Chapter 3

THE STABILITY BALL IN COEFFICIENT SPACE

In this chapter we develop procedures to determine maximal stability regions in the space of coefficients of a polynomial. The central idea used is the Boundary Crossing Theorem and its alternative version the Zero Exclusion Theorem of Chapter 1. We begin by calculating the largest ℓ_2 stability ball centered at a given point in the space of coefficients of a polynomial. Explicit formulas are developed for the Schur and Hurwitz cases by applying the Orthogonal Projection Theorem. Following this, we present the graphical approach of Tsypkin and Polyak to calculate the largest ℓ_p stability ball, in coefficient space, for arbitrary p. Then we deal with the robust Hurwitz and Schur stability of a family of disc polynomials, namely complex polynomials whose coefficients lie in prescribed discs in the complex plane.

3.1 INTRODUCTION

In considering robust stability with respect to parametric uncertainty, one is naturally led to formulate the following problem: Given a stability region in the complex plane and a nominal stable polynomial, find the largest region of a prescribed shape in the coefficient space around the nominal polynomial where the stability property is maintained. In this chapter we present some neat solutions for the left half plane (Hurwitz stability) and the unit circle (Schur stability) considering both real and complex coefficients. Our development, however, will clearly show that a general answer can be formulated by invoking the Boundary Crossing Theorem of Chapter 1, at least when the region of interest can be associated with a norm. This will always be assumed in this chapter. We remark that Kharitonov's Theorem (Chapter 5) deals essentially with a special case of this problem when the stability region is the left-half plane and the coefficient uncertainty is interval. We start by calculating the largest ℓ_2 stability ball in the space of coefficients of a real polynomial, treating both Hurwitz and Schur stability.

3.2 THE BALL OF STABLE POLYNOMIALS

Recall the Boundary Crossing Theorem of Chapter 1. Let the stability region $\mathcal{S}$ be any given open set of the complex plane C, $\partial\mathcal{S}$ its boundary, and $\mathcal{U}^o$ the interior of the closed set $\mathcal{U} = C - \mathcal{S}$. Assume that these three sets $\mathcal{S}$, $\partial\mathcal{S}$, and $\mathcal{U}^o$ are nonempty. For any given n, the set $\mathcal{P}_n$ of real polynomials of degree less than or equal to n is a vector space of dimension $n+1$ and as usual we identify this with $\mathbb{R}^{n+1}$ to which it is isomorphic. Let $\|\cdot\|$ be an arbitrary norm defined on $\mathcal{P}_n$. The *open balls* induced by this norm are of the form,

$$B(P_o(s), r) = \{P(s) \in \mathcal{P}_n \ : \ \|P(s) - P_o(s)\| < r\}. \tag{3.1}$$

With such an open ball is associated the *hypersphere*,

$$S(P_o(s), r) = \{P(s) \in \mathcal{P}_n \ : \ \|P(s) - P_o(s)\| = r\}, \tag{3.2}$$

which is just the boundary of $B(P_o(s), r)$. Now, as mentioned in Chapter 1, the subset of $\mathcal{P}_n$ consisting of all polynomials $\delta(s)$ which are of degree n and which have all their roots in $\mathcal{S}$ is an open set. As a direct consequence, given a polynomial $\delta(s)$ of degree n with all its roots contained in $\mathcal{S}$, there exists a positive real number ϵ such that every polynomial contained in $B(\delta(s), \epsilon)$ is of degree n and has all its roots in $\mathcal{S}$. In other words, letting $d^o(\cdot)$ denote the degree of a polynomial, we have that ϵ satisfies the following property:

Property 3.1.

$$\|\beta(s) - \delta(s)\| < \epsilon \quad \Longrightarrow \quad \begin{cases} d^o(\beta(s)) = n \\ \beta(s) \quad \text{has all its roots in } \mathcal{S}. \end{cases}$$

As in the proof of the Boundary Crossing Theorem, it is then possible to consider, for the given stable polynomial $\delta(s)$, the subset of all positive real numbers having the Property 3.1:

$$R_\delta := \{t \ : \ t > 0, \ t \text{ satisfies Property 3.1}\}.$$

We have just seen that R_δ is not empty. But obviously the elements of R_δ satisfy

$$t_2 \in R_\delta \quad \text{and} \quad 0 < t_1 < t_2 \Longrightarrow t_1 \in R_\delta.$$

Therefore R_δ is in fact an interval

$$(0, \rho(\delta)] \qquad \text{where: } \rho(\delta) = \sup_{t \in R_\delta} t.$$

Clearly, $\rho(\delta)$ has to be finite and $\rho(\delta)$ also satisfies Property 3.1 (that is why we closed the interval on the right). We have just proved the existence and uniqueness of a real number $\rho(\delta)$ characterized by :

1. $\rho(\delta)$ satisfies Property 3.1.

2. No real r greater than $\rho(\delta)$ satisfies Property 3.1.

We now give a precise characterization of $\rho(\delta)$.

Theorem 3.1 *Given a polynomial $\delta(s)$, of degree n, having all its roots in $\mathcal{S}$, there exists a positive real number $\rho(\delta)$ such that:*

a) Every polynomial contained in $B(\delta(s), \rho(\delta))$ has all its roots in $\mathcal{S}$ and is of degree n.

b) At least one polynomial on the hypersphere $S(\delta(s), \rho(\delta))$ has one of its roots in $\partial\mathcal{S}$ or is of degree less than n.

c) However, no polynomial lying on the hypersphere (even those of degree $< n$) can ever have a root in $\mathcal{U}^o$.

Proof. Clearly a) is true since $\rho(\delta)$ satisfies Property 3.1. We now prove b) and c). Since no real r greater than $\rho(\delta)$ satisfies Property 3.1, then for every $n \geq 1$ there exists a polynomial of degree less than n or with a root in $\mathcal{U} = C - \mathcal{S}$, say $\gamma_n(s)$, contained in the ball $\mathcal{B}(\delta(s), \rho(\delta) + \frac{1}{n})$. Being contained in the closure of $B(\delta(s), \rho(\delta) + 1)$ which is a compact set, this sequence must then contain a convergent subsequence $\gamma_{\phi(n)}(s)$. Let $\gamma(s)$ be its limit. Then $\gamma(s)$ is necessarily lying on the hypersphere $S(\delta(s), \rho(\delta))$, and it is also necessarily of degree less than n or with a root in $\mathcal{U}$; otherwise the existence of $\rho(\gamma)$ would contradict the fact that $\gamma(s)$ is the limit of a sequence of polynomials of degree less than n or with a root in $\mathcal{U}$.

To proceed, we need to invoke Rouché's Theorem (Theorem 1.2, Chapter 1). Suppose that there is a polynomial lying on $S(\delta(s), \rho(\delta))$ say $\gamma(s)$, which is of degree n but has at least one root s_k in $\mathcal{U}^o$. A consequence is that the set of polynomials of degree n with at least one root in the open set $\mathcal{U}^o$ is itself open. Thus it would be possible to find a ball of radius $\epsilon > 0$ around $\gamma(s)$ containing only polynomials of degree n and with at least one root in $\mathcal{U}^o$. This would then result in a contradiction because since $\gamma(s)$ lies on the hypersphere $\mathcal{S}(\delta(s), \rho(\delta))$, the intersection

$$B(\gamma(s), \epsilon) \cap B(\delta(s), \rho(\delta))$$

is certainly nonempty.

On the other hand suppose that this polynomial $\gamma(s)$ with at least one root in $\mathcal{U}^o$ is of degree less than n. For $\epsilon > 0$ consider the polynomial,

$$\gamma_\epsilon(s) = \epsilon\delta(s) + (1 - \epsilon)\gamma(s).$$

It is clear that $\gamma_\epsilon(s)$ is always of degree n and is inside $B(\delta(s), \rho(\delta))$ since

$$\|\delta(s) - \gamma_\epsilon(s)\| = (1 - \epsilon)\|\delta(s) - \gamma(s)\| < \rho(\delta).$$

This means that $\gamma_\epsilon(s)$ has all its roots in $\mathcal{S}$.Now, a straightforward application of Rouché's theorem shows that for ϵ small enough $\gamma_\epsilon(s)$ also has at least one root in $\mathcal{U}^o$, and this is again a contradiction. ♣

Of course, this result can be applied to a number of situations depending on the region $\mathcal{S}$ of interest and on the norm $\|\cdot\|$ chosen on $\mathcal{P}_n$. Below we consider the two cases of Hurwitz and Schur stability with the Euclidean norm. In both these cases a neat expression can be given for the corresponding $\rho(\delta)$.

3.3 THE REAL ℓ_2 STABILITY BALL

On $\mathcal{P}_n$, the usual inner product and the associated Euclidean norm are defined as follows. If

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n,$$

and

$$R(s) = r_0 + r_1 s + \cdots + r_n s^n,$$

then the inner product of $P(s)$ and $R(s)$ is given by

$$< P(s), R(s) >= p_0 r_0 + p_1 r_1 + \cdots + p_n r_n = \sum_{i=0}^{n} p_i r_i.$$

The Euclidean norm of a polynomial $P(s)$ is then:

$$\|P(s)\|_2^2 =< P(s), P(s) >= p_0^2 + p_1^2 + \cdots + p_n^2.$$

We can now look at the left-half plane case.

3.3.1 Hurwitz Stability

For this case we first have to introduce some subspaces of $\mathcal{P}_n$. Let Δ_0 be the subset of all elements $P(s)$ of $\mathcal{P}_n$ such that

$$P(0) = 0.$$

Δ_0 is a subspace of dimension n generated by the basis vectors

$$s, s^2, s^3, \cdots, s^{n-1}, s^n.$$

Dually, let Δ_n be the subset of all elements $P(s)$ of $\mathcal{P}_n$ that are of degree less than n, that is such that $p_n = 0$. Δ_n is also a subspace of dimension n, generated by the basis vectors

$$1, s, s^2, \cdots, s^{n-2}, s^{n-1}.$$

Finally, for each real $\omega \geq 0$, we can consider the subset Δ_ω of all elements of $\mathcal{P}_n$ which are divisible by $s^2+\omega^2$. Equivalently, Δ_ω is the set of all elements of $\mathcal{P}_n$ that have $+j\omega$ and $-j\omega$ among their roots. Δ_ω is also a subspace. It is of dimension $n-1$ and is generated by the basis vectors

$$s^2+\omega^2, s^3+\omega^2 s, \cdots, s^{n-1}+\omega^2 s^{n-3}, s^n+\omega^2 s^{n-2}.$$

It is to be noted however that the Δ_ω's are only defined when $n \geq 2$; we will assume this in what follows, but will explain what happens when $n < 2$.

Now, since $(\mathcal{P}_n, \|\cdot\|_2)$ is a Euclidean vector space, it is possible to apply the Projection Theorem. For any element $P(s)$ of $\mathcal{P}_n$, and for any subspace $\mathcal{V}$, there is a unique vector $\pi_P(s)$ in $\mathcal{V}$ at which the distance between $P(s)$ and all elements of $\mathcal{V}$ is minimized; $\pi_P(s)$ is nothing but the orthogonal projection of $P(s)$ on $\mathcal{V}$, and $\|P(s) - \pi_P(s)\|$ is called the *distance* from $P(s)$ to the subspace $\mathcal{V}$. Given then a Hurwitz stable polynomial

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n,$$

we let d_0, d_n, and d_ω denote the distances from $\delta(s)$ to the subspaces Δ_0, Δ_n, and Δ_ω respectively. Finally let us define

$$d_{\min} := \inf_{\omega \geq 0} d_\omega.$$

Applying Theorem 3.1 here tells us that the radius $\rho(\delta)$ of the largest stability hypersphere around $\delta(s)$ is characterized by the fact that every polynomial in $B(\delta(s),\rho(\delta))$ is stable and of degree n whereas at least one polynomial on the hypersphere $S(\delta(s),\rho(\delta))$ is of degree less than n or has a root on the imaginary axis. In fact we have the following result.

Theorem 3.2 *The radius of the largest stability hypersphere around a stable polynomial $\delta(s)$ is given by*

$$\rho(\delta) = \min(d_0, d_n, d_{\min}) \tag{3.3}$$

Proof. Let

$$r = \min(d_o, d_n, d_{\min}).$$

Clearly the open ball $B(\delta(s), r)$ centered at $\delta(s)$ and of radius r cannot contain any polynomial having a root which is 0 or purely imaginary or any polynomial of degree less than n. From our characterization of the stability hypersphere given in Theorem 3.1 above we deduce that $\rho(\delta) \geq r$ necessarily. On the other hand from the definition of r we can see that any ball centered at $\delta(s)$ and of radius greater than r must contain at least one unstable polynomial or a polynomial of degree less than n. Thus, necessarily again, $\rho(\delta) \leq r$. As a consequence we get that

$$\rho(\delta) = r = \min(d_o, d_n, d_{\min}).$$

♣

In (3.3) it is very easy to prove that

$$d_0 = |\delta_0|, \quad \text{and} \quad d_n = |\delta_n|.$$

The main problem then is to compute $d_{\min}$. In the following we will show how d_ω can be obtained in closed-form for any degree n. We will then show how

$d_{\min} = \inf_{\omega \geq 0} d_\omega$ can also be computed very easily by carrying out two similar minimizations over the finite range $[0,1]$. Therefore let $\delta(s)$ be an arbitrary stable polynomial of degree n. As usual separate $\delta(s)$ into odd and even parts:

$$\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n = \underbrace{\delta^{\text{even}}(s)}_{\text{even degree terms}} + \underbrace{\delta^{\text{odd}}(s)}_{\text{odd degree terms}}.$$

Theorem 3.3 *The distance d_ω between $\delta(s)$ and Δ_ω is given by:*

i) $n = 2p$:

$$d_\omega^2 = \frac{[\delta^e(\omega)]^2}{1+\omega^4+\cdots+\omega^{4p}} + \frac{[\delta^o(\omega)]^2}{1+\omega^4+\cdots+\omega^{4(p-1)}}. \tag{3.4}$$

ii) $n = 2p+1$:

$$d_\omega^2 = \frac{[\delta^e(\omega)]^2 + [\delta^o(\omega)]^2}{1+\omega^4+\cdots+\omega^{4p}}. \tag{3.5}$$

Proof. We know that $\mathcal{P}_n$ is a vector space of dimension $n+1$. Δ_ω on the other hand is of dimension $n-1$ and is generated by the following elements:

$$s^2+\omega^2, s^3+\omega^2 s, s^4+\omega^2 s^2, \cdots, s^n+\omega^2 s^{n-2}. \tag{3.6}$$

