# Chapter 2

# STABILITY OF A LINE SEGMENT

In this chapter we develop some results on the stability of a line segment of polynomials. A complete analysis of this problem for both the Hurwitz and Schur cases is given and the results are summarized as the Segment Lemma. We also prove the Vertex Lemma and the Real and Complex Convex Direction Lemmas which give certain useful conditions under which the stability of a line segment of polynomials can be ascertained from the stability of its endpoints. These results are based on some fundamental properties of the phase of Hurwitz polynomials and segments which are also proved.

## 2.1 INTRODUCTION

In the previous chapter, we discussed the stability of a fixed polynomial by using the Boundary Crossing Theorem. In this chapter we focus on the problem of determining the stability of a line segment joining two fixed polynomials which we refer to as the endpoints. This line segment of polynomials is a convex combination of the two endpoints. This kind of problem arises in robust control problems containing a single uncertain parameter, such as a gain or a time constant, when stability of the system must be ascertained for the entire interval of uncertainty. We give some simple solutions to this problem for both the Hurwitz and Schur cases and collectively call these results the Segment Lemma.

In general, the stability of the endpoints does not guarantee that of the entire segment of polynomials. For example consider the segment joining the two polynomials

$$P_1(s) = 3s^4 + 3s^3 + 5s^2 + 2s + 1 \quad \text{and} \quad P_2(s) = s^4 + s^3 + 5s^2 + 2s + 5.$$

It can be checked that both $P_1(s)$ and $P_2(s)$ are Hurwitz stable and yet the polynomial at the midpoint

$$\frac{P_1(s) + P_2(s)}{2} \quad \text{has a root at} \quad s = j.$$

However, if the polynomial representing the difference of the endpoints assumes certain special forms, the stability of the endpoints does indeed guarantee stability of the entire segment. These forms which are frequency independent are described in the Vertex Lemma and again both Hurwitz and Schur versions of this lemma are given. The conditions specified by the Vertex Lemma are useful for reducing robust stability determinations over a continuum of polynomials to that of a discrete set of points. A related notion, that of *convex directions*, requires that segment stability hold for all stable endpoints and asks for conditions on the difference polynomial that guarantee this. Conditions are established here for convex directions in the real and complex cases. The proofs of the Vertex Lemma and the Convex Direction Lemmas depend on certain phase relations for Hurwitz polynomials and segments which are established here. These are also of independent interest.

## 2.2 BOUNDED PHASE CONDITIONS

Let $\mathcal{S}$ be an open set in the complex plane representing the stability region and let $\partial\mathcal{S}$ denote its boundary. Suppose $\delta_1(s)$ and $\delta_2(s)$ are polynomials (real or complex) of degree $n$. Let

$$\delta_\lambda(s) := \lambda\delta_1(s) + (1-\lambda)\delta_2(s)$$

and consider the following *one parameter family* of polynomials:

$$[\delta_1(s), \delta_2(s)] := \{\delta_\lambda(s) : \lambda \in [0,1]\}.$$

This family will be referred to as a *segment* of polynomials. We shall say that the segment is stable if and only if every polynomial on the segment is stable. This property is also referred to as *strong stability* of the pair $(\delta_1(s), \delta_2(s))$.

We begin with a lemma which follows directly from the Boundary Crossing Theorem (Chapter 1). Let $\phi_{\delta_i}(s_0)$ denote the argument of the complex number $\delta_i(s_0)$.

**Lemma 2.1 (Bounded Phase Lemma)**
*Let $\delta_1(s)$ and $\delta_2(s)$ be stable with respect to $\mathcal{S}$ and assume that the degree of $\delta_\lambda(s) = n$ for all $\lambda \in [0,1]$. Then the following are equivalent:*

a) *the segment $[\delta_1(s), \delta_2(s)]$ is stable with respect to $\mathcal{S}$*

b) $\delta_\lambda(s^*) \neq 0$, *for all* $s^* \in \partial\mathcal{S}$; $\lambda \in [0,1]$

c) $|\phi_{\delta_1}(s^*) - \phi_{\delta_2}(s^*)| \neq \pi$ *radians for all* $s^* \in \partial\mathcal{S}$,

d) *The complex plane plot of* $\dfrac{\delta_1(s^*)}{\delta_2(s^*)}$, *for* $s^* \in \partial\mathcal{S}$ *does not cut the negative real axis.*

**Proof.** The equivalence of a) and b) follows from the Boundary Crossing Theorem. The equivalence of b) and c) is best illustrated geometrically in Figure 2.1. In words this simply states that whenever $\delta_\lambda(s^*) = 0$ for some $\lambda \in [0,1]$ the phasors $\delta_1(s^*)$ and $\delta_2(s^*)$ must line up with the origin with their endpoints on opposite sides of it. This is expressed by the condition $|\phi_{\delta_1}(s^*) - \phi_{\delta_2}(s^*)| = \pi$ radians.

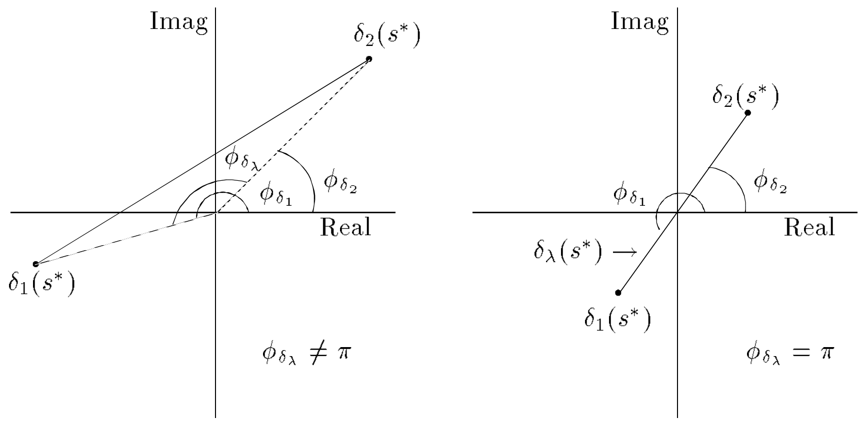



**Figure 2.1.** Image set of $\delta_\lambda(s^*)$ and $\phi_{\delta_\lambda}$

The equivalence of b) and d) follows from the fact that if

$$\lambda\delta_1(s^*) + (1-\lambda)\delta_2(s^*) = 0$$

then

$$\frac{\delta_1(s^*)}{\delta_2(s^*)} = -\left(\frac{1-\lambda}{\lambda}\right).$$

As $\lambda$ varies from 0 to 1, the right hand side of the above equation generates the negative real axis. Hence $\delta_\lambda(s^*) = 0$ for some $\lambda \in [0,1]$ if and only if $\frac{\delta_1(s^*)}{\delta_2(s^*)}$ is negative and real. ♣

This Lemma essentially states that the entire segment is stable provided the end points are, the degree remains invariant and the phase difference between the endpoints evaluated along the stability boundary is bounded by $\pi$. This condition will be referred to as the *Bounded Phase Condition*. We illustrate this result with some examples.

**Example 2.1. (Real Polynomials)** Consider the following feedback system shown in Figure 2.2. Suppose that we want to check the robust Hurwitz stability of the closed loop system for $\alpha \in [2,3]$. We first examine the stability of the two endpoints of the characteristic polynomial

$$\delta(s,\alpha) = s^3 + 2\alpha s^2 + (\alpha+1)s + (\alpha-1).$$

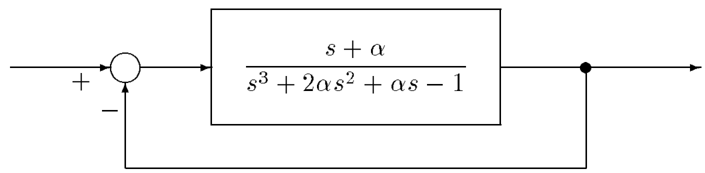



**Figure 2.2.** Feedback system (Example 2.1)

We let

$$\delta_1(s) := \delta(s,\alpha)|_{\alpha=2} = s^3 + 4s^2 + 3s + 1$$
$$\delta_2(s) := \delta(s,\alpha)|_{\alpha=3} = s^3 + 6s^2 + 4s + 2$$

Then $\delta_\lambda(s) = \lambda\delta_1(s) + (1-\lambda)\delta_2(s)$. We check that the endpoints $\delta_1(s)$ and $\delta_2(s)$ are stable. Then we verify the bounded phase condition, namely that the phase difference $|\phi_{\delta_1}(j\omega) - \phi_{\delta_2}(j\omega)|$ between these endpoints never reaches 180° as $\omega$ runs from 0 to $\infty$. Thus, the segment is robustly Hurwitz stable. This is shown in Figure 2.3.

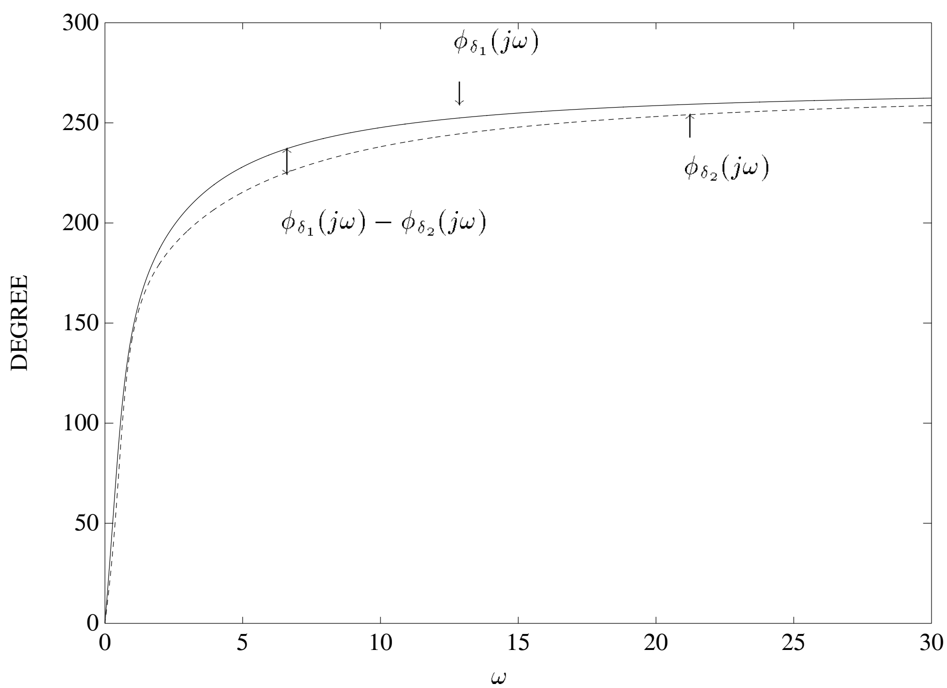



**Figure 2.3.** Phase difference of the endpoints of a stable segment (Example 2.1)

The condition d) of Lemma 2.1 can also be easily verified by drawing the polar plot of $\delta_1(j\omega)/\delta_2(j\omega)$.

**Example 2.2. (Complex Polynomials)** Consider the Hurwitz stability of the line segment joining the two complex polynomials:

$$\delta_1(s) = s^4 + (8-j)s^3 + (28-j5)s^2 + (50-j3)s + (33+j9)$$
$$\delta_2(s) = s^4 + (7+j4)s^3 + (46+j15)s^2 + (165+j168)s + (-19+j373).$$

We first verify that the two endpoints $\delta_1(s)$ and $\delta_2(s)$ are stable. Then we plot $\phi_{\delta_1}(j\omega)$ and $\phi_{\delta_2}(j\omega)$ with respect to $\omega$ (Figure 2.4). As we can see, the Bounded Phase Condition is satisfied, that is the phase difference $|\phi_{\delta_1}(j\omega) - \phi_{\delta_2}(j\omega)|$ never reaches 180°, so we conclude that the given segment $[\delta_1(s), \delta_2(s)]$ is stable.

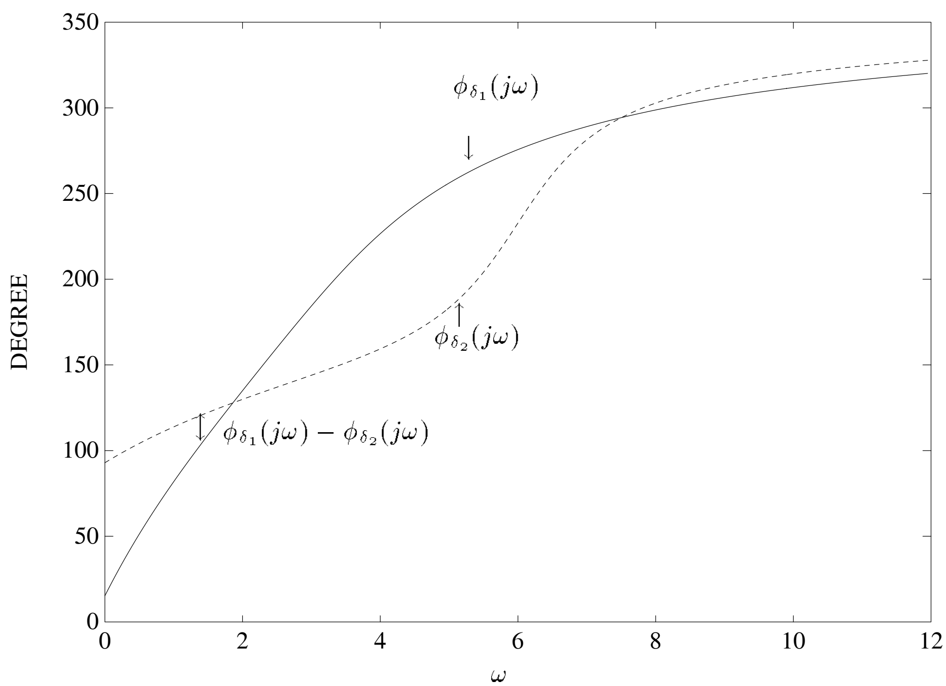



**Figure 2.4.** Phase difference vs $\omega$ for a complex segment (Example 2.2)

We can also use the condition d) of Lemma 2.1. As shown in Figure 2.5, the plot of $\delta_1(j\omega)/\delta_2(j\omega)$ does not cut the negative real axis of the complex plane. Therefore, the segment is stable.

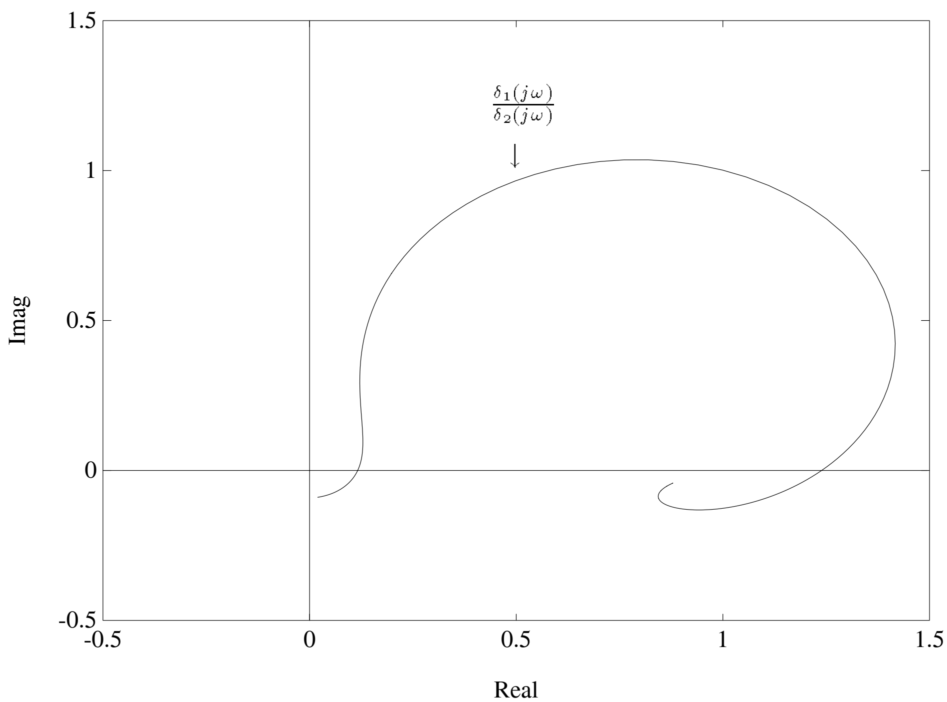



**Figure 2.5.** A Stable segment: $\frac{\delta_1(j\omega)}{\delta_2(j\omega)} \cap \mathbb{R}^- = \emptyset$ (Example 2.2)

**Example 2.3. (Schur Stability)** Let us consider the Schur stability of the segment joining the two polynomials

$$\delta_1(z) = z^5 + 0.4z^4 - 0.33z^3 + 0.058z^2 + 0.1266z + 0.059$$
$$\delta_2(z) = z^5 - 2.59z^4 + 2.8565z^3 - 1.4733z^2 + 0.2236z - 0.0121.$$

First we verify that the roots of both $\delta_1(z)$ and $\delta_2(z)$ lie inside the unit circle. In order to check the stability of the given segment, we simply evaluate the phases of $\delta_1(z)$ and $\delta_2(z)$ along the stability boundary, namely the unit circle. Figure 2.6 shows that the phase difference $\phi_{\delta_1}(e^{j\theta}) - \phi_{\delta_2}(e^{j\theta})$ reaches 180° at around $\theta = 0.81$ radians. Therefore, we conclude that there exists an unstable polynomial along the segment.

