sum_{i=1}^{l} |w_i \Delta p_i|^p\right]^{\frac{1}{p}} \\
\ell_\infty \text{ norm} &: \quad \|\Delta\mathbf{p}\|_\infty^w := \max_i w_i |\Delta p_i|.
\end{aligned}
$$

We will write $\|\triangle\mathbf{p}\|$ to refer to a generic weighted norm when the weight and type of norm are unimportant.

4.3 THE IMAGE SET APPROACH

The parametric stability margin may be calculated by using the complex plane image of the polynomial family $\mathbf{\Delta}_\rho(s)$ evaluated at each point on the stability boundary $\partial\mathcal{S}$. This is based on the following idea. Suppose that the family has constant degree n and $\delta(s, \mathbf{p}^0)$ is stable but $\mathbf{\Delta}_\rho(s)$ contains an unstable polynomial. Then the continuous dependence of the roots on $\mathbf{p}$ and the Boundary Crossing Theorem imply that there must also exist a polynomial in $\mathbf{\Delta}_\rho(s)$ such that it has a root, at a point, s^*, on the stability boundary $\partial\mathcal{S}$. In this case *the complex plane image set* $\mathbf{\Delta}_\rho(s^*)$ *must contain the origin of the complex plane.* This suggests that to detect the presence of instability in a family of polynomials of constant degree, we generate the image set of the family at each point of the stability boundary and determine if the origin is included in or excluded from this set. This fact was stated in Chapter 1 as the Zero Exclusion Principle, and we repeat it here for convenience.

Theorem 4.2 (Zero Exclusion Principle)
For given $\rho \geq 0$ and $\mathbf{p}^0$, suppose that the family of polynomials $\mathbf{\Delta}_\rho(s)$ is of constant degree and $\delta(s, \mathbf{p}^0)$ is stable. Then every polynomial in the family $\mathbf{\Delta}_\rho(s)$ is stable with respect to $\mathcal{S}$ if and only if the complex plane image set $\mathbf{\Delta}_\rho(s^*)$ excludes the origin for every $s^* \in \partial\mathcal{S}$.

Proof. As stated earlier this is simply a consequence of the continuity of the roots of $\delta(s, \mathbf{p})$ on $\mathbf{p}$ and the Boundary Crossing Theorem (Chapter 1). ♣

In fact, the above can be used as a computational tool to determine the maximum value ρ^* of ρ for which the family is stable. If $\delta(s, \mathbf{p}^0)$ is stable, it follows that there always exists an open stability ball around $\mathbf{p}^0$ since the stability region $\mathcal{S}$ is itself open. Therefore, for small values of ρ the image set $\mathbf{\Delta}_\rho(s^*)$ will exclude the origin for every point $s^* \in \partial\mathcal{S}$. As ρ is increased from zero, a *limiting* value ρ_0 may be reached where some polynomial in the corresponding family $\mathbf{\Delta}_{\rho_0}(s)$ loses degree or a polynomial in the family acquires a root s^* on the stability boundary. From Theorem 4.1 it is clear that this value ρ_0 is equal to ρ^*, the stability margin. In case the limiting value ρ_0 is never achieved, the stability margin ρ^* is infinity.

An alternative way to determine ρ^* is as follows: Fixing s^* at a point in the boundary of $\mathcal{S}$, let $\rho_0(s^*)$ denote the limiting value of ρ such that $0 \in \mathbf{\Delta}_\rho(s^*)$:

$$\rho_0(s^*) := \inf \{\rho : 0 \in \mathbf{\Delta}_\rho(s^*)\}.$$

Then, we define

$$\rho_b = \inf_{s^* \in \partial\mathcal{S}} \rho_0(s^*).$$

In other words, ρ_b is the limiting value of ρ for which some polynomial in the family $\mathbf{\Delta}_\rho(s)$ *acquires a root on the stability boundary* $\partial\mathcal{S}$. Also let the limiting value of ρ for which some polynomial in $\mathbf{\Delta}_\rho(s)$ *loses degree* be denoted by ρ_d:

$$\rho_d := \inf \left\{\rho : \delta_n(\mathbf{p}^0 + \Delta\mathbf{p}) = 0, \quad \|\Delta\mathbf{p}\| < \rho\right\}.$$

We have established the following theorem.

Theorem 4.3 *The parametric stability margin*

$$\rho^* = \min\{\rho_b, \rho_d\}.$$

Remark 4.1. We note that this theorem remains valid even when the stability region $\mathcal{S}$ is not connected. For instance, one may construct the stability region as a union of disconnected regions $\mathcal{S}_i$ surrounding each root of the nominal polynomial. In this case the stability boundary must also consist of the union of the individual boundary components $\partial\mathcal{S}_i$. The functional dependence of the coefficients δ_i on the parameters $\mathbf{p}$ is also not restricted in any way except for the assumption of continuity.

The above theorem shows that the problem of determining ρ^* can be reduced to the following steps:

A) determine the "local" stability margin $\rho(s^*)$ at each point s^* on the boundary of the stability region,

B) minimize the function $\rho(s^*)$ over the entire stability boundary and thus determine ρ_b

C) calculate ρ_d, and set

D) $\rho^* = \min\{\rho_b, \rho_d\}$.

In general the determination of ρ^* is a *difficult nonlinear optimization* problem. However, the breakdown of the problem into the steps described above, exploits the structure of the problem and has the advantage that the local stability margin calculation $\rho(s^*)$ with s^* frozen, can be simple. In particular, when the parameters enter linearly into the characteristic polynomial coefficients, this calculation can be done in closed form. It reduces to a least square problem for the ℓ_2 case, and equally simple linear programming or vertex problems for the ℓ_∞ or ℓ_1 cases. The dependence of ρ_b on s^* is in general highly nonlinear but this part of the minimization can be easily performed computationally because sweeping the boundary $\partial\mathcal{S}$ is a *one-dimensional search*. In the next section, we describe and develop this calculation in greater detail.

4.4 STABILITY MARGIN COMPUTATION IN THE LINEAR CASE

We develop explicit formulas for the parametric stability margin in the case in which the characteristic polynomial coefficients depend linearly on the uncertain parameters. In such cases we may write without loss of generality that

$$\delta(s, \mathbf{p}) = a_1(s)p_1 + \cdots + a_l(s)p_l + b(s) \qquad (4.6)$$

where $a_i(s)$ and $b(s)$ are real polynomials and the parameters p_i are real. As before, we write $\mathbf{p}$ for the vector of uncertain parameters, $\mathbf{p}^0$ denotes the nominal parameter vector and $\triangle\mathbf{p}$ the perturbation vector. In other words

$$\begin{aligned}\mathbf{p} &= [p_1, p_2, \cdots, p_l] \qquad \mathbf{p}^0 = [p_1^0, p_2^0, \cdots, p_l^0]\\ \triangle\mathbf{p} &= [p_1 - p_1^0, p_2 - p_2^0, \cdots, p_l - p_l^0]\\ &= [\triangle p_1, \triangle p_2, \cdots, \triangle p_l].\end{aligned}$$

Then the characteristic polynomial can be written as

$$\delta(s, \mathbf{p}^0 + \Delta\mathbf{p}) = \underbrace{\delta(s, \mathbf{p}^0)}_{\delta^0(s)} + \underbrace{a_1(s)\triangle p_1 + \cdots + a_l(s)\triangle p_l}_{\triangle\delta(s,\triangle\mathbf{p})}. \tag{4.7}$$

Now let s^* denote a point on the stability boundary $\partial\mathcal{S}$. For $s^* \in \partial\mathcal{S}$ to be a root of $\delta(s, \mathbf{p}^0 + \triangle\mathbf{p})$ we must have

$$\delta(s^*, \mathbf{p}^0) + a_1(s^*)\triangle p_1 + \cdots + a_l(s^*)\triangle p_l = 0. \tag{4.8}$$

We rewrite this equation introducing the weights $w_i > 0$.

$$\delta(s^*, \mathbf{p}^0) + \frac{a_1(s^*)}{w_1} w_1 \triangle p_1 + \cdots + \frac{a_l(s^*)}{w_l} w_l \triangle p_l = 0. \tag{4.9}$$

Obviously, the minimum $||\triangle\mathbf{p}||^w$ norm solution of this equation gives us $\rho(s^*)$, the calculation involved in step A in the last section.

$$\rho(s^*) = \inf\left\{ ||\triangle\mathbf{p}||^w \ : \ \delta(s^*, \mathbf{p}^0) + \frac{a_1(s^*)}{w_1} w_1 \triangle p_1 + \cdots + \frac{a_l(s^*)}{w_l} w_l \triangle p_l = 0 \right\}.$$

Similarly, corresponding to loss of degree we have the equation

$$\delta_n(\mathbf{p}^0 + \Delta\mathbf{p}) = 0. \tag{4.10}$$

Letting a_{in} denote the coefficient of the n^{th} degree term in the polynomial $a_i(s)$, $i = 1, 2, \cdots, l$ the above equation becomes

$$\underbrace{a_{1n}p_1^0 + a_{2n}p_2^0 + \cdots + a_{ln}p_l^0}_{\delta_n(\mathbf{p}^0)} + a_{1n}\triangle p_1 + a_{2n}\triangle p_2 + \cdots a_{ln}\triangle p_l = 0. \tag{4.11}$$

We can rewrite this after introducing the weight $w_i > 0$

$$\underbrace{a_{1n}p_1^0 + a_{2n}p_2^0 + \cdots + a_{ln}p_l^0}_{\delta_n(\mathbf{p}^0)} + \frac{a_{1n}}{w_1} w_1 \triangle p_1 + \frac{a_{2n}}{w_2} w_2 \triangle p_2 + \cdots + \frac{a_{ln}}{w_l} w_l \triangle p_l = 0. \tag{4.12}$$

The minimum norm $||\triangle\mathbf{p}||^w$ solution of this equation gives us ρ_d.

We consider the above equations in some more detail. Recall that the polynomials are assumed to be real. The equation (4.12) is real and can be rewritten in the form

$$\underbrace{\begin{bmatrix} \frac{a_{1n}}{w_1} & \cdots & \frac{a_{ln}}{w_l} \end{bmatrix}}_{A_n} \underbrace{\begin{bmatrix} w_1 \triangle p_1 \\ \vdots \\ w_l \triangle p_l \end{bmatrix}}_{t_n} = \underbrace{-\delta_n^0}_{b_n}. \tag{4.13}$$

In (4.9), two cases may occur depending on whether s^* is real or complex. If $s^* = s_r$ where s_r is real, we have the single equation

$$\underbrace{\begin{bmatrix} \frac{a_1(s_r)}{w_1} & \cdots & \frac{a_l(s_r)}{w_l} \end{bmatrix}}_{A(s_r)} \underbrace{\begin{bmatrix} w_1 \triangle p_1 \\ \vdots \\ w_l \triangle p_l \end{bmatrix}}_{t(s_r)} = \underbrace{-\delta^0(s_r)}_{b(s_r)}. \tag{4.14}$$

Let x_r and x_i denote the real and imaginary parts of a complex number x, i.e.

$$x = x_r + jx_i \qquad \text{with } x_r, x_i \text{ real.}$$

Using this notation, we will write

$$a_k(s^*) = a_{kr}(s^*) + ja_{ki}(s^*)$$

and

$$\delta^0(s^*) = \delta_r^0(s^*) + j\delta_i^0(s^*).$$

If $s^* = s_c$ where s_c is complex, (4.9) is equivalent to two equations which can be written as follows:

$$\underbrace{\begin{bmatrix} \frac{a_{1r}(s_c)}{w_1} & \cdots & \frac{a_{lr}(s_c)}{w_l} \\ \frac{a_{1i}(s_c)}{w_1} & \cdots & \frac{a_{li}(s_c)}{w_l} \end{bmatrix}}_{A(s_c)} \underbrace{\begin{bmatrix} w_1 \triangle p_1 \\ \vdots \\ w_l \triangle p_l \end{bmatrix}}_{t(s_c)} = \underbrace{\begin{bmatrix} -\delta_r^0(s_c) \\ -\delta_i^0(s_c) \end{bmatrix}}_{b(s_c)}. \tag{4.15}$$

These equations completely determine the parametric stability margin in any norm. Let $t^*(s_c)$, $t^*(s_r)$, and t_n^* denote the minimum norm solutions of (4.15), (4.14), and (4.13), respectively. Thus,

$$\|t^*(s_c)\| = \rho(s_c) \tag{4.16}$$

$$\|t^*(s_r)\| = \rho(s_r) \tag{4.17}$$

$$\|t_n^*\| = \rho_d. \tag{4.18}$$

If any of the above equations (4.13)-(4.15) do not have a solution, the corresponding value of $\rho(\cdot)$ is set equal to infinity.

Let $\partial\mathcal{S}_r$ and $\partial\mathcal{S}_c$ denote the real and complex subsets of $\partial\mathcal{S}$:

$$\partial\mathcal{S} = \partial\mathcal{S}_r \cup \partial\mathcal{S}_c.$$

$$\rho_r := \inf_{s_r \in \partial\mathcal{S}_r} \rho(s_r)$$
$$\rho_c := \inf_{s_c \in \partial\mathcal{S}_c} \rho(s_c),$$

Therefore,

$$\rho_b = \inf\{\rho_r, \rho_c\}. \tag{4.19}$$

We now consider the specific case of the ℓ_2 norm.

4.5 ℓ_2 STABILITY MARGIN

In this section we suppose that the length of the perturbation vector $\triangle\mathbf{p}$ is measured by a weighted ℓ_2 norm. That is, the minimum ℓ_2 norm solution of the equations (4.13), (4.14) and (4.15) are desired. Consider first (4.15). Assuming that $A(s_c)$ has full row rank=2, the minimum norm solution vector $t^*(s_c)$ can be calculated as follows:

$$t^*(s_c) = A^T(s_c)\left[A(s_c)A^T(s_c)\right]^{-1} b(s_c). \tag{4.20}$$

Similarly if (4.14) and (4.13) are consistent (i.e., $A(s_r)$ and A_n are nonzero vectors), we can calculate the solution as

$$t^*(s_r) = A^T(s_r)\left[A(s_r)A^T(s_r)\right]^{-1} b(s_r) \tag{4.21}$$
$$t_n^* = A_n^T\left[A_n A_n^T\right]^{-1} b_n. \tag{4.22}$$

If $A(s_c)$ has less than full rank the following two cases can occur.

Case 1: Rank $A(s_c) = 0$ In this case the equation is inconsistent since $b(s_c) \neq 0$ (otherwise $\delta^0(s_c) = 0$ and $\delta^0(s)$ would not be stable with respect to $\mathcal{S}$ since s_c lies on $\partial\mathcal{S}$). In this case (4.15) has no solution and we set

$$\rho(s_c) = \infty.$$

Case 2: Rank $A(s_c) = 1$ In this case the equation is consistent if and only if

$$\text{rank}\left[A(s_c), b(s_c)\right] = 1.$$

If the above rank condition for consistency is satisfied, we replace the two equations (4.15) by a single equation and determine the minimum norm solution of this latter equation. If the rank condition for consistency does not hold, equation (4.15) cannot be satisfied and we again set $\rho(s_c) = \infty$.

Example 4.1. (**ℓ_2 Schur stability margin**) Consider the discrete time control system with the controller and plant specified respectively by their transfer functions:

$$C(z) = \frac{z+1}{z^2}, \quad G(z,\mathbf{p}) = \frac{(-0.5-2p_0)z + (0.1+p_0)}{z^2 - (1+0.4p_2)z + (0.6+10p_1+2p_0)}.$$

The characteristic polynomial of the closed loop system is:

$$\delta(z,\mathbf{p}) = z^4 - (1+0.4p_2)z^3 + (0.1+10p_1)z^2 - (0.4+p_0)z + (0.1+p_0).$$

The nominal value of $\mathbf{p}^0 = [p_0^0 \; p_1^0 \; p_2^0] = [0, \; 0.1, \; 1]$. The perturbation is denoted as usual by the vector

$$\Delta\mathbf{p} = \begin{bmatrix} \Delta p_0 & \Delta p_1 & \Delta p_2 \end{bmatrix}.$$

The polynomial is Schur stable for the nominal parameter $\mathbf{p}^0$. We compute the ℓ_2 stability margin of the polynomial with weights $w_1 = w_2 = w_3 = 1$. Rewrite

$$\delta(z,\mathbf{p}^0+\Delta\mathbf{p}) = (-z+1)\Delta p_0 + 10z^2\Delta p_1 - 0.4z^3\Delta p_2 + (z^4 - 1.4z^3 + 1.1z^2 - 0.4z + 0.1)$$

and note that the degree remains invariant (=4) for all perturbations so that $\rho_d = \infty$. The stability region is the unit circle. For $z = 1$ to be a root of $\delta(z,\mathbf{p}^0+\Delta\mathbf{p})$ (see (4.14)), we have

$$\underbrace{\begin{bmatrix} 0 & 10 & -0.4 \end{bmatrix}}_{A(1)} \underbrace{\begin{bmatrix} \Delta p_0 \\ \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t(1)} = \underbrace{-0.4}_{b(1)}.$$

Thus,

$$\rho(1) = \|t^*(1)\|_2 = \left\| A^T(1)\left[A(1)A^T(1)\right]^{-1} b(1) \right\|_2 = 0.04\ .$$

Similarly, for the case of $z = -1$ (see (4.14)), we have

$$\underbrace{\begin{bmatrix} 2 & 10 & 0.4 \end{bmatrix}}_{A(-1)} \underbrace{\begin{bmatrix} \Delta p_0 \\ \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t(-1)} = \underbrace{-4}_{b(-1)}.$$

and $\rho(-1) = \|t^*(-1)\|_2 = 0.3919$. Thus $\rho_r = 0.04$.

For the case in which $\delta(z,\mathbf{p}^0+\Delta\mathbf{p})$ has a root at $z = e^{j\theta}$, $\theta \neq \pi$, $\theta \neq 0$, using (4.15), we have

$$\underbrace{\begin{bmatrix} -\cos\theta + 1 & 10\cos 2\theta & -0.4\cos 3\theta \\ -\sin\theta & 10\sin 2\theta & -0.4\sin 3\theta \end{bmatrix}}_{A(\theta)} \underbrace{\begin{bmatrix} \Delta p_0 \\ \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t(\theta)}$$

$$= \underbrace{-\begin{bmatrix} \cos 4\theta - 1.4\cos 3\theta + 1.1\cos 2\theta - 0.4\cos\theta + 0.1 \\ \sin 4\theta - 1.4\sin 3\theta + 1.1\sin 2\theta - 0.4\sin\theta \end{bmatrix}}_{b(\theta)}.$$

Thus,

$$\rho(e^{j\theta}) = \|t^*(\theta)\|_2 = \left\| A^T(\theta)\left[A(\theta)A^T(\theta)\right]^{-1} b(\theta)\right\|.$$

Figure 4.2 shows the plot of $\rho(e^{j\theta})$.