Therefore $\Delta_\omega^\perp$ is of dimension 2. Let $p_1(s)$ and $p_2(s)$ be an orthogonal basis for $\Delta_\omega^\perp$. Let $v_\delta^\omega(s)$ denote the orthogonal projection of $\delta(s)$ on Δ_ω. By definition of the orthogonal projection, $\delta - v_\delta^\omega$ is an element of $\Delta_\omega^\perp$ and we can write

$$\delta - v_\delta^\omega = \alpha_1 p_1 + \alpha_2 p_2. \tag{3.7}$$

Taking the inner product of both members of (3.7) with $p_1(s)$ and $p_2(s)$ successively and remembering that $p_1(s)$ and $p_2(s)$ are orthogonal to each other as well as to any element of Δ_ω, we get

$$\alpha_1 = \frac{<\delta, p_1>}{||p_1||^2} \quad \text{and} \quad \alpha_2 = \frac{<\delta, p_2>}{||p_2||^2}. \tag{3.8}$$

But the definition of the orthogonal projection also implies that

$$d_\omega^2 = ||\delta - v_\delta^\omega||^2. \tag{3.9}$$

Thus,combining (3.7), (3.8), (3.9) and again taking into account the fact that $p_1(s)$ and $p_2(s)$ are chosen to be orthogonal, we get

$$d_\omega^2 = \frac{<\delta, p_1>^2}{||p_1||^2} + \frac{<\delta, p_2>^2}{||p_2||^2}. \tag{3.10}$$

It just remains for us to find $p_1(s)$ and $p_2(s)$.

i) $n = 2p$: In this case we can choose

$$p_1(s) = 1 - \omega^2 s^2 + \omega^4 s^4 + \cdots + (-1)^p \omega^{2p} s^{2p},$$
$$p_2(s) = s - \omega^2 s^3 + \omega^4 s^5 + \cdots + (-1)^{(p-1)} \omega^{2(p-1)} s^{2p-1}.$$

One can check very easily that $p_1(s)$ and $p_2(s)$ are orthogonal to each element of (3.6) and also orthogonal to each other. Moreover they satisfy

$$< p_1, \delta > = \delta^e(\omega)$$
$$< p_2, \delta > = \delta^o(\omega)$$

and

$$\|p_1\|^2 = 1 + \omega^4 + \omega^8 + \cdots + \omega^{4p}$$
$$\|p_2\|^2 = 1 + \omega^4 + \omega^8 + \cdots + \omega^{4(p-1)}.$$

The expression in (3.4) for d_ω^2 then follows from (3.10) and the properties above.

ii) $n = 2p+1$: In this case $p_1(s)$ remains unchanged but $p_2(s)$ becomes

$$p_2(s) = s - \omega^2 s^3 + \omega^4 s^5 + \cdots + (-1)^{(p-1)} \omega^{2(p-1)} s^{2p-1} + (-1)^p \omega^{2p} s^{2p+1}.$$

$p_1(s)$ and $p_2(s)$ have then the same properties as when n is even except that

$$\|p_1\|^2 = \|p_2\|^2 = 1 + \omega^4 + \omega^8 + \cdots + \omega^{4p}$$

and therefore we have the expression in (3.5) for d_ω^2 when n is odd. Moreover, the formula in (3.7) tells us exactly what the projection $v_\delta^\omega(s)$ is:

$$v_\delta^\omega(s) = \delta(s) - \alpha_1 p_1(s) - \alpha_2 p_2(s).$$

♣

Having this expression for d_ω, the next step is to find:

$$d_{\min} = \inf_{\omega \geq 0} d_\omega.$$

A simple manipulation will show that there is no need to carry out a minimization over the infinite range $[0, \infty)$. We will consider the case when $n = 2p$, but a similar derivation holds if n is odd.

First it is clear that

$$d_{\min}^2 = \min\left(\inf_{\omega\in[0,1]} d_\omega^2, \inf_{\omega\in[0,1]} d_{\frac{1}{\omega}}^2\right).$$

We have

$$|\delta^e(\omega)|^2 = \left(\delta_0 - \delta_2\omega^2 + \delta_4\omega^4 + \cdots + (-1)^p \delta_{2p}\omega^{2p}\right)^2$$
$$|\delta^o(\omega)|^2 = \left(\delta_1 - \delta_3\omega^2 + \delta_5\omega^4 + \cdots + (-1)^{p-1} \delta_{2p-1}\omega^{2p-2}\right)^2$$

which yields:

$$\left|\delta^e\left(\frac{1}{\omega}\right)\right|^2 = \frac{1}{\omega^{4p}}\left[\delta_{2p} - \delta_{2p-2}\omega^2 + \delta_{2p-4}\omega^4 + \cdots + (-1)^p\delta_0\omega^{2p}\right]^2$$
$$\left|\delta^o\left(\frac{1}{\omega}\right)\right|^2 = \frac{1}{\omega^{4(p-1)}}\left[\delta_{2p-1} - \delta_{2p-3}\omega^2 + \delta_{2p-5}\omega^4 + \cdots + (-1)^{p-1}\delta_1\omega^{2p-2}\right]^2.$$

Now

$$d^2_{\frac{1}{\omega}} = \frac{\frac{1}{\omega^{4p}}\left[\delta_{2p} - \delta_{2p-2}\omega^2 + \delta_{2p-4}\omega^4 + \cdots + (-1)^p\delta_0\omega^{2p}\right]^2}{1 + \frac{1}{\omega^4} + \frac{1}{\omega^8} + \cdots + \frac{1}{\omega^{4p}}}$$
$$+ \frac{\frac{1}{\omega^{4(p-1)}}\left[\delta_{2p-1} - \delta_{2p-3}\omega^2 + \delta_{2p-5}\omega^4 + \cdots + (-1)^{p-1}\delta_1\omega^{2p-2}\right]^2}{1 + \frac{1}{\omega^4} + \frac{1}{\omega^8} + \cdots + \frac{1}{\omega^{4(p-1)}}}.$$

This last expression however is nothing but:

$$d^2_{\frac{1}{\omega}} = \frac{\left[\delta_{2p} - \delta_{2p-2}\omega^2 + \delta_{2p-4}\omega^4 + \cdots + (-1)^p\delta_0\omega^{2p}\right]^2}{1 + \omega^4 + \omega^8 + \cdots + \omega^{4p}}$$
$$+ \frac{\left[\delta_{2p-1} - \delta_{2p-3}\omega^2 + \delta_{2p-5}\omega^4 + \cdots + (-1)^{p-1}\delta_1\omega^{2p-2}\right]^2}{1 + \omega^4 + \omega^8 + \cdots + \omega^{4(p-1)}}$$

and we can see that $d^2_{\frac{1}{\omega}}$ has exactly the same structure as d^2_ω.

Can $d^2_{\frac{1}{\omega}}$ be considered as the "d^2_ω" of some other polynomial? The answer is yes. Consider $\delta'(s) = s^n\delta\left(\frac{1}{s}\right)$, the "reverse" polynomial which in our case is

$$\delta'(s) = s^{2p}\delta\left(\frac{1}{s}\right) = \delta_{2p} + \delta_{2p-1}s + \delta_{2p-2}s^2 + \cdots + \delta_2 s^{2p-2} + \delta_1 s^{2p-1} + \delta_0 s^{2p}.$$

Then

$$\delta'^e(\omega) = \delta'_{\text{even}}(j\omega) = \delta_{2p} - \delta_{2p-2}\omega^2 + \delta_{2p-4}\omega^4 + \cdots + (-1)^p\delta_0\omega^{2p}$$
$$\delta'^o(\omega) = \frac{\delta'_{\text{odd}}(j\omega)}{j\omega} = \delta_{2p-1} - \delta_{2p-3}\omega^2 + \delta_{2p-5}\omega^4 + \cdots + (-1)^{p-1}\delta_1\omega^{2p-2}.$$

Thus we see that in fact $d^2_{\frac{1}{\omega}}$ corresponds to d^2_ω computed for $\delta'(s)$. Suppose now that you have a subroutine DMIN ($\underline{\delta}$) that takes the vector of coefficients $\underline{\delta}$ as input and returns the minimum of d^2_ω over $[0,\ 1]$. Then the following algorithm will compute $d_{\min}$ by simply calling DMIN two times:

1. Set $\underline{\delta} = (\delta_0, \delta_1, \cdots, \delta_n)$.
2. First call: $d_1 = \text{DMIN}(\underline{\delta})$.

3. Switch: set $\underline{\delta} = (\delta_n, \delta_{n-1}, \cdots, \delta_0)$.

4. Second call: $d_2 = \text{DMIN}(\underline{\delta})$.

5. $d_{\min} = \min(d_1, d_2)$.

Moreover, d_ω^2 as given in (3.4) or (3.5) is a rational function of relative degree 0 with no real poles and therefore is extremely well behaved in $[0, 1]$. As a consequence the subroutine $DMIN$ is not particularly hard to implement.

Incidentally, we already knew that the two polynomials $\delta(s)$ and $\delta'(s) = s^n \delta\left(\frac{1}{s}\right)$ are stable together (i.e. one is stable if and only if the other one is stable). The development above tells us that moreover $\rho(\delta) = \rho(\delta')$.

In the case when $n = 1$, then the subspaces Δ_ω are not defined. However, if we apply the formula in (3.5) anyway, we find that

$$d_\omega^2 = \delta_0^2 + \delta_1^2, \quad \forall\, \omega \geq 0. \tag{3.11}$$

Therefore, $d_{\min}$ itself is given by the same expression, and when we compute $\rho(\delta)$, we find

$$[\rho(\delta)]^2 = \min(\delta_0^2, \delta_1^2, \delta_0^2 + \delta_1^2) = \min(\delta_0^2, \delta_1^2).$$

Thus even if we apply our formula for $n = 1$ we get the correct answer for $\rho(\delta)$. The same argument holds in the trivial case $n = 0$.

The Monic Case

In some applications it can happen that the leading coefficient is fixed and not subject to perturbation. Consider the situation where we are given a stable *monic* polynomial

$$\beta(s) = \beta_0 + \beta_1 s + \cdots + \beta_{n-1}s^{n-1} + s^n$$

and we want to find the closest *monic* polynomial with a root at $j\omega$. For convenience, the general distance d_ω that we computed in Theorem 3.2 will now be denoted $d_\omega^n[\delta(s)]$, to stress the fact that it applies to the polynomial $\delta(s)$ which is of degree n. Along the same lines, the distance for a monic polynomial of degree n will be denoted by $\bar{d}_\omega^n[\beta(s)]$. We then have the following result.

Theorem 3.4 *The distance $\bar{d}_\omega^n[\beta(s)]$ from the monic polynomial of degree n to the affine subspace of all monic polynomials with a root at $j\omega$ is given by*

$$\bar{d}_\omega^n[\beta(s)] = d_\omega^{n-1}[\beta(s) - s^n - \omega^2 s^{n-2}].$$

Proof. We know that Δ_ω is a vector space of dimension $n-1$ generated by the basis

$$s^2 + \omega^2, s^3 + \omega^2 s, \cdots, s^n + \omega^2 s^{n-2}.$$

The generic element of Δ_ω can be written as the linear combination

$$\lambda_1(s^2 + \omega^2) + \lambda_2(s^3 + \omega^2 s) + \cdots + \lambda_{n-1}(s^n + \omega^2 s^{n-2}).$$

Now, the generic monic polynomial of Δ_ω must satisfy $\lambda_{n-1} = 1$ and therefore can be written as,

$$\lambda_1(s^2+\omega^2) + \lambda_2(s^3+\omega^2 s) + \cdots + (s^n + \omega^2 s^{n-2}).$$

As a consequence, the distance that we are looking for can be expressed as

$$\bar{d}^n_\omega[\beta(s)] =$$
$$\inf_{\lambda_1,\lambda_2,\cdots,\lambda_{n-2}} \|\beta(s) - \lambda_1(s^2+\omega^2) - \lambda_2(s^3+\omega^2 s) - \cdots - (s^n+\omega^2 s^{n-2})\|.$$

But this can be rewritten as

$$d^{n\prime}_\omega[\beta(s)] =$$
$$\inf_{\lambda_1,\lambda_2,\cdots,\lambda_{n-2}} \|(\beta(s) - s^n - \omega^2 s^{n-2}) - \lambda_1(s^2+\omega^2) - \cdots - \lambda_{n-2}(s^{n-1}+\omega^2 s^{n-3})\|.$$

But since $\beta(s) - s^n - \omega^2 s^{n-2}$ is a polynomial of degree less than or equal to $n-1$, this last infimum is nothing but

$$d^{n-1}_\omega[\beta(s) - s^n - \omega^2 s^{n-2}].$$

♣

An example of the above calculation follows.

Example 3.1. Consider the Hurwitz polynomial

$$\delta(s) := 6 + 49s + 155s^2 + 280s^3 + 331s^4 + 266s^5 + 145s^6 + 52s^7 + 11s^8 + s^9.$$

and the problem of calculating $\rho(\delta)$ for it.

All coefficients are subject to perturbation If we assume that all coefficients of $\delta(s)$ including the leading coefficient are subject to perturbation, we need to apply Theorem 3.2 which deals with the stability margin for nonmonic polynomials. Consequently, we have

$$d^2_\omega = \frac{[\delta^e(\omega)]^2 + [\delta^o(\omega)]^2}{1+\omega^4+\cdots+\omega^{4p}}$$

where $p = 4$ and

$$\delta^e(\omega) = 6 - 155\omega^2 + 331\omega^4 - 145\omega^6 + 11\omega^8$$
$$\delta^o(\omega) = 49 - 280\omega^2 + 266\omega^4 - 52\omega^6 + \omega^8.$$

In order to compute $d_{\min}$ analytically we first compute real positive ω's satisfying

$$\frac{d(d_\omega)}{d\omega} = 0.$$

With these real positive ωs we evaluate d_ω. Then we have

ω	d_ω
3.2655	1.7662
1.8793	6.8778
1.6185	6.5478
0.7492	27.7509
0.4514	13.0165

From this, we have

$$d_{\min} = \inf_{\omega \geq 0} d_\omega = 1.7662 \text{ at } \omega = 3.2655$$

$$d_0 = 6 \quad \text{and} \quad d_n = 1.$$

Therefore, $\rho(\delta) = \min(d_0, d_n, d_{\min}) = 1$. Figure 3.1 shows the plot of d_ω with respect to ω. The graphical solution agrees with the analytical solution given earlier. This means that as we blow up the stability ball, the family first loses degree, then acquires a $j\omega$ root at $\omega = 3.2655$, and subsequently a root at the origin.