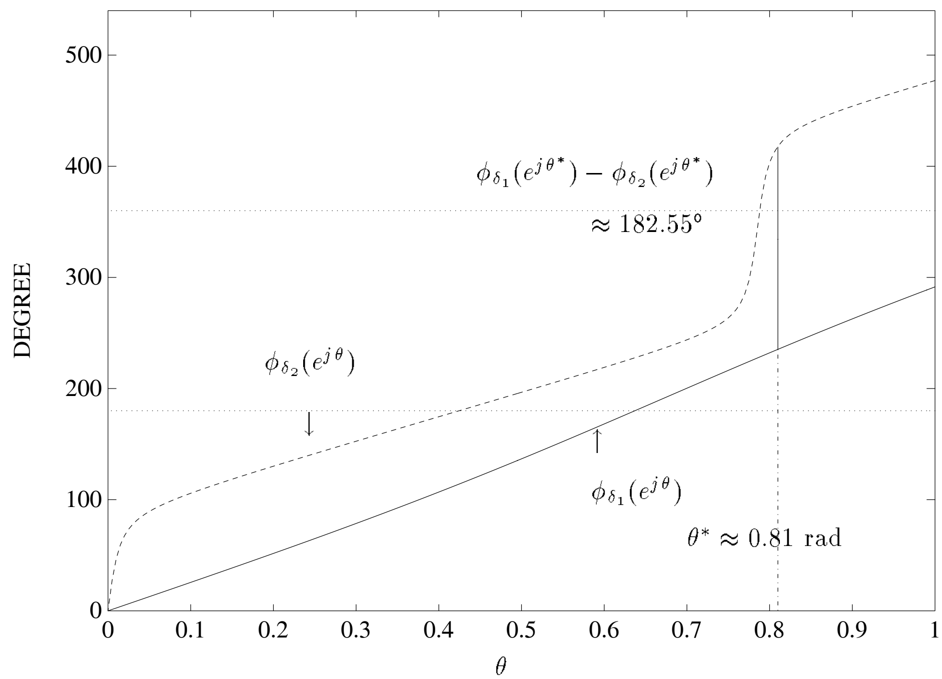



**Figure 2.6.** An unstable segment (Example 2.3)

Lemma 2.1 can be extended to a more general class of segments. In particular, let $\delta_1(s)$, $\delta_2(s)$ be quasipolynomials of the form

$$\begin{aligned} \delta_1(s) &= as^n + \sum e^{-sT_i} a_i(s) \\ \delta_2(s) &= bs^n + \sum e^{-sH_i} b_i(s) \end{aligned} \tag{2.1}$$

where $T_i, H_i \geq 0$, $a_i(s)$, $b_i(s)$ have degrees less than $n$ and $a$ and $b$ are arbitrary but nonzero and of the same sign. The Hurwitz stability of $\delta_1(s)$, $\delta_2(s)$ is then equivalent to their roots being in the left half plane.

**Lemma 2.2** *Lemma 2.1 holds for Hurwitz stability of the quasipolynomials of the form specified in (2.1).*

The proof of this lemma follows from the fact that Lemma 2.1 is equivalent to the Boundary Crossing Theorem which applies to Hurwitz stability of quasipolynomials $\delta_i(s)$ of the form given (see Theorem 1.14, Chapter 1).

**Example 2.4. (Quasi-polynomials)** Let us consider the Hurwitz stability of the line segment joining the following pair of quasi-polynomials:

$$\delta_1(s) = (s^2 + 3s + 2) + e^{-sT_1}(s + 1) + e^{-sT_2}(2s + 1)$$
$$\delta_2(s) = (s^2 + 5s + 3) + e^{-sT_1}(s + 2) + e^{-sT_2}(2s + 1)$$

where $T_1 = 1$ and $T_2 = 2$. We first check the stability of the endpoints by examining the frequency plots of

$$\frac{\delta_1(j\omega)}{(j\omega + 1)^2} \quad \text{and} \quad \frac{\delta_2(j\omega)}{(j\omega + 1)^2}.$$

Using the Principle of the Argument (equivalently, the Nyquist stability criterion) the condition for $\delta_1(s)$ (or $\delta_2(s)$) having all its roots in the left half plane is simply that the plot should not encircle the origin since the denominator term $(s + 1)^2$ does not have right half plane roots. Figure 2.7 shows that both endpoints $\delta_1(s)$ and $\delta_2(s)$ are stable.

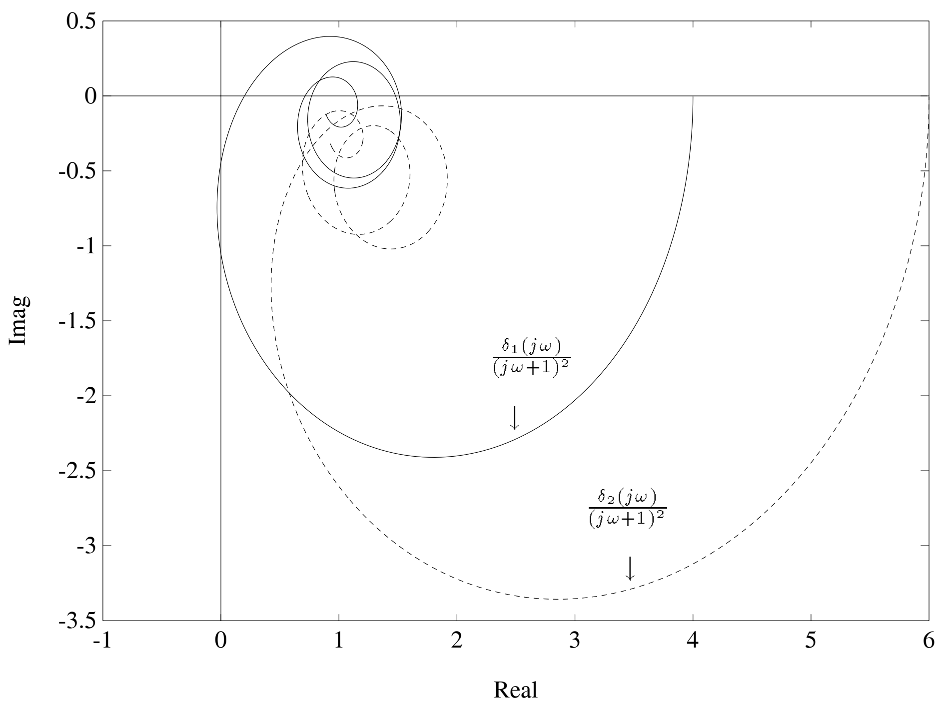



**Figure 2.7.** Stable quasi-polynomials (Example 2.4)

We generate the polar plot of $\delta_1(j\omega)/\delta_2(\omega)$ (Figure 2.8). As this plot does not cut the negative real axis the stability of the segment $[\delta_1(s), \delta_2(s)]$ is guaranteed by condition d) of Lemma 2.1.

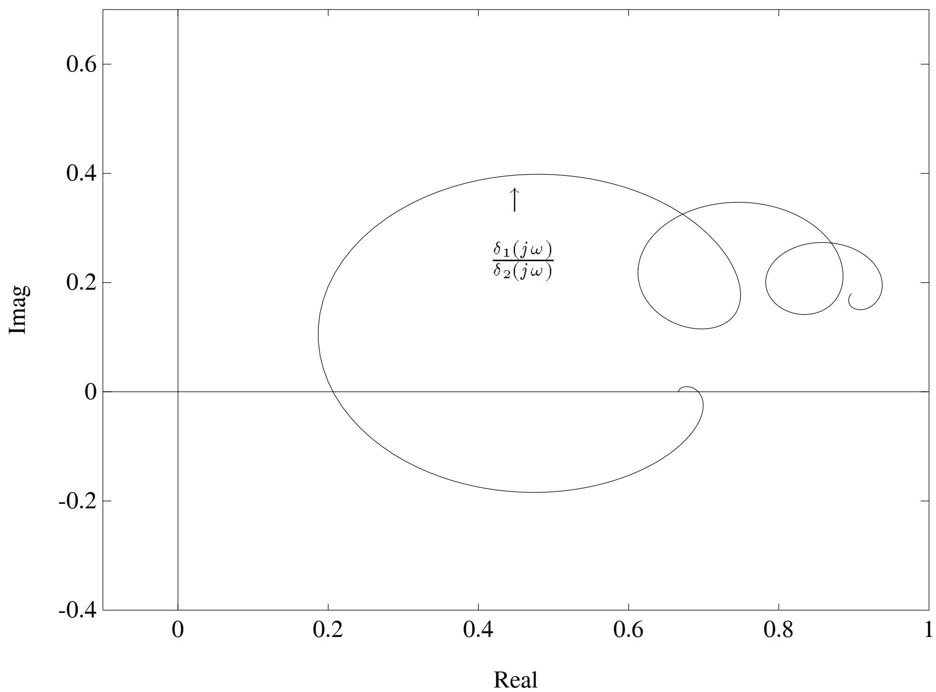



**Figure 2.8.** Stable segment of quasipolynomials (Example 2.4)

In the next section we focus specifically on the Hurwitz and Schur cases. We show how the frequency sweeping involved in using these results can always be *avoided* by isolating and checking only those frequencies where the phase difference can potentially reach 180°.

## 2.3 SEGMENT LEMMA

### 2.3.1 Hurwitz Case

In this subsection we are interested in strong stability of a line segment of polynomials joining two Hurwitz polynomials. We start by introducing a simple lemma which deals with convex combinations of two real polynomials, and finds the conditions under which one of these convex combinations can have a pure imaginary root. Recall the even-odd decomposition of a real polynomial $\delta(s)$ and the notation $\delta(j\omega) = \delta^e(\omega) + j\omega\delta^o(\omega)$ where $\delta^e(\omega)$ and $\delta^o(\omega)$ are real polynomials in $\omega^2$.

**Lemma 2.3** *Let $\delta_1(\cdot)$ and $\delta_2(\cdot)$ be two arbitrary real polynomials (not necessarily stable). Then there exists $\lambda \in [0,1]$ such that $(1-\lambda)\delta_1(\cdot) + \lambda\delta_2(\cdot)$ has a pure imaginary root $j\omega$, with $\omega > 0$ if and only if*

$$\begin{cases} \delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0 \\ \delta_1^e(\omega)\delta_2^e(\omega) \leq 0 \\ \delta_1^o(\omega)\delta_2^o(\omega) \leq 0 \end{cases}$$

**Proof.** Suppose first that there exists some $\lambda \in [0,1]$ and $\omega > 0$ such that

$$(1-\lambda)\delta_1(j\omega) + \lambda\delta_2(j\omega) = 0. \tag{2.2}$$

We can write,

$$\begin{aligned} \delta_i(j\omega) &= \delta_i^{\text{even}}(j\omega) + \delta_i^{\text{odd}}(j\omega) \\ &= \delta_i^e(\omega) + j\omega\delta_i^o(\omega), \qquad \text{for } \; i = 1,2. \end{aligned} \tag{2.3}$$

Thus, taking (2.3) and the fact that $\omega > 0$ into account, (2.2) is equivalent to

$$\begin{cases} (1-\lambda)\delta_1^e(\omega) + \lambda\delta_2^e(\omega) = 0 \\ (1-\lambda)\delta_1^o(\omega) + \lambda\delta_2^o(\omega) = 0. \end{cases} \tag{2.4}$$

But if (2.4) holds then necessarily

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0, \tag{2.5}$$

and since $\lambda$ and $1-\lambda$ are both nonnegative, (2.4) also implies that

$$\delta_1^e(\omega)\delta_2^e(\omega) \leq 0 \quad \text{and} \quad \delta_1^o(\omega)\delta_2^o(\omega) \leq 0, \tag{2.6}$$

and therefore this proves that the condition is necessary.
For the converse there are two cases:

c1) Suppose that

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0, \quad \delta_1^e(\omega)\delta_2^e(\omega) \leq 0, \quad \delta_1^o(\omega)\delta_2^o(\omega) \leq 0,$$

for some $\omega \geq 0$, but that we do not have $\delta_1^e(\omega) = \delta_2^e(\omega) = 0$, then

$$\lambda = \frac{\delta_1^e(\omega)}{\delta_1^e(\omega) - \delta_2^e(\omega)}$$

satisfies (2.4), and one can check easily that $\lambda$ is in $[0,1]$.

c2) Suppose now that

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0, \text{ and } \delta_1^e(\omega) = \delta_2^e(\omega) = 0. \tag{2.7}$$

Then we are left with,

$$\delta_1^o(\omega)\delta_2^o(\omega) \leq 0.$$

Here again, if we do not have $\delta_1^o(\omega) = \delta_2^o(\omega) = 0$, then the following value of $\lambda$ satisfies (2.4)

$$\lambda = \frac{\delta_1^o(\omega)}{\delta_1^o(\omega) - \delta_2^o(\omega)}.$$

If $\delta_1^o(\omega) = \delta_2^o(\omega) = 0$, then from (2.7) we conclude that both $\lambda = 0$ and $\lambda = 1$ satisfy (2.4) and this completes the proof. ♣

Based on this we may now state the Segment Lemma for the Hurwitz case.

**Lemma 2.4 (Segment Lemma: Hurwitz Case)**
*Let $\delta_1(s)$, $\delta_2(s)$ be real Hurwitz polynomials of degree $n$ with leading coefficients of the same sign. Then the line segment of polynomials $[\delta_1(s), \delta_2(s)]$ is Hurwitz stable if and only there exists no real $\omega > 0$ such that*

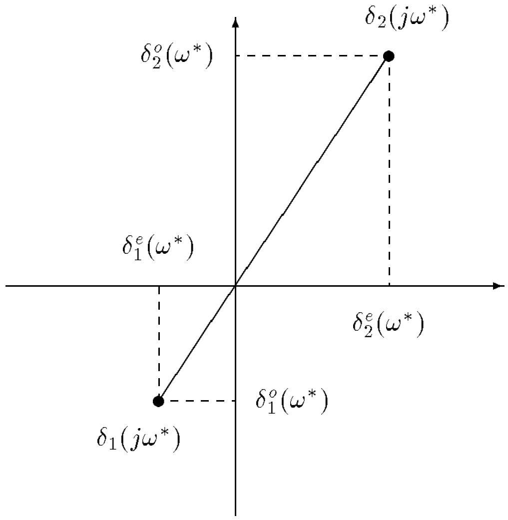
$$\begin{array}{ll} 1) & \delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0 \\ 2) & \delta_1^e(\omega)\delta_2^e(\omega) \leq 0 \\ 3) & \delta_1^o(\omega)\delta_2^o(\omega) \leq 0. \end{array} \tag{2.8}$$

**Proof.** The proof of this result again follows from the Boundary Crossing Theorem of Chapter 1. We note that since the two polynomials $\delta_1(s)$ and $\delta_2(s)$ are of degree $n$ with leading coefficients of the same sign, every polynomial on the segment is of degree $n$. Moreover, no polynomial on the segment has a real root at $s = 0$ because in such a case $\delta_1(0)\delta_2(0) \leq 0$, and this along with the assumption on the sign of the leading coefficients, contradicts the assumption that $\delta_1(s)$ and $\delta_2(s)$ are both Hurwitz. Therefore an unstable polynomial can occur on the segment if and only if a segment polynomial has a root at $s = j\omega$ with $\omega > 0$. By the previous lemma this can occur if and only if the conditions (2.8) hold. ♣

If we consider the image of the segment $[\delta_1(s), \delta_2(s)]$ evaluated at $s = j\omega$, we see that the conditions (2.8) of the Segment Lemma are the necessary and sufficient condition for the line segment $[\delta_1(j\omega), \delta_2(j\omega)]$ to pass through the origin of the complex plane. This in turn is equivalent to the phase difference condition $|\phi_{\delta_1}(j\omega) - \phi_{\delta_2}(j\omega)| = 180°$. We illustrate this in Figure 2.9.

**Example 2.5.** Consider the robust Hurwitz stability problem of the feedback system treated in Example 2.1. The characteristic polynomial is:

$$\delta(s, \alpha) = s^3 + 2\alpha s^2 + (\alpha + 1)s + (\alpha - 1).$$




$$\delta_1^e(\omega^*)\delta_2^e(\omega^*) < 0$$

$$\delta_1^o(\omega^*)\delta_2^o(\omega^*) < 0$$

$$\frac{\delta_1^o(\omega^*)}{\delta_1^e(\omega^*)} = \frac{\delta_2^o(\omega^*)}{\delta_2^e(\omega^*)}$$

$$\Updownarrow$$

$$\delta_1^e(\omega^*)\delta_2^o(\omega^*) - \delta_2^e(\omega^*)\delta_1^o(\omega^*) = 0$$

$$\Leftrightarrow \quad |\phi_{\delta_1}(j\omega^*) - \phi_{\delta_2}(j\omega^*)| = 180^\circ$$

**Figure 2.9.** Segment Lemma: geometric interpretation

We have already verified the stability of the endpoints

$$\delta_1(s) := \delta(s,\alpha)|_{\alpha=2} = s^3 + 4s^2 + 3s + 1$$
$$\delta_2(s) := \delta(s,\alpha)|_{\alpha=3} = s^3 + 6s^2 + 4s + 2.$$

Robust stability of the system is equivalent to that of the segment

$$\delta_\lambda(s) = \lambda\delta_1(s) + (1-\lambda)\delta_2(s),$$

To apply the Segment Lemma we compute the real positive roots of the polynomial

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) =$$
$$(-4\omega^2+1)(-\omega^2+4) - (-6\omega^2+2)(-\omega^2+3) = 0.$$

This equation has no real root in $\omega$ and thus there is no $j\omega$ root on the line segment. Thus, from the Segment Lemma, the segment $[\delta_1(s), \delta_2(s)]$ is stable and the closed loop system is robustly stable. Although frequency sweeping is unnecessary we have plotted in Figure 2.10 the image set of the line segment. As expected this plot avoids the origin of the complex plane for all $\omega$.