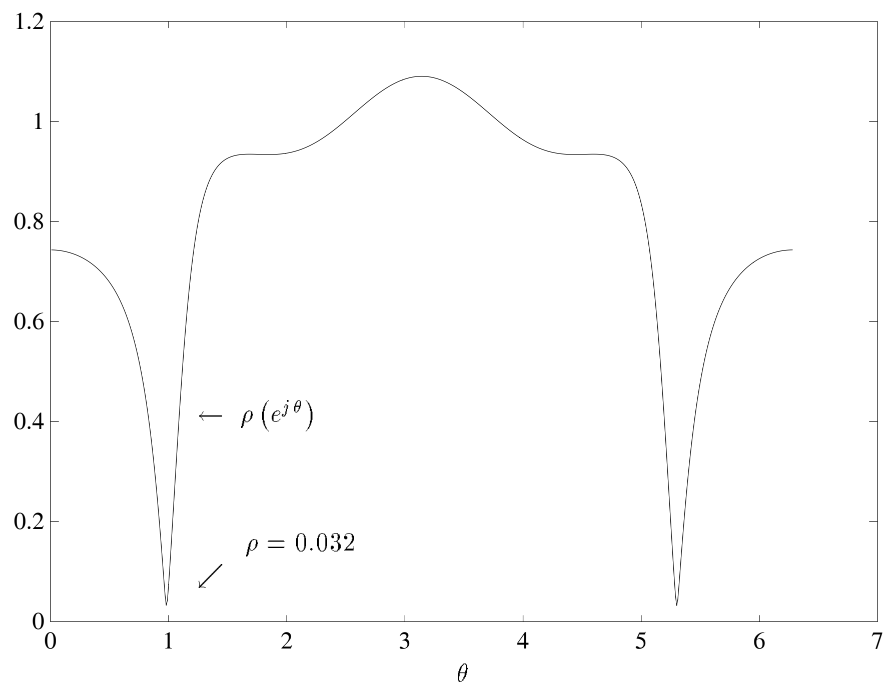

Figure 4.2. $\rho(\theta)$ (Example 4.1)

Therefore, the ℓ_2 parametric stability margin is

$$\rho_c = 0.032 = \rho_b = \rho^*.$$

Example 4.2. Consider the continuous time control system with the plant

$$G(s, \mathbf{p}) = \frac{2s + 3 - \frac{1}{3}p_1 - \frac{5}{3}p_2}{s^3 + (4 - p_2)s^2 + (-2 - 2p_1)s + (-9 + \frac{5}{3}p_1 + \frac{16}{3}p_2)}$$

and the proportional integral (PI) controller

$$C(s) = 5 + \frac{3}{s}.$$

The characteristic polynomial of the closed loop system is

$$\delta(s, \mathbf{p}) = s^4 + (4 - p_2)s^3 + (8 - 2p_1)s^2 + (12 - 3p_2)s + (9 - p_1 - 5p_2).$$

We see that the degree remains invariant under the given set of parameter variations and therefore $\rho_d = \infty$. The nominal values of the parameters are

$$\mathbf{p}^0 = [p_1^0 \;,\; p_2^0] = [0 \;,\; 0].$$

Then

$$\Delta\mathbf{p} = \left[\begin{array}{cc} \Delta p_1 & \Delta p_2 \end{array}\right] = \left[\begin{array}{cc} p_1 & p_2 \end{array}\right].$$

The polynomial is stable for the nominal parameter $\mathbf{p}^0$. Now we want to compute the ℓ_2 stability margin of this polynomial with weights $w_1 = w_2 = 1$. We first evaluate $\delta(s, \mathbf{p})$ at $s = j\omega$:

$$\delta(j\omega, \mathbf{p}^0 + \Delta\mathbf{p}) = (2\omega^2 - 1)\Delta p_1 + (j\omega^3 - 3j\omega - 5)\Delta p_2 + \omega^4 - 4j\omega^3 - 8\omega^2 + 12j\omega + 9.$$

For the case of a root at $s = 0$ (see (4.14)), we have

$$\underbrace{\left[\begin{array}{cc} -1 & -5 \end{array}\right]}_{A(0)} \underbrace{\left[\begin{array}{c} \Delta p_1 \\ \Delta p_2 \end{array}\right]}_{t(0)} = \underbrace{-9.}_{b(0)}$$

Thus,

$$\rho(0) = \|t^*(0)\|_2 = \left\| A^T(0) \left[\; A(0)A^T(0) \;\right]^{-1} b(0) \right\|_2 = \frac{9\sqrt{26}}{26}.$$

For a root at $s = j\omega$, $\omega > 0$, using the formula given in (4.15), we have with $w_1 = w_2 = 1$

$$\underbrace{\left[\begin{array}{cc} (2\omega^2 - 1) & -5 \\ 0 & (\omega^2 - 3) \end{array}\right]}_{A(j\omega)} \underbrace{\left[\begin{array}{c} \Delta p_1 \\ \Delta p_2 \end{array}\right]}_{t(j\omega)} = \underbrace{\left[\begin{array}{c} -\omega^4 + 8\omega^2 - 9 \\ 4\omega^2 - 12 \end{array}\right]}_{b(j\omega)}. \tag{4.23}$$

Here, we need to determine if there exists any ω for which the rank of the matrix $A(j\omega)$ drops. It is easy to see from the matrix $A(j\omega)$, that the rank drops when $\omega = \frac{1}{\sqrt{2}}$ and $\omega = \sqrt{3}$.

rank $A(j\omega) = 1$: For $\omega = \frac{1}{\sqrt{2}}$, we have rank $A(j\omega) = 1$ and rank $[A(j\omega), b(j\omega)] = 2$, and so there is no solution to (4.23). Thus

$$\rho\left(j\frac{1}{\sqrt{2}}\right) = \infty.$$

For $\omega = \sqrt{3}$, rank $A(j\omega) = \text{rank}\,[A(j\omega), b(j\omega)]$, and we do have a solution to (4.23). Therefore

$$\underbrace{\begin{bmatrix} 5 & -5 \end{bmatrix}}_{A(j\sqrt{3})} \underbrace{\begin{bmatrix} \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t(j\sqrt{3})} = \underbrace{6}_{b(j\sqrt{3})}.$$

Consequently,

$$\rho(j\sqrt{3}) = \left\| t^*(j\sqrt{3}) \right\|_2 = \left\| A^T(j\sqrt{3}) \left[\ A(j\sqrt{3})A^T(j\sqrt{3}) \ \right]^{-1} b(j\sqrt{3}) \right\|_2 = \frac{3\sqrt{2}}{5}.$$

rank $A(j\omega) = 2$: In this case (4.23) has a solution (which happens to be unique) and the length of the least square solution is found by

$$\begin{aligned} \rho(j\omega) = \|t^*(j\omega)\|_2 &= \left\| A^T(j\omega) \left[A(j\omega)A^T(j\omega) \right]^{-1} b(j\omega) \right\|_2 \\ &= \sqrt{\left(\frac{\omega^4 - 8\omega^2 - 11}{2\omega^2 - 1} \right)^2 + 16}. \end{aligned}$$

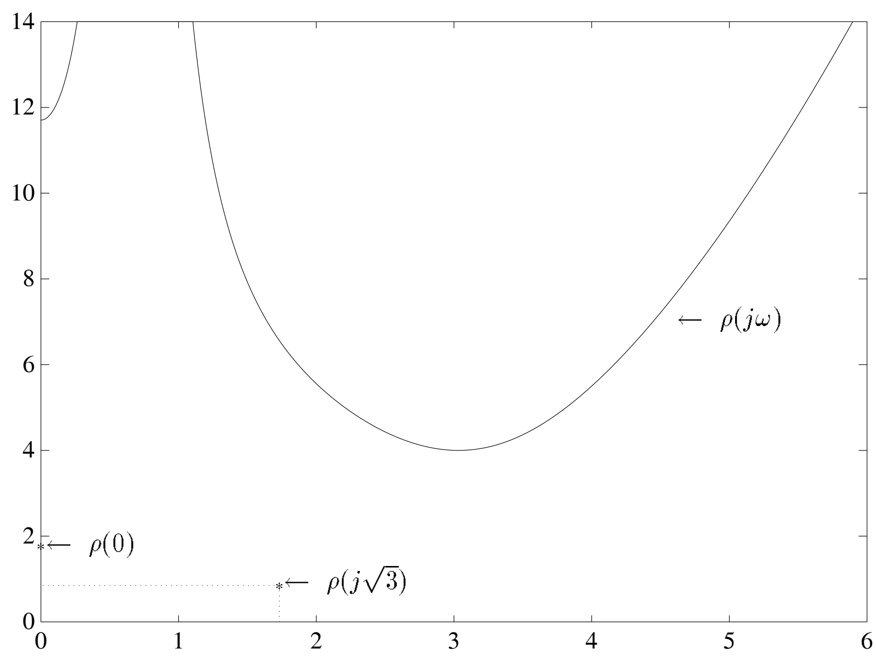

Figure 4.3. $\rho(j\omega)$, (Example 4.2)

Figure 4.3 shows the plot of $\rho(j\omega)$ for $\omega > 0$. The values of $\rho(0)$ and $\rho(j\sqrt{3})$ are also shown in Figure 4.3. Therefore,

$$\rho(j\sqrt{3}) = \rho_b = \frac{3\sqrt{2}}{5} = \rho^*$$

is the stability margin.

4.5.1 Discontinuity of the Stability Margin

In the last example we notice that the function $\rho(j\omega)$ has a discontinuity at $\omega = \omega^* = \sqrt{3}$. The reason for this discontinuity is that in the neighborhood of ω^*, the minimum norm solution of (4.15) is given by the formula for the rank 2 case. On the other hand, at $\omega = \omega^*$, the minimum norm solution is given by the formula for the rank 1 case. Thus the discontinuity of the function $\rho(j\omega)$ is due to the drop of rank from 2 to 1 of the coefficient matrix $A(j\omega)$ at ω^*. Furthermore, we have seen that if the rank of $A(j\omega^*)$ drops from 2 to 1 but the rank of $[A(j\omega^*), b(j\omega^*)]$ does not also drop, then the equation is inconsistent at ω^* and $\rho(j\omega^*)$ is infinity. In this case, this discontinuity in $\rho(j\omega)$ does not cause any problem in finding the global minimum of $\rho(j\omega)$. Therefore, the only values of ω^* that can cause a problem are those for which the rank of $[A(j\omega^*), b(j\omega^*)]$ drops to 1. Given the problem data, the occurrence of such a situation can be *predicted* by setting all 2×2 minors of the matrix $[A(j\omega), b(j\omega)]$ equal to zero and solving for the common real roots if any. These frequencies can then be treated separately in the calculation. Therefore, such discontinuities do not pose any problem from the computational point of view. Since the parameters for which rank dropping occurs lie on a proper algebraic variety, any slight and arbitrary perturbation of the parameters will dislodge them from this variety and restore the rank of the matrix. If the parameters correspond to physical quantities such arbitrary perturbations are natural and hence such discontinuities should not cause any problem from a physical point of view either.

4.5.2 ℓ_2 Stability Margin for Time-delay Systems

The results given above for determining the largest stability ellipsoid in parameter space for polynomials can be extended to quasipolynomials. This extension is useful when parameter uncertainty is present in systems containing time delays. As before, we deal with the case where the uncertain parameters appear linearly in the coefficients of the quasipolynomial.

Let us consider real quasipolynomials

$$\delta(s, \mathbf{p}) = p_1 Q_1(s) + p_2 Q_2(s) + \cdots + p_l Q_l(s) + Q_0(s) \tag{4.24}$$

where

$$Q_i(s) = s^{n_i} + \sum_{k=1}^{n_i} \sum_{j=1}^{m} a_{kj}^i s^{n_i - k} e^{-\tau_j^i s}, \qquad i = 0, 1, \cdots, l \tag{4.25}$$

and we assume that $n_0 > n_i$, $i = 1, 2, \cdots, l$ and that all parameters occurring in the equations (4.24) and (4.25) are real. Control systems containing time delays often have characteristic equations of this form (see Example 4.3).

The uncertain parameter vector is denoted $\mathbf{p} = [p_1, p_2, \cdots, p_l]$. The nominal value of the parameter vector is $\mathbf{p} = \mathbf{p}^0$, the nominal quasipolynomial $\delta(s, \mathbf{p}^0) = \delta^0(s)$ and $\mathbf{p} - \mathbf{p}^0 = \Delta\mathbf{p}$ denotes the deviation or perturbation from the nominal. The parameter vector is assumed to lie in the ball of radius ρ centered at $\mathbf{p}^0$:

$$\mathcal{B}(\rho, \mathbf{p}^0) = \{\mathbf{p} \ : \ \|\mathbf{p} - \mathbf{p}^0\|_2 < \rho\}. \tag{4.26}$$

The corresponding set of quasipolynomials is:

$$\mathbf{\Delta}_\rho(s) := \{\delta(s, \mathbf{p}^0 + \Delta\mathbf{p}) \ : \ \|\Delta\mathbf{p}\|_2 < \rho\}. \tag{4.27}$$

Recall the discussion in Chapter 1 regarding the Boundary Crossing Theorem applied to this class of quasipolynomials. From this discussion and the fact that in the family (4.24) the e^{-st} terms are associated only with the lower degree terms it follows that it is legitimate to say that each quasipolynomial defined above is Hurwitz stable if all its roots lie inside the left half of the complex plane. As before we shall say that the family is Hurwitz stable if each quasipolynomial in the family is Hurwitz. We observe that the "degree" of each quasipolynomial in $\mathbf{\Delta}_\rho(s)$ is the same since $n_0 > n_i$, and therefore by the Theorem 1.14 (Boundary Crossing Theorem applied to quasipolynomials), stability can be lost only by a root crossing the $j\omega$ axis. Accordingly, for every $-\infty < \omega < \infty$ we can introduce a set in the parameter space

$$\mathbf{\Pi}(\omega) = \{\mathbf{p} \ : \ \delta(j\omega, \mathbf{p}) = 0\}$$

This set corresponds to quasipolynomials that have $j\omega$ as a root. Of course for some particular ω this set may be empty. If $\mathbf{\Pi}(\omega)$ is nonempty we can define the distance between $\mathbf{\Pi}(\omega)$ and the nominal point $\mathbf{p}^0$:

$$\rho(\omega) = \inf_{\mathbf{p} \in \mathbf{\Pi}(\omega)} \{\|\mathbf{p} - \mathbf{p}^0\|\}.$$

If $\mathbf{\Pi}(\omega)$ is empty for some ω we set the corresponding $\rho(\omega) := \infty$. We also note that since all coefficients in (4.24) and (4.25) are assumed to be real, $\mathbf{\Pi}(\omega) = \mathbf{\Pi}(-\omega)$ and accordingly $\rho(\omega) = \rho(-\omega)$

Theorem 4.4 *The family of quasipolynomials $\mathbf{\Delta}_\rho(s)$ is Hurwitz stable if and only if the quasipolynomial $\delta^0(s)$ is stable and*

$$\rho < \rho^* = \inf_{0 \le \omega < \infty} \rho(\omega).$$

The proof of this theorem follows from the fact that the Boundary Crossing Theorem of Chapter 1 applies to the special class of degree invariant quasipolynomials $\mathbf{\Delta}_\rho(s)$. The problem of calculating $\rho(\omega)$, the ℓ_2 norm of the smallest length perturbation vector $\Delta\mathbf{p}$ for which $\delta(s, \mathbf{p}^0 + \Delta\mathbf{p})$ has a root at $j\omega$, can be solved exactly as in Section 4.5 where we dealt explicitly with the polynomial case. We illustrate this with an example.

Example 4.3. **(ℓ_2 stability margin for time delay system)** The model of a satellite attitude control system, Figure 4.4, containing a time delay in the loop, is shown.

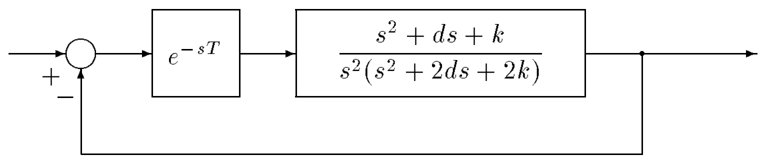

Figure 4.4. A satellite attitude control system with time-delay (Example 4.3)

The characteristic equation of the system is the quasipolynomial

$$\delta(s,\mathbf{p}) = s^4 + 2ds^3 + (e^{-sT}+2k)s^2 + e^{-sT}ds + e^{-sT}k.$$

The nominal parameters are

$$\mathbf{p}^0 = [k^0 \quad d^0] = [0.245 \quad 0.0218973] \quad \text{and} \quad T = 0.1.$$

The Hurwitz stability of the nominal system may be easily verified by applying the interlacing property. To compute the ℓ_2 real parametric stability margin around the nominal values of parameters, let

$$\Delta\mathbf{p} = [\Delta k \quad \Delta d].$$

We have

$$\begin{aligned}\delta(j\omega,\mathbf{p}^0+\Delta\mathbf{p}) = \omega^4 &- \omega^2\cos\omega T - 2k^0\omega^2 - 2\Delta k\omega^2 + d^0\omega\sin\omega T \\ &+\Delta d\omega\sin\omega T + k^0\cos\omega T + \Delta k\cos\omega T \\ &+j(-2d^0\omega^3 - 2\Delta d\omega^3 + \omega^2\sin\omega T + d^0\omega\cos\omega T \\ &+\Delta d\omega\cos\omega T - k^0\sin\omega T - \Delta k\sin\omega T)\end{aligned}$$

or

$$\underbrace{\begin{bmatrix} -2\omega^2+\cos\omega T & \omega\sin\omega T \\ -\sin\omega T & -2\omega^3+\omega\cos\omega T \end{bmatrix}}_{A(j\omega)} \underbrace{\begin{bmatrix}\Delta k \\ \Delta d\end{bmatrix}}_{t(j\omega)} =$$

$$\underbrace{\begin{bmatrix} -\omega^4+\omega^2\cos\omega T + 2k^0\omega^2 - d^0\omega\sin\omega T - k^0\cos\omega T \\ 2d^0\omega^3 - \omega^2\sin\omega T - d^0\omega\cos\omega T + k^0\sin\omega T \end{bmatrix}}_{b(j\omega)}.$$

Therefore

$$\begin{aligned}\rho(j\omega) &= \|t^*(j\omega)\|_2 \\ &= \left\| A^T(j\omega)\left[A(j\omega)A^T(j\omega)\right]^{-1} b(j\omega)\right\|_2 .\end{aligned}$$

From Figure 4.5, the minimum value of $\rho_b = 0.0146$.

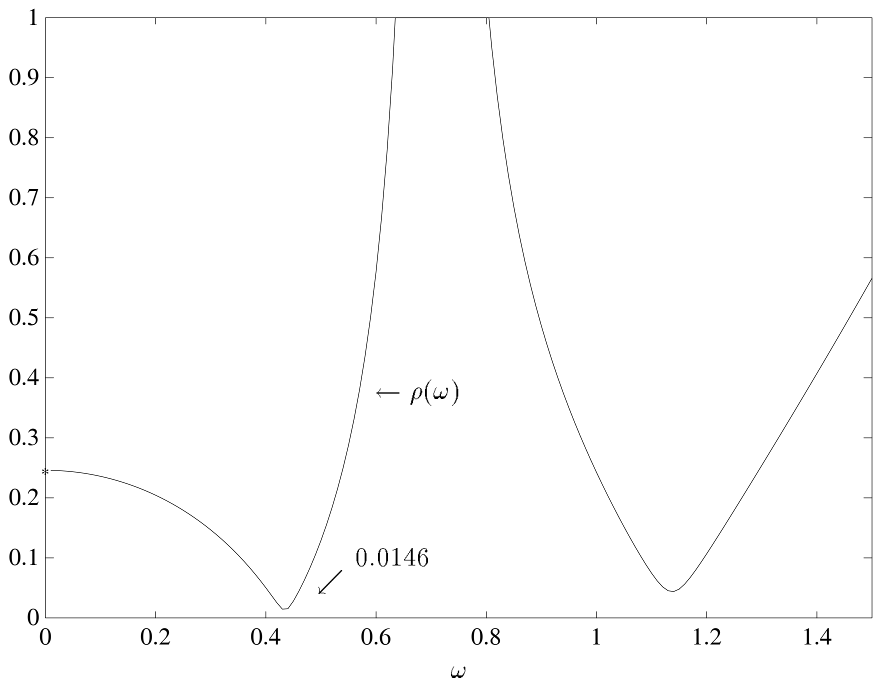

Figure 4.5. ℓ_2 stability margin for a quasipolynomial (Example 4.3)

The additional condition from the constant coefficient, corresponding to a root at $s = 0$ is

$$|\Delta k| < k^0 = 0.245.$$

Therefore, the ℓ_2 stability margin is 0.0146 .