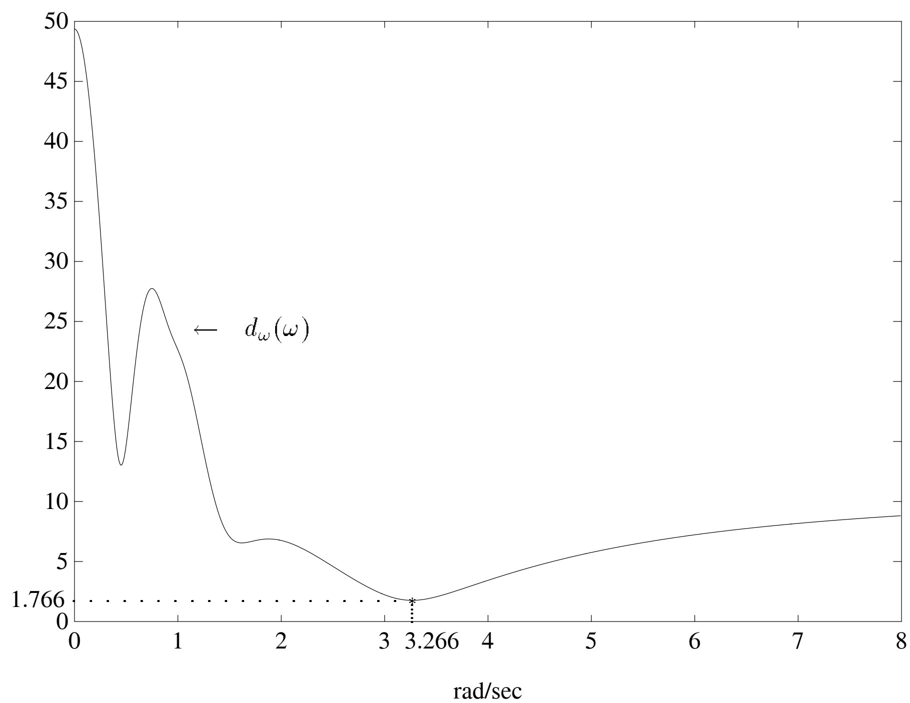

Figure 3.1. $d_\omega(\omega)$ vs. ω (all coefficients perturbing) (Example 3.1)

The leading coefficient is fixed Now let us assume that the leading coefficient of the polynomial family remains fixed at the value 1:

$$\beta(s) := 6 + 49s + 155s^2 + 280s^3 + 331s^4 + 266s^5 + 145s^6 + 52s^7 + 11s^8 + s^9.$$

Then we need to apply Theorem 3.4. We have

$$\bar{d}_\omega^n[\beta(s)]_{n=9} = d_\omega^8[\beta(s) - s^9 - \omega^2 s^7] := d_\omega^8[\alpha(s)]$$

and

$$d_\omega^2 = \frac{[\alpha^e(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{16}} + \frac{[\alpha^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{12}}$$

where

$$\begin{aligned}\alpha^e(\omega) &= 6 - 155\omega^2 + 331\omega^4 - 145\omega^6 + 11\omega^8 \\ \alpha^o(\omega) &= 49 - 280\omega^2 + 266\omega^4 + (-52 + \omega^2)\omega^6 \\ &= 49 - 280\omega^2 + 266\omega^4 - 52\omega^6 + \omega^8.\end{aligned}$$

As before, we first compute real positive ω's satisfying

$$\frac{d(d_\omega)}{d\omega} = 0.$$

With these real positive ω's we evaluate d_ω. Then we have

ω	d_ω
6.7639	8.0055
3.9692	20.6671
2.0908	6.5621
0.7537	27.8492
0.4514	13.0165

From this, we have

$$d_{\min} = \inf_{\omega \geq 0} d_\omega = 6.5621 \text{ at } \omega = 2.0908.$$

Therefore, $\rho(\delta) = \min(d_0, d_{\min}) = 6$. Figure 3.2 show the plot of d_ω with respect to ω. The graphical solution agrees with the analytical solution given earlier. This means that as the family is enlarged it first acquires a root at the origin and subsequently a $j\omega$ root at $\omega = 2.0908$.

3.3.2 Schur Stability

In this section we calculate the ℓ_2 stability ball in coefficient space for Schur stability. Let $P(z)$ denote a generic real polynomial of degree n and $\mathcal{P}_n$ denote the vector space over the reals of polynomials of degree n or less. We start by introducing the

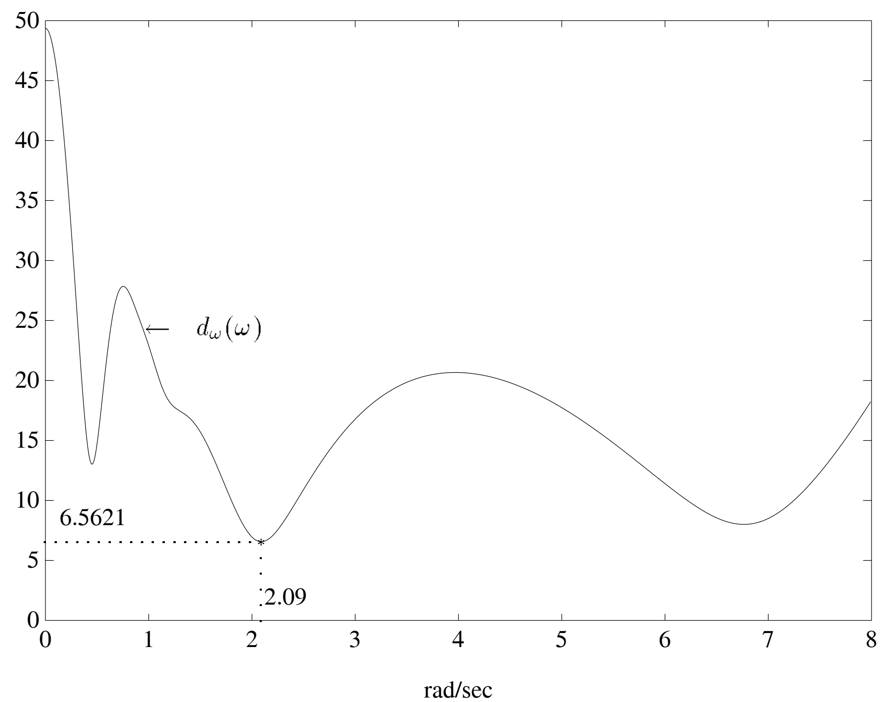

Figure 3.2. $d_\omega(\omega)$ vs. ω (leading coefficient fixed) (Example 3.1)

relevant subspaces of $\mathcal{P}_n$. Let Δ_{+1} be the subset of all elements $P(z)$ of $\mathcal{P}_n$ with a root at $z = +1$. Δ_{+1} is a subspace of dimension n and a basis for Δ_{+1} is the set

$$z-1, z(z-1), z^2(z-1), \cdots z^{n-1}(z-1).$$

It is easy to see that $\Delta_{+1}^\perp$ is generated by the polynomial,

$$P_{+1}(z) = 1 + z + z^2 + \cdots + z^n. \tag{3.12}$$

Let Δ_{-1} be the subset of all elements $P(z)$ of $\mathcal{P}_n$ with a root at $z = -1$. Δ_{-1} is also a subspace of dimension n and a basis is the set

$$z+1, z(z+1), z^2(z+1), \cdots z^{n-1}(z+1).$$

$\Delta_{-1}^\perp$ is generated by the polynomial,

$$P_{-1}(z) = 1 - z + z^2 - \cdots + (-1)^n z^n. \tag{3.13}$$

For each $0 < \theta < \pi$, we can introduce the set Δ_θ of all polynomials in $\mathcal{P}_n$ with a pair of roots at $e^{j\theta}$ and $e^{-j\theta}$. Δ_θ is also a subspace of dimension $n-1$, and a basis for Δ_θ is given by

$$z^2 - z2\cos\theta + 1,\, z(z^2 - z2\cos\theta + 1),\, z^2(z^2 - z2\cos\theta + 1), \cdots z^{n-2}(z^2 - z2\cos\theta + 1). \tag{3.14}$$

It is easy to verify that a basis for $\Delta_\theta^\perp$ is given by:

$$\begin{aligned} q_\theta^1(z) &= 1 + (\cos\theta)z + (\cos 2\theta)z^2 + \cdots + (\cos(n-1)\theta)z^{n-1} + (\cos n\theta)z^n \\ q_\theta^2(z) &= (\sin\theta)z + (\sin 2\theta)z^2 + \cdots + (\sin(n-1)\theta)z^{n-1} + (\sin n\theta)z^n. \end{aligned}$$

Now let

$$P(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n$$

be a real Schur polynomial of degree n. Let d_{+1}, d_{-1} and d_θ designate the distances from $P(z)$ to the subspaces Δ_{+1}, Δ_{-1}, Δ_θ, respectively, and let us define

$$d_{\min} = \inf_{0<\theta<\pi} d_\theta.$$

Just as in the continuous case we have the following theorem.

Theorem 3.5 *The radius of the largest stability hypersphere around a Schur polynomial $P(z)$ of degree n is given by:*

$$\rho(P) = \min(d_{+1}, d_{-1}, d_{\min}).$$

The distances from $P(z)$ to the subspaces Δ_{+1}, and Δ_{-1}, are respectively:

$$d_{+1} = \frac{|< P(z), P_{+1}(z) >|}{\|P_{+1}(z)\|} = \frac{|P(1)|}{\sqrt{n+1}}$$

and

$$d_{-1} = \frac{|< P(z), P_{-1}(z) >|}{\|P_{-1}(z)\|} = \frac{|P(-1)|}{\sqrt{n+1}}.$$

The distance d_θ between P(z) and the subspace Δ_θ is given by:

$$d_\theta^2 = \frac{\lambda_2^2\|q_\theta^1(z)\|^2 - 2\lambda_1\lambda_2 < q_\theta^1(z), q_\theta^2(z) > + \lambda_1^2\|q_\theta^2(z)\|^2}{\|q_\theta^1(z)\|^2\|q_\theta^2(z)\|^2 - < q_\theta^1(z), q_\theta^2(z) >^2}.$$

where

$$\begin{aligned} \lambda_1 &= < q_\theta^1(z), P(z) > = \mathrm{Re}[P(e^{j\theta})] \\ \lambda_2 &= < q_\theta^2(z), P(z) > = \mathrm{Im}[P(e^{j\theta})] \end{aligned}$$

and,

$$\|q_\theta^1(z)\|^2 = \sum_{k=0}^{n} \cos^2 k\theta = \frac{1}{2}\left[\sum_{k=0}^{n}(1 + \cos 2k\theta)\right]$$

$$= \frac{n+1}{2} + \frac{1}{2}(\cos n\theta)\left(\frac{\sin(n+1)\theta}{\sin\theta}\right)$$

$$\|q_\theta^2(z)\|^2 = \sum_{k=0}^{n}(\sin k\theta)^2 = \frac{1}{2}\left[\sum_{k=0}^{n}(1-\cos 2k\theta)\right]$$

$$= \frac{n+1}{2} - \frac{1}{2}(\cos n\theta)\left(\frac{\sin(n+1)\theta}{\sin\theta}\right)$$

$$< q_\theta^1(z), q_\theta^2(z) > = \sum_{k=0}^{n}(\cos k\theta)(\sin k\theta) = \frac{1}{2}\left[\sum_{k=0}^{n}\sin 2k\theta\right]$$

$$= \frac{1}{2}(\sin n\theta)\left(\frac{\sin(n+1)\theta}{\sin\theta}\right).$$

Proof. The proof is exactly similar to the Hurwitz case and uses the Orthogonal Projection Theorem. We know that $\Delta_\theta^\perp$ is generated by $q_\theta^1(z)$ and $q_\theta^2(z)$. The formula that results is slightly more complicated due to the fact that $q_\theta^1(z)$ and $q_\theta^2(z)$ are not orthogonal to each other. However one can easily compute that the denominator of d_θ^2 is

$$\frac{\left[(n+1)^2 - \frac{\sin^2(n+1)\theta}{\sin^2\theta}\right]}{4}.$$

♣

For the discrete time case also it is of interest to consider the case of monic polynomials, and to compute the distance $\bar{d}_{+1}$, $\bar{d}_{-1}$ and $\bar{d}_\theta$ from a given monic polynomial

$$Q(z) = q_0 + q_1 z + \cdots + q_{n-1}z^{n-1} + z^n$$

to the set of all monic polynomials with a root at $z = 1$, or $z = -1$, or a pair of roots at $e^{j\theta}$, $e^{-j\theta}$ respectively. We have the following theorem.

Theorem 3.6 *The distances $\bar{d}_{+1}$, $\bar{d}_{-1}$ and $\bar{d}_\theta^n[Q(z)]$ are given by,*

$$\bar{d}_{+1} = \frac{|Q(1)|}{\sqrt{n}}$$

$$\bar{d}_{-1} = \frac{|Q(-1)|}{\sqrt{n}}$$

$$\bar{d}_\theta^n[Q(z)] = d_\theta^{n-1}\left[Q(z) - z^n + 2\cos\theta z^{n-1} - z^{n-2}\right].$$

The proof of this result is left to the reader, but it basically uses the same ideas as for the Hurwitz case. We illustrate the use of the above formulas with an example.

Example 3.2. Consider the Schur stable polynomial

$$\delta(z) := 0.1 + 0.2z + 0.4z^2 + 0.3z^3 + z^4$$

and the problem of computing $\rho(\delta)$ for it.