### 2.3.2 Schur Case

Let us now consider the problem of checking the Schur stability of the line joining two real polynomials $P_1(z)$ and $P_2(z)$. We wish to know whether every polynomial of the form

$$\lambda P_1(z) + (1-\lambda)P_2(z), \quad \lambda \in [0,1]$$

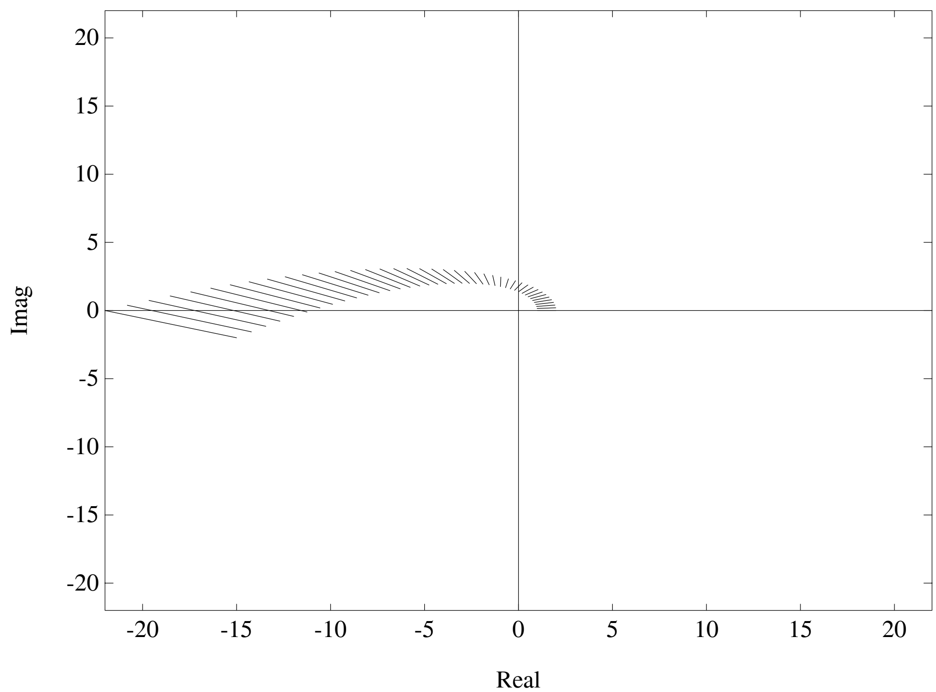



**Figure 2.10.** Image set of a stable segment (Example 2.5)

is Schur stable. Assume that there is at least one stable polynomial along the segment and that the degree remains constant along the segment. It follows then from the Boundary Crossing Theorem that if there exists a polynomial on the segment with an unstable root outside the unit circle, by continuity there must exist a polynomial on the segment with a root located on the boundary of the stability region, namely on the unit circle. In the following we give some simple and systematic tests for checking this. We will assume that the leading coefficients of the two extreme polynomials are of same sign. This is a necessary and sufficient condition for every polynomial on the segment to be of the same degree.

We begin with the following result.

**Lemma 2.5 (Schur Segment Lemma 1)**
*Let $P_1(z)$ and $P_2(z)$ be two real Schur polynomials of degree $n$, with the leading coefficients of the same sign. A polynomial on the line segment $[P_1(z), P_2(z)]$ has a root on the unit circle if and only if there exists $z_0$ with $|z_0| = 1$ such that*

$$P_1(z_0)P_2(z_0^{-1}) - P_2(z_0)P_1(z_0^{-1}) = 0 \tag{2.9}$$

$$\mathrm{Im}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] = 0 \tag{2.10}$$

*and*

$$\mathrm{Re}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] \le 0 \tag{2.11}$$

**Proof.**
*Necessity* Suppose that there exists $z_0$ with $|z_0| = 1$ and $\lambda \in [0,1]$ such that

$$\lambda P_1(z_0) + (1-\lambda)P_2(z_0) = 0 \tag{2.12}$$

Since $|z_0| = 1$ and (2.12) is real, we have

$$\lambda P_1(z_0^{-1}) + (1-\lambda)P_2(z_0^{-1}) = 0. \tag{2.13}$$

From (2.12) and (2.13) it follows that

$$P_1(z_0)P_2(z_0^{-1}) - P_1(z_0^{-1})P_2(z_0) = 0. \tag{2.14}$$

Separating (2.12) into real and imaginary parts and using the fact that $\lambda \in [0,1]$, it follows that

$$\mathrm{Re}[P_1(z_0)]\mathrm{Im}[P_2(z_0)] - \mathrm{Im}[P_1(z_0)]\mathrm{Re}[P_2(z_0)] = 0 \tag{2.15}$$

and

$$\mathrm{Re}[P_1(z_0)]\mathrm{Re}[P_2(z_0)] \le 0 \quad \text{and} \quad \mathrm{Im}[P_1(z_0)]\mathrm{Im}[P_2(z_0)] \le 0. \tag{2.16}$$

This is equivalent to

$$\mathrm{Im}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] = 0 \tag{2.17}$$

and

$$\mathrm{Re}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] \le 0 \tag{2.18}$$

proving the necessity of the conditions.

*Sufficiency* Suppose that

$$P_1(z_0)P_2(z_0^{-1}) - P_2(z_0)P_1(z_0^{-1}) = 0 \tag{2.19}$$

$$\mathrm{Im}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] = 0 \tag{2.20}$$

and

$$\mathrm{Re}\left[\frac{P_1(z_0)}{P_2(z_0)}\right] \le 0. \tag{2.21}$$

We note that $P_1(z_0) \neq P_2(z_0)$, otherwise (2.19)-(2.21) would imply that $P_1(z_0) = P_2(z_0) = 0$ which contradicts the assumption that $P_1(z)$ and $P_2(z)$ are both Schur. Then

$$\lambda = \frac{P_2(z_0)}{P_2(z_0) - P_1(z_0)} \tag{2.22}$$

satisfies (2.12) and one can easily check that $\lambda \in [0,1]$. ♣

In the above lemma, we need to compute the zeros of (2.9) which is not a polynomial. Instead we can form the polynomial

$$P_1(z_0)z_0^n P_2(z_0^{-1}) - P_2(z_0)z_0^n P_1(z_0^{-1}) = 0$$

whose unit circle roots coincide with those of (2.9). Note that $z^n P_i(z^{-1})$ is simply the "reverse" polynomial of $P_i(z)$.

### Symmetric-antisymmetric Decomposition

Recall the symmetric-antisymmetric decomposition introduced in Chapter 1. A given real polynomial can be decomposed into a symmetric part $h(z)$ and an anti-symmetric part $g(z)$ defined as follows:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = h(z) + g(z) \tag{2.23}$$

and

$$h(z) = \frac{1}{2}\left(P(z) + z^n P(z^{-1})\right), \qquad g(z) = \frac{1}{2}\left(P(z) - z^n P(z^{-1})\right).$$

We also have

$$h(z) = \alpha_n z^n + \cdots + \alpha_0 \quad \text{and} \quad g(z) = \beta_n z^n + \cdots + \beta_0$$

where

$$\alpha_i = \frac{a_i + a_{n-i}}{2}, \qquad \beta_i = \frac{a_i - a_{n-i}}{2}, \qquad i = 0, \cdots, n. \tag{2.24}$$

This decomposition plays a role similar to the even-odd decomposition in the Hurwitz case. We can enunciate the following lemma.

**Lemma 2.6 (Schur Segment Lemma 2)**
*Let $P_1(z)$ and $P_2(z)$ be two real polynomials of degree $n$ with the following symmetric-antisymmetric decomposition:*

$$P_1(z) = h_1(z) + g_1(z) \quad \text{and} \quad P_2(z) = h_2(z) + g_2(z)$$

*There exists $\lambda \in [0,1]$ and $z_0$ with $|z_0| = 1$ such that*

$$\lambda P_1(z_0) + (1-\lambda)P_2(z_0) = 0$$

*if and only if*

$$h_1(z_0)g_2(z_0) - g_1(z_0)h_2(z_0) = 0, \quad z_0^{-n} h_1(z_0)h_2(z_0) \leq 0, \quad z_0^{-n} g_1(z_0)g_2(z_0) \geq 0.$$

**Proof.**
*Necessity* Assume $z_0 = e^{j\theta}$ is a root of $\lambda P_1(z) + (1-\lambda)P_2(z) = 0$. Then

$$\lambda\,(h_1(z_0) + g_1(z_0)) + (1-\lambda)(h_2(z_0) + g_2(z_0)) = 0. \tag{2.25}$$

Since $|z_0| = 1$ this is equivalent to

$$\lambda\left(z_0^{-\frac{n}{2}}h_1(z_0) + z_0^{-\frac{n}{2}}g_1(z_0)\right) + (1-\lambda)\left(z_0^{-\frac{n}{2}}h_2(z_0) + z_0^{-\frac{n}{2}}g_2(z_0)\right) = 0. \tag{2.26}$$

Now observe that

$$\begin{aligned} z_0^{-\frac{n}{2}}h(z_0) &= \frac{1}{2}\left[(a_n + a_0)z_0^{\frac{n}{2}} + (a_{n-1} + a_1)z_0^{\frac{n}{2}-1} + \cdots + (a_0 + a_n)z_0^{-\frac{n}{2}}\right] \\ &= \frac{1}{2}\left[(a_n + a_0)\left(\cos\frac{n}{2}\theta + j\sin\frac{n}{2}\theta\right) + \cdots + (a_0 + a_n)\left(\cos\frac{n}{2}\theta - j\sin\frac{n}{2}\theta\right)\right] \\ &= (a_n + a_0)\cos\frac{n}{2}\theta + (a_{n-1} + a_1)\cos\left(\frac{n}{2} - 1\right)\theta + \cdots \end{aligned}$$

is real and

$$\begin{aligned} z_0^{-\frac{n}{2}}g(z_0) &= \frac{1}{2}\left[(a_n - a_0)z_0^{\frac{n}{2}} + (a_{n-1} - a_1)z_0^{\frac{n}{2}-1} + \cdots + (a_0 - a_n)z_0^{-\frac{n}{2}}\right] \\ &= \frac{1}{2}\left[(a_n - a_0)\left(\cos\frac{n}{2}\theta + j\sin\frac{n}{2}\theta\right) + \cdots + (a_0 - a_n)\left(\cos\frac{n}{2}\theta - j\sin\frac{n}{2}\theta\right)\right] \\ &= j\left[(a_n - a_0)\sin\frac{n}{2}\theta + (a_{n-1} + a_1)\sin\left(\frac{n}{2} - 1\right)\theta + \cdots\right] \end{aligned}$$

is imaginary. Thus, we rewrite (2.26) as follows:

$$\lambda\left(z_0^{-\frac{n}{2}}h_1(z_0) + jz_0^{-\frac{n}{2}}\frac{g_1(z_0)}{j}\right) + (1-\lambda)\left(z_0^{-\frac{n}{2}}h_2(z_0) + z_0^{-\frac{n}{2}}\frac{g_2(z_0)}{j}\right) = 0. \tag{2.27}$$

Note that $z_0^{-\frac{n}{2}}h_1(z_0)$ and $z_0^{-\frac{n}{2}}h_2(z_0)$ are real parts, and $z_0^{-\frac{n}{2}}\frac{g_1(z_0)}{j}$ and $z_0^{-\frac{n}{2}}\frac{g_1(z_0)}{j}$ are imaginary parts. Therefore, (2.27) is equivalent to:

$$\lambda z_0^{-\frac{n}{2}}h_1(z_0) + (1-\lambda)z_0^{-\frac{n}{2}}h_2(z_0) = 0 \tag{2.28}$$

and

$$\lambda z_0^{-\frac{n}{2}}\frac{g_1(z_0)}{j} + (1-\lambda)z_0^{-\frac{n}{2}}\frac{g_2(z_0)}{j} = 0. \tag{2.29}$$

As $z_0 \neq 0$ and both $\lambda$ and $(1-\lambda)$ are nonnegative we get

$$h_1(z_0)g_2(z_0) - g_1(z_0)h_2(z_0) = 0, \quad z_0^{-n}h_1(z_0)h_2(z_0) \leq 0, \quad z_0^{-n}g_1(z_0)g_2(z_0) \geq 0$$

which proves the necessity of these conditions.

*Sufficiency* For the converse we have two cases:

a) Suppose

$$h_1(z_0)g_2(z_0)-g_1(z_0)h_2(z_0) = 0, \quad z_0^{-n}h_1(z_0)h_2(z_0) \leq 0, \quad z_0^{-n}g_1(z_0)g_2(z_0) \geq 0$$

but we do not have $h_1(z_0) = h_2(z_0) = 0$. Then

$$\lambda = \frac{h_2(z_0)}{h_2(z_0) - h_1(z_0)} \tag{2.30}$$

satisfies (2.25) and one can verify that $\lambda \in [0, 1]$.

b) Now assume that

$$z_0^{-n}(h_1(z_0)g_2(z_0) - g_1(z_0)h_2(z_0)) = 0,$$

$$h_1(z_0) = h_2(z_0) = 0, \quad z_0^{-n}g_1(z_0)g_2(z_0) \geq 0$$

but we do not have $g_1(z_0) = g_2(z_0) = 0$. In this case

$$\lambda = \frac{g_2(z_0)}{g_2(z_0) - g_1(z_0)} \in [0, 1] \tag{2.31}$$

satisfies (2.25).

If $g_1(z_0) = g_2(z_0) = 0$, then $\lambda = 0$ or $\lambda = 1$ satisfies (2.25), which concludes the proof of the lemma. ♣

**Example 2.6.** Consider the segment joining the two polynomials

$$P_1(z) = z^3 + 1.5z^2 + 1.2z + 0.5 \quad \text{and} \quad P_2(z) = z^3 - 1.2z^2 + 1.1z - 0.4.$$

The symmetric-antisymmetric decomposition

$$P_1(z) = h_1(z) + g_1(z) \quad \text{and} \quad P_2(z) = h_2(z) + g_2(z)$$

where

$$\begin{aligned} h_1(z) &= 0.75z^3 + 1.35z^2 + 1.35z + 0.75 \\ g_1(z) &= 0.25z^3 + 0.15z^2 - 0.15z - 0.25 \\ h_2(z) &= 0.3z^3 - 0.05z^2 - 0.05z + 0.3 \\ g_1(z) &= 0.7z^3 - 1.15z^2 + 1.15z - 0.7 \end{aligned}$$

The polynomial

$$h_1(z)g_2(z) - g_1(z)h_2(z)$$

has four roots $z_0$ on the unit circle such that

$$z_0^{-n}h_1(z_0)h_2(z_0) \leq 0 \quad \text{and} \quad z_0^{-n}g_1(z_0)g_2(z_0) \geq 0$$

| $z_0$ | $z_0^{-n}h_1(z_0)h_2(z_0)$ | $z_0^{-n}g_1(z_0)g_2(z_0)$ |
|---|---|---|
| $z_{01} = -0.298 - 0.9546j$ | $-0.1136$ | $0.5651$ |
| $z_{02} = -0.298 + 0.9546j$ | $-0.1136$ | $0.5651$ |
| $z_{03} = 0.2424 - 0.9702j$ | $-0.4898$ | $0.0874$ |
| $z_{04} = 0.2424 + 0.9702j$ | $-0.4898$ | $0.0874$ |

For both $z_0 = z_{01}$ and $z_0 = z_{02}$ we find

$$\lambda_1 = \frac{h_2(z_0)}{h_2(z_0) - h_1(z_0)} = 0.7755$$

yielding

$$\lambda_1 P_1(z) + (1 - \lambda_1)P_2(z) = z^3 + 0.8939z^2 + 1.1775z + 0.2979$$

and one can check that the above polynomial has a pair of complex conjugate roots at $z_{01}$ and $z_{02}$.

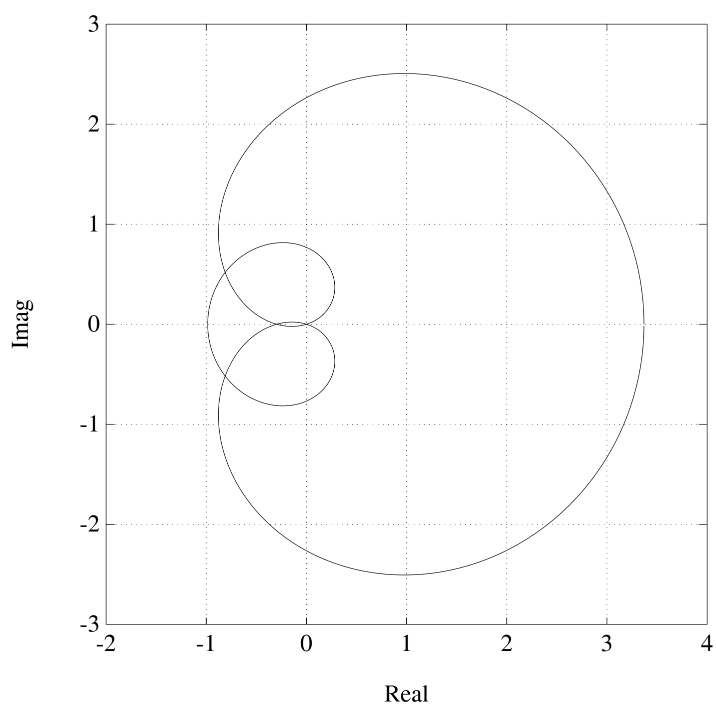



**Figure 2.11.** Unit circle image of the polynomial corresponding to $\lambda_1 = 0.7755$ (Example 2.6)

In Figure 2.11 we can check that the complex plane plot of $\lambda_1 P_1(e^{j\theta}) + (1 - \lambda_1)P_2(e^{j\theta})$ contains the origin, which implies that the convex combination of $P_1(z)$ and $P_2(z)$ has roots on the unit circle for $\lambda = \lambda_1$.

Similarly for $z_0 = z_{03}$ and $z_0 = z_{04}$

$$\lambda_2 = \frac{h_2(z_0)}{h_2(z_0) - h_1(z_0)} = 0.1751$$

yielding

$$\lambda_2 P_1(z) + (1 - \lambda_2)P_2(z) = z^3 - 0.7272z^2 + 1.1175z - 0.2424$$

which has roots at $z_{03}$ and $z_{04}$. The image set of $\lambda P_1(e^{j\theta}) + (1 - \lambda)P_2(e^{j\theta})$ for $\lambda = \lambda_2$ in Figure 2.12 contains the origin, which means that $\lambda_2 P_1(z) + (1 - \lambda_2)P_2(z)$ has a root on the unit circle for $\lambda = \lambda_2$.