Remark 4.2. In any specific example, the minimum of $\rho(\omega)$ needs to be searched only over a finite range of ω rather than over $[0, \infty)$. This is because the function $\rho(\omega)$ will be increasing for higher frequencies ω. This fact follows from the assumption regarding degrees $(n_0 > n_i)$ and Theorem 1.14 (Chapter 1).

4.6 ℓ_∞ AND ℓ_1 STABILITY MARGINS

If $\triangle \mathbf{p}$ is measured in the ℓ_∞ or ℓ_1 norm, we face the problem of determining the corresponding minimum norm solution of a linear equation of the type $At = b$ at each point on the stability boundary. Problems of this type can always be converted to a suitable linear programming problem. For instance, in the ℓ_∞ case, this problem is equivalent to the following optimization problem:

$$\text{Minimize } \beta$$

subject to the constraints:

$$At = b$$

$$-\beta \leq t_i \leq \beta, \qquad i = 1, 2, \cdots, l. \tag{4.28}$$

This is a standard linear programming (LP) problem and can therefore be solved by existing, efficient algorithms.

For the ℓ_1 case, we can similarly formulate a linear programming problem by introducing the variables,

$$|t_i| := y_i^+ - y_i^-, \qquad i = 1, 2, \cdots, l$$

Then we have the LP problem:

$$\text{Minimize} \sum_{i=1}^{l} \{y_i^+ - y_i^-\} \tag{4.29}$$

subject to

$$At = b$$

$$y_i^- - y_i^+ \leq t_i \leq y_i^+ - y_i^- \qquad i = 1, 2, \cdots, l.$$

We do not elaborate further on this approach. The reason is that ℓ_∞ and ℓ_1 cases are special cases of polytopic families of perturbations. We deal with this general class next.

4.7 POLYTOPIC FAMILIES

In this section we deal with the case where each component p_i of the real parameter vector $\mathbf{p} := [p_1, p_2, \cdots, p_l]^T$ can vary independently of the other components. In other words, we assume that $\mathbf{p}$ lies in an uncertainty set which is box-like:

$$\mathbf{\Pi} := \{\mathbf{p} : p_i^- \leq p_i \leq p_i^+, \quad i = 1, 2, \cdots, l\}. \tag{4.30}$$

Represent the system characteristic polynomial

$$\delta(s) := \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n \tag{4.31}$$

by the vector $\underline{\delta} := [\delta_0, \delta_1, \cdots, \delta_n]^T$. We assume that each component δ_i of $\underline{\delta}$, is a *linear* function of $\mathbf{p}$. To be explicit,

$$\begin{aligned}\delta(s,\mathbf{p}) &:= \delta_0(\mathbf{p}) + \delta_1(\mathbf{p})s + \delta_2(\mathbf{p})s^2 + \cdots + \delta_n(\mathbf{p})s^n \\ &:= p_1 Q_1(s) + p_2 Q_2(s) + \cdots + p_l Q_l(s) + Q_0(s).\end{aligned}$$

By equating coefficients of like powers of s we can write this as

$$\underline{\delta} = T\mathbf{p} + b,$$

where $\underline{\delta}$, $\mathbf{p}$ and b are column vectors, and

$$T : \mathbb{R}^l \longrightarrow \mathbb{R}^{n+1}$$

is a linear map. In other words, $\underline{\delta}$ is a linear (or affine) transformation of $\mathbf{p}$. Now introduce the coefficient set

$$\mathbf{\Delta} := \{\underline{\delta} : \underline{\delta} = T\mathbf{p} + b, \quad \mathbf{p} \in \mathbf{\Pi}\} \tag{4.32}$$

of the set of polynomials

$$\mathbf{\Delta}(s) := \{\delta(s,\mathbf{p}) : \mathbf{p} \in \mathbf{\Pi}\}. \tag{4.33}$$

4.7.1 Exposed Edges and Vertices

To describe the geometry of the set $\mathbf{\Delta}$ (equivalently $\mathbf{\Delta}(s)$) we introduce some basic facts regarding polytopes. Note that the set $\mathbf{\Pi}$ is in fact an example of a special kind of *polytope*. In general, a polytope in n-dimensional space is the convex hull of a set of points called *generators* in this space. A set of generators is *minimal* if removal of any point from the set alters the convex hull. A minimal set of generators is unique and constitutes the *vertex set* of the polytope. Consider the special polytope $\mathbf{\Pi}$, which we call a *box*. The vertices $\mathbf{V}$ of $\mathbf{\Pi}$ are obtained by setting each p_i to p_i^+ or p_i^-:

$$\mathbf{V} := \{\mathbf{p} : p_i = p_i^- \text{ or } p_i = p_i^+, \quad i = 1, 2, \cdots, l\}. \tag{4.34}$$

The *exposed edges* $\mathbf{E}$ of the box $\mathbf{\Pi}$ are defined as follows. For fixed i,

$$\mathbf{E}_i := \{\mathbf{p} : p_i^- \leq p_i \leq p_i^+, \quad p_j = p_j^- \text{ or } p_j^+, \quad \text{for all } j \neq i\} \tag{4.35}$$

then

$$\mathbf{E} := \cup_{i=1}^{l} \mathbf{E}_i. \tag{4.36}$$

We use the notational convention

$$\mathbf{\Delta} = T\mathbf{\Pi} + b \tag{4.37}$$

as an alternative to (4.32).

Lemma 4.1 *Let* $\mathbf{\Pi}$ *be a box and* T *a linear map, then* $\mathbf{\Delta}$ *is a polytope. If* $\mathbf{\Delta}_{\mathrm{V}}$ *and* $\mathbf{\Delta}_{\mathrm{E}}$ *denote the vertices and exposed edges of* $\mathbf{\Delta}$ *and* $\mathbf{E}$ *and* $\mathbf{V}$ *denote the exposed edges and vertices of* $\mathbf{\Pi}$*, we have*

$$\mathbf{\Delta}_{\mathrm{V}} \subset T\mathbf{V} + b$$
$$\mathbf{\Delta}_{\mathrm{E}} \subset T\mathbf{E} + b.$$

This lemma shows us that the polynomial set $\mathbf{\Delta}(s)$ is a *polytopic family* and that the vertices and exposed edges of $\mathbf{\Delta}$ can be obtained by mapping the vertices and exposed edges of $\mathbf{\Pi}$. Since $\mathbf{\Pi}$ is an axis parallel box its vertices and exposed edges are easy to identify. For an arbitrary polytope in n dimensions, it is difficult to distinguish the "exposed edges" computationally. This lemma is therefore useful even though the mapped edges and vertices of $\mathbf{\Pi}$ contain more elements than only the vertices and exposed edges of $\mathbf{\Delta}$. These facts are now used to characterize the image set of the family $\mathbf{\Delta}(s)$. Fixing $s = s^*$, we let $\mathbf{\Delta}(s^*)$ denote the *image set* of the points $\delta(s^*, \mathbf{p})$ in the complex plane obtained by letting $\mathbf{p}$ range over $\mathbf{\Pi}$:

$$\mathbf{\Delta}(s^*) := \{\delta(s^*, \mathbf{p}) : \mathbf{p} \in \mathbf{\Pi}\}. \tag{4.38}$$

Introduce the *vertex polynomials*:

$$\mathbf{\Delta}_{\mathrm{V}}(s) := \{\delta(s, \mathbf{p}) : \mathbf{p} \in \mathbf{V}\}$$

and the *edge polynomials*:

$$\mathbf{\Delta}_{\mathrm{E}}(s) := \{\delta(s, \mathbf{p}) : \mathbf{p} \in \mathbf{E}\}$$

Their respective images at $s = s^*$ are

$$\mathbf{\Delta}_{\mathrm{V}}(s^*) := \{\delta(s^*, \mathbf{p}) : \mathbf{p} \in \mathbf{V}\} \tag{4.39}$$

and

$$\mathbf{\Delta}_{\mathrm{E}}(s^*) := \{\delta(s^*, \mathbf{p}) : \mathbf{p} \in \mathbf{E}\}. \tag{4.40}$$

The set $\mathbf{\Delta}(s^*)$ is a convex polygon in the complex plane whose vertices and exposed edges originate from the vertices and edges of $\mathbf{\Pi}$. More precisely, we have the following lemma which describes the geometry of the image set in this polytopic case. Let *co* $(\cdot)$ denote the convex hull of the set $(\cdot)$.

Lemma 4.2

1) $\mathbf{\Delta}(s^*) = co\ (\mathbf{\Delta}_{\mathrm{V}}(s^*))$,

2) The vertices of $\mathbf{\Delta}(s^*)$ *are contained in* $\mathbf{\Delta}_{\mathrm{V}}(s^*)$,

3) The exposed edges of $\mathbf{\Delta}(s^*)$ *are contained in* $\mathbf{\Delta}_{\mathrm{E}}(s^*)$.

This lemma states that the vertices and exposed edges of the complex plane polygon $\mathbf{\Delta}(s^*)$ are contained in the images at s^* of the mapped vertices and edges of the box $\mathbf{\Pi}$. We illustrate this in the Figure 4.6.

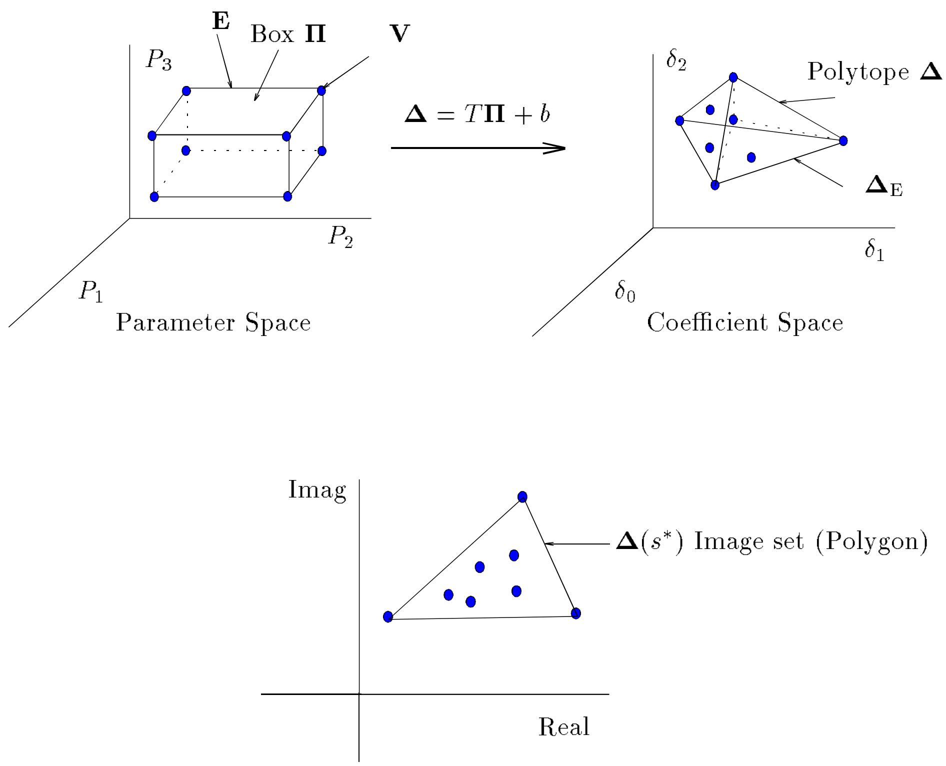

Figure 4.6. Vertices and edges of $\mathbf{\Pi}, \mathbf{\Delta}$ and $\mathbf{\Delta}(s^*)$

The above results will be useful in determining the robust stability of the family $\mathbf{\Delta}(s)$ with respect to an open stability region $\mathcal{S}$. In fact they are the key to establishing the important result that the stability of a polytopic family can be determined from its edges.

Assumption 4.1. Assume the family of polynomials $\mathbf{\Delta}(s)$ is of

1) constant degree, and

2) there exists at least one point $s^0 \in \partial\mathcal{S}$ such that $0 \notin \mathbf{\Delta}(s^0)$.

Theorem 4.5 *Under the above assumptions $\mathbf{\Delta}(s)$ is stable with respect to $\mathcal{S}$ if and only if $\mathbf{\Delta}_{\mathrm{E}}(s)$ is stable with respect to $\mathcal{S}$.*

Proof. Following the image set approach described in Section 4.3, it is clear that under the assumption of constant degree and the existence of at least one stable polynomial in the family, the stability of $\mathbf{\Delta}(s)$ is guaranteed if the image set $\mathbf{\Delta}(s^*)$ excludes the origin for every $s^* \in \partial\mathcal{S}$. Since $\mathbf{\Delta}(s^0)$ excludes the origin it follows that $\mathbf{\Delta}(s^*)$ excludes the origin for every $s^* \in \partial\mathcal{S}$ if and only if the edges of $\mathbf{\Delta}(s^*)$ exclude the origin. Here we have implicitly used the assumption that the image set moves continuously with respect to s^*. From Lemma 4.2, this is equivalent to the condition that $\mathbf{\Delta}_{\mathrm{E}}(s^*)$ excludes the origin for every $s^* \in \partial\mathcal{S}$. This condition is finally equivalent to the stability of the set $\mathbf{\Delta}_{\mathrm{E}}(s)$. ♣

Theorem 4.5 has established that to determine the stability of $\mathbf{\Delta}(s)$, it suffices to check the stability of the exposed edges. The stability verification of a multiparameter family is therefore reduced to that of a set of one parameter families. In fact this is precisely the kind of problem that was solved in Chapter 2 by the Segment Lemma and the Bounded Phase Lemma. In the next subsection we elaborate on the latter approach.

4.7.2 Bounded Phase Conditions for Checking Robust Stability of Polytopes

From a computational point of view it is desirable to introduce some further simplification into the problem of verifying the stability of a polytope of polynomials. Note that from Theorem 4.2 (Zero Exclusion Principle), to verify robust stability we need to determine whether or not the image set $\mathbf{\Delta}(s^*)$ *excludes* the origin for every point $s^* \in \partial\mathcal{S}$. This computation is particularly easy since $\mathbf{\Delta}(s^*)$ is a *convex polygon*. In fact, a convex polygon in the complex plane excludes the origin if and only if the angle subtended at the origin by its vertices is less than π radians (180°).

Consider the convex polygon $\mathcal{P}$ in the complex plane with vertices $[v_1, v_2, \cdots] := \mathbf{V}$. Let p_0 be an arbitrary point in $\mathcal{P}$ and define

$$\phi_{v_i} := \arg\left(\frac{v_i}{p_0}\right). \tag{4.41}$$

We adopt the convention that every angle lies between $-\pi$ and $+\pi$. Now define

$$\phi^+ := \sup_{v_i \in \mathbf{V}} \phi_{v_i}, \qquad 0 \le \phi^+ \le \pi$$
$$\phi^- := \inf_{v_i \in \mathbf{V}} \phi_{v_i}, \qquad -\pi < \phi^- \le 0.$$

The *angle subtended* at the origin by $\mathcal{P}$ is given by

$$\Phi_{\mathcal{P}} := \phi^+ - \phi^-. \tag{4.42}$$

$\mathcal{P}$ excludes the origin if and only if $\Phi_{\mathcal{P}} < \pi$. Applying this fact to the convex polygon $\mathbf{\Delta}(s^*)$, we can easily conclude the following.

Theorem 4.6 (Bounded Phase Theorem)
Under the assumptions

a) every polynomial in $\mathbf{\Delta}(s)$ *is of the same degree* ($\delta_n(\mathbf{p}) \neq 0, \mathbf{p} \in \mathbf{\Pi}$),

b) at least one polynomial in $\mathbf{\Delta}(s)$ *is stable with respect to* $\mathcal{S}$,

the set of polynomials $\mathbf{\Delta}(s)$ *is stable with respect to the open stability region* $\mathcal{S}$ *if and only if*

$$\Phi_{\mathbf{\Delta}_{\mathrm{V}}}(s^*) < \pi, \qquad \textit{for all } s^* \in \partial\mathcal{S}. \tag{4.43}$$

The computational burden imposed by this theorem is that we need to evaluate the maximal phase difference across the vertex of $\mathbf{\Pi}$. The condition (4.43) will be referred to conveniently as the Bounded Phase Condition. We illustrate Theorems 4.5 and 4.6 in the example below.

Example 4.4. Consider the standard feedback control system with the plant

$$G(s) = \frac{s+a}{s^2+bs+c}$$

where the parameters vary within the ranges:

$$a \in [1,2] = [a^-, a^+],$$
$$b \in [9,11] = [b^-, b^+],$$
$$c \in [15,18] = [c^-, c^+]$$

The controller is

$$C(s) = \frac{3s+2}{s+5}.$$

We want to examine whether this controller robustly stabilizes the plant. The closed loop characteristic polynomial is:

$$\delta(s) = a(3s+2) + b(s^2+5s) + c(s+5) + (s^3+8s^2+2s).$$

The polytope of polynomials whose stability is to be checked is:

$$\mathbf{\Delta}(s) = \left\{\delta(s) \ : \ a \in [a^-, a^+], \ \ b \in [b^-, b^+], \ \ c \in [c^-, c^+]\right\}$$

We see that the degree is invariant over the uncertainty set. From Theorem 4.5 this polytope is stable if and only if the exposed edges are. Here, we write the 12 edge polynomials $\mathbf{\Delta}_{\mathrm{E}}(s)$:

$$\delta_{E_1}(s) = \left(\lambda a^- + (1-\lambda)a^+\right)(3s+2) + b^-(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_2}(s) = \left(\lambda a^- + (1-\lambda)a^+\right)(3s+2) + b^-(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_3}(s) = \left(\lambda a^- + (1-\lambda)a^+\right)(3s+2) + b^+(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_4}(s) = \left(\lambda a^- + (1-\lambda)a^+\right)(3s+2) + b^+(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_5}(s) = a^-(3s+2) + \left(\lambda b^- + (1-\lambda)b^+\right)(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_6}(s) = a^-(3s+2) + \left(\lambda b^- + (1-\lambda)b^+\right)(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_7}(s) = a^+(3s+2) + \left(\lambda b^- + (1-\lambda)b^+\right)(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_8}(s) = a^+(3s+2) + \left(\lambda b^- + (1-\lambda)b^+\right)(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_9}(s) = a^-(3s+2) + b^-(s^2+5s) + \left(\lambda c^- + (1-\lambda)c^+\right)(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_{10}}(s) = a^-(3s+2) + b^+(s^2+5s) + \left(\lambda c^- + (1-\lambda)c^+\right)(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_{11}}(s) = a^+(3s+2) + b^-(s^2+5s) + \left(\lambda c^- + (1-\lambda)c^+\right)(s+5) + (s^3+8s^2+2s)$$
$$\delta_{E_{12}}(s) = a^+(3s+2) + b^+(s^2+5s)\left(\lambda c^- + (1-\lambda)c^+\right)(s+5) + (s^3+8s^2+2s).$$

The robust stability of $\mathbf{\Delta}(s)$ is equivalent to the stability of the above 12 edges. From Theorem 4.6, it is also equivalent to the condition that $\mathbf{\Delta}(s)$ has at least one stable polynomial, that $0 \notin \mathbf{\Delta}(j\omega)$ for at least one ω and that the *maximum phase difference* over the vertex set is less than $180°$ for any frequency ω (bounded phase condition). At $\omega = 0$ we have

$$0 \notin \mathbf{\Delta}(j0) = 2a + 5c = 2[1,2] + 5[15,18].$$

Also it may be verified that the center point of the box is stable. Thus we examine the maximum phase difference,evaluated at each frequency, over the following eight vertex polynomials:

$$\delta_{v_1}(s) = a^-(3s+2) + b^-(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_2}(s) = a^-(3s+2) + b^-(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_3}(s) = a^-(3s+2) + b^+(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_4}(s) = a^-(3s+2) + b^+(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_5}(s) = a^+(3s+2) + b^-(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_6}(s) = a^+(3s+2) + b^-(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_7}(s) = a^+(3s+2) + b^+(s^2+5s) + c^-(s+5) + (s^3+8s^2+2s)$$
$$\delta_{v_8}(s) = a^+(3s+2) + b^+(s^2+5s) + c^+(s+5) + (s^3+8s^2+2s).$$

Figure 4.7 shows that the maximum phase difference over all vertices does not reach $180°$ at any ω. Therefore, we conclude that the controller $C(s)$ robustly stabilizes the plant $G(s)$.