All coefficients are subject to perturbation Let us assume that all coefficients of $\delta(z)$ including the leading coefficient are subject to perturbation. Then we need to apply Theorem 3.5. Consequently, we have

$$d_\theta^2 = \frac{\lambda_2^2 \|q_\theta^1(z)\|^2 - 2\lambda_1\lambda_2 < q_\theta^1(z), q_\theta^2(z) > + \lambda_1^2 \|q_\theta^2(z)\|^2}{\|q_\theta^1(z)\|^2 \|q_\theta^2(z)\|^2 - < q_\theta^1(z), q_\theta^2(z) >^2}.$$

where

$$\begin{aligned}
\lambda_1 &= \mathrm{Re}[\delta(e^{j\theta})] \\
&= 0.1 + 0.2\cos\theta + 0.4\cos 2\theta + 0.3\cos 3\theta + \cos 4\theta \\
\lambda_2 &= \mathrm{Im}[\delta(e^{j\theta})] \\
&= 0.2\sin\theta + 0.4\sin 2\theta + 0.3\sin 3\theta + \sin 4\theta \\
\|q_\theta^1(z)\|^2 &= \frac{5}{2} + \frac{1}{2}(\cos 4\theta)\left(\frac{\sin 5\theta}{\sin\theta}\right) \\
\|q_\theta^2(z)\|^2 &= \frac{5}{2} - \frac{1}{2}(\cos 4\theta)\left(\frac{\sin(5\theta)}{\sin\theta}\right) \\
< q_\theta^1(z), q_\theta^2(z) > &= \frac{1}{2}(\sin 4\theta)\left(\frac{\sin(5\theta)}{\sin\theta}\right).
\end{aligned}$$

The graph of d_θ is depicted in Figure 3.3. We find that

$$d_{\min} = \inf_{0 \le \theta \le \pi} d_\theta = 0.4094.$$

Since

$$\begin{aligned}
d_{+1} &= \frac{|\delta(1)|}{\sqrt{5}} = 0.8944 \\
d_{-1} &= \frac{|\delta(-1)|}{\sqrt{5}} = 0.4472,
\end{aligned}$$

we have the first encounter with instability at $z = e^{j\theta}$, $\theta = 1.54$, and

$$\rho(\delta) = \min(d_{+1}, d_{-1}, d_{\min}) = 0.4094.$$

The leading coefficient is fixed Write

$$\beta(z) := 0.1 + 0.2z + 0.4z^2 + 0.3z^3 + z^4.$$

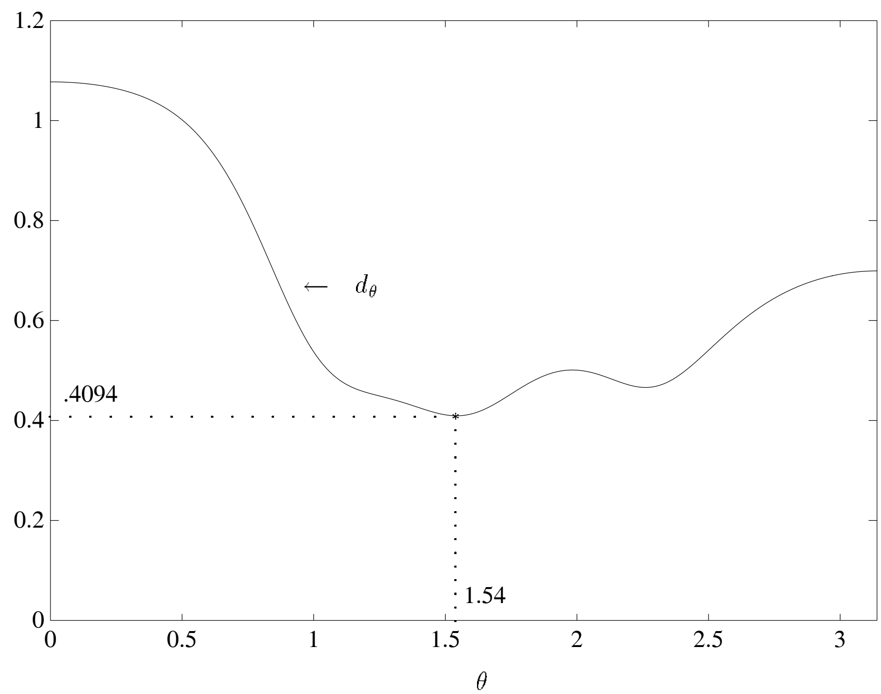

Figure 3.3. d_θ vs. θ (non-monic discrete polynomial: Example 3.2)

Applying Theorem 3.6 which deals with the stability margin for monic polynomials, we have

$$\begin{aligned}\bar{d}_\theta^n[\beta(z)]_{n=4} &= d_\theta^n[\beta(z) - z^4 + 2\cos\theta z^3 - z^2]_{n=3} \\ &= d_\theta^n[(.3 + 2\cos\theta)z^3 - .6z^2 + .2z + .1]_{n=3}.\end{aligned}$$

Now

$$[d_\theta]^2 = \frac{\lambda_2^2 \|q_\theta^1(z)\|^2 - 2\lambda_1\lambda_2 < q_\theta^1(z), q_\theta^2(z) > + \lambda_1^2 \|q_\theta^2(z)\|^2}{\|q_\theta^1(z)\|^2 \|q_\theta^2(z)\|^2 - < q_\theta^1(z), q_\theta^2(z) >^2}.$$

where

$$\begin{aligned}\lambda_1 &= .1 + .2\cos\theta - .6\cos 2\theta + (.3 + 2\cos\theta)\cos 3\theta \\ \lambda_2 &= .2\sin\theta - 0.6\sin 2\theta + (0.3 + 2\cos\theta)\sin 3\theta \\ \|q_\theta^1(z)\|^2 &= 2 + \frac{1}{2}(\cos 3\theta)\left(\frac{\sin 4\theta}{\sin\theta}\right) \\ \|q_\theta^2(z)\|^2 &= 2 - .5(\cos 3\theta)\left(\frac{\sin 4\theta}{\sin\theta}\right)\end{aligned}$$

$$< q^1\theta(z), q_\theta^2(z) > = .5(\sin 3\theta)\left(\frac{\sin 4\theta}{\sin\theta}\right).$$

The graph of d_θ is shown in Figure 3.4.

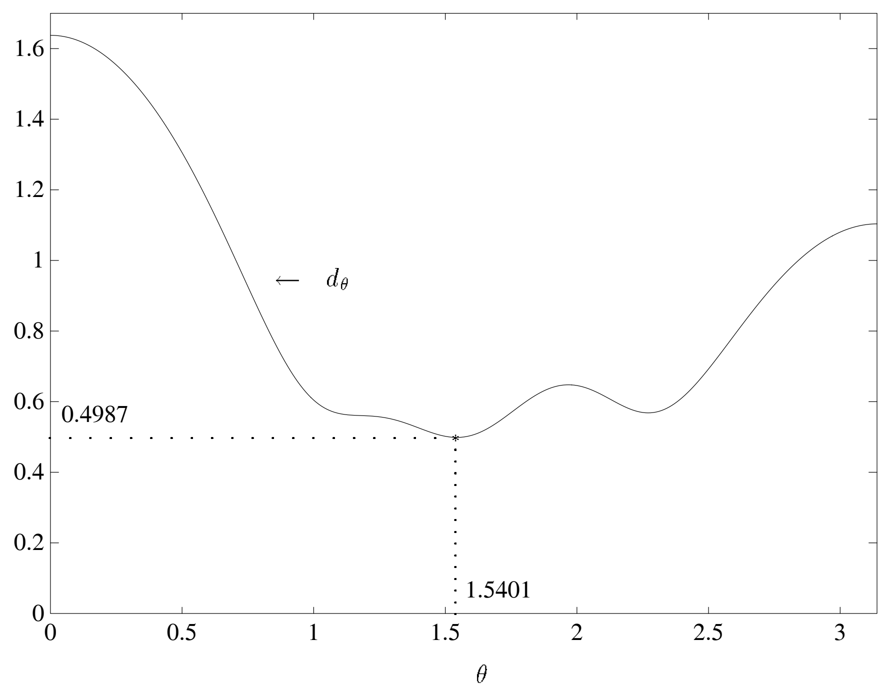

Figure 3.4. d_θ vs. θ (monic discrete polynomial: Example 3.2)

We find that

$$d_{\min} = \inf_{0\leq\theta\leq\pi} d_\theta = 0.4987.$$

Also

$$\bar{d}_{+1} = \frac{|\beta(1)|}{\sqrt{4}} = 1$$
$$\bar{d}_{-1} = \frac{|\beta(-1)|}{\sqrt{4}} = 0.5.$$

Therefore,

$$\rho(\delta) = \min(\bar{d}_{+1}, \bar{d}_{-1}, d_{\min}) = 0.4987.$$

3.4 THE TSYPKIN-POLYAK LOCUS: ℓ_p STABILITY BALL

In this section we consider the problem of determining the Hurwitz stability of a ball of polynomials specified by a weighted ℓ_p norm in the coefficient space for an arbitrary positive integer p. The solution given here was developed by Tsypkin and Polyak and is graphical in nature. It is based on a complex plane frequency domain plot and is therefore particularly suitable for computer implementation. Three cases are considered in some detail : $p = \infty$ (interval uncertainty), $p = 2$ (ellipsoidal uncertainty) and $p = 1$ (octahedral uncertainty). As we shall see the fundamental idea underlying the solution is, once again, a systematic use of the Boundary Crossing Theorem.

Let us parametrize the real polynomial

$$A(s) = a_0 + a_1 s + \cdots + a_n s^n \tag{3.15}$$

by its coefficient vector $\mathbf{a} = [a_0, a_1, \cdots, a_n]$. We consider a family of polynomials $A(s)$ centered at a nominal point $\mathbf{a}^0 = [a_0^0, a_1^0, \cdots, a_n^0]$ with the coefficients lying in the weighted ℓ_p ball of radius ρ,

$$\mathcal{B}_p(\mathbf{a}^0, \rho) := \left\{ \mathbf{a} \ : \ \left[\sum_{k=0}^{n} \left| \frac{a_k - a_k^0}{\alpha_k} \right|^p \right]^{\frac{1}{p}} \leq \rho \right\}. \tag{3.16}$$

In (3.16) $\alpha_k > 0$ are given weights, $1 \leq p \leq \infty$ is a fixed integer, $\rho \geq 0$ is a prescribed common margin for the perturbations. The case $p = \infty$ corresponds to the *box*

$$\mathcal{B}_\infty(\mathbf{a}^0, \rho) := \left\{ \mathbf{a} \ : \ \left[\max_k \left| \frac{a_k - a_k^0}{\alpha_k} \right| \right] \leq \rho \right\}. \tag{3.17}$$

We assume that $a_n^0 > 0$. For $\alpha_k = \alpha$ the set $\mathcal{B}_p(\mathbf{a}^0, \rho)$ is a ball with radius ρ in ℓ_p-space. For $p = 1$ the set $\mathcal{B}_1(\mathbf{a}^0, \rho)$ is a *weighted diamond*. Since there is a one to one correspondence between the point $\mathbf{a}$ and the corresponding polynomial $A(s)$, we loosely refer to the set $\mathcal{B}_p(\mathbf{a}^0, \rho)$ as an ℓ_p ball of polynomials. *Robust stability* will mean here that a ball of prescribed radius in a certain norm contains only Hurwitz polynomials of degree n. Our objective in this section is to derive a procedure for checking whether or not all polynomials in the ball $\mathcal{B}_p(\mathbf{a}^0, \rho)$ are Hurwitz for a prescribed value of ρ and also to determine the maximal ρ which guarantees robust stability.

Write $A_0(j\omega) = U_0(\omega) + j\omega V_0(\omega)$ where

$$U_0(\omega) = a_0^0 - a_2^0 \omega^2 + a_4^0 \omega^4 - \cdots, \tag{3.18}$$

$$V_0(\omega) = a_1^0 - a_3^0 \omega^2 + a_5^0 \omega^4 - \cdots \tag{3.19}$$

Let q be the index conjugate to p:

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{3.20}$$

For $1 < p < \infty$ introduce

$$S_p(\omega) := \left[\alpha_0^q + (\alpha_2\omega^2)^q + (\alpha_4\omega^4)^q + \cdots\right]^{\frac{1}{q}} \tag{3.21}$$

$$T_p(\omega) := \left[\alpha_1^q + (\alpha_3\omega^2)^q + (\alpha_5\omega^4)^q + \cdots\right]^{\frac{1}{q}}. \tag{3.22}$$

For $p = 1$ define

$$S_1(\omega) := \max_{k \text{ even}} \alpha_k\omega^k,$$
$$T_1(\omega) := \max_{k \text{ odd}} \alpha_k\omega^{k-1}$$

and for $p = \infty$ define

$$S_\infty(\omega) := \alpha_0 + \alpha_2\omega^2 + \cdots, \qquad T_\infty(\omega) := \alpha_1 + \alpha_3\omega^2 + \cdots. \tag{3.23}$$

Now, for each p let

$$x(\omega) := \frac{U_0(\omega)}{S_p(\omega)}, \qquad y(\omega) := \frac{V_0(\omega)}{T_p(\omega)} \tag{3.24}$$

and

$$z(\omega) := x(\omega) + jy(\omega). \tag{3.25}$$

In the procedure developed below we require the frequency plot of $z(\omega) = x(\omega) + jy(\omega)$ as ω runs from 0 to ∞. This plot is bounded and its endpoints $z(0)$, $z(\infty)$ are given by

$$\begin{aligned}
x(0) &= \frac{a_0^0}{\alpha_0} \\
x(\infty) &= \begin{cases} (-1)^{\frac{n}{2}}\dfrac{a_n^0}{\alpha_n}, & \text{n even} \\[2ex] (-1)^{\frac{n-1}{2}}\dfrac{a_{n-1}^0}{\alpha_{n-1}}, & \text{n odd} \end{cases} \\
y(0) &= \frac{a_1^0}{\alpha_1} \\
y(\infty) &= \begin{cases} (-1)^{\frac{n}{2}-1}\dfrac{a_{n-1}^0}{\alpha_{n-1}}, & \text{n even} \\[2ex] (-1)^{\frac{n-1}{2}}\dfrac{a_n^0}{\alpha_n}, & \text{n odd} \end{cases}
\end{aligned} \tag{3.26}$$

Now, in the complex plane we introduce the ℓ_p disc with radius ρ:

$$\mathcal{D}_p(\rho) := \left\{z = x + jy \ : \ \left[|x|^p + |y|^p\right]^{\frac{1}{p}} \leq \rho\right\}. \tag{3.27}$$

Theorem 3.7 *Each polynomial in the ball $\mathcal{B}_p(\mathbf{a}^0, \rho)$ is Hurwitz stable if and only if the plot of $z(\omega)$:*

A) goes through n quadrants in the counterclockwise direction,

B) does not intersect the ℓ_p disc with radius ρ, $\mathcal{D}_p(\rho)$, and

C) its boundary points $z(0)$, $z(\infty)$ have coordinates with absolute values greater than ρ.

Conditions B and C are respectively equivalent to the requirements

$$[|x(\omega)|^p + |y(\omega)|^p]^{\frac{1}{p}} > \rho, \qquad \text{for all } 0 \leq \omega < \infty \tag{3.28}$$

and

$$|x(0)| > \rho, \quad |y(0)| > \rho, \quad |x(\infty)| > \rho, \quad |y(\infty)| > \rho. \tag{3.29}$$

We also remark that condition C may be replaced by the equivalent:

$$a_0^0 > \rho\alpha_0, \quad a_n^0 > \rho\alpha_n, \quad a_1^0 > \rho\alpha_1, \quad a_{n-1}^0 > \rho\alpha_{n-1}. \tag{3.30}$$

From this theorem it is clear that the maximal ρ preserving robust Hurwitz stability of the ball $\mathcal{B}_p(\mathbf{a}^0, \rho)$ can be found by finding the radius of the maximal ℓ_p disc that can be inscribed in the frequency plot $z(\omega)$ without violating the boundary conditions in (3.29).