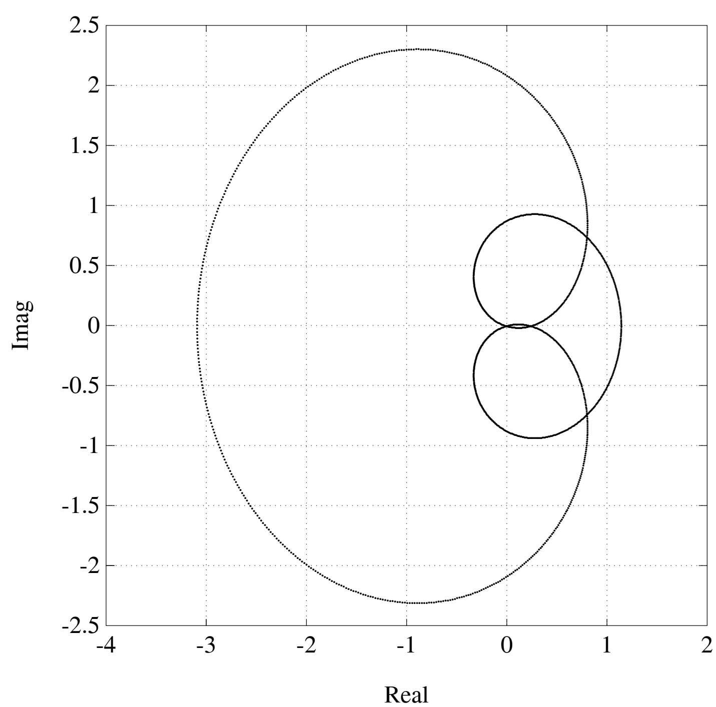



**Figure 2.12.** Unit circle image set of the polynomial corresponding to $\lambda_2 = 0.1751$ (Example 2.6)

In order to find the stable-unstable regions on the line $[P_1(z), P_2(z)]$, let us take some test polynomials corresponding to different values of $\lambda$.

$$\begin{array}{lll} \lambda = 0.1: & \lambda P_1(z) + (1-\lambda)P_2(z) = z^3 - 0.93z^2 + 1.11z - 0.31 & \text{stable.} \\ \lambda = 0.5: & \lambda P_1(z) + (1-\lambda)P_2(z) = z^3 + 0.15z^2 + 1.15z + 0.05 & \text{unstable.} \\ \lambda = 0.8: & \lambda P_1(z) + (1-\lambda)P_2(z) = z^3 + 0.96z^2 + 1.18z + 0.32 & \text{stable.} \end{array}$$

The partitioning of the segment into stable and unstable pieces is shown in Figure 2.13.

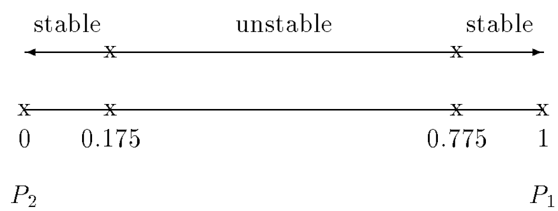



**Figure 2.13.** Stability regions along the segment (Example 2.6)

### Trigonometric Version

Let us again consider a real polynomial $P(z)$ of degree $n$, and write

$$P(z) = h(z) + g(z) \tag{2.32}$$

$$P(z) = 2z^{\frac{n}{2}}\left(\frac{z^{-\frac{n}{2}}h(z)}{2} + \frac{z^{-\frac{n}{2}}g(z)}{2}\right). \tag{2.33}$$

Note that

$$\frac{1}{2}z^{-\frac{n}{2}}h(z) = \frac{\alpha_n}{2}z^{\frac{n}{2}} + \cdots + \frac{\alpha_1}{2}z^{-\left(\frac{n}{2}-1\right)} + \frac{\alpha_0}{2}z^{-\frac{n}{2}} \tag{2.34}$$

$$\frac{1}{2}z^{-\frac{n}{2}}g(z) = \frac{\beta_n}{2}z^{\frac{n}{2}} + \cdots + \frac{\beta_1}{2}z^{-\left(\frac{n}{2}-1\right)} + \frac{\beta_0}{2}z^{-\frac{n}{2}} \tag{2.35}$$

where $\alpha_i, \beta_j$ are real with $\alpha_i = \alpha_{n-i}$ and $\beta_i = -\beta_{n-i}$ for $i = 0, \cdots, n$. When $z = e^{j\theta}$ we get

$$\frac{e^{-j\frac{n\theta}{2}}h(e^{j\theta})}{2} = h^*(\theta)$$

$$\frac{e^{-j\frac{n\theta}{2}}g(e^{j\theta})}{2} = jg^*(\theta)$$

where

$$h^*(\theta) = \begin{cases} \alpha_n \cos\frac{n}{2}\theta + \alpha_{n-1}\cos\left(\frac{n}{2}-1\right)\theta + \cdots + \alpha_{\frac{n}{2}} & \text{if } n \text{ even} \\ \alpha_n \cos\frac{n}{2}\theta + \alpha_{n-1}\cos\left(\frac{n}{2}-1\right)\theta + \cdots + \alpha_{\frac{n+1}{2}}\cos\frac{\theta}{2} & \text{if } n \text{ odd} \end{cases} \tag{2.36}$$

and

$$g^*(\theta) = \begin{cases} \left(\beta_n \sin \frac{n}{2}\theta + \beta_{n-1} \sin\left(\frac{n}{2}-1\right)\theta + \cdots + \beta_{\frac{n}{2}+1} \sin\theta\right) & \text{if } n \text{ even} \\ \left(\beta_n \sin \frac{n}{2}\theta + \beta_{n-1} \sin\left(\frac{n}{2}-1\right)\theta + \cdots + \beta_{\frac{n}{2}+1} \sin\frac{\theta}{2}\right) & \text{if } n \text{ odd} \end{cases} \tag{2.37}$$

We write

$$P(e^{j\theta}) = 2e^{jn\frac{\theta}{2}}\left(h^*(\theta) + jg^*(\theta)\right) \tag{2.38}$$

$$P(e^{j\theta}) = 2e^{jn\frac{\theta}{2}}\delta(\theta). \tag{2.39}$$

We can then enunciate another version of the Segment Lemma.

**Lemma 2.7 (Schur Segment Lemma 3)**
*Let $P_1(z)$ and $P_2(z)$ be two polynomials of degree $n$, with*

$$P_1(e^{j\theta}) = 2e^{jn\frac{\theta}{2}}\left(h_1^*(\theta) + jg_1^*(\theta)\right)$$
$$P_2(e^{j\theta}) = 2e^{jn\frac{\theta}{2}}\left(h_2^*(\theta) + jg_2^*(\theta)\right)$$

*On the segment joining $P_1(z)$ and $P_2(z)$ there exists a polynomial with a root on the unit circle if and only if there exists $\theta \in [0, 2\pi)$ such that*

$$h_1^*(\theta)g_2^*(\theta) - g_1^*(\theta)h_2^*(\theta) = 0$$
$$h_1^*(\theta)h_2^*(\theta) \leq 0$$
$$g_1^*(\theta)g_2^*(\theta) \leq 0.$$

The proof is omitted as it is similar to the previous cases.

## 2.4 SOME FUNDAMENTAL PHASE RELATIONS

In this section, we develop some auxiliary results that will aid us in establishing the Convex Direction Lemma and the Vertex Lemma, which deal with conditions under which vertex stability implies segment stability. The above results depend heavily on some fundamental formulas for the rate of change of phase with respect to frequency for fixed Hurwitz polynomials and for a segment of polynomials. These are derived in this section.

### 2.4.1 Phase Properties of Hurwitz Polynomials

Let $\delta(s)$ be a real or complex polynomial and write

$$\delta(j\omega) = p(\omega) + jq(\omega) \tag{2.40}$$

where $p(\omega)$ and $q(\omega)$ are real functions. Also let

$$X(\omega) := \frac{q(\omega)}{p(\omega)} \tag{2.41}$$

and

$$\varphi_\delta(\omega) := \tan^{-1}\frac{q(\omega)}{p(\omega)} = \tan^{-1} X(\omega). \tag{2.42}$$

Let Im[$x$] and Re[$x$] denote the imaginary and real parts of the complex number $x$.

**Lemma 2.8** *If $\delta(s)$ is a real or complex Hurwitz polynomial*

$$\frac{dX(\omega)}{d\omega} > 0, \quad \text{for all } \omega \in [-\infty, +\infty]. \tag{2.43}$$

*Equivalently*

$$\mathrm{Im}\left[\frac{1}{\delta(j\omega)}\frac{d\delta(j\omega)}{d\omega}\right] > 0 \quad \text{for all } \omega \in [-\infty, +\infty]. \tag{2.44}$$

**Proof.** A Hurwitz polynomial satisfies the monotonic phase increase property

$$\frac{d\varphi_\delta(\omega)}{d\omega} = \frac{1}{1+X^2(\omega)}\frac{dX(\omega)}{d\omega} > 0 \ \text{ for all } \omega \in [-\infty, +\infty]$$

and this implies

$$\frac{dX(\omega)}{d\omega} > 0 \ \text{ for all } \omega \in [-\infty, +\infty].$$

The formula

$$\frac{d\varphi_\delta(\omega)}{d\omega} = \mathrm{Im}\left[\frac{1}{\delta(j\omega)}\frac{d\delta(j\omega)}{d\omega}\right] \tag{2.45}$$

follows from the relations

$$\begin{aligned}\frac{1}{\delta(j\omega)}\frac{d\delta(j\omega)}{d\omega} &= \frac{1}{p(\omega)+jq(\omega)}\left(\frac{dp(\omega)}{d\omega} + j\frac{dq(\omega)}{d\omega}\right) \\ &= \frac{\left[p(\omega)\frac{dp(\omega)}{d\omega} + q(\omega)\frac{dq(\omega)}{d\omega}\right] + j\left[p(\omega)\frac{dq(\omega)}{d\omega} - q(\omega)\frac{dp(\omega)}{d\omega}\right]}{p^2(\omega)+q^2(\omega)}\end{aligned} \tag{2.46}$$

and

$$\frac{d\varphi_\delta(\omega)}{d\omega} = \frac{\left[p(\omega)\frac{dq(\omega)}{d\omega} - q(\omega)\frac{dp(\omega)}{d\omega}\right]}{p^2(\omega)+q^2(\omega)}. \tag{2.47}$$

♣

We shall see later that inequality (2.43) can be strengthened when $\delta(s)$ is a *real* Hurwitz polynomial.

Now let $\delta(s)$ be a *real* polynomial of degree $n$. We write:

$$\delta(s) = \delta^{\mathrm{even}}(s) + \delta^{\mathrm{odd}}(s) = h(s^2) + sg(s^2) \tag{2.48}$$

where $h$ and $g$ are real polynomials in $s^2$. Then

$$\begin{aligned}\delta(j\omega) &= h(-\omega^2) + j\omega g(-\omega^2) \\ &= \rho_\delta(\omega)e^{j\varphi_\delta(\omega)}.\end{aligned} \tag{2.49}$$

We associate with the real polynomial $\delta(s)$ the two auxiliary even degree complex polynomials

$$\underline{\delta}(s) := h(s^2) + jg(s^2) \tag{2.50}$$

and

$$\bar{\delta}(s) := h(s^2) - js^2 g(s^2) \tag{2.51}$$

and write formulas analogous to (2.49) for $\underline{\delta}(j\omega)$ and $\bar{\delta}(j\omega)$. $\delta(s)$ is *antiHurwitz* if and only if all its zeros lie in the open right half plane ($\mathrm{Re}[s] > 0$). Let $t = s^2$ be a new complex variable.

**Lemma 2.9** *Consider*

$$h(t) + jg(t) = 0 \tag{2.52}$$

*and*

$$h(t) - jtg(t) = 0 \tag{2.53}$$

*as equations in the complex variable $t$. If $\delta(s) = h(s^2)+sg(s^2)$ is Hurwitz and degree $\delta(s) \geq 2$ each of these equations has all its roots in the lower-half of the complex plane* ($\mathrm{Im}[t] \leq 0$). *When degree $[\delta(s)] = 1$ (2.53) has all its roots in* $\mathrm{Im}[t] \leq 0$.

**Proof.** The statement regarding the case when degree$[\delta(s)] = 1$ can be directly checked. We therefore proceed with the assumption that degree$[\delta(s)] > 1$. Let

$$\delta(s) = a_0 + a_1 s + \cdots + a_n s^n. \tag{2.54}$$

Since $\delta(s)$ is Hurwitz we can assume without loss of generality that $a_i > 0$, $i = 0, \cdots, n$. We have

$$\begin{aligned} h(-\omega^2) &= a_0 + a_2(-\omega^2) + a_4(-\omega^2)^2 + \cdots \\ \omega g(-\omega^2) &= \omega\left[a_1 + a_3(-\omega^2) + a_5(-\omega^2)^2 + \cdots\right]. \end{aligned} \tag{2.55}$$

As $s$ runs from 0 to $+j\infty$, $-\omega^2$ runs from 0 to $-\infty$. We first recall the Hermite-Biehler Theorem of Chapter 1. According to this Theorem if $\delta(s)$ is Hurwitz stable, all the roots of the two equations

$$h(t) = 0 \quad \text{and} \quad g(t) = 0 \tag{2.56}$$

are distinct, real and negative. Furthermore the interlacing property holds and the maximum of the roots is one of $h(t) = 0$. For the rest of the proof we will assume that $\delta(s)$ is of odd degree. A similar proof will hold for the case that $\delta(s)$ is of even degree. Let degree$[\delta(s)] = 2m + 1$ with $m \geq 1$. Note that the solutions of

$$h(t) + jg(t) = 0 \tag{2.57}$$

are identical with the solutions of

$$\frac{g(t)}{h(t)} = j. \tag{2.58}$$

Let us denote the roots of $h(t) = 0$ by $\lambda_1, \lambda_2, \cdots, \lambda_m$ where $\lambda_1 < \lambda_2 < \cdots < \lambda_m$. The sign of $h(t)$ changes alternately in each interval $]\lambda_i, \lambda_{i+1}[$, $(i = 1, \cdots, m-1)$.

If $\frac{g(t)}{h(t)}$ is expressed by partial fractions as

$$\frac{g(t)}{h(t)} = \frac{c_1}{t-\lambda_1} + \frac{c_2}{t-\lambda_2} + \cdots + \frac{c_m}{t-\lambda_m}, \tag{2.59}$$

then each $c_i, i = 1, \cdots, m$ should be positive. This is because when $t = -\omega^2$ passes increasingly (from left to right) through $\lambda_i$, the sign of $\frac{g(t)}{h(t)}$ changes from $-$ to $+$. This follows from the fact that $g(t)$ has just one root in each interval and $a_0 > 0$, $a_1 > 0$.

If we suppose

$$\mathrm{Im}[t] \geq 0, \tag{2.60}$$

then

$$\mathrm{Im}\left[\frac{c_i}{t-\lambda_i}\right] \leq 0 \quad i = 1, \cdots, m \tag{2.61}$$

and consequently we obtain

$$\mathrm{Im}\left[\frac{g(t)}{h(t)}\right] = \sum_{1\leq i\leq m} \mathrm{Im}\left[\frac{c_i}{t-\lambda_i}\right] \leq 0. \tag{2.62}$$

Such a $t$ cannot satisfy the relation in (2.58). This implies that the equation

$$h(t) + jg(t) = 0 \tag{2.63}$$

in $t$ has all its roots in the lower-half of the complex plane $\mathrm{Im}[t] \leq 0$. We can treat $[h(t) - jtg(t)]$ similarly. ♣

This lemma leads to a key monotonic phase property.

**Lemma 2.10** *If $\delta(s)$ is Hurwitz and of degree $\geq 2$ then*

$$\frac{d\varphi_\delta}{d\omega} > 0 \tag{2.64}$$

*and*

$$\frac{d\varphi_{\bar\delta}}{d\omega} > 0. \tag{2.65}$$

**Proof.** From the previous lemma we have by factorizing $h(-\omega^2) + jg(-\omega^2)$

$$h(-\omega^2) + jg(-\omega^2) = a_n(-\omega^2 - \alpha_1)\cdots(-\omega^2 - \alpha_m) \tag{2.66}$$

with some $\alpha_1, \alpha_2, \cdots, \alpha_m$ whose imaginary parts are negative. Now

$$\arg\left[h(-\omega^2) + jg(-\omega^2)\right] = \sum_{i=1}^{m} \arg(-\omega^2 - \alpha_i). \tag{2.67}$$

When $(-\omega^2)$ runs from 0 to $(-\infty)$, each component of the form $\arg(-\omega^2 - \alpha_i)$ is monotonically increasing. Consequently $\arg\left[h(-\omega^2) + jg(-\omega^2)\right]$ is monotonically increasing as $(-\omega^2)$ runs from 0 to $(-\infty)$. In other words, $\arg\left[h(s^2) + jg(s^2)\right]$ is monotonically increasing as $s(= j\omega)$ runs from 0 to $j\infty$. This proves (2.64); (2.65) is proved in like manner. ♣

The dual result is given without proof.

**Lemma 2.11** *If $\delta(s)$ is antiHurwitz*

$$\frac{d\varphi_{\underline{\delta}}}{d\omega} < 0 \tag{2.68}$$

*and*

$$\frac{d\varphi_{\bar{\delta}}}{d\omega} < 0 \tag{2.69}$$

We remark that in Lemma 2.10 $\underline{\delta}(s)$ and $\bar{\delta}(s)$ are *not* Hurwitz even though they enjoy the respective monotonic phase properties in (2.64) and (2.65). Similarly in Lemma 2.11 $\underline{\delta}(s)$ and $\bar{\delta}(s)$ are not antiHurwitz even though they enjoy the monotonic phase properties in (2.68) and (2.69), respectively.

The above results allow us to tighten the bound given in Lemma 2.8 on the rate of change of phase of a real Hurwitz polynomial.