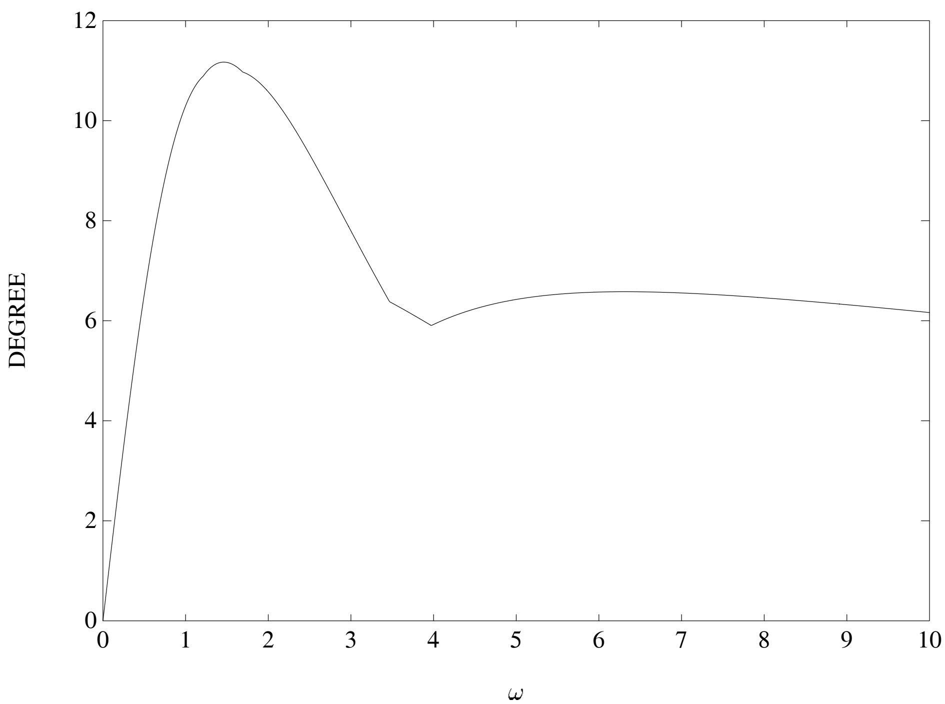

Figure 4.7. Maximum phase differences of vertices (Example 4.4)

In the next example we show how the maximal stability box can be found using the Bounded Phase Condition.

Example 4.5. Continuing with the previous example (Example 4.4), suppose that we now want to expand the range of parameter perturbations and determine the largest size box of parameters stabilized by the controller given. We define a variable sized parameter box as follows:

$$a \in [a^- - \epsilon, a^+ + \epsilon], \quad b \in [b^- - \epsilon, b^+ + \epsilon], \quad c \in [c^- - \epsilon, c^+ + \epsilon].$$

We can easily determine the maximum value of ϵ for which robust stability is preserved by simply applying the Bounded Phase Condition while increasing the value of ϵ. Figure 4.8 shows that the maximum phase difference over the vertex set (attained at some frequency) plotted as a function of ϵ. We find that at $\epsilon = 6.5121$, the phase difference over the vertex set reaches 180° at $\omega = 1.3537$. Therefore, we conclude that the maximum value of ϵ is 6.5120. This means that the controller $C(s)$ can robustly stabilize the family of plants $G(s, \mathbf{p})$ where the ranges of the parameters are

$$a \in [-5.512, 8.512], \quad b \in [2.488, 17.512], \quad c \in [8.488, 24.512].$$

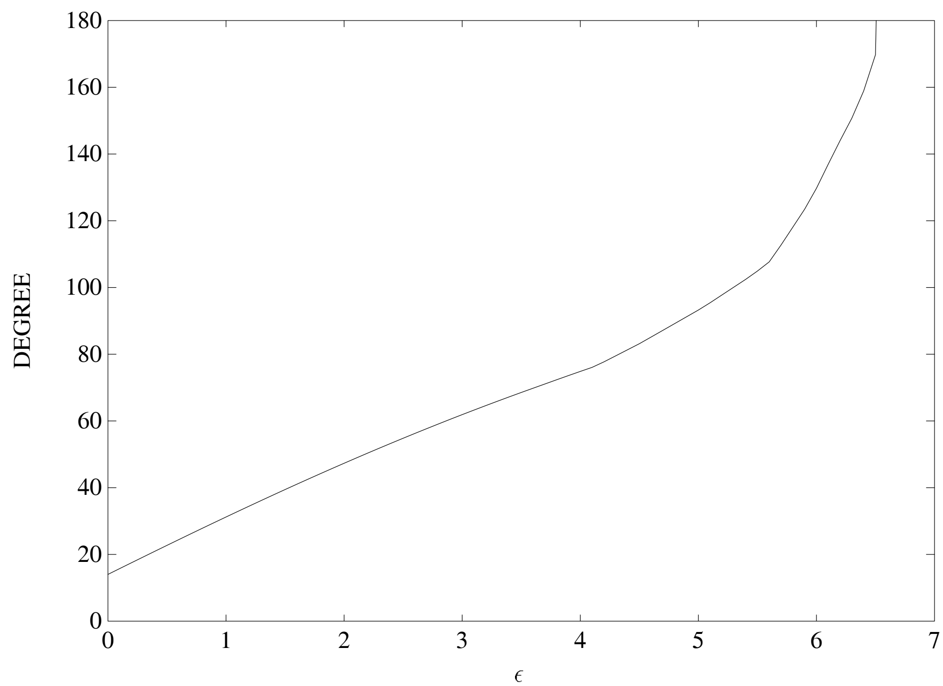

Figure 4.8. Maximum phase differences of vertices vs. ϵ (Example 4.5)

The purpose of the next example is to draw the reader's attention to the importance of the Assumptions 4.1 under which Theorem 4.5 is valid. The example illustrates that a polytopic family can have exposed edges that are stable even though the family is unstable, when the second item in Assumptions 4.1 fails to hold. In particular it cautions the reader that it is important to verify that the image set excludes the origin at least at one point on the stability boundary.

Example 4.6. Consider the family of polynomials

$$\mathbf{\Delta}(s, \mathbf{p}) := s + p_1 - jp_2$$

where $p_1 \in [1,2]$ and $p_2 \in [0,1]$. Suppose that the stability region is the *outside* of the shaded circle in Figure 4.9:

$$\mathcal{S} := \{s \in \mathbb{C} \ : \ |s + (-1.5 + j0.25)| > 0.2\}$$

The vertex polynomials are

$$\mathbf{\Delta}_{\mathrm{V}}(s) := \{s+1, s+1-j, s+2, s+2-j\}.$$

The root loci of the elements of $\mathbf{\Delta}_{\mathrm{E}}(s)$ is shown in Figure 4.9 along with the instability region (shaded region). Figure 4.9 shows that all elements of $\mathbf{\Delta}_{\mathrm{E}}(s)$ are stable with respect to the stability region $\mathcal{S}$. One might conclude from this, in view of the theorem, that the set $\mathbf{\Delta}(s)$ is stable.

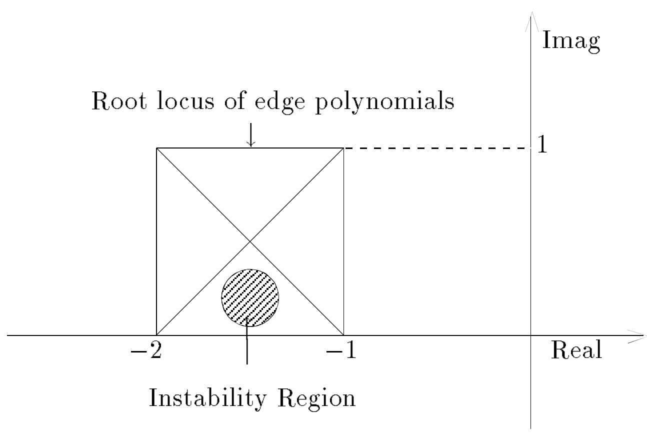

Figure 4.9. Instability region and edges (Example 4.6)

However, this is not true as $\delta(s, \mathbf{p})$ is unstable at $p_1 = 1.5$, $p_2 = 0.25$. The apparent contradiction with the conclusion of the theorem is because the theorem attempts to ascertain the presence of unstable elements in the family by detecting Boundary Crossing. In this particular example, no Boundary Crossing occurs as we can see from the plots of the image sets of the exposed edges evaluated along the boundary of $\mathcal{S}$ (see Figure 4.10). However, we can easily see that item 2) of Assumptions 4.1 is violated since there is *no frequency* at which the origin is excluded from the image set. Therefore, there is no contradiction with the theorem.

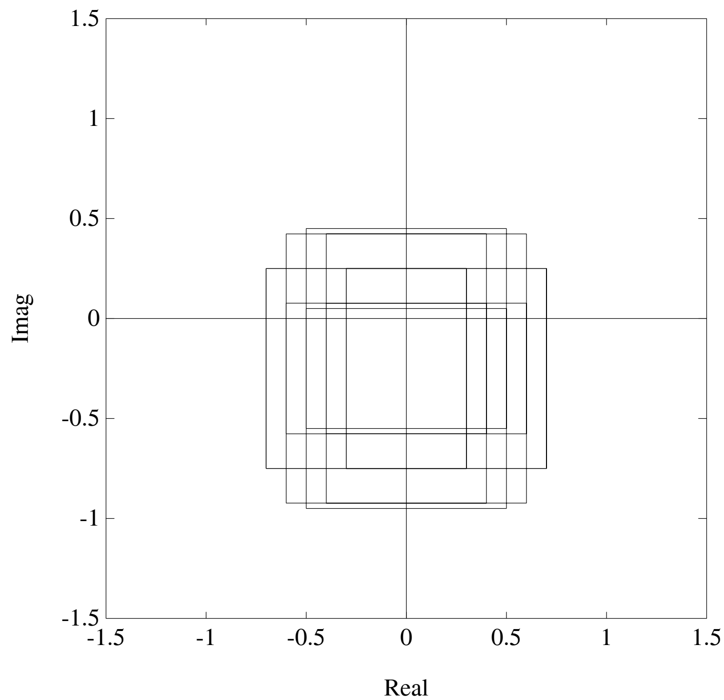

Figure 4.10. Image sets of edges evaluated along $\partial\mathcal{S}$ include the origin (Example 4.6)

The next example deals with the determination of robust Schur stability.

Example 4.7. Consider the discrete time control system, in the standard feedback configuration with controller and plant transfer functions given as

$$C(z) = \frac{1}{z^2} \text{ and } G(z, \mathbf{p}) = \frac{p_2 - p_1 z}{z^2 - (p_1 + 0.23)z - 0.37}.$$

The characteristic polynomial of the closed loop system is

$$\delta(z, \mathbf{p}) = z^4 - (p_1 + 0.23)z^3 - 0.37z^2 - p_1 z + p_2.$$

The nominal values of parameters are $p_1^0 = 0.17$ and $p_2^0 = 0.265$. It is verified that the polynomial $\delta(z, p_1^0, p_2^0)$ is Schur stable. We want now to determine the maximum perturbation that the parameters can undergo without losing stability. One way to estimate this is to determine the maximum value of ϵ (stability margin) such that $\delta(z, \mathbf{p})$ remains stable for all

$$p_1 \in [p_1^0 - \epsilon, p_1^0 + \epsilon], \qquad p_2 \in [p_2^0 - \epsilon, p_2^0 + \epsilon].$$

This can be accomplished by checking the Schur stability of the exposed edges for each fixed value of ϵ. It can also be done by computing the maximum phase difference over the 4 vertices at each point on the stability boundary, namely the unit circle. The smallest value of ϵ that makes an edge unstable is the stability margin. Figure 4.11 shows that at $\epsilon = 0.1084$ the Bounded Phase Condition is violated and the maximum phase difference of the vertices reaches 180°. This implies that an edge becomes unstable for $\epsilon = 0.1084$. Therefore, the stability margin is < 0.1084.

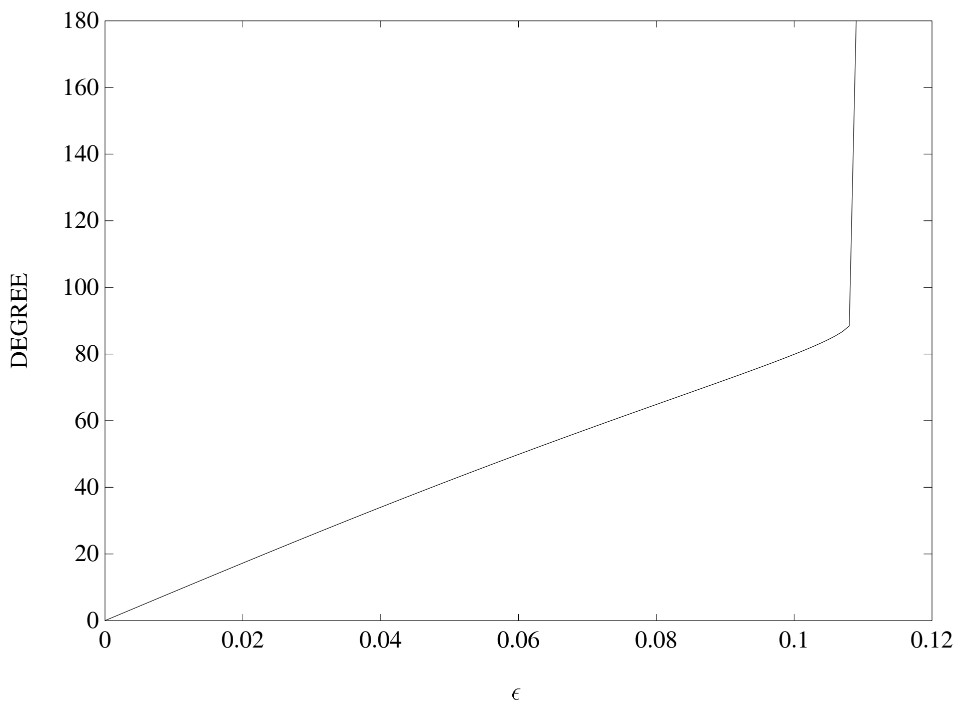

Figure 4.11. Maximum phase difference vs. ϵ (Example 4.7)

The next example deals with a control system where each root is required to remain confined in a stability region. It also shows how the controller can be chosen to maximize the stability margin.

Example 4.8. In a digital tape transport system, the open loop transfer function between tape position and driving motor supply voltage is

$$G(s) = \frac{n_0 + n_1 s}{d_0 + d_1 s + d_2 s^2 + d_3 s^3 + d_4 s^4 + d_5 s^5}.$$

The coefficients n_i and d_i depend linearly on two uncertain physical parameters, K (elastic constant of the tape) and D (friction coefficient). The parameter vector is

$$\mathbf{p} = [D, K]^T$$

with the nominal parameter values

$$\mathbf{p}^0 = [D^0, K^0]^t = [20, 4 \times 10^4]^T.$$

Considering a proportional controller

$$C(s) = K_p$$

the closed loop characteristic polynomial is written as

$$\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_5 s^5.$$

The equation connecting $\underline{\delta} = [\delta_5, \cdots, \delta_0]^T$ and $\mathbf{p}$ is

$$\underbrace{\begin{bmatrix} \delta_5 \\ \delta_4 \\ \delta_3 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{bmatrix}}_{\underline{\delta}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0.025 & 0 \\ 0.035 & 0.25 \times 10^{-4} \\ 0.01045 & 0.35 \times 10^{-4} \\ 0.03K_p & 0.1045 \times 10^{-4} \\ 0 & 0.3 \times 10^{-4} K_p \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} D \\ K \end{bmatrix}}_{t} + \underbrace{\begin{bmatrix} 1 \\ 2.25 \\ 1.5225 \\ 0.2725 \\ 0 \\ 0 \end{bmatrix}}_{b}.$$

In the following, we choose as the "stability" region $\mathcal{S}$ the union of two regions in the complex plane ($s = \sigma + j\omega$):

1) a disc $\mathcal{D}$ centered on the negative real axis at $\sigma = -0.20$ and with radius $R = 0.15$ (containing closed loop dominant poles);

2) a half plane $\mathcal{H}$: $\sigma < -0.50$.

The boundary ∂S of the "stability region" is the union of two curves which are described (in the upper half plane) by the following maps $B_{\mathcal{D}}$ and $B_{\mathcal{H}}$:

$$\partial\mathcal{D} : s = B_{\mathcal{D}}(\gamma) = (-0.20 + 0.15\cos\gamma) + j(0.15\sin\gamma), \qquad 0 \le \gamma \le \pi$$
$$\partial\mathcal{H} : s = B_{\mathcal{H}}(\omega) = -0.50 + j\omega, \qquad 0 \le \omega < \infty$$

Our objective is to obtain the parameter K_p of the maximally robust controller, in the sense that it guarantees "stability" with respect to the above region while maximizing the size of a perturbation box centered at $\mathbf{p}^0$. In other words we seek to maximize the perturbation range tolerated by the system subject to the requirement that the closed loop poles remain in the prescribed region $\mathcal{S}$.