Proof.
Necessity. Assume that all polynomials in $\mathcal{B}_p(\mathbf{a}^0, \rho)$ are Hurwitz. Then $A_0(s)$ is also Hurwitz, and it follows from the monotonic phase property of Hurwitz polynomials discussed in Chapter 1 with $S(\omega) = S_p(\omega)$, $T(\omega) = T_p(\omega)$ that condition A must hold. To show that condition C must also hold we observe that a necessary condition for a polynomial $A(s) = \sum_{k=0}^{n} a_k s^k$ with $a_0 > 0$ to be Hurwitz is that $a_k > 0$, $k = 0, 1, \cdots$. If we choose $a_k = a_k^0 - \rho\alpha_k$ for some k and $a_i = a_i^0$, for $i \neq k$ it follows that the corresponding polynomial $A(s)$ lies in $\mathcal{B}_p(\mathbf{a}^0, \rho)$, is therefore Hurwitz and hence $a_k > 0$ or $a_k^0/\alpha_k > \rho$. With $k = 0, 1, n-1$ and n we get (3.29) and (3.30).

Suppose now that condition B fails. Then there exists $0 \leq \omega_0 < \infty$ such that

$$z(\omega_0) = x(\omega_0) + jy(\omega_0) \in \mathcal{D}_p(\rho) \tag{3.31}$$

We complete the proof of necessity of Condition B by showing that under these conditions $\mathcal{B}_p(\mathbf{a}^0, \rho)$ contains an unstable polynomial. We treat the three cases $1 < p < \infty$, $p = 1$ and $p = \infty$ separately. In each case a contradiction is developed by displaying a polynomial $A_1(s) \in \mathcal{B}_p(\mathbf{a}^0, \rho)$ which has a root at $s = j\omega_0$ and is therefore not Hurwitz. Write $x(\omega_0) = x_0$, $y(\omega_0) = y_0$.

Case 1: $1 < p < \infty$.

In this case the condition $z(\omega_0) = x(\omega_0) + jy(\omega_0) \in \mathcal{D}_p(\rho)$ is equivalent to

$$|x_0|^p + |y_0|^p \leq \rho^p. \tag{3.32}$$

Now write $S_0 := S_p(\omega_0)$, $T_0 := T_p(\omega_0)$ and consider the polynomial $A_1(s)$ with coefficients a_k^1 defined as follows:

$$a_{2i}^1 = a_{2i}^0 - (-1)^i x_0 S_0^{-\frac{q}{p}} \alpha_{2i}^q \omega_0^{\frac{2iq}{p}}, \qquad i = 0, 1, \cdots$$

$$a_{2i+1}^1 = a_{2i+1}^0 - (-1)^i y_0 T_0^{-\frac{q}{p}} \alpha_{2i+1}^q \omega_0^{\frac{2iq}{p}}, \qquad i = 0, 1, \cdots.$$

Then

$$\sum_{k \text{ even}} \left| \frac{a_k^1 - a_k^0}{\alpha_k} \right|^p = |x_0|^p S_0^{-q} \sum_{k \text{ even}} (\alpha_k \omega_0^k)^q = |x_0|^p$$

$$\sum_{k \text{ odd}} \left| \frac{a_k^1 - a_k^0}{\alpha_k} \right|^p = |y_0|^p T_0^{-q} \sum_{k \text{ odd}} (\alpha_k \omega_0^{k-1})^q = |y_0|^p$$

so that

$$\sum_{k=0}^{n} \left| \frac{a_k^1 - a_k^0}{\alpha_k} \right|^p = |x_0|^p + |y_0|^p \leq \rho^p. \tag{3.33}$$

Thus $A_1(s) \in \mathcal{B}_p(\mathbf{a}^0, \rho)$. But

$$\begin{aligned} A_1(j\omega_0) &= U_0(\omega_0) - x_0 S_0^{-\frac{q}{p}} \sum_{k \text{ even}} \alpha_k^q \omega_0^{\frac{kq}{p}} \omega_0^k \\ &\quad + j\omega_0 \left[V_0(\omega_0) - y_0 T_0^{-\frac{q}{p}} \sum_{k \text{ odd}} \alpha_k^q \omega_0^{\frac{(k-1)q}{p}} \omega_0^{k-1} \right] \\ &= [U_0(\omega_0) - x_0 S_p(\omega_0)] + j\omega_0 [V_0(\omega_0) - y_0 T_p(\omega_0)] \\ &= 0. \end{aligned}$$

Hence $A_1(s)$ has the imaginary root $j\omega_0$ and is not Hurwitz.

Case 2: $p = 1$.

Here

$$\mathcal{B}_1(\mathbf{a}^0, \rho) = \left\{ \mathbf{a} : \sum_{k=0}^{n} \left| \frac{a_k^1 - a_k^0}{\alpha_k} \right| \leq \rho \right\} \tag{3.34}$$

$$S_1(\omega_0) = \max_{k \text{ even}} \alpha_k \omega_0^k \; := \; \alpha_m \omega_0^m$$

$$T_1(\omega_0) = \max_{k \text{ odd}} \alpha_k \omega_0^{k-1} \; := \; \alpha_t \omega_0^{t-1}.$$

and the condition $z(\omega_0) \in \mathcal{D}_1(\rho)$ is equivalent to

$$|x_0| + |y_0| \leq \rho. \tag{3.35}$$

Now construct the polynomial $A_1(s) = \sum_{k=0}^{n} a_k^1 s^k$ with

$$a_k^1 = a_k^0, \quad k \neq m, \quad k \neq t \tag{3.36}$$
$$a_m^1 = a_m^0 - (-1)^{\frac{m}{2}} x_0 \alpha_m \tag{3.37}$$
$$a_t^1 = a_t^0 - (-1)^{\frac{(t-1)}{2}} y_0 \alpha_t. \tag{3.38}$$

Then

$$\sum_{k=0}^{n} \left| \frac{(a_k^1 - a_k^0)}{\alpha_k} \right| = |x_0| + |y_0| \leq \rho \tag{3.39}$$

so that $A_1(s) \in \mathcal{B}_1(\mathbf{a}^0, \rho)$. However

$$A_1(j\omega_0) = \left[U^0(\omega_0) - x_0 S_1(\omega_0)\right] + j\omega_0 \left[V^0(\omega_0) - y_0 T_1(\omega_0)\right] = 0. \tag{3.40}$$

Hence $A_1(s)$ has the imaginary root $j\omega_0$, i.e. A_1 is not Hurwitz.

Case 3: $p = \infty$.

In this case we have

$$\mathcal{B}_\infty(\mathbf{a}^0, \rho) = \left\{ \mathbf{a} : \frac{|a_k^1 - a_k^0|}{\alpha_k} \leq \rho, \quad k = 0, 1, 2 \cdots \right\} \tag{3.41}$$

and

$$S_\infty(\omega) = \alpha_0 + \alpha_2 \omega^2 + \cdots, \qquad T_\infty(\omega) = \alpha_1 + \alpha_3 \omega^2 + \cdots \tag{3.42}$$

The ℓ_∞ disc is given by

$$\mathcal{D}_\infty(\rho) = \{ z = x + jy : |x| \leq \rho, \quad |y| \leq \rho \} \tag{3.43}$$

and Condition B is violated if and only if

$$|x_0| \leq \rho, \qquad |y_0| \leq \rho. \tag{3.44}$$

Now consider the polynomial $A_1(s) = \sum_{k=0}^{n} a_k^1 s^k$ with the coefficients chosen as

$$a_k^1 = a_k^0 - (-1)^{\frac{k}{2}} x_0 \alpha_k, \qquad k \text{ even} \tag{3.45}$$
$$a_k^1 = a_k^0 - (-1)^{\frac{(k-1)}{2}} y_0 \alpha_k, \qquad k \text{ odd.} \tag{3.46}$$

Then

$$\frac{|a_k^1 - a_k^0|}{\alpha_k} = |x_0| \leq \rho, \quad k \text{ even}, \qquad \frac{|a_k^1 - a_k^0|}{\alpha_k} = |y_0| \leq \rho, \quad k \text{ odd} \tag{3.47}$$

so that $A_1(s) \in \mathcal{B}_\infty(\mathbf{a}^0, \rho)$. But

$$A_1(j\omega_0) = [U^0(\omega_0) - x_0 S_\infty(\omega_0)] + j\omega_0 \left[V^0(\omega_0) - y_0 T_\infty(\omega_0)\right] = 0. \qquad (3.48)$$

showing that $A_1(s)$ has an imaginary axis root and is therefore not Hurwitz.

Thus we have shown for all $1 \leq p \leq \infty$ that the assumption that B does not hold leads to a contradiction. This completes the proof of necessity of the conditions A, B and C.

Sufficiency. Suppose now that conditions A, B and C hold but there exists a polynomial $\tilde{A}(s) = \sum_{k=0}^{n} \tilde{a}_k s^k \in \mathcal{B}_p(\mathbf{a}^0, \rho)$ which is not Hurwitz. Consider the convex combination $A_\lambda(s) = \lambda A_0(s) + (1-\lambda)\tilde{A}(s), \quad 0 \leq \lambda \leq 1$. The leading coefficient a_n^λ of $A_\lambda(s)$ is positive since $a_n^0 > 0$, $\tilde{a}_n \geq a_n^0 - \rho\alpha_n > 0$ while $a_n^\lambda = \lambda a_n^0 + (1-\lambda)\tilde{a}_n, \quad 0 \leq \lambda \leq 1$. The roots of $A_\lambda(s)$ are then continuous functions of λ. Since $A_0(s)$ is Hurwitz and $\tilde{A}(s)$ is not, by the Boundary Crossing Theorem of Chapter 1 there exists some $\bar{\lambda}, \ 0 \leq \bar{\lambda} < 1$ such that $A_{\bar{\lambda}}(s)$ has a root on the imaginary axis. Moreover since $A_0(s)$ and $\tilde{A}(s)$ are in the convex set $\mathcal{B}_p(\mathbf{a}^0, \rho)$ it follows that $A_{\bar{\lambda}}(s)$ is also in $\mathcal{B}_p(\mathbf{a}^0, \rho)$.

Denote the coefficients of $A_{\bar{\lambda}}(s)$ as $\bar{a}_k$ and its imaginary root as $j\omega_0$. We can write

$$\bar{a}_k := a_k^0 + \mu_k \alpha_k,$$

where

$$\left[\sum_{k=0}^{n} |\mu_k|^p\right]^{\frac{1}{p}} \leq \rho.$$

Writing

$$\bar{A}(j\omega_0) \stackrel{\triangle}{=} \bar{U}(\omega_0) + j\omega_0 \bar{V}(\omega_0) = 0$$

it follows that

$$\bar{U}(\omega_0) = U_0(\omega_0) + \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \mu_k \alpha_k \omega_0^k,$$
$$\bar{V}(\omega_0) = V_0(\omega_0) + \sum_{k \text{ odd}} (-1)^{\frac{(k-1)}{2}} \mu_k \alpha_k \omega_0^{k-1}.$$

If $\omega_0 = 0$ then $\bar{U}(0) = 0, \ \ 0 = U_0(0) + \mu_0\alpha_0, \ \ a_0^0 = |U_0(0)| = |\mu_0|\alpha_0 \leq \rho\alpha_0$, and this contradicts condition C. If $\omega_0 \neq 0$, then $\bar{A}(j\omega_0) = 0$ implies that

$$\bar{U}(\omega_0) = 0, \qquad \bar{V}(\omega_0) = 0. \qquad (3.49)$$

From $\bar{U}(\omega_0) = 0$, we have

$$|U_0(\omega_0)| = \left|\sum_{k \text{ even}} (-1)^{1+\frac{k}{2}} \mu_k \alpha_k \omega_0^k\right|$$

$$
\begin{aligned}
&\leq \left[\sum_{k \text{ even}} |\mu_k|^p\right]^{\frac{1}{p}} \left[\sum_{k \text{ even}} (\alpha_k \omega_0^k)^q\right]^{\frac{1}{q}} \quad \text{(Hölder's inequality)} \\
&= \left[\sum_{k \text{ even}} |\mu_k|^p\right]^{\frac{1}{p}} S_p(\omega_0).
\end{aligned}
$$

Similarly, from $\bar{V}(\omega_0) = 0$, we have

$$
\begin{aligned}
|V_0(\omega_0)| &= \left|\sum_{k \text{ odd}} (-1)^{1+\frac{k-1}{2}} \mu_k \alpha_k \omega_0^{k-1}\right| \\
&\leq \left[\sum_{k \text{ odd}} |\mu_k|^p\right]^{\frac{1}{p}} T_p(\omega_0).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left[|x(\omega_0)|^p + |y(\omega_0)|^p\right]^{\frac{1}{p}} &= \left[\frac{|U_0(\omega_0)|^p}{|S_p(\omega_0)|^p} + \frac{|V_0(\omega_0)|^p}{|T_p(\omega_0)|^p}\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{k \text{ even}} |\mu_k|^p + \sum_{k \text{ odd}} |\mu_k|^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{k=0}^{n} |\mu_k|^p\right]^{\frac{1}{p}} \\
&\leq \rho.
\end{aligned}
$$

which shows that condition B is violated. Thus the assumption that $\tilde{A}(s)$ is non-Hurwitz leads to a contradiction. This completes the proof of sufficiency and of the theorem. ♣

Discussion

(1) It is useful to interpret the above theorem in terms of the complex plane image of the ball of polynomials $A(s) \in \mathcal{B}_p(\mathbf{a}^0, \rho)$. For all $A(s) \in \mathcal{B}_p(\mathbf{a}^0, \rho)$ the set $A(j\omega) = U(\omega) + j\omega V(\omega)$ is described by the inequality

$$
\left[\left|\frac{U(\omega) - U_0(\omega)}{S_p(\omega)}\right|^p + \left|\frac{V(\omega) - V_0(\omega)}{T_p(\omega)}\right|^p\right]^{\frac{1}{p}} \leq \rho,
$$

and the conditions of the Theorem specify that this set should not contain the origin for all $\omega > 0$. This coincides with the Zero Exclusion Theorem of Chapter 1.

(2) The assumption $\alpha_k > 0$, $k = 0, 1, \cdots, n$ may be dropped although in this case the frequency plot $z(\omega)$ will be unbounded for $\alpha_0 = 0$ or $\alpha_n = 0$.

(3) If conditions B and C hold but A fails then the nominal polynomial $A^0(s)$ is not Hurwitz. In this case all polynomials in $\mathcal{B}_p(\mathbf{a}^0, \rho)$ have the same number of roots in the open left half-plane as the nominal polynomial $A_0(s)$.