**Theorem 2.1** *For a real Hurwitz polynomial*

$$\delta(s) = h(s^2) + sg(s^2),$$

*the rate of change of the argument of $\delta(j\omega)$ is bounded below by:*

$$\frac{d\varphi_\delta(\omega)}{d\omega} \geq \left|\frac{\sin\left(2\varphi_\delta(\omega)\right)}{2\omega}\right|, \quad \text{for all } \omega > 0. \tag{2.70}$$

*Equivalently with*

$$X(\omega) = \frac{\omega g\left(-\omega^2\right)}{h\left(-\omega^2\right)}$$

*we have*

$$\frac{dX(\omega)}{d\omega} \geq \left|\frac{X(\omega)}{\omega}\right|, \quad \text{for all } \omega > 0. \tag{2.71}$$

*In (2.70) and (2.71) equality holds only when degree $[\delta(s)] = 1$.*

**Proof.** The equivalence of the two conditions (2.70) and (2.71) follows from

$$\frac{d\varphi_\delta(\omega)}{d\omega} = \frac{1}{1 + X^2(\omega)}\frac{dX(\omega)}{d\omega}$$

and

$$\begin{aligned}
\frac{1}{1+X^2(\omega)}\left|\frac{X(\omega)}{\omega}\right| &= \left|\frac{1}{1+X^2(\omega)}\frac{X(\omega)}{\omega}\right| \\
&= \left|\frac{h^2\left(-\omega^2\right)}{h^2\left(-\omega^2\right)+\omega^2 g^2\left(-\omega^2\right)}\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right| \\
&= \left|\cos^2\left(\varphi_\delta(\omega)\right)\frac{1}{\omega}\tan\left(\varphi_\delta(\omega)\right)\right| \\
&= \left|\frac{1}{\omega}\cos\left(\varphi_\delta(\omega)\right)\sin\left(\varphi_\delta(\omega)\right)\right| \\
&= \left|\frac{\sin\left(2\varphi_\delta(\omega)\right)}{2\omega}\right|.
\end{aligned}$$

We now prove (2.71). The fact that equality holds in (2.71) in the case where $\delta(s)$ has degree equal to 1 can be easily verified directly. We therefore proceed with the assumption that degree$[\delta(s)] \geq 2$. From Lemma 2.8, we know that

$$\frac{dX(\omega)}{d\omega} > 0$$

so that

$$\begin{aligned}
\frac{dX(\omega)}{d\omega} &= \frac{\frac{d\left(\omega g\left(-\omega^2\right)\right)}{d\omega} h\left(-\omega^2\right) - \frac{d\left(h\left(-\omega^2\right)\right)}{d\omega}\left(\omega g\left(-\omega^2\right)\right)}{h^2\left(-\omega^2\right)} \\
&= \frac{g\left(-\omega^2\right) h\left(-\omega^2\right) + \omega \dot{g}\left(-\omega^2\right) h\left(-\omega^2\right) - \omega \dot{h}\left(-\omega^2\right) g\left(-\omega^2\right)}{h^2\left(-\omega^2\right)} \\
&= \underbrace{\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}}_{\frac{X(\omega)}{\omega}} + \omega \underbrace{\frac{\dot{g}\left(-\omega^2\right) h\left(-\omega^2\right) - \dot{h}\left(-\omega^2\right) g\left(-\omega^2\right)}{h^2\left(-\omega^2\right)}}_{\frac{d}{d\omega}\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right)} > 0.
\end{aligned}$$

From Lemma 2.10 we have

$$\frac{d\varphi_{\underline{\delta}}}{d\omega} > 0 \quad \text{and} \quad \frac{d\varphi_{\bar{\delta}}}{d\omega} > 0$$

where

$$\underline{\delta}(s) = h\left(s^2\right) + jg\left(s^2\right) \quad \text{and} \quad \bar{\delta}(s) = h\left(s^2\right) - js^2 g\left(s^2\right).$$

First consider

$$\frac{d\varphi_{\underline{\delta}}}{d\omega} = \frac{1}{1+\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right)^2}\frac{d}{d\omega}\left(\frac{g(-\omega^2)}{h(-\omega^2)}\right) > 0.$$

Since

$$\frac{1}{1+\left(\frac{g(-\omega^2)}{h(-\omega^2)}\right)^2} > 0$$

we have

$$\frac{d}{d\omega}\left(\frac{g(-\omega^2)}{h(-\omega^2)}\right) > 0.$$

Thus, for $\omega > 0$ we have

$$\frac{dX(\omega)}{d\omega} = \frac{X(\omega)}{\omega} + \omega\frac{d}{d\omega}\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right) > \frac{X(\omega)}{\omega}. \tag{2.72}$$

Now consider

$$\frac{d\varphi_{\bar{\delta}}}{d\omega} = \frac{1}{1+\left(\frac{\omega^2 g(-\omega^2)}{h(-\omega^2)}\right)^2}\frac{d}{d\omega}\left(\frac{\omega^2 g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right) > 0.$$

Here, we have

$$\begin{aligned}\frac{d}{d\omega}\left(\frac{\omega^2 g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right) &= \omega\left[2\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)} + \frac{\omega\left[h\left(-\omega^2\right)\dot{g}\left(-\omega^2\right) - g\left(-\omega^2\right)\dot{h}\left(-\omega^2\right)\right]}{h^2\left(-\omega^2\right)}\right] \\ &= \omega\left[2\frac{X(\omega)}{\omega} + \omega\frac{d}{d\omega}\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right)\right] > 0.\end{aligned}$$

With $\omega > 0$, it follows that

$$2\frac{X(\omega)}{\omega} + \omega\frac{d}{d\omega}\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right) > 0$$

and therefore

$$\underbrace{\frac{X(\omega)}{\omega} + \omega\frac{d}{d\omega}\left(\frac{g\left(-\omega^2\right)}{h\left(-\omega^2\right)}\right)}_{\frac{dX(\omega)}{d\omega}} > -\frac{X(\omega)}{\omega}. \tag{2.73}$$

Combining (2.72) and (2.73), we have, when degree$[\delta(s)] \geq 2$,

$$\frac{dX(\omega)}{d\omega} > \left|\frac{X(\omega)}{\omega}\right|, \quad \text{for all } \omega > 0.$$

♣

A useful technical result can be derived from the above Theorem.

**Lemma 2.12** *Let $\omega_0 > 0$ and the constraint*

$$\varphi_\delta(\omega_0) = \theta \tag{2.74}$$

*be given. The infimum value of*

$$\left.\frac{d\varphi_\delta(\omega)}{d\omega}\right|_{\omega=\omega_0} \tag{2.75}$$

*taken over all real Hurwitz polynomials $\delta(s)$ of a prescribed degree satisfying (2.74) is given by*

$$\left|\frac{\sin(2\theta)}{2\omega_0}\right|. \tag{2.76}$$

*The infimum is actually attained only when $0 < \theta < \frac{\pi}{2}$ and $\delta(s)$ is of degree one, by polynomials of the form*

$$\delta(s) = K(s\tan\theta + \omega_0). \tag{2.77}$$

*For polynomials of degree ($> 1$) the infimum can be approximated to arbitrary accuracy.*

**Proof.** The lower bound on the infimum given in (2.76) is an immediate consequence of (2.70) in Theorem 2.1. The fact that (2.77) attains the bound (2.76) can be checked directly. To prove that the infimum can be approximated to arbitrary accuracy it suffices to construct a sequence of Hurwitz polynomials $\delta_k(s)$ of prescribed degree $n$ each satisfying (2.74) and such that

$$\lim_{k\to\infty}\left.\frac{d\varphi_{\delta_k}(\omega)}{d\omega}\right|_{\omega=\omega_0} = \left|\frac{\sin(2\theta)}{2\omega_0}\right|. \tag{2.78}$$

For example when $0 < \theta < \frac{\pi}{2}$ we can take

$$\delta_k(s) = K\left(s\tan\theta + \omega_0 + \frac{1}{k}\right)(\epsilon_k s + 1)^{n-1}, \qquad k = 1, 2, \ldots. \tag{2.79}$$

where $\epsilon_k > 0$ is adjusted to satisfy the constraint (2.74) for each $k$:

$$\varphi_{\delta_k}(\omega_0) = \theta. \tag{2.80}$$

It is easy to see that $\epsilon_k \to 0$ and (2.78) holds. A similar construction can be carried out for other values of $\theta$ provided that the degree $n$ is large enough that the the constraint (2.80) can be satisfied; in particular $n \geq 4$ is always sufficient for arbitrary $\theta$. ♣

**Remark 2.1.** In the case of complex Hurwitz polynomials the lower bound on the rate of change of phase with respect to $\omega$ is zero and the corresponding statement is that

$$\left.\frac{d\varphi_{\delta_k}(\omega)}{d\omega}\right|_{\omega=\omega_0} \tag{2.81}$$

can be made as small as desired by choosing complex Hurwitz polynomials $\delta_k(s)$ satisfying the constraint (2.80).

These technical results are useful in certain constructions related to convex directions.

### 2.4.2 Phase Relations for a Segment

Consider now a line segment $\lambda\delta_1(s) + (1-\lambda)\delta_2(s)$, $\lambda \in [0,1]$ generated by the two real polynomials $\delta_1(s)$ and $\delta_2(s)$ of degree $n$ with leading coefficients and constant coefficients of the same sign. The necessary and sufficient condition for a polynomial in the interior of this segment to acquire a root at $s = j\omega_0$ is that

$$\begin{aligned} \lambda_0\delta_1^e(\omega_0) + (1-\lambda_0)\delta_2^e(\omega_0) &= 0 \\ \lambda_0\delta_1^o(\omega_0) + (1-\lambda_0)\delta_2^o(\omega_0) &= 0 \end{aligned} \tag{2.82}$$

for some $\lambda_0 \in (0,1)$. Since the segment is real and the constant coefficients are of the same sign it is sufficient to verify the above relations for $\omega_0 > 0$. Therefore the above equations are equivalent to

$$\lambda_0\delta_1^e(\omega_0) + (1-\lambda_0)\delta_2^e(\omega_0) = 0 \tag{2.83}$$

$$\lambda_0\omega_0\delta_1^o(\omega_0) + (1-\lambda_0)\omega_0\delta_2^o(\omega_0) = 0. \tag{2.84}$$

and also to

$$\begin{aligned} \lambda_0\delta_1^e(\omega_0) + (1-\lambda_0)\delta_2^e(\omega_0) &= 0 \\ \lambda_0\omega_0^2\delta_1^o(\omega_0) + (1-\lambda_0)\omega_0^2\delta_2^o(\omega_0) &= 0 \end{aligned} \tag{2.85}$$

since $\omega_0 > 0$. Noting that

$$\begin{aligned} \underline{\delta}_1(j\omega) &= \delta_1^e(\omega) + j\delta_1^o(\omega) = \rho_{\underline{\delta}_1}(\omega)e^{j\varphi_{\underline{\delta}_1}(\omega)} \\ \underline{\delta}_2(j\omega) &= \delta_2^e(\omega) + j\delta_2^o(\omega) = \rho_{\underline{\delta}_2}(\omega)e^{j\varphi_{\underline{\delta}_2}(\omega)} \\ \bar{\delta}_1(j\omega) &= \delta_1^e(\omega) + j\omega^2\delta_1^o(\omega) = \varphi_{\bar{\delta}_1}(\omega)e^{j\varphi_{\bar{\delta}_1}(\omega)} \\ \bar{\delta}_2(j\omega) &= \delta_2^e(\omega) + j\omega^2\delta_2^o(\omega) = \rho_{\bar{\delta}_2}(\omega)e^{j\varphi_{\bar{\delta}_2}(\omega)} \end{aligned} \tag{2.86}$$

we can write (2.82), (2.84), and (2.85), respectively in the equivalent forms

$$\lambda_0\delta_1(j\omega_0) + (1-\lambda_0)\delta_2(j\omega_0) = 0 \tag{2.87}$$

$$\lambda_0\underline{\delta}_1(j\omega_0) + (1-\lambda_0)\underline{\delta}_2(j\omega_0) = 0 \tag{2.88}$$

and

$$\lambda_0\bar{\delta}_1(j\omega_0) + (1-\lambda_0)\bar{\delta}_2(j\omega_0) = 0. \tag{2.89}$$

Now let

$$\begin{aligned} \delta_1(j\omega) &= \delta_1^e(\omega) + j\omega\delta_1^o(\omega) = \rho_{\delta_1}(\omega)e^{j\varphi_{\delta_1}(\omega)} \\ \delta_2(j\omega) &= \delta_2^e(\omega) + j\omega\delta_2^o(\omega) = \rho_{\delta_2}(\omega)e^{j\varphi_{\delta_2}(\omega)} \end{aligned} \tag{2.90}$$

$$\delta_0(s) := \delta_1(s) - \delta_2(s) \tag{2.91}$$

$$\begin{aligned} \underline{\delta}_0(j\omega) &:= \underline{\delta}_1(j\omega) - \underline{\delta}_2(j\omega) \\ \bar{\delta}_0(j\omega) &:= \bar{\delta}_1(j\omega) - \bar{\delta}_2(j\omega). \end{aligned} \tag{2.92}$$

We now state a key technical lemma.

**Lemma 2.13** *Let $\delta_1(s), \delta_2(s)$ be real polynomials of degree $n$ with leading coefficients of the same sign and assume that $\lambda_0 \in (0,1)$ and $\omega_0 > 0$ satisfy (2.85)-(2.89). Then*

$$\left.\frac{d\varphi_{\delta_0}}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{d\varphi_{\delta_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\delta_1}}{d\omega}\right|_{\omega=\omega_0} \tag{2.93}$$

$$\left.\frac{d\varphi_{\underline{\delta}_0}}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{d\varphi_{\underline{\delta}_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\underline{\delta}_1}}{d\omega}\right|_{\omega=\omega_0} \tag{2.94}$$

*and*

$$\left.\frac{d\varphi_{\bar{\delta}_0}}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{d\varphi_{\bar{\delta}_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\bar{\delta}_1}}{d\omega}\right|_{\omega=\omega_0}. \tag{2.95}$$

**Proof.** We prove only (2.93) in detail. If

$$\delta(j\omega) = p(\omega) + jq(\omega) \tag{2.96}$$

then

$$\tan\varphi_\delta(\omega) = \frac{q(\omega)}{p(\omega)}. \tag{2.97}$$

Let $\dot{q}(\omega) := \frac{dq(\omega)}{d\omega}$ and differentiate (2.97) with respect to $\omega$ to get

$$(1 + \tan^2\varphi_\delta(\omega))\frac{d\varphi_\delta}{d\omega} = \frac{p(\omega)\dot{q}(\omega) - q(\omega)\dot{p}(\omega)}{p^2(\omega)} \tag{2.98}$$

and

$$\frac{d\varphi_\delta}{d\omega} = \frac{p(\omega)\dot{q}(\omega) - q(\omega)\dot{p}(\omega)}{p^2(\omega) + q^2(\omega)}. \tag{2.99}$$

We apply the formula in (2.99) to

$$\delta_0(j\omega) = (p_1(\omega) - p_2(\omega)) + j(q_1(\omega) - q_2(\omega)) \tag{2.100}$$

to get

$$\frac{d\varphi_{\delta_0}}{d\omega} = \frac{(p_1 - p_2)(\dot{q}_1 - \dot{q}_2) - (q_1 - q_2)(\dot{p}_1 - \dot{p}_2)}{(p_1 - p_2)^2 + (q_1 - q_2)^2}. \tag{2.101}$$

Using (2.87)

$$\lambda_0 p_1(\omega_0) + (1 - \lambda_0)p_2(\omega_0) = 0 \tag{2.102}$$

and

$$\lambda_0 q_1(\omega_0) + (1-\lambda_0) q_2(\omega_0) = 0. \tag{2.103}$$

Since $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz $\lambda_0 \neq 0$, and $\lambda_0 \neq 1$, so that

$$p_1(\omega_0) - p_2(\omega_0) = -\frac{p_2(\omega_0)}{\lambda_0} = \frac{p_1(\omega_0)}{1-\lambda_0} \tag{2.104}$$

and

$$q_1(\omega_0) - q_2(\omega_0) = -\frac{q_2(\omega_0)}{\lambda_0} = \frac{q_1(\omega_0)}{1-\lambda_0}. \tag{2.105}$$

Substituting these relations in (2.101) we have

$$\begin{aligned}
\left.\frac{d\varphi_{\delta_0}}{d\omega}\right|_{\omega=\omega_0} &= \left.\frac{\frac{1}{1-\lambda_0}p_1\dot{q}_1 + \frac{1}{\lambda_0}p_2\dot{q}_2 - \frac{1}{1-\lambda_0}q_1\dot{p}_1 - \frac{1}{\lambda_0}q_2\dot{p}_2}{(p_1-p_2)^2 + (q_1-q_2)^2}\right|_{\omega=\omega_0} \\
&= \left.\frac{\frac{1}{1-\lambda_0}(p_1\dot{q}_1 - q_1\dot{p}_1)}{\frac{p_1^2+q_1^2}{(1-\lambda_0)^2}}\right|_{\omega=\omega_0} + \left.\frac{\frac{1}{\lambda_0}(p_2\dot{q}_2 - q_2\dot{p}_2)}{\frac{p_2^2+q_2^2}{\lambda_0^2}}\right|_{\omega=\omega_0} \\
&= (1-\lambda_0)\left.\frac{p_1\dot{q}_1 - q_1\dot{p}_1}{p_1^2+q_1^2}\right|_{\omega=\omega_0} + \lambda_0\left.\frac{p_2\dot{q}_2 - q_2\dot{p}_2}{p_2^2+q_2^2}\right|_{\omega=\omega_0} \\
&= (1-\lambda_0)\left.\frac{d\varphi_{\delta_1}}{d\omega}\right|_{\omega=\omega_0} + \lambda_0\left.\frac{d\varphi_{\delta_2}}{d\omega}\right|_{\omega=\omega_0}.
\end{aligned} \tag{2.106}$$

This proves (2.93). The proofs of (2.94) and (2.95) are identical starting from (2.88) and (2.89), respectively. ♣

The relation (2.93) holds for complex segments also. Suppose that $\delta_i(s), i=1,2$ are complex Hurwitz polynomials of degree $n$ and consider the complex segment $\delta_2(s) + \lambda\delta_0(s), \lambda \in [0,1]$ with $\delta_0(s) = \delta_1(s) - \delta_2(s)$. The condition for a polynomial in the interior of this segment to have a root at $s = j\omega_0$ is

$$\delta_2(j\omega_0) + \lambda_0\delta_0(j\omega_0) = 0, \quad \lambda_0 \in (0,1). \tag{2.107}$$

It is straightforward to derive from the above, just as in the real case that

$$\left.\frac{d\varphi_{\delta_0}}{d\omega}\right|_{\omega=\omega_0} = \lambda_0\left.\frac{d\varphi_{\delta_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\delta_1}}{d\omega}\right|_{\omega=\omega_0}. \tag{2.108}$$

The relationship (2.108) can be stated in terms of

$$X_i(\omega) := \tan\ \varphi_{\delta_i}(\omega), \quad i = 0,1,2. \tag{2.109}$$

Using the fact that

$$\frac{d\varphi_{\delta_i}(\omega)}{d\omega} = \frac{1}{(1+X_i^2(\omega))}\frac{dX_i(\omega)}{d\omega}, \quad i = 0,1,2 \tag{2.110}$$

(2.108) can be written in the equivalent form

$$\frac{1}{(1+X_0^2(\omega))}\frac{dX_0(\omega)}{d\omega}\bigg|_{\omega=\omega_0} =$$

$$\lambda_0\frac{1}{(1+X_2^2(\omega))}\frac{dX_2(\omega)}{d\omega}\bigg|_{\omega=\omega_0} + (1-\lambda_0)\frac{1}{(1+X_1^2(\omega))}\frac{dX_1(\omega)}{d\omega}\bigg|_{\omega=\omega_0}. \qquad (2.111)$$

Geometric reasoning (the image set of the segment at $s = j\omega_0$ passes through the origin) shows that

$$|X_0(\omega)|_{\omega=\omega_0} = |X_1(\omega)|_{\omega=\omega_0} = |X_2(\omega)|_{\omega=\omega_0}. \qquad (2.112)$$

Using (2.112) in (2.111) we obtain the following result.