To do this we first determine the range of controller parameter values which ensure that the nominal system has roots in $\mathcal{S}$. This can be found by a root locus plot of the system with respect to the controller parameter K_p with the plant parameters held at their nominal values. This root locus plot is given in Figure 4.12. The range is found to be

$$K_p \in [0.0177, 0.0397].$$

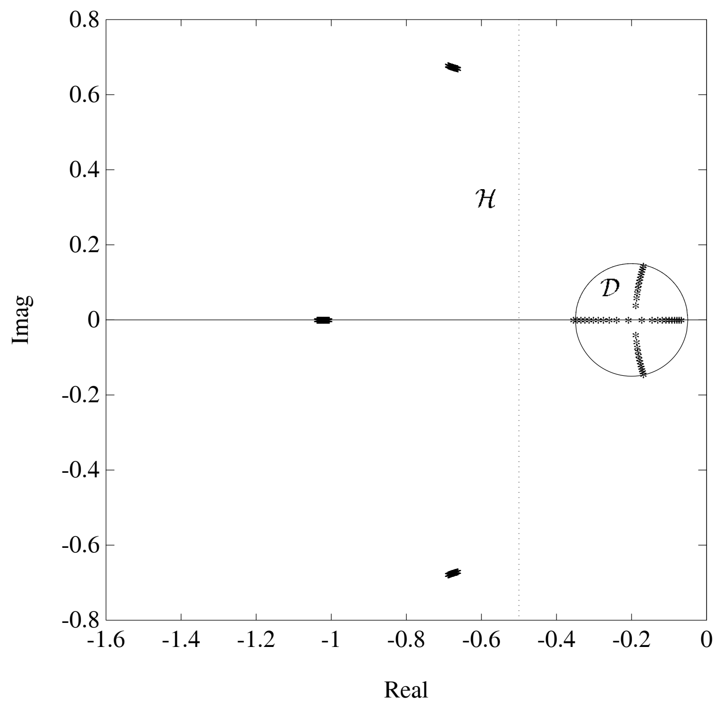

Figure 4.12. Root locus of the nominal closed system with respect to K_p (Example 4.8)

Then for each setting of the controller parameter in this range we determine the ℓ_∞ margin by determining the maximal phase difference over the vertex set evaluated over the stability boundary defined above. In this manner we sweep over the controller parameter range. The maximally robust controller is found to be

$$C(s) = K_p^* \approx 0.0397.$$

The maximal ℓ_∞ stability margin obtained is

$$\rho^* \approx 14.76.$$

Figure 4.13 shows that the optimal parameter K_p^* which produces the largest ρ ($\rho^* \approx 14.76$) locates the closed loop poles, evaluated at the nominal point, *on the stability boundary.* As the plant parameters perturb away from the nominal the roots move inward into the stability region. However this is true only under the assumption that the controller parameter K_p is *not subject to perturbation.* In fact a slight perturbation of the controller parameter set at $K_p = K_p^*$ sends the roots out of the "stability" region as the root locus plot has shown. This highlights the fact that *robustification with respect to the plant parameters* has been obtained at the expense of *enhanced sensitivity* with respect to the controller parameters. This could have been avoided by including the controller parameter K_p as an additional uncertain parameter and searching for a nominal point in the augmented parameter space where the stability margin is large enough.

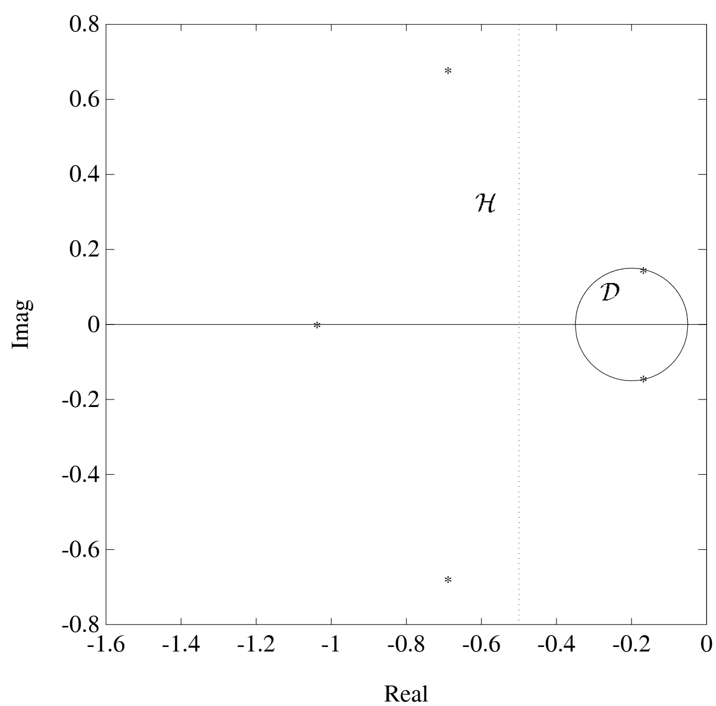

Figure 4.13. Root locations of the closed loop system at K_p^* (Example 4.8)

In the next example we again consider the problem of adjusting a controller gain to enhance the robustness with respect to plant parameters.

Example 4.9. Consider the plant and feedback controller

$$G(s) = \frac{1}{s^2 + p_1 s + p_2}, \qquad C(s) = \frac{K}{s}$$

where the nominal parameters are

$$\mathbf{p}^0 = [p_1^0 \quad p_2^0] = [5 \quad 1].$$

We find that $0 < K < 5$ stabilizes the nominal plant $G(s)$. In this example we want to find the optimum value of K such that the ℓ_2 stability margin ρ^* with respect to the parameters p_1, p_2 is the largest possible We first evaluate the ℓ_2 stability margin in terms of K. The characteristic polynomial is

$$\delta(s, \mathbf{p}, K) = s^3 + p_1 s^2 + p_2 s + K.$$

Thus we have

$$\delta(j\omega, \mathbf{p}, K) = -j\omega^3 - p_1\omega^2 + jp_2\omega + K = 0$$

and

$$\underbrace{\begin{bmatrix} -\omega^2 & 0 \\ 0 & \omega \end{bmatrix}}_{A(\omega)} \underbrace{\begin{bmatrix} \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t} = \underbrace{\begin{bmatrix} -K + p_1^0\omega^2 \\ \omega^3 - \omega p_2^0 \end{bmatrix}}_{b(\omega, \mathbf{p}^0, K)}.$$

Therefore,

$$\rho(K) = \min_{\omega} \left\| A(\omega)^T \left(A(\omega)A(\omega)^T \right)^{-1} b(\omega, \mathbf{p}^0, K) \right\|_2$$

and the optimum value K^* corresponds to

$$\rho^* = \max_{K} \rho(K).$$

Figure 4.14 shows that the maximum value of the stability margin ρ^* is achieved as K tends to 0.

Again, it is interesting to observe that as K tends to 0 the system has a real root that tends to the imaginary axis. Therefore, at the same time that the real parametric stability margin is *increasing to its maximal value*, the robustness with respect to the parameter K is *decreasing drastically*. In fact as K tends to 0 the "optimum" value with respect to perturbations in $\mathbf{p} = [p_1, p_2]$, it will also be true that the slightest perturbation of the parameter K itself will destabilize the system. This is similar to what was observed in the previous example. It suggests the general precautionary guideline, that robustness with respect to plant parameters is often obtained at the expense of enhanced sensitivity with respect to the controller parameters.

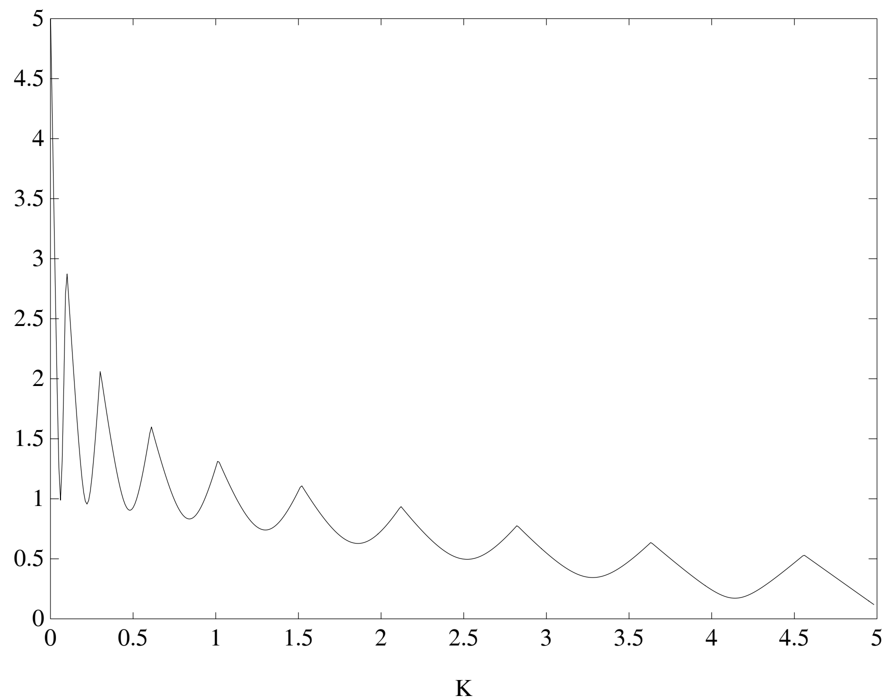

Figure 4.14. $\rho(K)$ vs. K (Example 4.9)

4.7.3 Polytopes of Complex Polynomials

The edge and vertex results developed above for a *real* polytope $\mathbf{\Delta}(s)$ carry over to a polytope of complex polynomials. This is because in either case the image set $\mathbf{\Delta}(s^*)$ is a polygon in the complex plane whose edges are contained in the set of images of the exposed edges of $\mathbf{\Delta}(s)$. Therefore when the degree remains invariant, stability of the exposed edges implies that of the entire polytope. Thus the stability of a complex polytopic family

$$a_1Q_1(s) + a_2Q_2(s) + \cdots + a_mQ_m(s)$$

with $a_i = \alpha_i + j\beta_i$ for $\alpha_i^- \leq \alpha_i \leq \alpha_i^+$, $\beta_i^- \leq \beta_i \leq \beta_i^+$, $i = 1, \cdots, m$ and $Q_i(s)$ being complex polynomials can be tested by considering the parameter vector $\mathbf{p} := [\alpha_1, \beta_1, \cdots, \alpha_m, \beta_m]$ and testing the exposed edges of the complex polynomials corresponding to the exposed edges of this box.

Example 4.10. Consider the parametrized polynomial

$$\delta(s, \mathbf{p}) = s^3 + (4 - j\alpha)s^2 + (\beta - j3)s + (\gamma - j(2 + \alpha))$$

where the parameters vary as:

$$\alpha \in [-0.7, 0.7], \ \ \beta \in [6.3, 7.7], \ \ \gamma \in [5.3, 6.7].$$

We verify that polynomial is Hurwitz stable for $(\alpha^0, \beta^0, \gamma^0) = (0, 7, 6)$. In order to check the robust stability of this polynomial we need to verify the stability of the set of edges connecting the following eight vertex polynomials:

$$\begin{aligned}
\delta_{V_1}(s) &= s^3 + (4 + j0.7)s^2 + (6.3 - j3)s + (5.3 - j1.3) \\
\delta_{V_2}(s) &= s^3 + (4 + j0.7)s^2 + (6.3 - j3)s + (6.7 - j1.3) \\
\delta_{V_3}(s) &= s^3 + (4 + j0.7)s^2 + (7.7 - j3)s + (5.3 - j1.3) \\
\delta_{V_4}(s) &= s^3 + (4 + j0.7)s^2 + (7.7 - j3)s + (6.7 - j1.3) \\
\delta_{V_5}(s) &= s^3 + (4 - j0.7)s^2 + (6.3 - j3)s + (5.3 - j2.7) \\
\delta_{V_6}(s) &= s^3 + (4 - j0.7)s^2 + (6.3 - j3)s + (6.7 - j2.7) \\
\delta_{V_7}(s) &= s^3 + (4 - j0.7)s^2 + (7.7 - j3)s + (5.3 - j2.7) \\
\delta_{V_8}(s) &= s^3 + (4 - j0.7)s^2 + (7.7 - j3)s + (6.7 - j2.7)
\end{aligned}$$

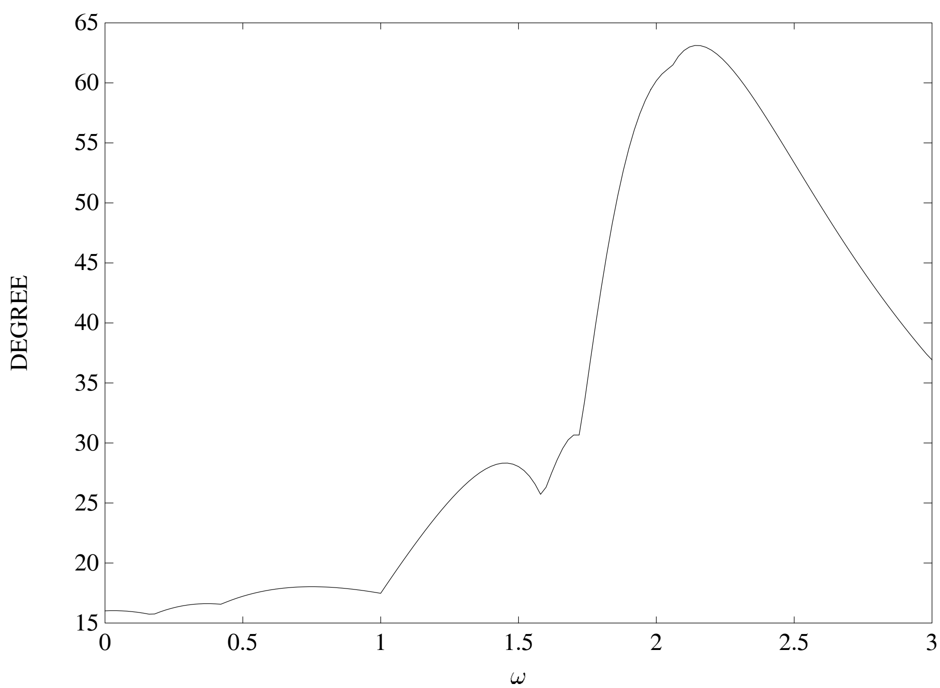

Figure 4.15. Maximum phase differences of vertices (checking the stability of edges) (Example 4.10)

The stability of these edges may be checked by verifying the Bounded Phase Condition. We examine whether at any frequency ω the maximum phase difference of the vertices reaches $180°$. Figure 4.15 shows that the maximum phase difference over this vertex set never reaches $180°$ for any ω. Thus, we conclude that all edges are Hurwitz stable, and so is the given polynomial family.

4.7.4 Polytopes of Quasipolynomials

In fact the above idea can also be extended to determining the robust stability of time-delay systems containing parameter uncertainty. Consider the quasipolynomial family

$$P(s) = a_n s^n + a_{n-1}e^{-sT_{n-1}}P_{n-1}(s) + a_{n-2}e^{-sT_{n-2}}P_{n-2}(s) + \cdots + a_0 e^{-sT_0}P_0(s)$$

where each polynomial $P_i(s)$ is of degree less than n, may be real or complex,

$$0 < T_0 < T_1 < \cdots < T_{n-1}$$

are real and $a_i \in [a_i^-, a_i^+]$ are real parameters varying in the box

$$\mathbf{A} = \left\{ \mathbf{a} \ : \ a_i^- \leq a_i \leq a_i^+, \quad i = 0, 1, \cdots, n \right\}.$$

This corresponds to a polytope of quasipolynomials. The image of this polytope at $s = s^*$, is a polygon whose exposed edges originate from the exposed edges of $\mathbf{A}$. Under the assumption that the "degree" of the family remains invariant (a_n does not pass through zero) the Boundary Crossing Theorem applies to this family and we conclude from image set arguments that the stability of the exposed edges implies stability of the entire polytope. This is illustrated in the example below.

Example 4.11. Consider the feedback system with time delay as shown in Figure 4.16.

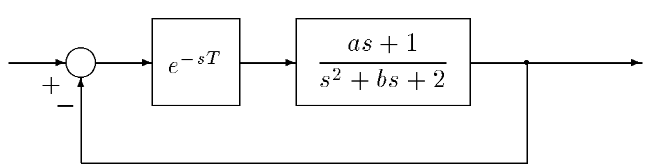

Figure 4.16. A time-delay system (Example 4.11)

The characteristic equation of the closed loop system is

$$\begin{aligned} \delta(s) &= (s^2 + bs + 2) + e^{-sT}(as + 1) \\ &= s^2 + bs + 2 + ae^{-sT}s + e^{-sT}. \end{aligned}$$

Suppose that the parameters a and b are bounded by $a \in [a^-, a^+]$ and $b \in [b^-, b^+]$. We have the following four vertex quasipolynomials

$$\begin{aligned} p_0(s) &= s^2 + b^- s + 2 + a^- e^{-sT} s + e^{-sT} \\ p_1(s) &= s^2 + b^- s + 2 + a^+ e^{-sT} s + e^{-sT} \\ p_2(s) &= s^2 + b^+ s + 2 + a^- e^{-sT} s + e^{-sT} \\ p_3(s) &= s^2 + b^+ s + 2 + a^+ e^{-sT} s + e^{-sT}. \end{aligned}$$

and four edges as follows:

$$[p_0(s), p_1(s)], \quad [p_0(s), p_2(s)], \quad [p_1(s), p_3(s)], \quad [p_2(s), p_3(s)].$$

Let us first take the edge joining $p_0(s)$ and $p_1(s)$. The stability of $p_0(s)$ can be easily checked by examining the frequency plot of

$$\frac{p_0(j\omega)}{(j\omega + 1)^n}$$

where n is the degree of $p_0(s)$. Using the Principle of the Argument (Chapter 1), the condition for $p_0(s)$ to have all left half plane roots is simply that the plot should not encircle the origin since the denominator term $(s+1)^n$ does not have right half plane roots. Once we verify the stability of the vertex polynomials $p_0(s)$, we need to check the stability of the edge $\lambda p_1(s) + (1-\lambda)p_0(s)$. For the stability of this edge, it is necessary that

$$\lambda p_1(j\omega) + (1-\lambda)p_0(j\omega) \neq 0, \quad \text{for} \quad 0 < \lambda \leq 1 \quad \text{and for all} \quad \omega \in \mathbb{R}.$$

This is also equivalent to

$$\frac{1-\lambda}{\lambda} + \frac{p_1(j\omega)}{p_0(j\omega)} \neq 0.$$

Since $(1-\lambda)/\lambda$ takes values in $[0, \infty)$ when λ varies in $(0, 1]$, it is necessary that $p_1(j\omega)/p_0(j\omega)$ should not take values in $[0, -\infty)$. Equivalently, the plot of $p_1(j\omega)/p_0(j\omega)$ should not cross the negative real axis of the complex plane.

For the choice of $a \in [0.5, 1.5]$, $b \in [1.5, 2.5]$, and $T = 0.5$, Figure 4.17 shows the frequency plots required to determine the stability of the remaining segments and thus the robust stability of the entire family of systems. From these figures we see that each of the frequency plots $\dfrac{p_i(j\omega)}{p_k(j\omega)}$ does not cut the negative real axis. Thus we conclude that the system is robustly stable.