(4) If $\rho_{\max}$ is the radius of the largest ℓ_p disc inscribed in the plot $z(\omega)$ with $z(\omega_0)$ being a point of contact and conditions A, B and C hold, then the polynomial $A_1(s)$ constructed in the proof is the critical polynomial that destroys stability.

(5) The theorem requires us to generate the plot of $z(\omega)$ for $0 < \omega < \infty$, verify that it goes through n quadrants in the counterclockwise direction, avoids the ℓ_p disc $\mathcal{D}_p(\rho)$ and ensure that the endpoints $z(0)$ and $z(\infty)$ lie outside the square of side 2ρ centered at the origin in the complex plane. Obviously this is ideally suited for visual representation, with the complex plane represented by a computer screen.

Summary of Computations

We summarize the results of the theorem for the special cases $p = 1$, $p = 2$ and $p = \infty$ in terms of what needs to be computed.

In the case $p = 1$, $z(\omega)$ is given by

$$x(\omega) = \frac{a_0^0 - a_2^0\omega^2 + a_4\omega^4 - \cdots}{\max_{k \text{ even}} \alpha_k \omega^k},$$

$$y(\omega) = \frac{a_1^0 - a_3^0\omega^2 + a^5\omega^4 - \cdots}{\max_{k \text{ odd}} \alpha_k \omega^{k-1}}$$

and the plot should not intersect the rhombus $|x| + |y| \leq \rho$.

When $p = 2$, $z(\omega)$ is given by

$$x(\omega) = \frac{a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \cdots}{(\alpha_0^2 + \alpha_2^2\omega^4 + \alpha_4^2\omega^8 + \cdots)^{\frac{1}{2}}},$$

$$y(\omega) = \frac{a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \cdots}{(\alpha_1^2 + \alpha_3^2\omega^4 + \alpha_5^2\omega^8 + \cdots)^{\frac{1}{2}}}$$

and this plot must not intersect the circle $|x|^2 + |y|^2 \leq \rho^2$.

When $p = \infty$, the frequency plot $z(\omega) = x(\omega) + jy(\omega)$ is given by

$$x(\omega) = \frac{a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \cdots}{\alpha_0 + \alpha_2\omega^2 + \alpha_4\omega^4 + \cdots},$$

$$y(\omega) = \frac{a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \cdots}{\alpha_1 + \alpha_3\omega^2 + \alpha_5\omega^4 + \cdots}$$

and must not intersect the square $|x| \le \rho$, $|y| \le \rho$.

We illustrate the above Theorem with an example.

Example 3.3. Consider the polynomial

$$A(s) = s^6 + 14s^5 + 80.25s^4 + 251.25s^3 + 502.25s^2 + 667.25s + 433.5.$$

With the choice of

$$\alpha = [0.1,\ 1.4,\ 5.6175,\ 15.075,\ 25.137,\ 33.36,\ 43.35]$$

we have the following stability margins:

$$\begin{aligned} p &= 1, & \rho &= 3.6252 \\ p &= 2, & \rho &= 2.8313 \\ p &= \infty, & \rho &= 1.2336. \end{aligned}$$

The required plot of $z(\omega)$ and the discs $\mathcal{D}_1(\rho), \mathcal{D}_2(\rho)$ and $\mathcal{D}_\infty(\rho)$ are shown in Figures 3.5, 3.6, and 3.7.

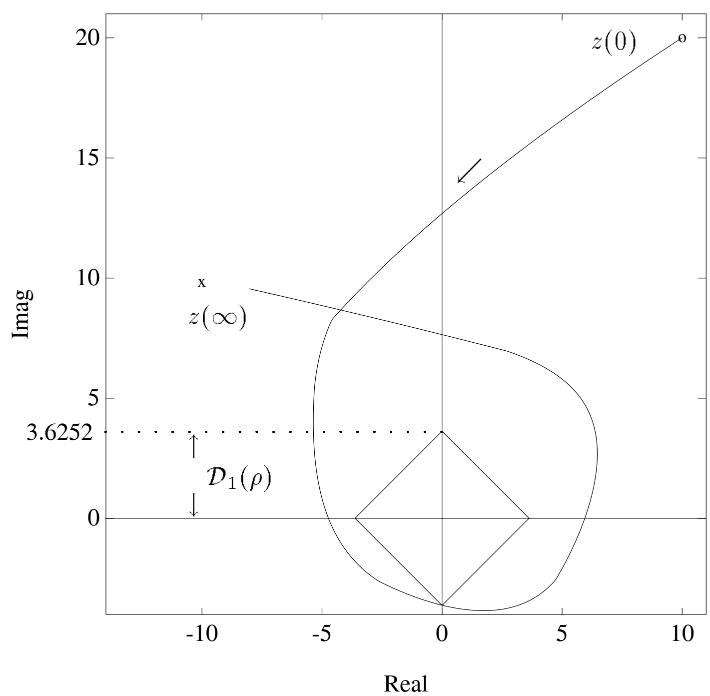

Figure 3.5. Tsypkin - Polyak locus: ℓ_1 stability margin (Example 3.3)

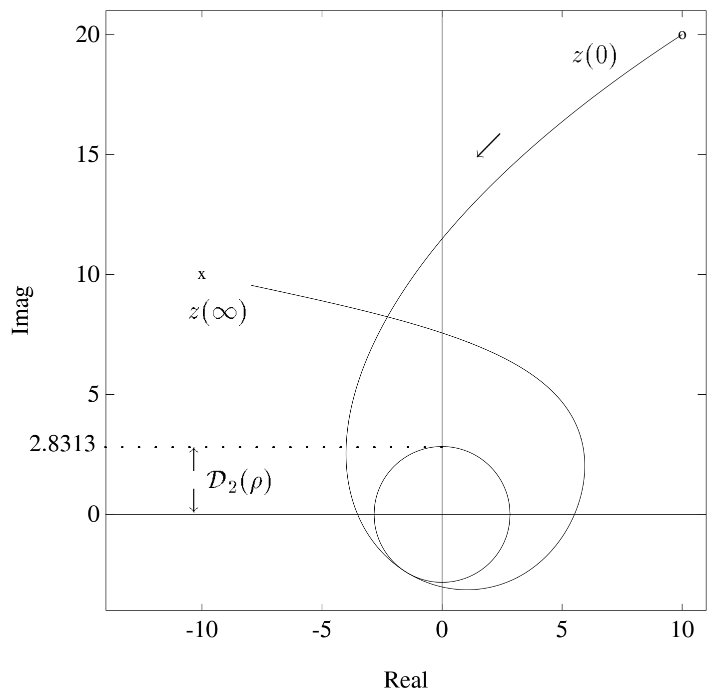

Figure 3.6. Tsypkin - Polyak locus: ℓ_2 stability margin (Example 3.3)

3.5 ROBUST STABILITY OF DISC POLYNOMIALS

In this section we consider an alternative model of uncertainty. We deal with the robust stability of a set $\mathcal{F}_D$ of *disc* polynomials, which are characterized by the fact that each coefficient of a typical element $P(s)$ in $\mathcal{F}_D$ can be any complex number in an arbitrary but fixed disc of the complex plane. The motivation for considering robust stability of disc polynomials is the same as the one for considering robust stability of the ℓ_p ball of polynomials: namely it is a device for taking into account variation of parameters in prescribed ranges. We let $\mathcal{S}$ be the stability region of interest and consider $n+1$ arbitrary discs D_i, $i = 0, \cdots, n$ in the complex plane. Each disc D_i is centered at the point β_i and has radius $r_i \geq 0$. Now, let $\mathcal{F}_D$ be the family of all complex polynomials,

$$\delta(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n,$$

such that

$$\delta_j \in D_j, \quad \text{for } j = 0, \cdots, n \; . \tag{3.50}$$

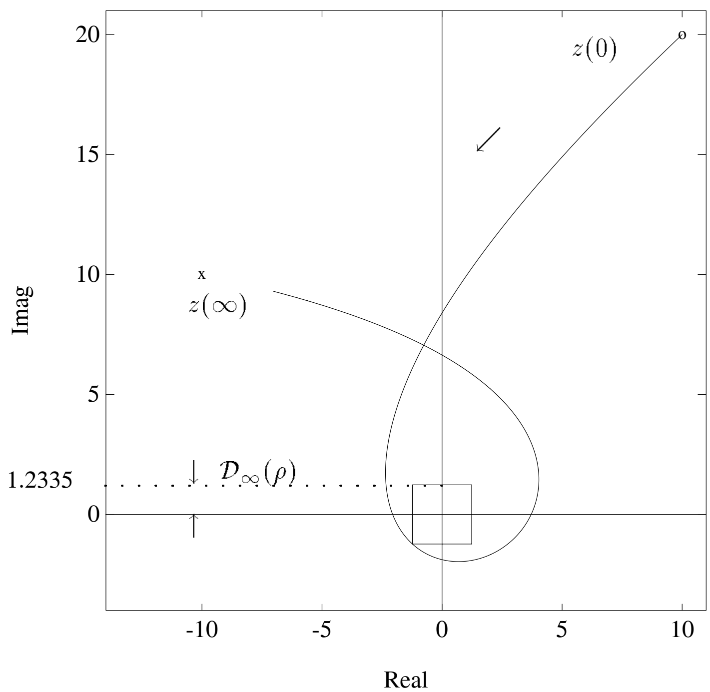

Figure 3.7. Tsypkin - Polyak locus: ℓ_∞ stability margin (Example 3.3)

In other words every coefficient δ_j of the polynomial $\delta(z)$ in $\mathcal{F}_D$ satisfies

$$|\delta_j - \beta_j| \le r_j. \tag{3.51}$$

We assume that every polynomial in $\mathcal{F}_D$ is of degree n:

$$0 \notin D_n. \tag{3.52}$$

Figure 3.8 illustrates an example of such a family of complex polynomials. Each disc D_i describes the possible range of values for the coefficient δ_i. The problem here is to give necessary and sufficient conditions under which it can be established that all polynomials in $\mathcal{F}_D$ have their roots in $\mathcal{S}$. This problem is solved for both Hurwitz and Schur stability . For example, we prove that the Hurwitz stability of $\mathcal{F}_D$ is equivalent to the stability of the *center* polynomial together with the fact that two specific stable proper rational functions must have an H_∞-norm less than 1.

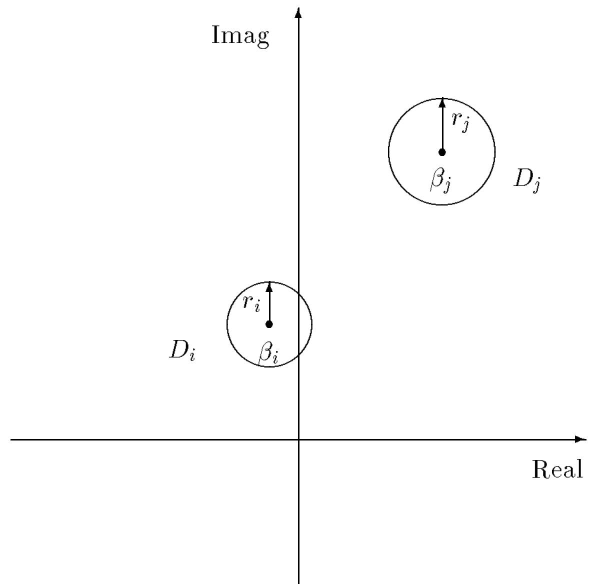

Figure 3.8. Disks around the coefficients $(\beta_i,\ \beta_j)$

3.5.1 Hurwitz Case

Let $g(s) = \frac{n(s)}{d(s)}$ where $n(s)$ and $d(s)$ are complex polynomials such that the degree of $n(s)$ is less than or equal to the degree of $d(s)$ and such that $d(s)$ is Hurwitz. In other words $g(s)$ is a proper, stable, complex, rational function. The H_∞-norm of $g(s)$ is then defined as

$$\|g\|_\infty := \sup_{\omega \in \mathbf{R}} \left| \frac{n(j\omega)}{d(j\omega)} \right|. \tag{3.53}$$

Now, let $\beta(s)$ be the *center* polynomial, that is the polynomial whose coefficients are the centers of the discs D_j:

$$\beta(s) = \beta_0 + \beta_1 s + \cdots + \beta_n s^n. \tag{3.54}$$

Construct polynomials $\gamma_1(s)$ and $\gamma_2(s)$ as follows:

$$\gamma_1(s) := r_0 - jr_1 s - r_2 s^2 + jr_3 s^3 + r_4 s^4 - jr_5 s^5 - \cdots,$$
$$\gamma_2(s) := r_0 + jr_1 s - r_2 s^2 - jr_3 s^3 + r_4 s^4 + jr_5 s^5 - \cdots$$

and let $g_1(s)$ and $g_2(s)$ be the two proper rational functions

$$g_1(s) = \frac{\gamma_1(s)}{\beta(s)}, \qquad g_2(s) = \frac{\gamma_2(s)}{\beta(s)}. \tag{3.55}$$

The main result on Hurwitz stability of disc polynomials is the following.

Theorem 3.8 *Each member of the family of polynomials $\mathcal{F}_D$ is Hurwitz if and only if*

1) $\beta(s)$ is Hurwitz, and,

2) $\|g_1\|_\infty < 1$ and $\|g_2\|_\infty < 1$.

For the proof we first need to establish two simple lemmas. The first lemma characterizes proper, stable, complex rational functions with H_∞-norm less than 1 and is useful in its own right.

Lemma 3.1 *If $g(s) = \frac{n(s)}{d(s)}$ is a proper stable complex rational function with $\deg(d(s)) = q$, then $\|g\|_\infty < 1$ if and only if the following hold:*

a1) $|n_q| < |d_q|$,

b1) $d(s) + e^{j\theta} n(s)$ *is Hurwitz for all θ in $[0, 2\pi)$.*

Proof. Condition a1) is obviously necessary because when ω goes to infinity, the ratio in (3.53) tends to the limit $\left|\frac{n_q}{d_q}\right|$. The necessity of condition b1) on the other hand follows from the Boundary Crossing Theorem and Rouché's theorem on analytic functions (Chapter 1) since $|d(j\omega)| > |e^{j\theta} n(j\omega)| = |n(j\omega)|$.