**Lemma 2.14** *Let* $[\lambda\delta_1(s) + (1-\lambda)\delta_2(s)]$, $\lambda \in [0,1]$ *be a real or complex segment of polynomials. If a polynomial in the interior of this segment, corresponding to* $\lambda = \lambda_0$ *has a root at* $s = j\omega_0$ *then*

$$\frac{dX_0(\omega)}{d\omega}\bigg|_{\omega=\omega_0} = \lambda_0\,\frac{dX_2(\omega)}{d\omega}\bigg|_{\omega=\omega_0} + (1-\lambda_0)\,\frac{dX_1(\omega)}{d\omega}\bigg|_{\omega=\omega_0}. \qquad (2.113)$$

These auxiliary results will help us to establish the Convex Direction and Vertex Lemmas in the following sections.

## 2.5 CONVEX DIRECTIONS

It turns out that it is possible to give necessary and sufficient conditions on $\delta_0(s)$ under which strong stability of the pair $(\delta_2(s), \delta_0(s)+\delta_2(s))$ will hold for *every* $\delta_2(s)$ and $\delta_0(s)+\delta_2(s)$ that are Hurwitz. This is accomplished using the notion of *convex directions.*

As before let $\delta_1(s)$ and $\delta_2(s)$ be polynomials of degree $n$. Write

$$\delta_0(s) := \delta_1(s) - \delta_2(s)$$

and let

$$\delta_\lambda(s) = \lambda\delta_1(s) + (1-\lambda)\delta_2(s) = \delta_2(s) + \lambda\delta_0(s) \qquad (2.114)$$

and let us assume that the degree of every polynomial on the segment $\{\delta_\lambda(s) : \lambda \in [0,1]\}$ is $n$. Now, the problem of interest is: Give necessary and sufficient conditions on $\delta_0(s)$ under which stability of the segment in (2.114) is guaranteed whenever the endpoints are Hurwitz stable? A polynomial $\delta_0(s)$ satisfying the above property is called *a convex direction.* There are two distinct results on convex directions corresponding to the real and complex cases. We begin with the complex case.

### Complex Convex Directions

In the complex case we have the following result.

**Lemma 2.15 (Complex Convex Direction Lemma)**
*Let $\delta_\lambda(s) : \lambda \in [0,1]$ be a complex segment of polynomials of degree $n$ defined as in (2.114). The complex polynomial $\delta_0(s)$ is a convex direction if and only if*

$$\frac{d\varphi_{\delta_0}(\omega)}{d\omega} \leq 0 \tag{2.115}$$

*for every frequency $\omega \in \mathbb{R}$ such that $\delta_0(j\omega) \neq 0$. Equivalently*

$$\frac{dX_0(\omega)}{d\omega} \leq 0 \tag{2.116}$$

*for every frequency $\omega \in \mathbb{R}$ such that $\delta_0(j\omega) \neq 0$.*

**Proof.** The equivalence of the two conditions (2.115) and (2.116) is obvious. Suppose now that (2.116) is true. In the first place if $\omega_0$ is such that $\delta_0(j\omega_0) = 0$, it follows that $\delta_2(j\omega_0) + \lambda_0\delta_0(j\omega_0) \neq 0$ for any real $\lambda_0 \in [0,1]$ as this would contradict the fact that $\delta_2(s)$ is Hurwitz. Now from Lemma (2.14) we see that the segment has a polynomial with a root at $s = j\omega_0$ only if

$$\left.\frac{dX_0(\omega)}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{dX_2(\omega)}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{dX_1(\omega)}{d\omega}\right|_{\omega=\omega_0}. \tag{2.117}$$

Since $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz it follows from Lemma 2.8 that

$$\frac{dX_i(\omega)}{d\omega} > 0, \quad \omega \in \mathbb{R}; \quad i = 1, 2. \tag{2.118}$$

and $\lambda_0 \in (0,1)$. Therefore the right hand side of (2.117) is strictly positive whereas the left hand side is nonpositive by hypothesis. This proves that there cannot exist any $\omega_0 \in \mathbb{R}$ for which (2.117) holds. The stability of the segment follows from the Boundary Crossing Theorem (Chapter 1).

The proof of necessity is based on showing that if the condition (2.115) fails to hold it is possible to construct a Hurwitz polynomial $p_2(s)$ such that the end point $p_1(s) = p_2(s) + \delta_0(s)$ is Hurwitz stable and the segment joining them is of constant degree but contains unstable polynomials. The proof is omitted as it is similar to the real case which is proved in detail in the next lemma. It suffices to mention that when $\omega = \omega^*$ is such that

$$\left.\frac{d\varphi_{\delta_0}(\omega)}{d\omega}\right|_{\omega=\omega^*} > 0 \tag{2.119}$$

we can take

$$p_1(s) = (s - j\omega^*)t(s) + \mu\delta_0(s) \tag{2.120}$$

$$p_2(s) = (s - j\omega^*)t(s) - \mu\delta_0(s) \tag{2.121}$$

where $t(s)$ is chosen to be a complex Hurwitz polynomial of degree greater than the degree of $\delta_0(s)$, and satisfying the conditions:

$$\begin{aligned} &|X_0(\omega^*)| = |X_t(\omega^*)| \\ &\left.\frac{d\varphi_{\delta_0}(\omega)}{d\omega}\right|_{\omega=\omega^*} > \left.\frac{d\varphi_t(\omega)}{d\omega}\right|_{\omega=\omega^*} > 0. \end{aligned} \tag{2.122}$$

The existence of such $t(s)$ is clear from Remark 2.1 following Lemma 2.12. The proof is completed by noting that $p_i(s)$ are Hurwitz, the segment joining $p_1(s)$ and $p_2(s)$ is of constant degree (for small enough $|\mu|$), but the segment polynomial $\frac{1}{2}(p_1(s) + p_2(s))$ has $s = j\omega^*$ as a root. ♣

### Real Convex Directions

The following Lemma gives the necessary and sufficient condition for $\delta_0(s)$ to be a convex direction in the real case.

**Lemma 2.16 (Real Convex Direction Lemma)**
*Consider the real segment $\{\delta_\lambda(s) : \lambda \in [0,1]\}$ of degree $n$. The real polynomial $\delta_0(s)$ is a convex direction if and only if*

$$\frac{d\varphi_{\delta_0}(\omega)}{d\omega} \leq \left|\frac{\sin(2\varphi_{\delta_0}(\omega))}{2\omega}\right| \tag{2.123}$$

*is satisfied for every frequency $\omega > 0$ such that $\delta_0(j\omega) \neq 0$. Equivalently*

$$\frac{dX_0(\omega)}{d\omega} \leq \left|\frac{X_0(\omega)}{\omega}\right| \tag{2.124}$$

*for every frequency $\omega > 0$ such that $\delta_0(j\omega) \neq 0$.*

**Proof.** The equivalence of the conditions (2.124) and (2.123) has already been shown (see the proof of Theorem 2.1) and so it suffices to prove (2.124). If degree $\delta_i(s) = 1$ for $i = 1, 2$, degree $\delta_0(s) \leq 1$ and (2.124) holds. In this case it is straightforward to verify from the requirements that the degree along the segment is 1 and $\delta_i(s)$, $i = 1, 2$ are Hurwitz, that no polynomial on the segment has a root at $s = 0$. Hence such a segment is stable by the Boundary Crossing Theorem. We assume henceforth that degree$[\delta_i(s)] \geq 2$ for $i = 1, 2$. In general the assumption of invariant degree along the segment (the leading coefficients of $\delta_i(s)$ are of the same sign) along with the requirement that $\delta_i(s), i = 1, 2$ are Hurwitz imply that the constant coefficients of $\delta_i(s), i = 1, 2$ are also of the same sign. This rules out the possibility of any polynomial in the segment having a root at $s = 0$.

Now if $\omega_0 > 0$ is such that $\delta_0(j\omega_0) = 0$, and $\delta_2(j\omega_0) + \lambda_0\delta_0(j\omega_0) = 0$ for some real $\lambda_0 \in (0, 1)$ it follows that $\delta_2(j\omega_0) = 0$. However this would contradict the fact that $\delta_2(s)$ is Hurwitz. Thus such a $j\omega_0$ also cannot be a root of any polynomial on the segment. To proceed let us first consider the case where $\delta_0(s) = as + b$ with $b \neq 0$. Here (2.124) is again seen to hold. From Lemma 2.13 it follows that $s = j\omega_0$ is a root of a polynomial on the segment only if for some $\lambda_0 \in (0, 1)$

$$\left.\frac{d\varphi_{\underline{\delta}_0}}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{d\varphi_{\underline{\delta}_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\underline{\delta}_1}}{d\omega}\right|_{\omega=\omega_0}. \tag{2.125}$$

In the present case we have

$$\tan\varphi_{\underline{\delta}_0}(\omega) = \frac{a}{b} \tag{2.126}$$

and therefore the left hand side of (2.125)

$$\left.\frac{d\varphi_{\underline{\delta}_0}}{d\omega}\right|_{\omega=\omega_0} = 0. \tag{2.127}$$

Since $\delta_i(s), i = 1, 2$ are Hurwitz and of degree $\geq 2$ we have from Lemma 2.10 that

$$\left.\frac{d\varphi_{\underline{\delta}_i}}{d\omega}\right|_{\omega=\omega_0} > 0 \tag{2.128}$$

so that the right hand side of (2.125)

$$\lambda_0 \left.\frac{d\varphi_{\underline{\delta}_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\underline{\delta}_1}}{d\omega}\right|_{\omega=\omega_0} > 0. \tag{2.129}$$

This contradiction shows that such a $j\omega_0$ cannot be a root of any polynomial on the segment, which must therefore be stable.

We now consider the general case where degree$[\delta_0(s)] \geq 2$ or $\delta_0(s) = as$. From Lemma 2.14 we see that the segment has a polynomial with a root at $s = j\omega_0$ only if

$$\left.\frac{dX_0(\omega)}{d\omega}\right|_{\omega=\omega_0} = \lambda_0 \left.\frac{dX_2(\omega)}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0) \left.\frac{dX_1(\omega)}{d\omega}\right|_{\omega=\omega_0}. \tag{2.130}$$

Since $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz it follows from Theorem 2.1 that we have

$$\frac{dX_i(\omega)}{d\omega} > \left|\frac{X_i(\omega)}{\omega}\right|, \quad \omega > 0; \quad i = 1, 2 \tag{2.131}$$

and $\lambda_0 \in (0, 1)$. Furthermore we have

$$|X_0(\omega)|_{\omega=\omega_0} = |X_1(\omega)|_{\omega=\omega_0} = |X_2(\omega)|_{\omega=\omega_0} \tag{2.132}$$

so that the right hand side of (2.130) satisfies

$$\lambda_0 \left.\frac{dX_2(\omega)}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0) \left.\frac{dX_1(\omega)}{d\omega}\right|_{\omega=\omega_0} \tag{2.133}$$

$$> \lambda_0 \left| \frac{X_2(\omega)}{\omega} \right|_{\omega=\omega_0} + (1-\lambda_0) \left| \frac{X_1(\omega)}{\omega} \right|_{\omega=\omega_0} = \left| \frac{X_0(\omega_0)}{\omega_0} \right|. \tag{2.134}$$

On the other hand the left hand side of (2.130) satisfies

$$\left. \frac{dX_0(\omega)}{d\omega} \right|_{\omega=\omega_0} \leq \left| \frac{X_0(\omega_0)}{\omega_0} \right|. \tag{2.135}$$

This contradiction proves that there cannot exist any $\omega_0 \in \mathbb{R}$ for which (2.130) holds. Thus no polynomial on the segment has a root at $s = j\omega_0$, $\omega_0 > 0$ and the stability of the entire segment follows from the Boundary Crossing Theorem of Chapter 1. This completes the proof of sufficiency.

The proof of necessity requires us to show that if the condition (2.123) fails there exists a Hurwitz polynomial $r_2(s)$ such that $r_1(s) = r_2(s) + \delta_0(s)$ is also Hurwitz stable but the segment joining them is not. Suppose then that $\delta_0(s)$ is a given polynomial of degree $n$ and $\omega^* > 0$ is such that $\delta_0(j\omega^*) \neq 0$ but

$$\frac{d\varphi_{\delta_0}(\omega)}{d\omega} > \left| \frac{\sin(2\varphi_{\delta_0}(\omega))}{2\omega} \right| \tag{2.136}$$

for some $\omega^* > 0$. It is then possible to construct a real Hurwitz polynomial $t(s)$ of degree $\geq \ n-2$ such that

$$r_1(s) := (s^2 + {\omega^*}^2)t(s) + \mu\delta_0(s) \tag{2.137}$$

and

$$r_2(s) := (s^2 + {\omega^*}^2)t(s) - \mu\delta_0(s) \tag{2.138}$$

are Hurwitz and have leading coefficients of the same sign for sufficiently small $|\mu|$. It suffices to choose $t(s)$ so that

$$\left. \frac{d\varphi_{\delta_0}}{d\omega} \right|_{\omega=\omega^*} > \left. \frac{d\varphi_t}{d\omega} \right|_{\omega=\omega^*} > \left| \frac{\sin 2\varphi_{\delta_0}(\omega^*)}{2\omega^*} \right| = \left| \frac{\sin 2\varphi_t(\omega^*)}{2\omega^*} \right|. \tag{2.139}$$

The fact that such $t(s)$ exists is guaranteed by Lemma 2.12. It remains to prove that $r_i(s)$, $i = 1, 2$ can be made Hurwitz stable by choice of $\mu$.

For sufficiently small $|\mu|$, $n-2$ of the zeros of $r_i(s)$, $i = 1, 2$ are close to those of $t(s)$, and hence in the open left half plane while the remaining two zeros are close to $\pm j\omega^*$. To prove that that the roots lying close to $\pm j\omega^*$ are in the open left half plane we let $s(\mu)$ denote the root close to $j\omega^*$ and analyse the behaviour of the real part of $s(\mu)$ for small values of $\mu$. We already know that

$$\text{Re}[s(\mu)]|_{\mu=0} = 0. \tag{2.140}$$

We will establish the fact that $\text{Re}[s(\mu)]$ has a local maximum at $\mu = 0$ and this together with (2.140) will show that $\text{Re}[s(\mu)]$ is negative in a neighbourhood of

$\mu = 0$, proving that $r_i(s)$, $i = 1, 2$ are stable. To prove that $\mathrm{Re}[s(\mu)]$ has a local maximum at $\mu = 0$ it suffices to establish that

$$\left.\frac{d}{d\mu}\mathrm{Re}\left[s(\mu)\right]\right|_{\mu=0} = 0 \tag{2.141}$$

and

$$\left.\frac{d^2}{d\mu^2}\mathrm{Re}\left[s(\mu)\right]\right|_{\mu=0} < 0. \tag{2.142}$$

Now since $s(\mu)$ is a root of $r_1(s)$ we have

$$r_1\left(s(\mu)\right) = \left(s(\mu) - j\omega^*\right) u\left(s(\mu)\right) + \mu\delta_0\left(s(\mu)\right) = 0 \tag{2.143}$$

where

$$u(s) := (s + j\omega^*)t(s). \tag{2.144}$$

By differentiating (2.143) with respect to $\mu$ we get

$$\frac{ds(\mu)}{d\mu} = -\frac{\delta_0\left(s(\mu)\right)}{u\left(s(\mu)\right) + \left(s(\mu) - j\omega^*\right)\frac{du(s(\mu))}{ds(\mu)} + \mu\frac{d\delta_0(s(\mu))}{ds(\mu)}} \tag{2.145}$$

and hence

$$\left.\frac{ds(\mu)}{d\mu}\right|_{\mu=0} = -\frac{\delta_0(j\omega^*)}{u(j\omega^*)} = -\frac{\delta_0(j\omega^*)}{2j\omega^* t(j\omega^*)}. \tag{2.146}$$

From the fact that

$$\left|\frac{\sin 2\varphi_{\delta_0}(\omega^*)}{2\omega^*}\right| = \left|\frac{\sin 2\varphi_t(\omega^*)}{2\omega^*}\right| \tag{2.147}$$