The Bounded Phase Condition described earlier can also be applied to polytopes of complex polynomials and quasipolynomials. Essentially, we need to check that one member of the family is stable, the "degree" does not drop, and that the image set evaluated along the stability boundary excludes the origin. Since the image set is a convex polygon, the bounded phase condition

$$\Phi_{\mathbf{\Delta}_{\mathrm{V}}} < \pi$$

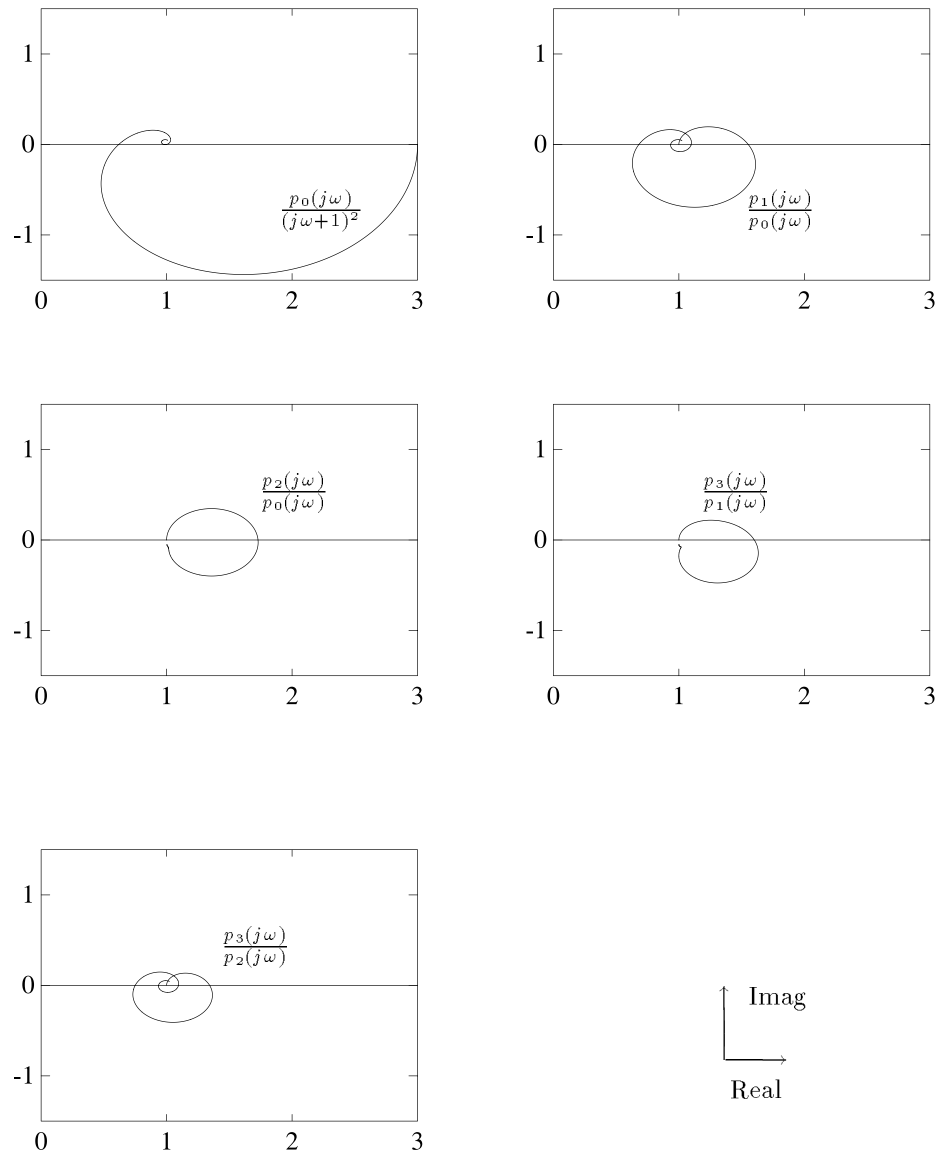

Figure 4.17. All frequency plots do not cut negative real axis (Example 4.11)

can be used to check robust stability. To illustrate this we now repeat Example 4.11 by using the Bounded Phase Condition.

Example 4.12. Consider the previous example. After verifying the stability of one member of the family and the degree invariance, we need to compute the maximum phase difference over the vertices at each frequency ω as follows. Let

$$\phi(\omega) = \phi_{\max}(\omega) - \phi_{\min}(\omega)$$

where

$$\phi_{\max}(\omega) = \max\left\{0, \arg\left(\frac{p_1(j\omega)}{p_0(j\omega)}\right), \arg\left(\frac{p_2(j\omega)}{p_0(j\omega)}\right), \arg\left(\frac{p_3(j\omega)}{p_0(j\omega)}\right)\right\}$$
$$\phi_{\min}(\omega) = \min\left\{0, \arg\left(\frac{p_1(j\omega)}{p_0(j\omega)}\right), \arg\left(\frac{p_2(j\omega)}{p_0(j\omega)}\right), \arg\left(\frac{p_3(j\omega)}{p_0(j\omega)}\right)\right\}$$

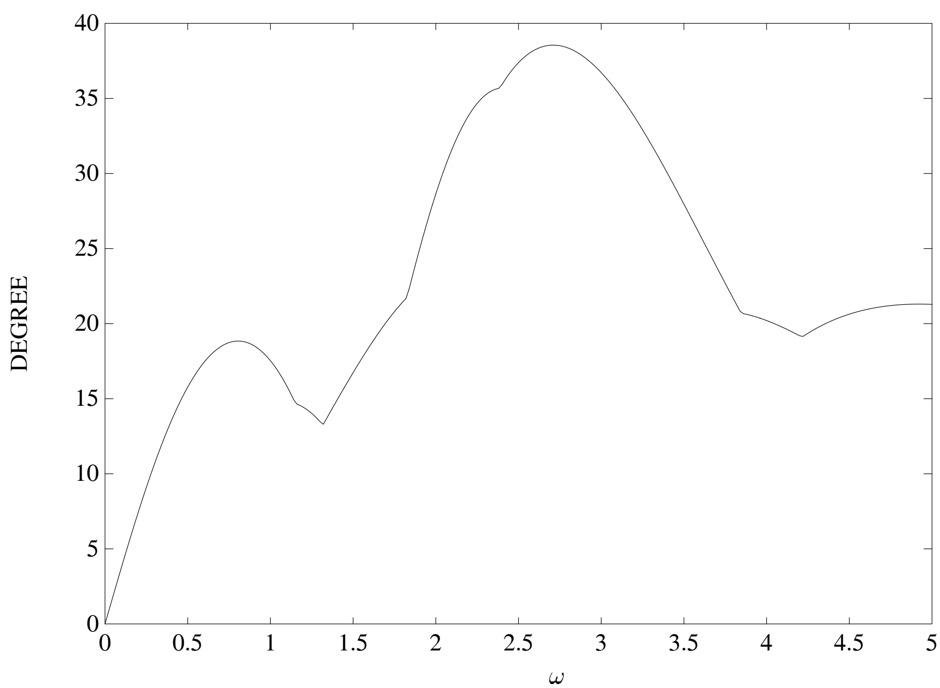

Figure 4.18. Maximum phase differences of vertices (Example 4.12)

Figure 4.18 shows the maximum phase difference $\phi(\omega)$ for $\omega \geq 0$. Since $\phi(\omega)$ never reaches 180°, at least one polynomial in the family is Hurwitz stable and the image set excludes the origin at some frequency, we conclude that the family is robustly stable.

4.8 EXTREMAL PROPERTIES OF EDGES AND VERTICES

In this section we deal with the problem of finding the *worst case stability margin* over a polytope of stable polynomials. This value of the worst case stability margin is a measure of the *robust performance* of the system or associated controller. As mentioned earlier this is a formidably difficult problem in the general case. In the linear case that is being treated in this chapter, it is feasible to determine robust performance by exploiting the image set boundary generating property of the exposed edges. Indeed for such families it is shown below that the worst case stability margins occur in general over the exposed edges. In special cases it can occur over certain vertices.

Consider the polytopic family of polynomials

$$\delta(s, \mathbf{p}) = p_1 Q_1(s) + p_2 Q_2(s) + \cdots + p_l Q_l(s) + Q_0(s) \tag{4.44}$$

with associated uncertainty set $\mathbf{\Pi}$ as defined in (4.30) and the exposed edges $\mathbf{E}$. Specifically, let

$$\mathbf{E}_i := \left\{ \mathbf{p} : p_i^- \leq p_i \leq p_i^+, \quad p_j = p_j^- \text{ or } p_j^+, \quad \text{for all } i \neq j \right\} \tag{4.45}$$

then

$$\mathbf{E} := \cup_{i=1}^{l} \mathbf{E}_i. \tag{4.46}$$

As before, we assume that the degree of the polynomial family remains invariant as $\mathbf{p}$ varies over $\mathbf{\Pi}$. Suppose that the stability of the family with respect to a stability region $\mathcal{S}$ has been proved. Then, with each point $\mathbf{p} \in \mathbf{\Pi}$ we can associate a stability ball of radius $\rho(\mathbf{p})$ in some norm $\|\mathbf{p}\|$. A natural question to ask at this point is: What is the minimum value of $\rho(\mathbf{p})$ as $\mathbf{p}$ ranges over $\mathbf{\Pi}$? It turns out that the minimum value of $\rho(\mathbf{p})$ is attained over the exposed edge set $\mathbf{E}$.

Theorem 4.7

$$\inf_{\mathbf{p} \in \mathbf{\Pi}} \rho(\mathbf{p}) = \inf_{\mathbf{p} \in \mathbf{E}} \rho(\mathbf{p}). \tag{4.47}$$

Proof. From the property that the edge set generates the boundary of the image set $\mathbf{\Delta}(s^*)$, we have

$$\begin{aligned}
\inf_{\mathbf{p} \in \mathbf{\Pi}} \rho(\mathbf{p}) &= \inf \{ \|\Delta \mathbf{p}\| : \delta(s, \mathbf{p} + \Delta \mathbf{p}) \text{ unstable for } \mathbf{p} \in \mathbf{\Pi} \} \\
&= \inf \{ \|\Delta \mathbf{p}\| : \delta(s^*, \mathbf{p} + \Delta \mathbf{p}) = 0, \quad s^* \in \partial \mathcal{S}, \quad \mathbf{p} \in \mathbf{\Pi} \} \\
&= \inf \{ \|\Delta \mathbf{p}\| : \delta(s^*, \mathbf{p} + \Delta \mathbf{p}) = 0, \quad s^* \in \partial \mathcal{S}, \quad \mathbf{p} \in \mathbf{E} \} \\
&= \inf_{\mathbf{p} \in \mathbf{E}} \rho(\mathbf{p}).
\end{aligned}$$

♣

This theorem is useful for determining the worst case parametric stability margin over an uncertainty box. Essentially it again reduces a multiparameter optimization problem to a set of one parameter optimization problems.

Further simplification can be obtained when the stability of the polytope can be guaranteed from the vertices. In this case we can find the worst case stability margin by evaluating $\rho(\mathbf{p})$ over the vertex set. For example, for the case of Hurwitz stability we have the following result which depends on the Vertex Lemma of Chapter 2.

Theorem 4.8 *Let $\delta(s)$ be a polytope of real Hurwitz stable polynomials of degree n of the form (4.44) with $Q_i(s)$ of the form*

$$Q_i(s) = s_i^t(a_i s + b_i)A_i(s)P_i(s) \tag{4.48}$$

where $t_i \geq 0$ are integers, a_i and b_i are arbitrary real numbers, $A_i(s)$ are antiHurwitz, and $P_i(s)$ are even or odd polynomials. Then

$$\inf_{\mathbf{p}\in\mathbf{\Pi}} \rho(\mathbf{p}) = \inf_{\mathbf{p}\in\mathbf{V}} \rho(\mathbf{p}). \tag{4.49}$$

The proof utilizes the Vertex Lemma of Chapter 2 but is otherwise identical to the argument used in the previous case and is therefore omitted. A similar result holds for the Schur case and can be derived using the corresponding Vertex Lemma from Chapter 2. We illustrate the usefulness of this result in determining the worst case performance evaluation and optimization of controllers in the examples below.

Example 4.13. Consider the plant

$$G(s) = \frac{p_2 s + 1}{p_1 s + p_0}$$

where the parameters $\mathbf{p} = [p_0 \quad p_1 \quad p_2]$ vary in

$$\mathbf{\Pi} := \{p_0 \in [2,4], \quad p_1 \in [4,6], \quad p_2 \in [10,15]\}.$$

This plant is robustly stabilized by the controller

$$C(s) = K\frac{s+5}{s(s-1)}$$

when $0.5 \leq K \leq 2$. The characteristic polynomial is

$$\delta(s,\mathbf{p},K) = s^2(s-1)p_1 + s(s-1)p_0 + Ks(s+5)p_2 + K(s+5)$$

and

$$\delta(j\omega,\mathbf{p},K) = -j\omega^3 p_1 - \omega^2(p_0 - p_1 - p_2 K) + j\omega(5Kp_2 - p_0 + K) + 5K.$$

This leads to

$$\underbrace{\begin{bmatrix} -\omega^2 & \omega^2 & -\omega^2 K \\ -\omega & -\omega^3 & 5\omega K \end{bmatrix}}_{A(\omega,K)} \underbrace{\begin{bmatrix} \Delta p_0 \\ \Delta p_1 \\ \Delta p_2 \end{bmatrix}}_{t} = \underbrace{\begin{bmatrix} \omega^2 p_0 - \omega^2 p_1 + \omega^2 K p_2 - 5K \\ \omega p_0 + \omega^3 p_1 - 5\omega K p_2 - \omega K \end{bmatrix}}_{b(\omega,\mathbf{p}^o,K)}$$

and the ℓ_2 stability margin around $\mathbf{p}^0$ is

$$\begin{aligned}\rho^*(\mathbf{p}^0, K) &= \min_\omega \rho(\omega, \mathbf{p}^0, K)\\ &= \min_\omega \left\| A^T(\omega, K)\left(A(\omega, K)A^T(\omega, K)\right)^{-1} b(\omega, \mathbf{p}^0, K)\right\|_2.\end{aligned}$$

To determine the worst case ℓ_2 stability margin over $\mathbf{\Pi}$ for fixed $K = K_0$ we apply Theorem 4.7. This tells us that the worst case stability margin occurs on one of the edges $\mathbf{\Pi}_{\mathrm{E}}$ of $\mathbf{\Pi}$. The characteristic polynomial of the closed loop system is

$$\delta(s, \mathbf{p}) = \underbrace{s^2(s-1)}_{Q_1(s)} p_1 + \underbrace{s(s-1)}_{Q_2(s)} p_0 + \underbrace{K_0 s(s+5)}_{Q_3(s)} p_2 + K_0(s+5)$$

which shows that the vertex condition of Theorem 4.8 is satisfied so that the worst case stability margin occurs on one of the vertices of $\mathbf{\Pi}$. Therefore, the worst case ℓ_2 stability margin for $K = K_0$ is

$$\rho^*(K_0) = \min_{\mathbf{p} \in \mathbf{\Pi}_{\mathrm{V}}} \rho^*(\mathbf{p}, K_0).$$

Suppose that we wish to determine the value of $K \in [0.5, 2]$ that possesses the *maximum* value of the worst case ℓ_2 stability margin. This problem can be solved by repeating the procedure of determining $\rho^*(K)$ for each K in $[0.5, 2]$ and selecting the value of K whose corresponding $\rho^*(K)$ is maximum:

$$\rho^* = \max_K \rho^*(K)$$

From Figure 4.19, we find that the maximum worst case ℓ_2 stability margin is 5.8878 when $K = 2$.

4.9 THE TSYPKIN-POLYAK PLOT

In this section we deal with stability margin calculations for characteristic polynomials of the form:

$$\delta(s, \mathbf{p}) = F_1(s)P_1(s) + F_2(s)P_2(s) + \cdots + F_m(s)P_m(s)$$

where $F_i(s)$ are fixed real or complex polynomials and the coefficients of the real polynomials $P_i(s)$, are the uncertain parameters. The uncertain real parameter vector $\mathbf{p} = [p_1, p_2 \cdots]$ represents these coefficients. An elegant graphical solution to this problem has been given by Tsypkin and Polyak based on evaluation of the image set. This method is described below.

We begin by considering the polynomial family

$$\mathbf{Q}(s) = \left\{ A(s) + \rho \sum_{i=1}^{l} r_i B_i(s) \ : \quad |r_i| \le 1, \quad i = 1, \cdots, l \right\} \tag{4.50}$$

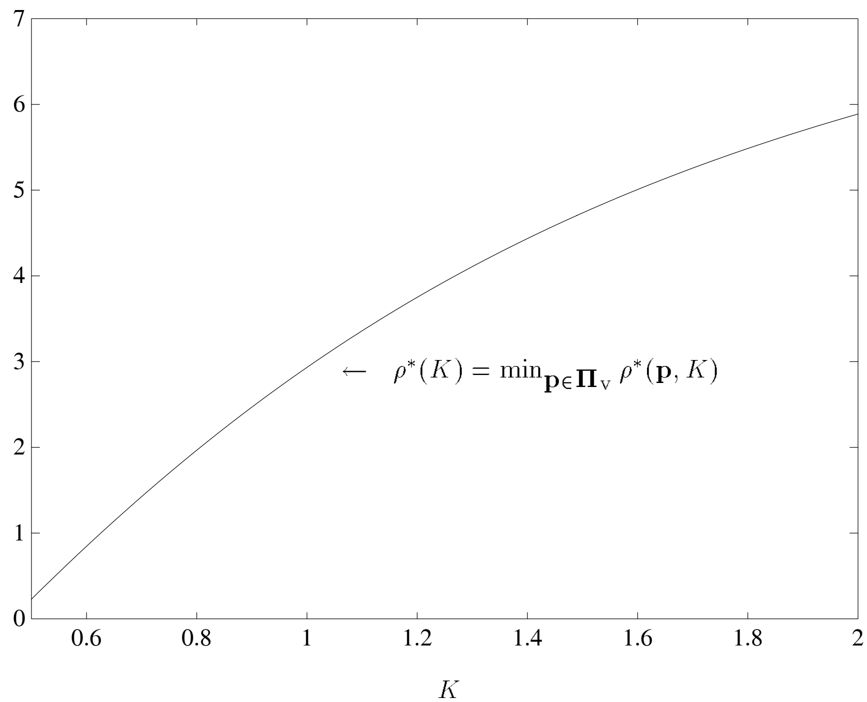

Figure 4.19. K vs. $\rho^*(K)$ (Example 4.13)

where $A(s)$ is a given real or complex polynomial of degree n and $B_1(s), \cdots, B_l(s)$ are given real or complex polynomials of degree $\leq n$, r_i are uncertain real parameters and the real parameter ρ is a common margin of perturbations or a dilation parameter. If we evaluate the above polynomial family at a point $s = s^*$ in the complex plane we obtain the image set of the family at that point. Because of the interval nature of the uncertain parameters, r_i, this set is a polygon. This polygon is determined by the complex numbers $A(s^*), B_1(s^*), \cdots, B_l(s^*)$. As discussed in the previous section, robust stability of a control system of which the polynomial above is the characteristic polynomial, requires that the family $\mathbf{Q}(s)$ be of constant degree. Under this assumption, the family $\mathbf{Q}(s)$ is stable if it contains at least one stable polynomial and the image set evaluated at each point of the stability boundary excludes the origin. The lemma given below gives an explicit geometric characterization of this condition.

Let us consider fixed nonzero complex numbers $A, B_1, \cdots, B_l$, and let

$$B_k = |B_k|e^{j\phi_k}, k = 1, 2, \cdots, l$$

and $A = |A|e^{j\theta}$. Introduce the set

$$\mathcal{B} = \left\{ \sum_{i=1}^{l} r_i B_i : |r_i| \leq 1 \right\},$$

which is a polygon in the complex plane with $2l$ or fewer vertices and opposite edges parallel to $B_i, i = 1, \cdots, l$. Consider the complex plane set $A + \rho\mathcal{B}$. This set is depicted in Figures 4.20 and 4.21.