For sufficiency suppose that conditions a1) and b1) are true, and let us assume by contradiction that $\|g\|_\infty \geq 1$. Since $|g(j\omega)|$ is a continuous function of ω and since its limit, as ω goes to infinity, is $\left|\frac{n_q}{d_q}\right| < 1$, then there must exist at least one ω_o in $\mathbb{R}$ for which

$$|g(j\omega_o)| = \left|\frac{n(j\omega_o)}{d(j\omega_o)}\right| = 1.$$

But this implies that $n(j\omega_o)$ and $d(j\omega_o)$ differ only by a complex number of modulus 1 and therefore it is possible to find θ_o in $[0, 2\pi)$ such that:

$$n(j\omega_o) + e^{j\theta_o} d(j\omega_o) = 0$$

and this obviously contradicts condition b1). ♣

Now, using the definition of $\beta(s)$ given in (3.54), it is easy to see that a typical polynomial $\delta(s)$ in $\mathcal{F}_D$ can be written as:

$$\delta(s) = \beta(s) + \sum_{k=0}^{n} z_k r_k s^k \qquad (3.56)$$

where the z_k, $k = 0, \cdots, n$, are arbitrary complex numbers of modulus less than or equal to 1.

The next lemma gives a first set of necessary and sufficient conditions under which stability of $\mathcal{F}_D$ can be ascertained.

Lemma 3.2 *The family of complex polynomials $\mathcal{F}_D$ contains only Hurwitz polynomials if and only if the following conditions hold:*

a2) $\beta(s)$ is Hurwitz,

b2) For all complex numbers $z_0, \cdots, z_n$ of modulus less than or equal to one we have:

$$\left\| \frac{\sum_{k=0}^{n} z_k r_k s^k}{\beta(s)} \right\|_\infty < 1.$$

Proof. We start by proving that conditions a2) and b2) are sufficient. Here again, if $\beta(s)$ is known to be stable, and if condition b2) holds then a straightforward application of Rouché's theorem yields that any polynomial $\delta(s)$ in (3.56) is also Hurwitz.

Conversely, it is clear again from Rouché's Theorem, that condition a2) is necessary, since $\beta(s)$ is in $\mathcal{F}_D$. Thus, let us assume that $\beta(s)$ is stable and let us prove that condition b2) is also satisfied.

To do so we use the characterization derived in Lemma 3.1. First of all, remember that one of our assumptions on the family $\mathcal{F}_D$ is that the n^{th} disc D_n does not contain the origin of the complex plane (see (3.52)). This implies in particular that for any complex number z_n of modulus less than or equal to 1:

$$\left| \frac{z_n r_n}{\beta_n} \right| \leq \frac{r_n}{|\beta_n|} < 1. \tag{3.57}$$

Now let us assume by contradiction that condition b2) is not satisfied. In this case there would exist at least one set $\{z_0, \cdots, z_n\}$, of $n+1$ complex numbers of modulus less than or equal to one, for which:

$$\left\| \frac{\sum_{k=0}^{n} z_k r_k s^k}{\beta(s)} \right\|_\infty \geq 1. \tag{3.58}$$

Condition (3.57) shows that condition a1) of Lemma 3.1 is always satisfied with

$$\beta(s) = d(s) \quad \text{and} \quad \sum_{k=0}^{n} z_k r_k s^k = n(s).$$

Now (3.58) implies by Lemma 3.1 that for this particular set $\{z_0, \cdots, z_n\}$ it is possible to find at least one real θ_o in $[0, 2\pi)$ such that:

$$\beta(s) + e^{j\theta_o} \sum_{k=0}^{n} z_k r_k s^k \quad \text{is unstable.} \tag{3.59}$$

However if we let $z'_k = e^{j\theta_o} z_k$ for $k = 0$ to $k = n$, then all the complex numbers z'_k also have modulus less than or equal to one, which implies that the polynomial in (3.59) is an element of $\mathcal{F}_D$ and this contradicts the fact that $\mathcal{F}_D$ contains only Hurwitz polynomials. ♣

We can now complete the proof of Theorem 3.8.

Proof of Theorem 3.8 Let us start with Lemma 3.2 and let us consider condition b2). Let $z_0, \cdots, z_n$ be $n+1$ arbitrary complex numbers of modulus less than or equal to 1, and let,

$$n(s) := z_0 r_0 + z_1 r_1 s + \cdots + z_n r_n s^n.$$

We can write

$$n(j\omega) = \sum_{k=0}^{n} z_k r_k (j\omega)^k = \sum_{k=0}^{n} z_k r_k e^{jk\frac{\pi}{2}} \omega^k. \tag{3.60}$$

For all k in $\{0, \cdots, n\}$ it is possible to write

$$z_k = t_k e^{j\theta_k}, \quad \text{where} \quad t_k \in [0,1], \quad \text{and} \quad \theta_k \in [0, 2\pi). \tag{3.61}$$

Using (3.60) and (3.61), we see that

$$n(j\omega) = \sum_{k=0}^{n} t_k r_k e^{j\left(\theta_k + k\frac{\pi}{2}\right)} \omega^k. \tag{3.62}$$

Therefore we always have the following inequalities:

$$\text{If } \omega \geq 0 \text{ then } \quad |n(j\omega)| \leq \sum_{k=0}^{n} r_k \omega^k \tag{3.63}$$

and,

$$\text{If } \omega \leq 0 \text{ then } \quad |n(j\omega)| \leq \sum_{k=0}^{k=n} (-1)^k r_k \omega^k. \tag{3.64}$$

However, it is clear that the upper bound in (3.63) is achieved for the particular choice of z_k determined by $t_k = 1, \theta_k = -k\frac{\pi}{2}$, for which $n(s) = \gamma_1(s)$.

On the other hand the upper bound in (3.64) is also achieved for the particular choice $t_k = 1, \theta_k = k\frac{\pi}{2}$, leading this time to $n(s) = \gamma_2(s)$.

As a direct consequence, condition b2) is then satisfied if and only if condition 2) in Theorem 3.8 is true, and this completes the proof of the theorem. ♣

3.5.2 Schur Case

We again consider the family of disc polynomials $\mathcal{F}_D$ defined in (3.51) and (3.52). This time the problem is to derive necessary and sufficient conditions for Schur stability of the entire family. In this case we call $g(z) = \frac{n(z)}{d(z)}$ a proper, stable, complex rational function if $n(z)$ and $d(z)$ are complex polynomials such that the degree of $n(z)$ is less than or equal to the degree of $d(z)$ and if $d(z)$ is Schur (i.e.

$d(z)$ has all its roots in the open unit disc). The H_∞-norm of $g(z)$ is then defined as:

$$\|g\|_\infty := \sup_{\theta\in[0,2\pi)} \left|\frac{n(e^{j\theta})}{d(e^{j\theta})}\right|. \tag{3.65}$$

Again, let the center polynomial be:

$$\beta(z) = \beta_0 + \beta_1 z + \cdots + \beta_n z^n. \tag{3.66}$$

Then we have the following main result on Schur stability of disc polynomials.

Theorem 3.9 *The family of complex disc polynomials $\mathcal{F}_D$ contains only Schur polynomials if and only if the following conditions hold:*

1) $\beta(z)$ is Schur, and

2) The following inequality holds,

$$\sum_{k=0}^{k=n} r_k < \inf_{\theta\in[0,2\pi)} |\beta(e^{j\theta})|.$$

To prove this result, we need the following lemma which is completely analogous to Lemma 3.2.

Lemma 3.3 *The family of complex polynomials $\mathcal{F}_D$ contains only Schur polynomials if and only if the following conditions hold:*

a3) $\beta(z)$ is Schur,

b3) For any complex numbers $z_0, \cdots, z_n$ of modulus less than or equal to one we have:

$$\left\|\frac{\sum_{k=0}^n z_k r_k z^k}{\beta(z)}\right\|_\infty < 1.$$

Proof. The sufficiency of conditions a3) and b3) is quite straightforward and follows again from Rouché's Theorem. The necessity of condition a3) is also clear. Thus assume that $\beta(z)$ is stable and suppose by contradiction that for some set $\{z_0, \cdots, z_n\}$ of complex numbers of modulus less than or equal to 1, we have,

$$\left\|\frac{\sum_{k=0}^n z_k r_k z^k}{\beta(z)}\right\|_\infty \geq 1. \tag{3.67}$$

Let

$$n(z) = \sum_{k=0}^{n} z_k r_k z^k. \tag{3.68}$$

It is clear that as $\lambda \longrightarrow 0$:

$$\left\|\frac{\lambda n(z)}{\beta(z)}\right\|_\infty \longrightarrow 0.$$

Therefore, by continuity of the norm, it is possible to find λ_o in $(0,1)$ such that

$$\left\| \frac{\lambda_o n(z)}{\beta(z)} \right\|_\infty = 1. \tag{3.69}$$

Since the unit circle is a compact set, we deduce from (3.69) and the definition of the H_∞ norm in (3.53), that there exists θ_o in $[0, 2\pi)$ for which:

$$|\lambda_o n(e^{j\theta_o})| = |\beta(e^{j\theta_o})|.$$

Thus it is possible to find a complex number of modulus one, say $e^{j\phi}$, such that:

$$\beta(e^{j\theta_o}) + e^{j\phi}\lambda_o n(e^{j\theta_o}) = 0. \tag{3.70}$$

Therefore, the polynomial

$$\beta(z) + e^{j\phi}\lambda_o n(z),$$

is not Schur, and yet as is easy to see, it belongs to $\mathcal{F}_D$. Thus we have reached a contradiction. ♣

Proof of Theorem 3.9 To prove Theorem 3.9, we just need to look at condition b3) of Lemma 3.3.

For any complex numbers z_i of modulus less than or equal to one, we have the following inequality:

$$\left| \frac{\sum_{k=0}^{n} z_k r_k e^{jk\theta}}{\beta(e^{j\theta})} \right| \leq \frac{\sum_{k=0}^{n} r_k}{|\beta(e^{j\theta})|}. \tag{3.71}$$

If we consider the continuous function of θ which appears on the right hand side of this inequality, then we know that it reaches its maximum for some value θ_o in $[0, 2\pi)$. At this maximum, it suffices to choose $z_k = e^{-jk\theta_o}$ to transform (3.71) into an equality. It follows that condition b3) is satisfied if and only if this maximum is strictly less than one, and this is equivalent to condition 2) of Theorem 3.9. ♣

3.5.3 Some Extensions

We focus on the Hurwitz stability case below, but similar results obviously hold for the Schur case.

1) **Case of real centers:** In the case where the discs D_j are centered on the real axis, that is when $\beta(s)$ is a polynomial with real coefficients, then $g_1(s)$ and $g_2(s)$ have the same H_∞-norm, and therefore one has to check the norm of only one rational function.

2) **Maximal Hurwitz disc families:** Consider a stable nominal polynomial

$$\beta^o(s) = \beta_0^o + \beta_1^o s + \cdots + \beta_n^o s^n,$$

and let $r_0, \cdots, r_n$ be $n+1$ fixed real numbers which are greater than or equal to 0. Using our result, it is very easy to find the largest positive number

$\epsilon_{\max}$ of all positive numbers ϵ such that the family of disc polynomials whose coefficients are contained in the open discs with centers β_j^o and of radii ϵr_j, is entirely stable. To do so let:

$$\gamma_1(s) = r_0 - jr_1 s - r_2 s^2 + jr_3 s^3 + r_4 s^4 - jr_5 s^5 - \cdots,$$
$$\gamma_2(s) = r_0 + jr_1 s - r_2 s^2 - jr_3 s^3 + r_4 s^4 + jr_5 s^5 - \cdots,$$

and form

$$g_1(s) = \frac{\gamma_1(s)}{\beta^o(s)}, \qquad g_2(s) = \frac{\gamma_2(s)}{\beta^o(s)}.$$

Now, if $\|g_1\|_\infty = \eta_1$ and $\|g_2\|_\infty = \eta_2$, then

$$\epsilon_{\max} = \min\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right).$$

The quantities η_1 and η_2 can of course be found from the polar plots of $g_1(j\omega)$ and $g_2(j\omega)$ respectively.

We illustrate the above results with an example.

Example 3.4. Consider the nominal polynomial

$$\beta^o(s) = (2 - j3.5) + (1.5 - j6)s + (9 - j27)s^2 + (3.5 - j18)s^3 + (-1 - j11)s^4$$

Suppose that each coefficient δ_i perturbs inside a corresponding disc with radius r_i and

$$r_0 = 2, \quad r_1 = 1, \quad r_2 = 8, \quad r_3 = 3, \quad r_4 = 1.$$

We want to check the robust Hurwitz stability of this family of disc polynomials. We first verify the stability of the nominal polynomial. It is found that $\beta^o(s)$ has roots at

$$-0.4789 - j1.5034, \quad -0.9792 + j1.0068, \quad -0.1011 + j0.4163, \quad -0.0351 - j0.3828\ .$$

and is therefore Hurwitz. Now form

$$\begin{aligned}\gamma_1(s) &= r_0 - jr_1 s - r_2 s^2 + jr_3 s^3 + r_4 s^4 \\ &= 2 - js - 8s^2 + j3s^3 + s^4 \\ \gamma_2(s) &= r_0 + jr_1 s - r_2 s^2 - jr_3 s^3 + r_4 s^4 \\ &= 2 + js - 8s^2 - j3s^3 + s^4,\end{aligned}$$

and

$$g_1(s) = \frac{\gamma_1(s)}{\beta^o(s)} \quad \text{and} \quad g_2(s) = \frac{\gamma_2(s)}{\beta^o(s)}.$$

From the frequency plot of $g_i(j\omega)$ (Figure 3.9) we can see that $\|g_1\|, \|g_2\| > 1$. Thus the condition of Theorem 3.8 is violated. Therefore, the given disc polynomial family contains unstable elements.