(see (2.139)) it follows that $\delta_0(j\omega^*)$ and $t(j\omega^*)$ have arguments that are equal or differ by $\pi$ radians so that

$$\frac{\delta_0(j\omega^*)}{t(j\omega^*)} \tag{2.148}$$

is purely real. Therefore we have

$$\left.\frac{d}{d\mu}\mathrm{Re}\left[s(\mu)\right]\right|_{\mu=0} = 0.$$

To complete the proof we need to establish that

$$\left.\frac{d^2}{d\mu^2}\mathrm{Re}\left[s(\mu)\right]\right|_{\mu=0} < 0.$$

By differentiating (2.145) once again with respect to $\mu$ we can obtain the second derivative. After some calculation we get:

$$\left.\frac{d^2 s(\mu)}{d\mu^2}\right|_{\mu=0} = -\frac{j}{2(\omega^*)^2}\frac{\delta_0^2(j\omega^*)}{t^2(j\omega^*)}\left(\frac{1}{u(j\omega^*)}\left.\frac{du(j\omega)}{d\omega}\right|_{\omega=\omega^*} - \frac{1}{\delta_0(j\omega^*)}\left.\frac{d\delta_0(j\omega)}{d\omega}\right|_{\omega=\omega^*}\right).$$

Using the fact that $\frac{\delta_0(j\omega^*)}{t(j\omega^*)}$ is purely real and the formulas (see (2.45))

$$\operatorname{Im}\left[\frac{1}{u(j\omega^*)}\frac{du(j\omega)}{d\omega}\right]\Bigg|_{\omega=\omega^*} = \frac{d\varphi_u}{d\omega}\Bigg|_{\omega=\omega^*} \tag{2.149}$$

$$\operatorname{Im}\left[\frac{1}{\delta_0(j\omega^*)}\frac{d\delta_0(j\omega)}{d\omega}\right]\Bigg|_{\omega=\omega^*} = \frac{d\varphi_{\delta_0}}{d\omega}\Bigg|_{\omega=\omega^*} \tag{2.150}$$

we get

$$\frac{d^2}{d\mu^2}\operatorname{Re}[s(\mu)]\Bigg|_{\mu=0} = -\frac{1}{2(\omega^*)^2}\frac{\delta_0^2(j\omega^*)}{t^2(j\omega^*)}\left(\frac{d\varphi_{\delta_0}}{d\omega}\Bigg|_{\omega=\omega^*} - \frac{d\varphi_u}{d\omega}\Bigg|_{\omega=\omega^*}\right). \tag{2.151}$$

Now

$$\frac{d\varphi_u}{d\omega}\Bigg|_{\omega=\omega^*} = \frac{d\varphi_t}{d\omega}\Bigg|_{\omega=\omega^*} \tag{2.152}$$

and by construction

$$\frac{d\varphi_{\delta_0}}{d\omega}\Bigg|_{\omega=\omega^*} > \frac{d\varphi_t}{d\omega}\Bigg|_{\omega=\omega^*}.$$

Once again using the fact that $\frac{\delta_0(j\omega^*)}{t(j\omega^*)}$ is real we finally have from (2.151):

$$\frac{d^2}{d\mu^2}\operatorname{Re}[s(\mu)]\Bigg|_{\mu=0} < 0. \tag{2.153}$$

This proves that the real part of $s(\mu)$ is negative for $\mu$ in the neighbourhood of $\mu = 0$ and therefore $r_1(s)$ must be stable as claimed. An identical argument shows that $r_2(s)$ is stable. The proof is now completed by the fact that $r_i(s)$, $i = 1, 2$ are Hurwitz and the segment joining them is of constant degree but the segment polynomial $\frac{1}{2}(r_1(s) + r_2(s))$ has $s = j\omega^*$ as a root. Thus $\delta_0(s)$ is not a convex direction. ♣

We illustrate the usefulness of convex directions by some examples.

**Example 2.7.** Consider the line segment joining the following two endpoints which are Hurwitz:

$$\begin{aligned}\delta_1(s) &:= s^4 + 12.1s^3 + 8.46s^2 + 11.744s + 2.688\\ \delta_2(s) &:= 2s^4 + 9s^3 + 12s^2 + 10s + 3.\end{aligned}$$

We first verify the stability of the segment by using the Segment Lemma. The positive real roots of the polynomial

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) = 0$$

that is

$$(\omega^4 - 8.46\omega^2 + 2.688)(-9\omega^2 + 10) - (2\omega^4 - 12\omega^2 + 3)(-12.1\omega^2 + 11.744) = 0$$

are

$$2.1085, \qquad 0.9150, \qquad 0.3842.$$

However, none of these $\omega$s satisfy the conditions 2) and 3) of the Segment Lemma (Lemma 2.4). Thus, we conclude that the entire line segment is Hurwitz.

Next we apply the Real Convex Direction Lemma to the difference polynomial

$$\begin{aligned}\delta_0(s) :&= \delta_2(s) - \delta_1(s) \\ &= s^4 - 3.1s^3 + 3.54s^2 - 1.744s + 0.312\end{aligned}$$

so that

$$\delta_0(j\omega) = \underbrace{(\omega^4 - 3.54\omega^2 + 0.312)}_{\delta_0^r(\omega)} + j\,\underbrace{(3.1\omega^3 - 1.744\omega)}_{\delta_0^i(\omega)}$$

and the two functions that need to be evaluated are

$$\begin{aligned}\frac{d}{d\omega}\varphi_{\delta_0}(\omega) &= \frac{\delta_0^r(\omega)\left(\frac{d\delta_0^i(\omega)}{d\omega}\right) - \left(\frac{d\delta_0^r(\omega)}{d\omega}\right)\delta_0^i(\omega)}{\left(\delta_0^r(\omega)\right)^2 + \left(\delta_0^i(\omega)\right)^2} \\ &= \frac{(\omega^4 - 3.54\omega^2 + 0.312)(9.3\omega^2 - 1.744) - (4\omega^3 - 7.08\omega)(3.1\omega^3 - 1.744\omega)}{(\omega^4 - 3.54\omega^2 + 0.312)^2 + (3.1\omega^3 - 1.744\omega)^2}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\sin(2\varphi_{\delta_0}(\omega))}{2\omega}\right| &= \left|\frac{\delta_0^r(\omega)\delta_0^i(\omega)}{\omega\left[(\delta_0^r(\omega))^2 + (\delta_0^i(\omega))^2\right]}\right| \\ &= \left|\frac{(\omega^4 - 3.54\omega^2 + 0.312)(3.1\omega^2 - 1.744)}{(\omega^4 - 3.54\omega^2 + 0.312)^2 + (3.1\omega^3 - 1.744\omega)^2}\right|.\end{aligned}$$

These two functions are depicted in Figure 2.14. Since the second function dominates the first for each $\omega$ the plots show that $\delta_0(s)$ is a convex direction. Consequently, the line segment joining the given $\delta_1(s)$ and $\delta_2(s)$ is Hurwitz. Furthermore since $\delta_0(s)$ is a convex direction, we know, in addition, that every line segment of the form $\delta(s) + \lambda\delta_0(s)$ for $\lambda \in [0, 1]$ is Hurwitz for *an arbitrary* Hurwitz polynomial $\delta(s)$ of degree 4 with positive leading coefficient, provided $\delta(s) + \delta_0(s)$ is stable. This additional information is not furnished by the Segment Lemma.

**Example 2.8.** Consider the Hurwitz stability of the segment joining the following two polynomials:

$$\begin{aligned}\delta_1(s) &= 1.4s^4 + 6s^3 + 2.2s^2 + 1.6s + 0.2 \\ \delta_2(s) &= 0.4s^4 + 1.6s^3 + 2s^2 + 1.6s + 0.4.\end{aligned}$$

Since $\delta_1(s)$ and $\delta_2(s)$ are stable, we can apply the Segment Lemma to check the stability of the line segment $[\delta_1(s), \delta_2(s)]$. First we compute the roots of the polynomial equation:

$$\delta_1^e(\omega)\delta_2^o(\omega) - \delta_2^e(\omega)\delta_1^o(\omega) =$$

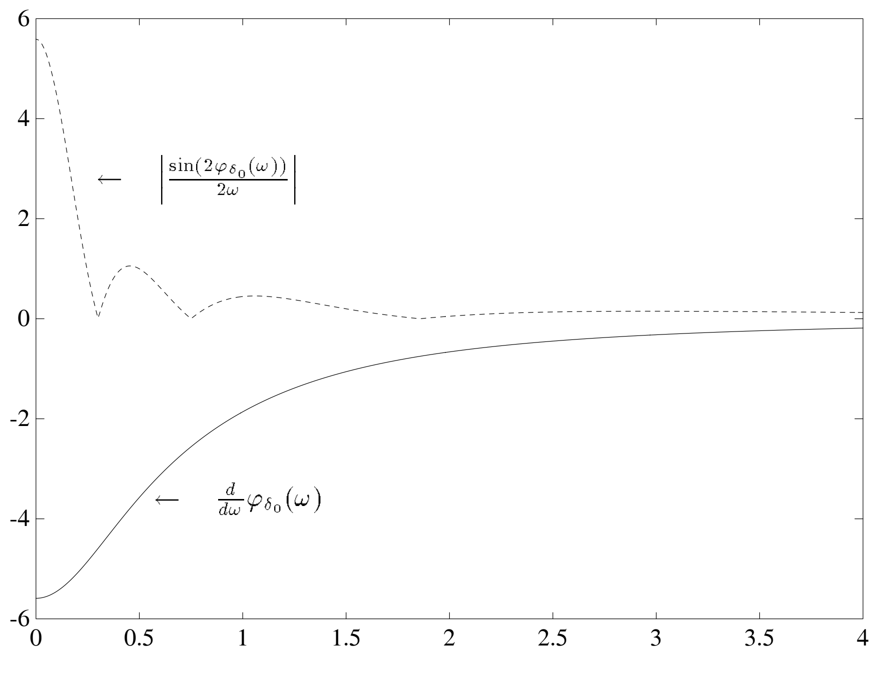



**Figure 2.14.** $\delta_0(s)$ is a convex direction (Example 2.8)

$$(1.4\omega^4 - 2.2\omega^2 + 0.2)(-1.6\omega^2 + 1.6) - (0.4\omega^4 - 2\omega^2 + 0.4)(-6\omega^2 + 1.6) = 0.$$

There is one positive real root $\omega \approx 6.53787$. We proceed to check the conditions 2) and 3) of the Segment Lemma (Lemma 2.4):

$$(1.4\omega^4 - 2.2\omega^2 + 0.2)(0.4\omega^4 - 2\omega^2 + 0.4)|_{\omega\approx 6.53787} > 0$$
$$(-6\omega^2 + 1.6)(-1.6\omega^2 + 1.6)|_{\omega\approx 6.53787} > 0.$$

Thus, we conclude that the segment $[\delta_1(s), \delta_2(s)]$ is stable.

Now let us apply the Real Convex Direction Lemma to the difference polynomial

$$\begin{aligned}\delta_0(s) &= \delta_1(s) - \delta_2(s)\\ &= s^4 + 4.4s^3 + 0.2s^2 - 0.2.\end{aligned}$$

We have

$$\delta_0(j\omega) = \underbrace{(\omega^4 - 0.2\omega^2 - 0.2)}_{\delta_0^r(\omega)} + j\underbrace{(-4.4\omega^3)}_{\delta_0^i(\omega)}.$$

The two functions that need to be evaluated are

$$\frac{d\varphi_{\delta_0}(\omega)}{d\omega} =$$
$$\frac{(\omega^4 - 0.2\omega^2 - 0.2)(-13.2\omega^2) - (4\omega^3 - 0.4\omega)(-4.4\omega^3)}{(\omega^4 - 0.2\omega^2 - 0.2)^2 + (-4.4\omega^3)^2}$$

and

$$\left|\frac{\sin(2\varphi_{\delta_0}(\omega))}{2\omega}\right| = \left|\frac{(\omega^4 - 0.2\omega^2 - 0.2)(-4.4\omega^2)}{(\omega^4 - 0.2\omega^2 - 0.2)^2 + (-4.4\omega^3)^2}\right|.$$

These two functions are depicted in Figure 2.15. Since the second function does not dominate the first at each $\omega$ we conclude that $\delta_0(s)$ is not a convex direction.

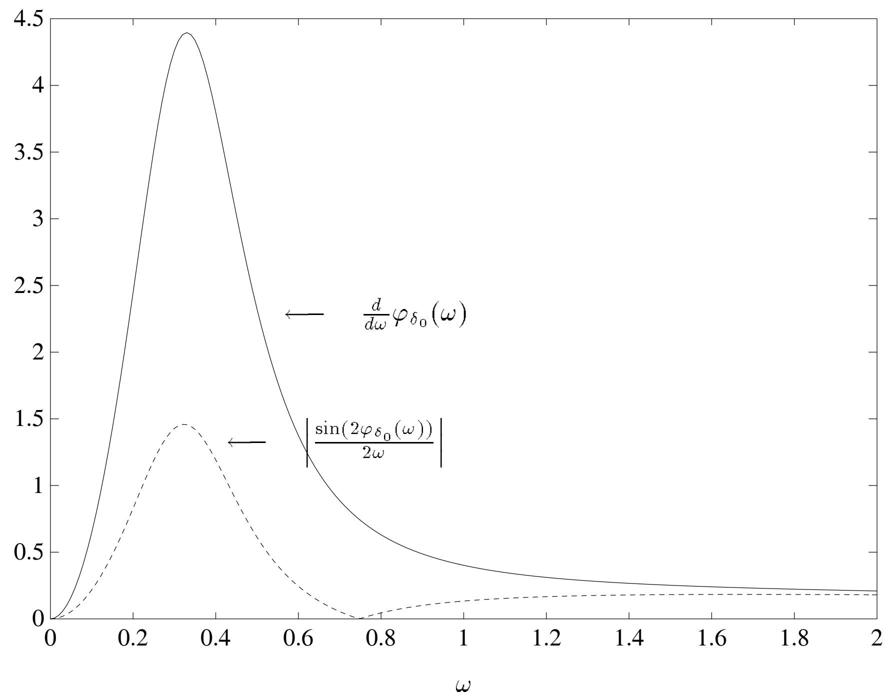



**Figure 2.15.** $\delta_0(s)$ is a nonconvex direction (Example 2.9)

**Remark 2.2.** This example reinforces the fact that the segment joining $\delta_1(s)$ and $\delta_2(s)$ can be stable even though $\delta_0(s) = \delta_1(s) - \delta_2(s)$ is not a convex direction. On the other hand, even though this particular segment is stable, there exists at least one Hurwitz polynomial $\delta_2(s)$ of degree 4 such that the segment $[\delta_2(s), \delta_2(s)+\delta_0(s)]$ is not Hurwitz even though $\delta_2(s)$ and $\delta_2(s) + \delta_0(s)$ are.

## 2.6 THE VERTEX LEMMA

The conditions given by the Convex Direction Lemmas are frequency dependent. It is possible to give frequency-independent conditions on $\delta_0(s)$ under which Hurwitz stability of the vertices implies stability of every polynomial on the segment $[\delta_1(s), \delta_2(s)]$. In this section we first consider various special forms of the difference polynomial $\delta_0(s)$ for which this is possible. In each case we use Lemma 2.13 and Hurwitz stability of the vertices to contradict the hypothesis that the segment has unstable polynomials. We then combine the special cases to obtain the general result. This main result is presented as the Vertex Lemma.

We shall assume throughout this subsection that each polynomial on the segment $[\delta_1(s), \delta_2(s)]$ is of degree $n$. This will be true if and only if $\delta_1(s)$ and $\delta_2(s)$ are of degree $n$ and their leading coefficients are of the same sign. We shall assume this without loss of generality.

We first consider real polynomials of the form

$$\delta_0(s) = s^t(as+b)P(s)$$

where $t$ is a nonnegative integer and $P(s)$ is odd or even. Suppose arbitrarily, that $t$ is even and $P(s) = E(s)$ an even polynomial. Then

$$\delta_0(s) = \underbrace{s^t E(s) b}_{\delta_0^{\mathrm{even}}(s)} + \underbrace{s^{t+1} E(s) a}_{\delta_0^{\mathrm{odd}}(s)}. \tag{2.154}$$

Defining $\underline{\delta}_0(j\omega)$ as before we see that

$$\tan \underline{\delta}_0(\omega) = \frac{a}{b} \tag{2.155}$$

so that

$$\frac{d\varphi_{\underline{\delta}_0}}{d\omega} = 0. \tag{2.156}$$

From Lemma 2.13 (i.e., (2.94)), we see that

$$\lambda_0 \left.\frac{d\varphi_{\underline{\delta}_2}}{d\omega}\right|_{\omega=\omega_0} + (1-\lambda_0)\left.\frac{d\varphi_{\underline{\delta}_1}}{d\omega}\right|_{\omega=\omega_0} = 0 \tag{2.157}$$

and from Lemma 2.10 we see that if $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz, then

$$\frac{d\varphi_{\underline{\delta}_2}}{d\omega} > 0 \tag{2.158}$$

and

$$\frac{d\varphi_{\underline{\delta}_1}}{d\omega} > 0 \tag{2.159}$$

so that (2.157) cannot be satisfied for $\lambda_0 \in [0,1]$. An identical argument works when $t$ is odd. The case when $P(s) = O(s)$ is an odd polynomial can be handled similarly by using (2.95) in Lemma 2.13. The details are left to the reader. Thus we are led to the following result.

**Lemma 2.17** *If $\delta_0(s) = s^t(as+b)P(s)$ where $t \geq 0$ is an integer, $a$ and $b$ are arbitrary real numbers and $P(s)$ is an even or odd polynomial, then stability of the segment $[\delta_1(s), \delta_2(s)]$ is implied by those of the endpoints $\delta_1(s)$, $\delta_2(s)$.*

We can now prove the following general result.