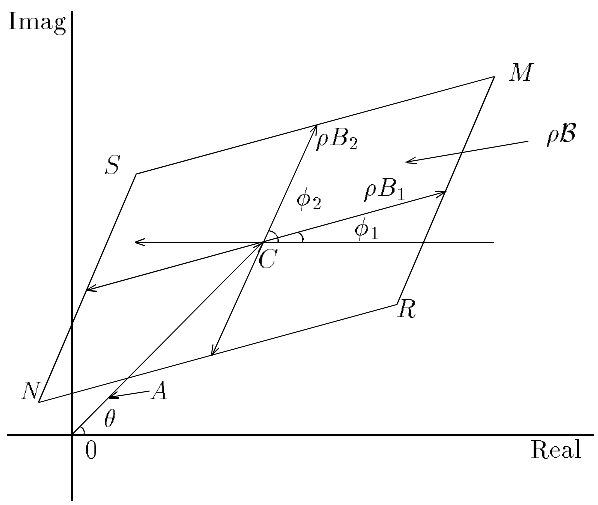

Figure 4.20. The set $A + \rho\mathcal{B}$

Lemma 4.3 *The zero exclusion condition*

$$0 \notin A + \rho\mathcal{B}, \qquad \rho > 0$$

is equivalent to

$$\max_{1 \leq k \leq l} \frac{|A||\sin(\theta - \phi_k)|}{\sum_{i=1}^{l} |B_i||\sin(\phi_i - \phi_k)|} > \rho, \quad \text{if } \sin(\phi_i - \phi_k) \neq 0 \text{ for some } i, k \quad (4.51)$$

$$\max_{1 \leq k \leq l} \frac{|A|}{\sum_{i=1}^{l} |B_i|} > \rho, \quad \text{if } \sin(\phi_i - \phi_k) \equiv 0 \text{ and } \sin(\theta - \phi_k) \equiv 0, \quad \forall\ i, k. \ (4.52)$$

Proof. The origin is excluded from the convex set $A + \rho\mathcal{B}$ if and only if there exists a line L' which separates the set from the origin. Consider the projection of the set $A + \rho\mathcal{B}$ onto the orthogonal complement L' of the line L in the complex

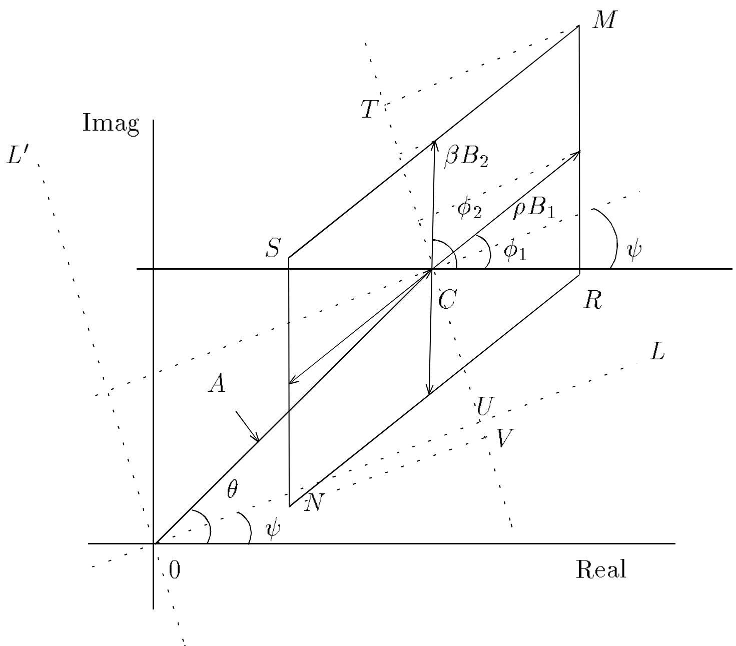

Figure 4.21. Projection of $A + \rho\mathcal{B}$ onto L'

plane passing through the origin at an angle ψ with the real axis (see Figure 4.21). This projection is of length

$$2\rho(|B_1||\sin(\phi_1 - \psi)| + \cdots + |B_l||\sin(\phi_l - \psi)|)$$

and is shown in the Figure 4.21 as TV. On the other hand the projection of the vector A on to L' is of length $|A||\sin(\theta - \psi)|$ and is shown as the line CU in Figure 4.21. A line separating the origin from the set exists if and only if there exists some $\psi \in [0, 2\pi)$ for which

$$|A||\sin(\theta - \psi)| > \rho(|B_1||\sin(\phi_1 - \psi)| + \cdots + |B_l||\sin(\phi_l - \psi)|).$$

Since the set $\mathcal{B}$ is a polygon it suffices to choose L to be parallel to one of its sides. This can be visualized as sliding the set along lines parallel to its sides and checking if the origin enters the set at any position. Thus it is enough to test that the above inequality holds at one of the values $\psi = \phi_i$, $i = 1, 2, \cdots, l$. This is precisely what is expressed by the formula (4.51) above. The formula (4.52) deals with the degenerate case when the set collapses to a line even though $l \neq 1$. ♣

The expressions (4.51) and (4.52) can also be written in the alternative form:

$$\max_{1\le k\le l} \frac{\left|\mathrm{Im}\left(\frac{A}{B_k}\right)\right|}{\sum_{i=1}^{l}\left|\mathrm{Im}\left(\frac{B_i}{B_k}\right)\right|} > \rho, \quad \text{if } \mathrm{Im}\left(\frac{B_i}{B_k}\right) \neq 0 \text{ for some } i,k \tag{4.53}$$

$$\max_{1\le k\le l} \frac{|A|}{\sum_{i=1}^{l}|B_i|} > \rho, \quad \text{if } \mathrm{Im}\left(\frac{B_i}{B_k}\right) \equiv 0 \text{ and } \left(\frac{A}{B_k}\right) \equiv 0, \quad \forall\ i,k. \tag{4.54}$$

We now apply this result to the special form of the characteristic polynomial (4.2).

Linear Interval Family

We consider the Hurwitz stability of the family of constant degree polynomials

$$\begin{aligned} \delta(s,\mathbf{p}) &= F_1(s)P_1(s) + \cdots + F_m(s)P_m(s), \\ P_i(s) &= p_{i0} + p_{i1}s + \cdots, \qquad\qquad |p_{ik} - p_{ik}^0| \le \rho\alpha_{ik}, \\ F_i(s) &= f_{i0} + f_{i1}s + \cdots. \end{aligned} \tag{4.55}$$

Here p_{ik}^0 are the coefficients of the nominal polynomials

$$P_i^0(s) = p_{i0}^0 + p_{i1}^0 s + \cdots$$

$\alpha_{ik} \ge 0$ are given and $\rho \ge 0$ is a common margin of perturbations. Everywhere below stable means Hurwitz stable. Denote

$$\begin{aligned} A(s) &= F_1(s)P_1^0(s) + \cdots + F_m(s)P_m^0(s), \\ S_i(\omega) &= \alpha_{i0} + \alpha_{i2}\omega^2 + \cdots \\ T_i(\omega) &= \omega(\alpha_{i1} + \alpha_{i3}\omega^2 + \cdots), \\ \theta(\omega) &= \arg\ A(j\omega), \\ \psi_i(\omega) &= \arg\ F_i(j\omega), \qquad i = 1,\cdots,m, \end{aligned} \tag{4.56}$$

where $0 \le \omega \le \infty$. The image set of the family has the form

$$A(j\omega) + \rho\mathcal{Q}(\omega)$$

where

$$\mathcal{Q}(\omega) = \left\{ \sum_{i=1}^{m} \left[s_i(\omega) + jt_i(\omega)\right] F_i(j\omega) : \quad |s_i(\omega)| \le S_i(\omega), \quad |t_i(\omega)| \le T_i(\omega) \right\}$$

This set may be rewritten as

$$\mathcal{Q}(\omega) = \tag{4.57}$$

$$\left\{ \sum_{i=1}^{m} \left[r_i S_i(\omega)F_i(j\omega) + r_{m+i}T_i(\omega)jF_i(j\omega)\right], \quad |r_i| \le 1, \quad |r_{m+i}| \le 1, \quad i = 1,\cdots,m. \right\}.$$

We apply Lemma 4.3 (see (4.51)) to the above set by setting $l = 2m$, $B_i = S_i(\omega)F_i(j\omega)$, $i = 1, 2, \cdots, m$ and $B_i = T_i(\omega)jF_i(j\omega)$, $i = m+1, \cdots, 2m$. Since

$$\begin{aligned}
\arg F_i(j\omega) &= \psi_i(\omega), \\
\arg jF_i(j\omega) &= \psi_{m+i}(\omega) = \psi_i(\omega) + \frac{\pi}{2},
\end{aligned}$$

for $i = 1, 2, \cdots, m$, we can rewrite (4.51) as the maximum of two terms:

$$\begin{aligned}
\max_{1\le k\le m} \frac{|A(j\omega)||\sin(\theta(\omega)-\psi_k(\omega))|}{\sum_{i=1}^{2m}|B_i||\sin(\psi_i(\omega)-\psi_k(\omega))|} &= \max_{1\le k\le m} \frac{|A(j\omega)||\sin(\theta(\omega)-\psi_k(\omega))|}{v_k(\omega)} \\
\max_{m+1\le k\le 2m} \frac{|A(j\omega)||\sin(\theta(\omega)-\psi_k(\omega))|}{\sum_{i=1}^{2m}|B_i||\sin(\psi_i(\omega)-\psi_k(\omega))|} &= \max_{1\le k\le m} \frac{|A(j\omega)||\cos(\theta(\omega)-\psi_k(\omega))|}{u_k(\omega)}
\end{aligned}$$

where $u_k(\omega)$ and $v_k(\omega)$ are defined as

$$\begin{aligned}
u_k(\omega) := \sum_{i=1}^{m} &[S_i(\omega)|F_i(j\omega)||\cos(\psi_i(\omega)-\psi_k(\omega))| \\
&+T_i(\omega)|F_i(j\omega)||\sin(\psi_i(\omega)-\psi_k(\omega))|], \qquad k = 1, 2, \cdots, m \\
v_k(\omega) := \sum_{i=1}^{m} &[S_i(\omega)|F_i(j\omega)||\sin(\psi_i(\omega)-\psi_k(\omega))| \\
&+T_i(\omega)|F_i(j\omega)||\cos(\psi_i(\omega)-\psi_k(\omega))|], \qquad k = 1, 2, \cdots, m.
\end{aligned}$$

Let Γ be defined as:

$$\begin{aligned}
\Gamma := \{\omega : \cos(\psi_i(\omega)-\psi_k(\omega)) = \sin(\psi_i(\omega)-\psi_k(\omega)) = 0, \\
\omega \in [0,\infty), \quad i, k = 1, \cdots, m\}.
\end{aligned}$$

We note that $u_k(\omega) = 0$ if and only if $\omega \in \Gamma$. Likewise $v_k(\omega) = 0$ if and only if $\omega \in \Gamma$. Now define

$$\begin{aligned}
x_k(\omega) &:= \frac{|A(j\omega)||\cos(\theta(\omega)-\psi_k(\omega))|}{u_k(\omega)}, \qquad \omega \notin \Gamma \\
x_k(\omega) &:= \frac{|A(j\omega)|}{\sum_{i=1}^{m}[S_i(\omega)|F_i(j\omega)| + T_i(\omega)|F_i(j\omega)|]}, \qquad \omega \in \Gamma
\end{aligned} \tag{4.58}$$

for $k = 1, 2, \cdots, m$ and

$$\begin{aligned}
y_k(\omega) &:= \frac{|A(j\omega)||\sin(\theta(\omega)-\psi_k(\omega))|}{v_k(\omega)}, \qquad \omega \notin \Gamma \\
y_k(\omega) &:= \frac{|A(j\omega)|}{\sum_{i=1}^{m}[S_i(\omega)|F_i(j\omega)| + T_i(\omega)|F_i(j\omega)|]}, \qquad \omega \in \Gamma
\end{aligned} \tag{4.59}$$

for $k = 1, 2, \cdots, m$. Finally let

$$\mu(\omega) := \max_{1\le k\le m} \max\{x_k(\omega), y_k(\omega)\}, \qquad 0 \le \omega \le \infty. \tag{4.60}$$

Notice that $\mu(\omega)$ is well defined for all $\omega \in [0, \infty]$. However the frequencies $0, \infty$ and the points $\omega \in \Gamma$ may be points of discontinuity of the function $\mu(\omega)$. For $\omega = 0$, it is easy to see that

$$\mu(0) = \max_{1\le k\le m} \{x_k(0)\} = \frac{|\sum_{i=1}^m p_{i0}^0 f_{i0}|}{\sum_{i=1}^m \alpha_{i0}|f_{i0}|} := \mu_0. \tag{4.61}$$

We also have

$$\mu_n = \frac{\left|\sum_{i=1}^m \sum_{k+l=n} p_{ik}^0 f_{il}\right|}{\sum_{i=1}^m \sum_{k+l=n} \alpha_{ik}|f_{il}|}. \tag{4.62}$$

To state the equivalent nontrignometric expressions for x_k and y_k let

$$\frac{F_i(j\omega)}{F_k(j\omega)} = U_{ik}(\omega) + jV_{ik}(\omega)$$
$$\frac{A(j\omega)}{F_k(j\omega)} = U_{0k}(\omega) + jV_{0k}(\omega).$$

As before let $\omega \in \Gamma$ denote the set of common roots of the equations

$$U_{ik}(\omega) = V_{ik}(\omega) = 0, \quad i = 1, \cdots, m. \tag{4.63}$$

We have

$$x_k(\omega) = \frac{|U_{0k}(\omega)|}{\sum_{i=1}^m [S_i(\omega)|U_{ik}(\omega)| + T_i(\omega)|V_{ik}(\omega)|]}, \qquad \omega \notin \Gamma \tag{4.64}$$

$$y_k(\omega) = \frac{|V_{0k}(\omega)|}{\sum_{i=1}^m [S_i(\omega)|V_{ik}(\omega)| + T_i(\omega)|U_{ik}(\omega)|]}, \qquad \omega \notin \Gamma \tag{4.65}$$

and for $\omega \in \Gamma$, $x_k(\omega)$, $y_k(\omega)$ are defined as before in (4.58) and (4.59). Now let

$$\mu_{\min} := \inf_{0<\omega<\infty} \mu(\omega) \tag{4.66}$$

and

$$\rho^* = \min\{\mu_0, \mu_n, \mu_{\min}\}. \tag{4.67}$$

Theorem 4.9 *The family (4.55) is stable if and only if $A(s)$ is Hurwitz and*

$$\rho^* > \rho. \tag{4.68}$$

Proof. The theorem can be proved by evaluating the image set $\{\delta(j\omega, \mathbf{p})\}$ for the family (4.55) and applying Lemma 4.3 to it. Recall the image set of the family has the form

$$A(j\omega) + \rho \mathcal{Q}(\omega)$$

where $\mathcal{Q}(\omega)$ is of the form in (4.57). We again apply Lemma 4.3 to the above set by setting $l = 2m$, $B_i = S_i(\omega)F_i(j\omega)$, $i = 1, 2, \cdots, m$ and $B_i = T_i(\omega)jF_i(j\omega)$, $i = m+1, \cdots, 2m$. Then the condition that the polygon $A(j\omega)+\rho\mathcal{Q}(\omega)$ exclude the origin for $0 \leq \omega < \infty$ corresponds to the condition $\mu_{\min} > \rho$. The condition that $A(0) + \rho\mathcal{Q}(0)$ excludes the origin corresponds to $\mu_0 > \rho$ and the condition that all polynomials in the family be of degree n corresponds to $\mu_n > \rho$. ♣

The requirement that $A(s)$ be Hurwitz can be combined with the robustness condition, namely verification of the inequality (4.68). This leads us to the following criterion.

Theorem 4.10 *The family (4.55) is stable if and only if* $A(j\omega) \neq 0$, $\rho < \mu_0$, $\rho < \mu_n$ *and the complex plane plot*

$$z(\omega) = \frac{A(j\omega)}{|A(j\omega)|}\mu(\omega) \tag{4.69}$$

goes through n *quadrants as* ω *runs from* 0 *to* ∞ *and does not intersect the circle of radius* ρ*, centered at the origin.*

The plot of $z(\omega)$ is known as the *Tsypkin-Polyak* plot. The maximal value of ρ in (4.55) which preserves robust stability is equal to the radius ρ^* of the largest circle, enclosed in the $z(\omega)$ plot, subject of course to the conditions $\rho < \mu_0$, $\rho < \mu_n$.

We illustrate these results with an example.

Example 4.14. Consider the family of polynomials

$$\delta(s, \mathbf{p}) = \underbrace{(s^2 + 2s + 2)}_{F_1(s)} \underbrace{(p_{11}s + p_{10})}_{P_1(s)} + \underbrace{(s^4 + 2s^3 + 2s^2 + s)}_{F_2(s)} \underbrace{(p_{22}s^2 + p_{21}s + p_{20})}_{P_2(s)}$$

where the nominal values of parameters $\mathbf{p}$ in $P_1(s)$ and $P_2(s)$ are

$$\mathbf{p}^0 = [p_{11}^0 \;\; p_{10}^0 \;\; p_{22}^0 \;\; p_{21}^0 \;\; p_{20}^0] = [0.287 \;\; 0.265 \;\; 0.215 \;\; 2.06 \;\; 2.735].$$

We first verify that the nominal polynomial

$$\begin{aligned} A(s) &= F_1(s)P_1^0(s) + F_2(s)P_2^0(s) \\ &= 0.215s^6 + 2.49s^5 + 7.285s^4 + 10.092s^3 + 8.369s^2 + 3.839s + 0.53 \end{aligned}$$

is Hurwitz. We want to find the stability margin ρ such that the family remains Hurwitz for all parameter excursions satisfying

$$|p_{ik} - p_{ik}^0| \leq \rho\alpha_{ik}$$

with $\alpha_{ik} = 1$ for $ik = 10, 11, 20, 21, 22$. Here we use the nontrignometric expressions given in (4.64) and (4.65). We first compute

$$\begin{aligned}
\frac{F_1(j\omega)}{F_1(j\omega)} &= 1 + j0 = U_{11}(\omega) + jV_{11}(\omega) \\
\frac{F_2(j\omega)}{F_1(j\omega)} &= \frac{-\omega^6 - 2\omega^2}{\omega^2+4} + j\frac{-\omega^3 + 2\omega}{\omega^2+4} = U_{21}(\omega) + jV_{21}(\omega) \\
\frac{F_1(j\omega)}{F_2(j\omega)} &= \frac{-\omega^4 - 2}{\omega^6+1} + j\frac{\omega^2 - 2}{\omega^7+\omega} = U_{12}(\omega) + jV_{12}(\omega) \\
\frac{F_2(j\omega)}{F_2(j\omega)} &= 1 + j0 = U_{22}(\omega) + jV_{22}(\omega)
\end{aligned}$$

and

$$\begin{aligned}
\frac{A(j\omega)}{F_1(j\omega)} &= \frac{0.215\omega^8 - 2.735\omega^6 + 2.755\omega^4 - 9.59\omega^2 + 1.06}{\omega^2+4} \\
&\qquad + j\frac{-2.06\omega^7 + 0.502\omega^5 - 7.285\omega^3 + 6.618\omega}{\omega^2+4} \\
&= U_{01}(\omega) + jV_{01}(\omega) \\
\frac{A(j\omega)}{F_2(j\omega)} &= \frac{-0.215\omega^8 + 2.735\omega^6 - 0.265\omega^4 - 0.502\omega^2 + 2.779}{\omega^6+1} \\
&\qquad + j\frac{2.06\omega^8 - 0.287\omega^6 + 1.751\omega^3 - 0.53}{\omega^7+\omega} \\
&= U_{02}(\omega) + jV_{02}(\omega).
\end{aligned}$$