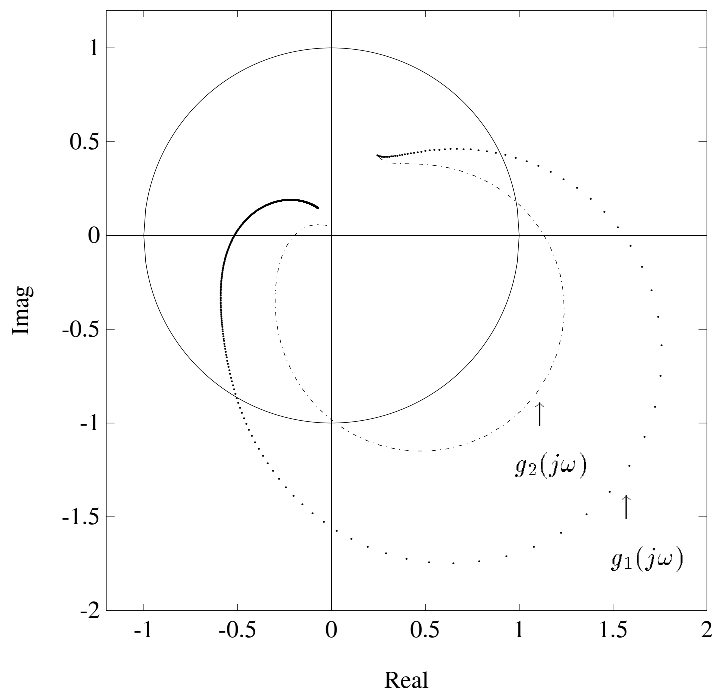

Figure 3.9. $g_1(j\omega)$ and $g_2(j\omega)$ (unstable family: Example 3.4)

Now suppose we want to proportionally reduce the size of the discs so that the resulting smaller family becomes robustly stable. We compute

$$\|g_1\|_\infty = 2.0159 := \eta_1 \quad \text{and} \quad \|g_2\|_\infty = 1.3787 := \eta_2$$

and we have

$$\epsilon_{\max} = \min\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right) = 0.4960 \ .$$

We now modify the radii of discs to $\hat{r}_i = \epsilon_{\max} r_i$ for $i = 0, 1, 2, 3, 4$. Consequently,

$$\hat{\gamma}_1(s) = 0.9921 - j0.4961s - 3.9684s^2 + j1.4882s^3 + 0.4961s^4$$
$$\hat{\gamma}_2(s) = 0.9921 + j0.4961s - 3.9684s^2 - j1.4882s^3 + 0.4961s^4,$$

and

$$\hat{g}_1(s) = \frac{\hat{\gamma}_1(s)}{\beta^o(s)} \quad \text{and} \quad \hat{g}_2(s) = \frac{\hat{\gamma}_2(s)}{\beta^o(s)}.$$

Figure 3.10 shows that $\|g_1\|_\infty < 1$ and $\|g_2\|_\infty < 1$ which verifies the robust stability of the disc polynomial family with the adjusted disc radii.

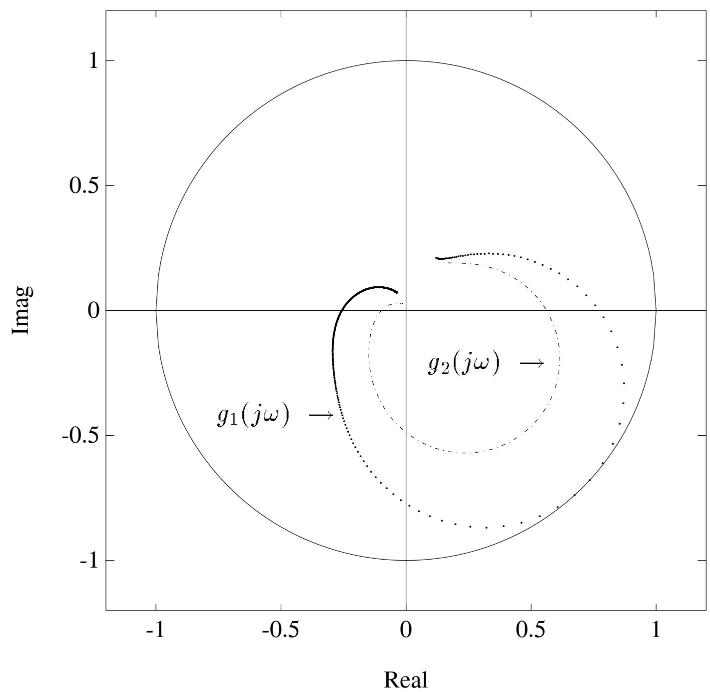

Figure 3.10. $\hat{g}_1(j\omega)$ and $\hat{g}_2(j\omega)$ (stable family: Example 3.4)

3.6 EXERCISES

3.1 Calculate the radius of the Hurwitz stability ball in the coefficient space for each of the polynomials

a) $(s+1)^3$

b) $(s+2)^3$

c) $(s+3)^3$

d) $(s+1)(s^2+s+1)$

e) $(s+2)(s^2+s+1)$

f) $(s+1)(s^2+2s+2)$

g) $(s+2)(s^2+2s+2)$

considering both the cases where the leading coefficient is fixed and subject to perturbation.

3.2 Derive a closed form expression for the radius of the Hurwitz stability ball for the polynomial $a_2 s^2 + a_1 s + a_0$.

3.3 Calculate the radius of the Schur stability ball in the coefficient space for the polynomials

a) $z^3(z+0.5)^3$

b) $(z-0.5)^3$

c) $z(z^2+z+0.5)$

d) $z(z+0.5)(z-0.5)$

e) $z(z^2-z+0.5)$

f) $z^2(z+0.5)$

considering both the cases where the leading coefficient is fixed and where it is subject to perturbation.

3.4 Consider the feedback system shown in Figure 3.11. The characteristic poly-

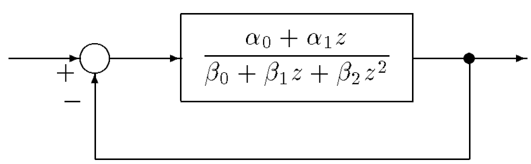

Figure 3.11. Feedback control system

nomial of the closed loop system is

$$\begin{aligned}\delta(z) &= (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) z + \beta_2 z^2 \\ &= \delta_0 + \delta_1 z + \delta_2 z^2\end{aligned}$$

with nominal value

$$\delta^0(z) = \frac{1}{2} - z + z^2.$$

Find $\epsilon_{\max}$ so that with $\delta_i \in [\delta_i^0 - \epsilon, \delta_i^0 + \epsilon]$ the closed loop system is robustly Schur stable.
Answer: $\epsilon_{\max} = 0.17$.

3.5 Consider the feedback system shown in Figure 3.12.

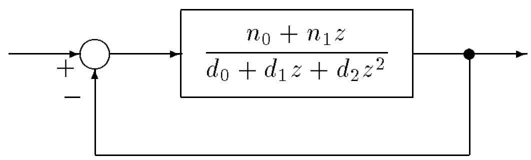

Figure 3.12. Feedback control system

Assume that the nominal values of the coefficients of the transfer function are

$$\left[n_0^0, n_1^0\right] = [-1, 2] \quad \text{and} \quad [d_0, d_1, d_2] = [-1, -2, 8].$$

Now let

$$n_i \in [n_i^0 - \epsilon, n_i^0 + \epsilon] \quad \text{and} \quad d_j \in [d_j^0 - \epsilon, d_j^0 + \epsilon]$$

and find $\epsilon_{\max}$ for robust Schur stability.
Answer: $\epsilon_{\max} = 1.2$.

3.6 Consider the third degree polynomial $P_r(s)$ given by

$$P_r(s) = \left(s - re^{j\frac{3\pi}{4}}\right)(s + r)\left(s - re^{j\frac{5\pi}{4}}\right).$$

In other words, the roots of $P_r(s)$ are equally distributed on the left-half of the circle of radius r in the complex plane. Now, let $m(r)$ be the sum of the squares of the coefficients of $P_r(s)$. One can easily compute that

$$m(r) = 2\left(1 + \sqrt{2}r + 2r^2 + \sqrt{2}r^3 + r^4\right).$$

We consider the normalized polynomial,

$$\delta_r(s) = \frac{1}{\sqrt{m(r)}} P_r(s).$$

Compute

$$\rho(r) = \inf_{\omega > 0} d_\omega(\delta_r),$$

for increasing values of r and plot $\rho(r)$ as a function of r.

3.7 In the Schur case consider the third degree polynomial $P_r(z)$ given by

$$P_r(z) = \left(z - re^{j\frac{2\pi}{3}}\right)\left(z - re^{j\frac{4\pi}{3}}\right)(z - r).$$

In other words, the roots of $P_r(s)$ are equally distributed on the circle of radius r in the complex plane. Again, let $m(r)$ be the sum of the squares of the coefficients of $P_r(z)$, and consider the normalized polynomial,

$$\delta_r(z) = \frac{1}{\sqrt{m(r)}} P_r(z).$$

Compute

$$\rho(r) = \inf_{\theta > 0} d_\theta(\delta_r),$$

for increasing values of r $(0 \leq r < 1)$ and plot $\rho(r)$ as a function of r.

3.8 The purpose of this problem is to calculate the distance $d_\omega(P)$ for different choices of norms on $\mathcal{P}_n$, the vector space of all real polynomials of degree less than or equal to n.

As usual we identify $\mathcal{P}_n$ with $\mathbb{R}^{n+1}$. Let $\|\cdot\|$ be a given norm on $\mathcal{P}_n$. For a fixed $\omega > 0$, we denote by Δ_ω the subspace of $\mathcal{P}_n$ which consists of all polynomials $\delta(s)$ which satisfy: $\delta(j\omega) = 0$. Let $P(s)$ be an arbitrary but fixed element of $\mathcal{P}_n$. We then define,

$$d_\omega(P) = \inf_{\delta \in \Delta_\omega} \|P - \delta\|.$$

1) Compute $d_\omega(P)$ when $\|\cdot\|$ is the ℓ_∞ norm $\|\cdot\|_\infty$, that is,

$$\|\delta_0 + \delta_1 s + \cdots + \delta_n s^n\|_\infty = \max_{0 \leq k \leq n} |\delta_k| \ .$$

2) Compute $d_\omega(P)$ when $\|\cdot\|$ is the ℓ_1 norm $\|\cdot\|_1$, that is,

$$\|\delta_0 + \delta_1 s + \cdots + \delta_n s^n\|_1 = \sum_{k=0}^{n} |\delta_k|.$$

3) More generally, compute $d_\omega(P)$ when $\|\cdot\|$ is the ℓ_p norm $\|\cdot\|_p$ $(1 < p < +\infty)$, that is,

$$\|\delta_0 + \delta_1 s + \cdots + \delta_n s^n\|_p = \left(\sum_{k=0}^{n} |\delta_k|^p \right)^{\frac{1}{p}} .$$

Hint: In all cases use the following result of Banach space theory, which plays an essential role in the study of extremal problems. Let X be an arbitrary Banach space and let $\|\cdot\|$ denote the norm on X. Let also M be a closed subspace of X. Then for any fixed x in X we have,

$$\inf_{m \in M} \|x - m\| = \max_{x^* \in M^\perp} | < x^*, x > |$$

where "max" indicates that the supremum is attained and where $M^{\perp}$ is the *annihilator* of M, that is the subspace of X^* which consists of all linear functionals x^* which satisfy,

$$\forall \;\; m \in M, \quad < x^*, m >= 0.$$

Note that in general for $1 \leq p \leq +\infty$,

$$(\mathbb{R}^n, \|\cdot\|_p)^* = (\mathbb{R}^n, \|\cdot\|_q)$$

where q is the *conjugate* of p which is defined by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

3.9 Repeat Exercise 3.8, but this time to calculate $d_\theta(P)$ where $\theta \in (0, \pi)$ and

$$d_\theta(P) = \inf_{\delta \in \Delta_\theta} \|P - \delta\|.$$

Here Δ_θ designates the subspace of $\mathcal{P}_n$ consisting of all polynomials $\delta(s)$ which are such that $\delta(e^{j\theta}) = 0$.

3.10 Prove the distance formulas in Theorems 3.6.

3.11 Consider the standard unity feedback control system with transfer functions $G(s)$ and $C(s)$ in the forward path. Let

$$G(s) = \frac{p_0 + p_1 s}{s^2(s + q_0)} \quad \text{and} \quad C(s) = 1.$$

Determine the stability margin in the space of parameters (p_0, p_1, q_0) assuming the nominal value $(p_0^0, p_1^0, q_0^0) = (1, 1, 2)$.

3.12 Using the Tsypkin-Polyak locus calculate the weighted ℓ_1, ℓ_2 and ℓ_∞ stability margins in the coefficient space for the polynomial

$$\delta(s) = s^4 + 6s^3 + 13s^2 + 12s + 4$$

choosing the weights proportional to the magnitude of each nominal value.

3.13 For the nominal polynomial

$$\delta(s) = s^3 + 5s^2 + (3 - j)s + 6 - 2j$$

find the largest number $\epsilon_{\max}$ such that all polynomials with coefficients centered at the nominal value and of radius $\epsilon_{\max}$ are Hurwitz stable.

3.14 Consider a feedback control system with the plant transfer function

$$G(s) = \frac{n_0 + n_1 s + \cdots + n_{n-1}s^{n-1} + n_n s^n}{d_0 + d_1 s + \cdots + d_{n-1}s^{n-1} + d_n s^n}$$

and the controller transfer function

$$C(s) = \frac{b_0 + b_1 s + \cdots + b_{m-1}s^{m-1} + b_m s^m}{a_0 + a_1 s + \cdots + a_{m-1}s^{m-1} + a_m s^m}.$$

Let $\delta(s)$ be the closed loop characteristic polynomial with coefficients

$$\underline{\delta} := \begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{n+m-1} & \delta_{n+m} \end{bmatrix}^T$$

and let $x := [b_0 \;\; b_1 \;\; \cdots \;\; b_m \;\; a_0 \;\; a_1 \;\; \cdots \;\; a_m]^T$. Then we can write $\delta = M_p x$ where

$$M_p := \underbrace{\begin{bmatrix} d_0 & & & n_0 & & \\ d_1 & d_0 & & n_1 & n_0 & \\ d_2 & d_1 & \ddots & n_2 & n_1 & \ddots \\ \vdots & d_2 & & \vdots & n_2 & \\ \vdots & \vdots & & \vdots & \vdots & \\ d_n & \vdots & & n_n & \vdots & \\ & d_n & & & n_n & \\ & & \ddots & & & \ddots \end{bmatrix}}_{M_p \in \mathbf{R}^{(n+m+1) \times 2(m+1)}}.$$

Prove that x stabilizes the plant if there exists δ such that

$$d(\delta, M_p) < \rho(\delta)$$

where $\rho(\delta)$ is the Euclidean radius of the largest stability hypersphere centered at δ and $d(\delta, M_p)$ is the Euclidean distance between δ and the subspace spanned by the columns of M_p.

3.7 NOTES AND REFERENCES

The ℓ_2 stability hypersphere in coefficient space was first calculated by Soh, Berger and Dabke [214]. Here we have endeavored to present this result for both the Hurwitz and Schur cases in its greatest generality. The calculation of the ℓ_p stability ball is due to Tsypkin and Polyak [225]. These calculations have been extended to the case of complex polynomials by Kogan [149] and bivariate polynomials by Polyak and Shmulyian [191]. A simplified proof of the Tsypkin-Polyak locus has been given by Mansour [170]. The problem of ascertaining the robust stability of disc polynomials was formulated and solved in Chapellat, Dahleh and Bhattacharyya [61]. Exercise 3.14 is taken from Bhattacharyya, Keel and Howze [34].