**Lemma 2.18 (Vertex Lemma: Hurwitz Case)**

a) *Let $\delta_1(s)$ and $\delta_2(s)$ be real polynomials of degree $n$ with leading coefficients of the same sign and let*

$$\begin{aligned}\delta_0(s) &= \delta_1(s) - \delta_2(s)\\ &= A(s)s^t(as+b)P(s) \end{aligned}\tag{2.160}$$

*where $A(s)$ is antiHurwitz, $t \geq 0$ is an integer, $a$, $b$ are arbitrary real numbers, and $P(s)$ is even or odd. Then stability of the segment $[\delta_1(s), \delta_2(s)]$ is implied by that of the endpoints $\delta_1(s)$, $\delta_2(s)$.*

b) *When $\delta_0(s)$ is not of the form specified in a), stability of the endpoints is not sufficient to guarantee that of the segment.*

**Proof.**

a) Write

$$A(s) = A^{\text{even}}(s) + A^{\text{odd}}(s) \tag{2.161}$$

and let

$$\bar{A}(s) := A^{\text{even}}(s) - A^{\text{odd}}(s). \tag{2.162}$$

Since $A(s)$ is antiHurwitz, $\bar{A}(s)$ is Hurwitz. Now consider the segment $[\bar{A}(s)\delta_1(s), \bar{A}(s)\delta_2(s)]$ which is Hurwitz if and only if $[\delta_1(s), \delta_2(s)]$ is Hurwitz. But

$$\begin{aligned}\bar{A}(s)\delta_0(s) &= \bar{A}(s)\delta_1(s) - \bar{A}(s)\delta_2(s)\\ &= \underbrace{\left[(A^{\text{even}}(s))^2 - (A^{\text{odd}}(s))^2\right]}_{T(s)} s^t(as+b)P(s). \end{aligned}\tag{2.163}$$

Since $T(s)$ is an even polynomial we may use Lemma 2.17 to conclude that the segment $[\bar{A}(s)\delta_1(s), \bar{A}(s)\delta_2(s)]$ is Hurwitz if and only if $\bar{A}(s)\delta_1(s)$ and $\bar{A}(s)\delta_2(s)$ are. Since $\bar{A}(s)$ is Hurwitz it follows that the segment $[\delta_1(s), \delta_2(s)]$ is Hurwitz if and only if the endpoints $\delta_1(s)$ and $\delta_2(s)$ are.

b) We prove this part by means of the following example. Consider the segment

$$\delta_\lambda(s) = (2+14\lambda)s^4 + (5+14\lambda)s^3 + (6+14\lambda)s^2 + 4s + 3.5. \tag{2.164}$$

Now set $\lambda = 0$ and $\lambda = 1$, then we have

$$\begin{aligned}\delta_\lambda|_{\lambda=0} &= \delta_1(s) = 2s^4 + 5s^3 + 6s^2 + 4s + 3.5\\ \delta_\lambda|_{\lambda=1} &= \delta_2(s) = 16s^4 + 19s^3 + 20s^2 + 4s + 3.5\end{aligned}$$

and consequently,

$$\begin{aligned}\delta_0(s) &= \delta_2(s) - \delta_1(s)\\ &= 14s^4 + 14s^3 + 14s^2\\ &= 14s^2(s^2 + s + 1).\end{aligned}$$

It can be verified that two endpoints $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz. Notice that since $(s^2 + s + 1)$ is Hurwitz with a pair of complex conjugate roots, $\delta_0(s)$ cannot be partitioned into the form of (2.160). Therefore, we conclude that when $\delta_0(s)$ is not of the form specified in (2.160), stability of the endpoints is not sufficient to guarantee that of the segment.

♣

**Remark 2.3.** We remark that the form of $\delta_0(s)$ given in (2.160) is a real convex direction.

**Example 2.9.** Suppose that the transfer function of a plant containing an uncertain parameter is written in the form:

$$P(s) = \frac{P_2}{P_1(s) + \lambda P_0(s)}$$

where the uncertain parameter $\lambda$ varies in $[0, 1]$, and the degree of $P_1(s)$ is greater than those of $P_0(s)$ or $P_2(s)$. Suppose that a unity feedback controller is to be designed so that the plant output follows step and ramp inputs and rejects sinusoidal disturbances of radian frequency $\omega_0$. Let us denote the controller by

$$C(s) = \frac{Q_2(s)}{Q_1(s)}.$$

A possible choice of $Q_1(s)$ which will meet the tracking and disturbance rejection requirements is

$$Q_1(s) = s^2(s^2 + \omega_0^2)(as + b)$$

with $Q_2(s)$ being of degree 5 or less. The stability of the closed loop requires that the segment

$$\delta_\lambda(s) = Q_2(s)P_2(s) + Q_1(s)(P_1(s) + \lambda P_0(s),$$

be Hurwitz stable. The corresponding difference polynomial $\delta_0(s)$ is

$$\delta_0(s) = Q_1(s)P_0(s).$$

With $Q_1(s)$ of the form shown above, it follows that $\delta_0(s)$ is of the form specified in the Vertex Lemma if $P_0(s)$ is anti-Hurwitz or even or odd or product thereof. Thus, in such a case robust stability of the closed loop would be equivalent to the stability of the two vertex polynomials

$$\begin{aligned}\delta_1(s) &= Q_2(s)P_2(s) + Q_1(s)P_1(s)\\ \delta_2(s) &= Q_2(s)P_2(s) + Q_1(s)P_1(s) + Q_1(s)P_0(s).\end{aligned}$$

Let $\omega_0 = 1$, $a = 1$, and $b = 1$ and

$$P_1(s) = s^2 + s + 1, \quad P_0(s) = s(s-1), \quad P_2(s) = s^2 + 2s + 1$$

$$Q_2(s) = s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1.$$

Since $P_0(s) = s(s-1)$ is the product of an odd and an antiHurwitz polynomial, the conditions of the Vertex Lemma are satisfied and robust stability is equivalent to that of the two vertex polynomials

$$\begin{aligned}\delta_1(s) &= 2s^7 + 9s^6 + 24s^5 + 38s^4 + 37s^3 + 22s^2 + 7s + 1\\ \delta_2(s) &= 3s^7 + 9s^6 + 24s^5 + 38s^4 + 36s^3 + 22s^2 + 7s + 1.\end{aligned}$$

Since $\delta_1(s)$ and $\delta_2(s)$ are Hurwitz, the controller

$$C(s) = \frac{Q_2(s)}{Q_1(s)} = \frac{s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1}{s^2(s^2+1)(s+1)}$$

robustly stabilizes the closed loop system and provides robust asymptotic tracking and disturbance rejection.

The Vertex Lemma can easily be extended to the case of Schur stability.

**Lemma 2.19 (Vertex Lemma: Schur Case)**

*a) Let $\delta_1(z)$ and $\delta_2(z)$ be polynomials of degree $n$ with $\delta_1(1)$ and $\delta_2(1)$ nonzero and of the same sign, and with leading coefficients of the same sign. Let*

$$\begin{aligned}\delta_0(z) &= \delta_1(z) - \delta_2(z)\\ &= A(z)(z-1)^{t_1}(z+1)^{t_2}(az+b)P(z)\end{aligned} \tag{2.165}$$

*where $A(z)$ is antiSchur, $t_1$, $t_2 \geq 0$ are integers, $a$, $b$ are arbitrary real numbers, and $P(z)$ is symmetric or antisymmetric. Then Schur stability of the segment $[\delta_1(z), \delta_2(z)]$ is implied by that of the endpoints $\delta_1(z)$, $\delta_2(z)$.*

*b) When $\delta_0(z)$ is not of the form specified in a), Schur stability of the endpoints is not sufficient to guarantee that of the segment.*

**Proof.** The proof is based on applying the bilinear transformation and using the corresponding results for the Hurwitz case. Let $P(z)$ be any polynomial and let

$$\hat{P}(s) := (s-1)^n P\left(\frac{s+1}{s-1}\right).$$

If $P(z)$ is of degree $n$, so is $\hat{P}(s)$ provided $P(1) \neq 0$. Now apply the bilinear transformation to the polynomials $\delta_0(z)$, $\delta_1(z)$ and $\delta_2(z)$ to get $\hat{\delta}_0(s)$, $\hat{\delta}_1(s)$ and $\hat{\delta}_2(s)$, where $\hat{\delta}_0(s) = \hat{\delta}_1(s) - \hat{\delta}_2(s)$. The proof consists of showing that under the assumption that $\delta_0(z)$ is of the form given in (2.165), $\hat{\delta}_0(s)$, $\hat{\delta}_1(s)$, and $\hat{\delta}_2(s)$ satisfy the conditions of the Vertex Lemma for the Hurwitz case. Since $\delta_1(1)$ and $\delta_2(1)$ are of the same sign, $\delta_\lambda(1) = \lambda\delta_1(1) + (1-\lambda)\delta_2(1) \neq 0$ for $\lambda \in [0,1]$. This in turn implies that $\hat{\delta}_\lambda(s)$ is of degree $n$ for all $\lambda \in [0,1]$. A straightforward calculation shows that

$$\hat{\delta}_0(s) = \hat{A}(s)2^{t_1}(2s)^{t_2}(cs+d)\hat{P}(s)$$

which is precisely the form required in the Vertex Lemma for the Hurwitz case. Thus, the segment $\hat{\delta}_\lambda(s)$ cannot have a $j\omega$ root and the segment $\delta_\lambda(z)$ cannot have a root on the unit circle. Therefore, Schur stability of $\delta_1(z)$ and $\delta_2(z)$ guarantees that of the segment. ♣

## 2.7 EXERCISES

**2.1** Consider the standard unity feedback control system given in Figure 2.16

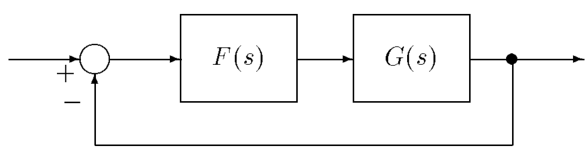



**Figure 2.16.** A unity feedback system

where

$$G(s) := \frac{s+1}{s^2(s+p)}, \qquad F(s) = \frac{(s-1)}{s(s+3)(s^2-2s+1.25)}$$

and the parameter $p$ varies in the interval $[1,5]$.

a) Verify the robust stability of the closed loop system. Is the Vertex Lemma applicable to this problem?

b) Verify your answer by the s-plane root locus (or Routh-Hurwitz criteria).

**2.2** Rework the problem in Exercise 2.1 by transforming via the bilinear transformation to the $z$ plane, and using the Schur version of the Segment or Vertex Lemma. Verify your answer by the z-plane root locus (or Jury's test).

**2.3** The closed loop characteristic polynomial of a missile of mass $M$ flying at constant speed is:

$$\delta(s) = \left(-83,200 + 108,110K - 9,909.6K\frac{1}{M}\right)$$

$$+\left(-3,328 + 10,208.2K + 167.6\frac{1}{M}\right)s$$
$$+\left(-1,547.79K\frac{1}{M} + 1,548 - 877.179K + 6.704\frac{1}{M} - 2.52497K\frac{1}{M}\right)s^2$$
$$+\left(64 - 24.1048K + 0.10475\frac{1}{M}\right)s^3 + s^4.$$

where the nominal value of $M$, $M^0 = 1$. Find the range of $K$ for robust stability if $\frac{1}{M} \in [1,4]$.
**Answer**: $K = [-0.8, 1.2]$.

**2.4** For the feedback system shown in Figure 2.17

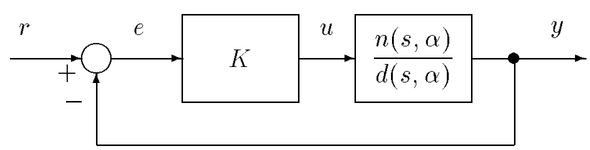



**Figure 2.17.** Feedback control system

where

$$n(s,\alpha) = s^2 + (3-\alpha)s + 1$$
$$d(s,\alpha) = s^3 + (4+\alpha)s^2 + 6s + 4 + \alpha.$$

Partition the $(K,\alpha)$ plane into stable and unstable regions. Show that the stable region is bounded by

$$5 + K(3-\alpha) > 0$$

and

$$K + \alpha + 4 > 0.$$

**2.5** Consider the feedback system shown in Figure 2.18.

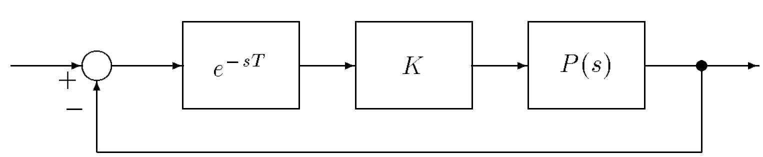



**Figure 2.18.** Feedback control system

Let

$$P(s) = \frac{n(s)}{d(s)} = \frac{6s^3 + 18s^2 + 30s + 25}{s^4 + 6s^3 + 18s^2 + 30s + 25}.$$

Determine the robust stability of the system for $T = 0.1$sec with $0 < K < 1$.
**Hint**: Check that

$$P_0(s) = d(s) \quad \text{and} \quad P_1(s) = d(s) + e^{-sT} n(s)$$

are stable and the plot $\dfrac{P_0(j\omega)}{P_1(j\omega)}$ does not cut the negative real axis.

**2.6** Consider the system given in Figure 2.19

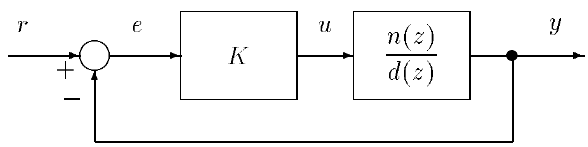



**Figure 2.19.** Feedback control system

and let

$$n(z) = \left(z - \frac{1+j}{4}\right)\left(z - \frac{1-j}{4}\right)\left(z + \frac{1}{2}\right)$$
$$d(z) = \left(z + \frac{3}{4}\right)\left(z - \frac{1}{2}\right)\left(z - \frac{-j-1}{2}\right)\left(z - \frac{j-1}{2}\right).$$

Find the range of stabilizing $K$ using the Schur Segment Lemma.
**Answer**: $K < -1.53$ and $K > -0.59$

**2.7** Show that $\delta_0(s)$ given in (2.160) is a convex direction.

**2.8** Show that the following polynomials are convex directions.

a) $\delta_0(s) = (s - r_1)(s + r_2)(s - r_3)(s + r_4)\cdots(s + (-1)^m r_m)$

where $0 < r_1 < r_2 < r_3 < \cdots < r_m$.

b) $\delta_0(s) = (s + r_1)(s - r_2)(s + r_3)(s - r_4)\cdots(s - (-1)^m r_m)$

where $0 \le r_1 \le r_2 \le r_3 \le \cdots \le r_m$.

**2.9** Is the following polynomial a convex direction?

$$\delta_0(s) = s^4 - 2s^3 - 13s^2 + 14s + 24$$

**2.10** Consider the two Schur polynomials:

$$P_1(z) = z^4 + 2.5z^3 + 2.56z^2 + 1.31z + 0.28$$
$$P_2(z) = z^4 + 0.2z^3 + 0.17z^2 + 0.052z + 0.0136$$

Check the Schur stability of the segment joining these two polynomials by using:

a) Schur Segment Lemma 1

b) Schur Segment Lemma 2

c) Schur Segment Lemma 3

d) Bounded Phase Lemma

**2.11** Consider the feedback control system shown in Figure 2.20

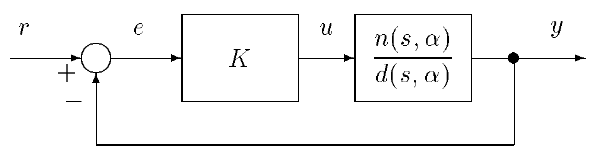



**Figure 2.20.** Feedback control system

where

$$n(s, \alpha) = s + \alpha$$
$$d(s, \alpha) = s^2 + (2\alpha)s^2 + \alpha s - 1$$

and $\alpha \in [2, 3]$. Partition the $K$ axis into robustly stabilizing and nonstabilizing regions.
**Answer**: Stabilizing for $K \in (-4.5, 0.33)$ only.

**2.12** Repeat Exercise 2.11 when there is a time delay of 1sec. in the feedback loop.

**2.13** Consider a feedback system with plant transfer function $G(s)$ and controller transfer function $C(s)$:

$$G(s) = \frac{N(s)}{D(s)} \quad C(s) = K. \tag{2.166}$$

Show that if $N(s)$ is a convex direction there exists at most one segment of stabilizing gains $K$.

**2.14** Prove that $\delta_0(s)$ given in the Vertex Lemma (2.160) is a real convex direction.

**2.15** Carry out the construction of the polynomial $\delta_k(s)$ required in the proof of Lemma 2.12 for the angle $\theta$ lying in the second, third and fourth quadrants.

## 2.8 NOTES AND REFERENCES

The Segment Lemma for the Hurwitz case was derived by Chapellat and Bhattacharyya [57]. An alternative result on segment stability involving the Hurwitz matrix has been given by Bialas [39]. Bose [46] has given analytical tests for the Hurwitz and Schur stability of convex combinations of polynomials. The Convex Direction Lemmas for the real and complex cases and Lemma 2.12 are due to Rantzer [195]. The Schur Segment Lemma 1 (Lemma 2.5) is due to Zeheb [246] and Schur Segment Lemma 2 (Lemma 2.6) and Schur Segment Lemma 3 (Lemma 2.7) are due to Tin [224]. The results leading up to the Vertex Lemma were developed by various researchers: the monotonic phase properties given in Lemmas 2.10 and 2.11 are due to Mansour and Kraus [173]. An alternative proof of Lemma 2.9 is given in Mansour [169]. In Bose [47] monotonicity results for Hurwitz polynomials are derived from the point of view of reactance functions and it is stated that Theorem 2.1 follows from Tellegen's Theorem in network theory. The direct proof of Theorem 2.1 given here as well as the proofs of the auxiliary Lemmas in Section 4 are due to Keel and Bhattacharyya [136]. Lemma 2.17 is due to Hollot and Yang [119] who first proved the vertex property of first order compensators. Mansour and Kraus gave an independent proof of the same lemma [173], and Peterson [189] dealt with the antiHurwitz case. The unified proof of the Vertex Lemma given here, based on Lemma 2.13 was first reported in Bhattacharyya [31] and Bhattacharyya and Keel [32]. The vertex result given in Exercise 2.8 was proved by Kang [129] using the alternating Hurwitz minor conditions. The polynomial used in Exercise 2.9 is taken from Barmish [12]. Vertex results for quasipolynomials have been developed by Kharitonov and Zhabko [147].