With the given choice of α_{ik}, we also have

$$S_1(\omega) = 1, \qquad S_2(\omega) = 1 + \omega^2, \qquad T_1(\omega) = \omega, \qquad T_2(\omega) = \omega.$$

From these we construct the function

$$z(\omega) = \frac{A(j\omega)}{|A(j\omega)|}\mu(\omega) = \frac{A(j\omega)}{|A(j\omega)|}\left[\max_{k=1,2} \max\{x_k(\omega), y_k(\omega)\}\right]$$

where $x_k(\omega)$ and $y_k(\omega)$ are given as follows:

$$x_1(\omega) = \frac{\left|\dfrac{0.215\omega^8 - 2.735\omega^6 + 2.755\omega^4 - 9.59\omega^2 + 1.06}{\omega^2+4}\right|}{1 + (1+\omega^2)\left|\dfrac{-\omega^6 - 2\omega^2}{\omega^2+4}\right| + \omega\left|\dfrac{-\omega^3 + 2\omega}{\omega^2+4}\right|}$$

$$x_2(\omega) = \frac{\left|\dfrac{-0.215\omega^8 + 2.735\omega^6 - 0.265\omega^4 - 0.502\omega^2 + 2.779}{\omega^6+1}\right|}{\left|\dfrac{-\omega^4 - 2}{\omega^6+1}\right| + \omega\left|\dfrac{\omega^2 - 2}{\omega^7+\omega}\right| + (1-\omega^2)}$$

$$y_1(\omega) = \frac{\left|\dfrac{-2.06\omega^7 + 0.502\omega^5 - 7.285\omega^3 + 6.618\omega}{\omega^2+4}\right|}{\omega + (1+\omega^2)\left|\dfrac{-\omega^3+2\omega}{\omega^2+4}\right| + \omega\left|\dfrac{-\omega^6 - 2\omega^2}{\omega^2+4}\right|}$$

$$y_2(\omega) = \frac{\left|\dfrac{2.06\omega^8 - 0.287\omega^6 + 1.751\omega^3 - 0.53}{\omega^7+\omega}\right|}{\left|\dfrac{\omega^2-2}{\omega^7+\omega}\right| + \omega\left|\dfrac{-\omega^4-2}{\omega^6+1}\right| + \omega}.$$

μ_0 and μ_n are given by

$$\mu_0 = \frac{\left|\sum_{i=1}^2 p_{i0}^0 f_{i0}\right|}{\sum_{i=1}^2 \alpha_{i0}|f_{i0}|} = \frac{|(0.265)(2)|}{(1)|(2)|} = 0.265$$

$$\mu_n = \frac{\left|\sum_{i=1}^2 \sum_{k+l=n} p_{ik}^0 f_{il}\right|}{\sum_{i=1}^2 \sum_{k+l=n} \alpha_{ik}|f_{il}|} = \frac{|(0.215)(1)|}{(1)|(1)|} = 0.215$$

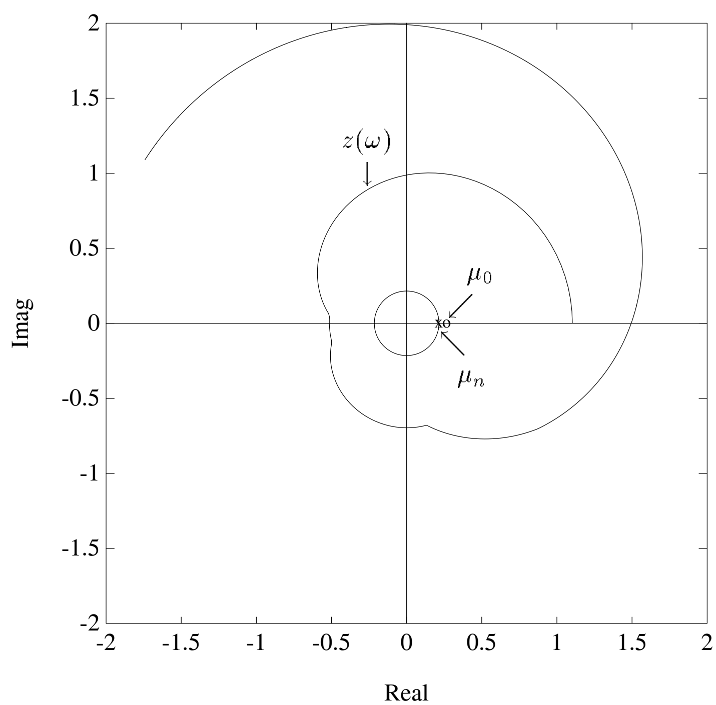

Figure 4.22. Tsypkin-Polyak locus for a polytopic family (Example 4.14)

Figure 4.22 displays the Tsypkin-Polyak plot $z(\omega)$ along with μ_0 and μ_n. The radius of the smallest circle which touches this plot is ρ^*. From the figure we find

$$\rho^* = 0.215.$$

4.10 ROBUST STABILITY OF LINEAR DISC POLYNOMIALS

In the last chapter we considered the robust stability of a family of *disc* polynomials. The main result derived there considered the coefficients themselves to be subject to *independent* perturbations. Here we consider interdependent perturbations by letting the coefficients depend linearly on a primary set of parameters δ_i.

Consider $n+1$ complex polynomials $P_0(s), \cdots, P_n(s)$, such that

$$\deg(P_0(s)) > \max_{i \in \{1,2,\cdots,n\}} \deg(P_i(s)). \tag{4.70}$$

Let also $D_1, \cdots, D_n$ be n arbitrary discs in the complex plane, where D_i is centered at the complex number d_i and has radius r_i.

Now consider the set $\mathcal{P}$ of all polynomials of the form:

$$\delta(s) = P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s), \quad \text{where } \delta_i \in D_i, \quad i = 1, 2, \cdots, n. \tag{4.71}$$

We refer to such a family as a *linear disc polynomial.* If we define $\beta(s)$ by

$$\beta(s) = P_0(s) + \sum_{i=1}^{n} d_i P_i(s) \tag{4.72}$$

then it is clear that Lemma 3.2 (Chapter 3) can be readily extended to the following result.

Lemma 4.4 *The family of complex polynomial $\mathcal{P}$ contains only Hurwitz polynomials if and only if the following conditions hold:*

a) $\beta(s)$ is Hurwitz

b) for any complex numbers $z_0, \cdots, z_n$ of modulus less than or equal to one we have:

$$\left\| \frac{\sum_{k=1}^{n} z_k r_k P_k(s)}{\beta(s)} \right\|_\infty < 1.$$

The proof of this Lemma follows from Rouché's Theorem in a manner identical to that of Lemma 3.2 in Chapter 3. Observe now, that for any complex numbers z_i of modulus less than or equal to one, we have the following inequality:

$$\left| \frac{\sum_{k=1}^{n} z_k r_k P_k(j\omega)}{\beta(j\omega)} \right| \leq f(\omega) \tag{4.73}$$

where

$$f(\omega) = \frac{\sum_{k=1}^{n} r_k |P_k(j\omega)|}{|\beta(j\omega)|}. \tag{4.74}$$

On the other hand, since $f(\omega)$ is a continuous function of ω which goes to 0 as ω goes to infinity, then $f(\omega)$ reaches its maximum for some finite value ω_0. At this point ω_0 it is always possible to write

$$P_k(j\omega_0) = e^{j\theta_k}|P_k(j\omega_0)|, \quad \text{with} \quad \theta_k \in [0, 2\pi) \quad \text{for} \quad k = 1, 2, \cdots, n$$

so we see that it is always possible to find a set of complex numbers $z_0, \cdots, z_n$ such that (4.73) becomes an equality at the point where $f(\omega)$ reaches its maximum. As a result we have the following generalization of the result on disc polynomials given in Chapter 3.

Theorem 4.11 *The family of complex polynomials $\mathcal{P}$ contains only Hurwitz polynomials if and only if the following conditions hold:*

a) $\beta(s)$ is Hurwitz

b) for all $\omega \in \mathbb{R}$,

$$\frac{\sum_{k=1}^{n} r_k |P_k(j\omega)|}{|\beta(j\omega)|} < 1.$$

A similar result holds also for the Schur case.

4.11 EXERCISES

4.1 In a standard unity feedback control system, the plant is

$$G(s) = \frac{a_1 s - 1}{s^2 + b_1 s + b_0}.$$

The parameters a_1, b_0 and b_1 have nominal values $a_1^0 = 1$, $b_1^0 = -1$, $b_0^0 = 2$ and are subject to perturbation. Consider the constant controller $C(s) = K$. Determine the range of values S_k of K for which the nominal system remains Hurwitz stable. For each value of $K \in S_k$ find the ℓ_2 stability margin $\rho(K)$ in the space of the three parameters subject to perturbation. Determine the optimally robust compensator value $K \in S_k$.

4.2 Repeat Exercise 4.1 using the ℓ_∞ stability margin.

4.3 Consider the polynomial

$$s^4 + s^3(12 + p_1 + p_3) + s^2(47 + 10.75p_1 + 0.75p_2 + 7p_3 + 0.25p_4)$$
$$+s(32.5p_1 + 7.5p_2 + 12p_3 + 0.5p_4) + 18.75p_1 + 18.75p_2 + 10p_3 + 0.5p_4.$$

Determine the ℓ_2 Hurwitz stability margin in the parameter space $[p_1, p_2, p_3, p_4]$.

4.4 The nominal polynomial (i.e. with $\Delta p_i = 0$) in Exercise 4.3 has roots -5, -5, $-1-j$, $-1+j$. Let the stability boundary consist of two circles of radii 0.25 centered at $-1+j$ and $-1-j$ respectively and a circle of radius 1 centered at -5. Determine the maximal ℓ_2 stability ball for this case.

4.5 Repeat Exercise 4.4 with the ℓ_∞ stability margin.

4.6 The transfer function of a plant in a unity feedback control system is given by

$$G(s) = \frac{1}{s^3 + d_2(\mathbf{p})s^2 + d_1(\mathbf{p})s + d_0(\mathbf{p})}$$

where

$$\mathbf{p} = [p_1, p_2, p_3]$$

and

$$d_2(\mathbf{p}) = p_1 + p_2, \quad d_1(\mathbf{p}) = 2p_1 - p_3, \quad d_0(\mathbf{p}) = 1 - p_2 + 2p_3.$$

The nominal parameter vector is $\mathbf{p}^0 = [3, 3, 1]$ and the controller proposed is

$$C(s) = K_c \frac{1 + 0.667s}{1 + 0.0667s}$$

with the nominal value of the gain being $K_c = 15$. Compute the weighted ℓ_∞ Hurwitz stability margin with weights $[w_1, w_2, w_3] = [1, 1, 0.1]$.

4.7 In Exercise 4.6 assume that K_c is the only adjustable design parameter and let $\rho(K_c)$ denote the weighted ℓ_∞ Hurwitz stability margin with weights as in Problem 4.6. Determine the range of values of K_c for which all perturbations within the unit weighted ℓ_∞ ball are stabilized, i.e. for which $\rho(K_c) > 1$.

4.8 Repeat Exercises 4.6 and 4.7 with the weighted ℓ_2 stability margin.

4.9 In Exercises 4.7 and 4.8 determine the optimally robust controllers in the weighted ℓ_2 or weighted ℓ_∞ stability margins, over the range of values of K_c which stabilizes the nominal plant.

4.10 Consider a complex plane polygon $\mathcal{P}$ with consecutive vertices $z_1, z_2, \cdots z_N$ with $N > 2$. Show that a point z is excluded from (does not belong to) $\mathcal{P}$ if and only if

$$\min_{1 \le n \le N} \left\{ \operatorname{Im} \frac{z - z_i}{z_{i+1} - z_1} \right\} < 0.$$

Use this result to develop a test for stability of a polytopic family.

4.11 In a discrete time control system the transfer function of the plant is:

$$G(z) = \frac{z-2}{(z+1)(z+2)}.$$

Determine the transfer function of a first order controller

$$C(z) = \frac{\alpha_0 + \alpha_1 z}{z + \beta_0}$$

which results in a closed loop system that is deadbeat, i.e. the characteristic roots are all at the origin. Determine the largest ℓ_2 and ℓ_∞ stability balls centered at this controller in the space of controller parameters α_0, α_1, β_0 for which closed loop Schur stability is preserved.

4.12 Consider the Hurwitz stability of the family

$$\begin{aligned} \mathbf{P}(s) &= \left\{ P_0(s) + \rho \sum_{i=1}^{m} r_i P_i(s) : |r_i| \leq 1 \right\} \\ P_i(s) &= a_{i0} + a_{i1}s + \cdots + a_{in}s^n, \quad a_{0n} \neq 0 \end{aligned} \tag{4.75}$$

Let ω_l denote a real root of the equations

$$\text{Im}\left[\frac{P_i(j\omega)}{P_0(j\omega)}\right] = 0$$

and define

$$\begin{aligned} u_i(\omega) &= |P_i(j\omega)|, \\ \psi_i(\omega) &= \arg P_i(j\omega) \\ \rho(\omega) &= \max_{1 \leq k \leq m} \frac{u_0(\omega)|\sin(\psi_0(\omega) - \psi_k(\omega))|}{\sum_{i=1}^{m} u_i(\omega)|\sin(\psi_i(\omega) - \psi_k(\omega))|}, \quad \omega \neq \omega_l, \end{aligned}$$

and

$$\begin{aligned} \rho(\omega_l) &= \frac{|P_0(j\omega_l)|}{\sum_{i=1}^{\infty} |P_i(j\omega_l)|} \\ \rho(\infty) &= \frac{|a_{0n}|}{\sum_{i=1}^{m} |a_{in}|} \end{aligned}$$

Show that the family (4.75) is stable if and only if $P_0(s)$ is stable and $\rho(\omega) > \rho$, $0 \leq \omega \leq \infty$.

4.13 Consider now the ℓ_p-analog of the above family:

$$P(s) = P_0(s) + \gamma \sum_{i=1}^{m} r_i P_i(s), \qquad \left[\sum_{i=1}^{m} |r_i|^p\right]^{\frac{1}{p}} \leq 1 \tag{4.76}$$

where $1 \leq p \leq \infty$. This set of polynomials is not a polytope when $p \neq 1$ or $p \neq \infty$. Write

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad P_i(j\omega) = u_i(\omega)e^{j\psi_i(\omega)}, \qquad i = 0, \cdots, m$$

$$\rho(\omega) = \max_{0 \leq \phi \leq 2\pi} \frac{u_0(\omega)\sin(\psi_0(\omega) - \phi)}{\left[\sum_{k=1}^{m} |u_k(\omega)\sin(\psi_k(\omega) - \phi)|^q\right]^{\frac{1}{q}}}$$

For $p = 2$ (ellipsoidal constraints) show that the condition for robust stability can be written in the form $\rho(\omega) > \rho$ by introducing:

$$P_i(j\omega) := U_i(\omega) + jV_i(\omega), \quad i = 0, \cdots, m$$

$$\Delta := \sum_{i=1}^{m} U_k^2 \sum_{i=1}^{m} V_k^2 - \left[\sum_{i=1}^{m} U_k V_k\right]^2$$

$$\alpha := \Delta^{-1}\left[V_0 \sum_{i=1}^{m} U_k V_k - U_0 \sum_{i=1}^{m} V_k^2\right]$$

$$\beta := \Delta^{-1}\left[U_0 \sum_{i=1}^{m} U_k V_k - V_0 \sum_{i=1}^{m} U_k^2\right]$$

$$\rho^2(\omega) \;=\; \alpha^2 \sum_{i=1}^{m} U_k^2 + 2\alpha\beta \sum_{i=1}^{m} U_k V_k + \beta^2 \sum_{i=1}^{m} V_k^2.$$

4.14 Consider the family of polynomials

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad |a_k - a_k^0| < \rho\alpha_k$$

and denote the nominal polynomial as $A_0(z)$. Show that all polynomials in this family have their roots inside the unit circle if and only if the plot

$$z(\theta) = \frac{A_0(e^{j\theta})}{|A_0(e^{j\theta})|}\mu(\theta)$$

$$\mu(\theta) = \max_{0 \leq i \leq n} \frac{|\sum_{k=1}^{m} a_k^0 \sin(k-i)\theta|}{\sum_{k=1}^{m} \alpha_k |\sin(k-i)\theta|}, \qquad 0 < \theta < \pi$$

$$\mu(0) = \frac{|\sum_{k=1}^{m} a_k^0|}{\sum_{k=1}^{m} \alpha_k}$$

$$\mu(\pi) = \frac{|\sum_{k=1}^{m} (-1)^k a_k^0|}{\sum_{k=1}^{m} \alpha_k}$$

makes n turns around the origin as θ runs from 0 to 2π and does not intersect the circle of radius ρ.

4.15 Consider the Hurwitz stability of the family of polynomials

$$\delta(s, \mathbf{p}) = p_1 s(s+9.5)(s-1) - p_2(6.5s+0.5)(s-2)$$

where $p_1 \in [1, 1.1]$ and $p_2 \in [1.2, 1.25]$. Show that the family is robustly stable. Determine the worst case stability margins over the given uncertainty set using the ℓ_2 and then the ℓ_∞ norm.

4.12 NOTES AND REFERENCES

The problem of calculating the ℓ_2 stability margin in parameter space for Hurwitz stability in the linear case was solved in Biernacki, Hwang and Bhattacharyya [40], Chapellat, Bhattacharyya and Keel [62], Chapellat and Bhattachar-yya [59]. The problem of computing the real stability margin has been dealt with by Hinrichsen and Pritchard [114, 112, 111], Tesi and Vicino [220, 218], Vicino, Tesi and Milanese [230], DeGaston and Safonov [80], and Sideris and Sánches Peña [210]. Real parametric stability margins for discrete time control systems were calculated in Aguirre, Chapellat, and Bhattacharyya [4]. The calculation of the ℓ_2 stability margin in time delay systems (Theorem 4.4) is due to Kharitonov [145]. The type of pitfall pointed out in Example 4.6 is due to Soh and Foo [216]. The stability detecting property of exposed edges was generalized by Soh and Foo [215] to arbitrary analytic functions. The reduction of the ℓ_∞ problem to a linear programming problem was pointed out by Tesi and Vicino in [220]. The Tsypkin-Polyak plot, Theorems 4.9 and 4.10, and the results described in Exercises 4.12-4.14 were developed in [225, 226]. The problem of discontinuity of the parametric stability margin was highlighted by Barmish, Khargonekar, Shi, and Tempo [16], and Example 4.2 is adapted from [16]. A thorough analysis of the problem of discontinuity of the parametric stability margin has been carried out by Vicino and Tesi in [229]. The role of exposed edges in determining the stability of polytopes originates from and follows from the Edge Theorem due to Bartlett, Hollot, and Lin [21] which we develop in Chapter 6. This idea was explored in Fu and Barmish [99] for polytopes of polynomials and by Fu, Olbrot, and Polis [100] for polytopic systems with time-delay. Exercise 4.10 is taken from Kogan [149]. Example 4.8 is due to Vicino and Tesi [220] and is taken from [220]. The *image set* referred to here is also called the *value set* in the literature in this field.