# Chapter 1

# STABILITY THEORY VIA THE BOUNDARY CROSSING THEOREM

In this chapter we introduce the Boundary Crossing Theorem for polynomials. Although intuitively obvious, this theorem, used systematically, can lead to many useful and nontrivial results in stability theory. In fact it plays a fundamental role in most of the results on robust stability. We illustrate its usefulness here by using it to give extremely simple derivations of the Hermite-Biehler Theorem and of the Routh and Jury tests.

## 1.1 INTRODUCTION

This chapter develops some results on stability theory for a given fixed polynomial. This theory has been extensively studied and has a vast body of literature. Instead of giving a complete account of all existing results we concentrate on a few fundamental results which will be used extensively in the remainder of this book to deal with stability problems related to families of polynomials. These results are presented in a unified and elementary fashion and the approach consists of a systematic use of the following fact: Given a parametrized family of polynomials and any continuous path in the parameter space leading from a stable to an unstable polynomial, then, the first unstable point that is encountered in traversing this path corresponds to a polynomial whose unstable roots lie on the boundary (and not in the interior) of the instability region in the complex plane.

The above result, called the Boundary Crossing Theorem, is established rigorously in the next section. The proof follows simply from the continuity of the roots of a polynomial with respect to its coefficients. The consequences of this result, however, are quite far reaching, and this is demonstrated in the subsequent sections by using it to give simple derivations of the classical Hermite-Biehler Theorem, the Routh test for left half plane stability and the Jury test for unit disc stability.

The purpose of this chapter is to give a simple exposition of these fundamental results which makes them particularly easy to learn and understand. Moreover the Boundary Crossing Theorem will play an important role in later chapters dealing with Kharitonov's Theorem and its generalization, the stability of families of polynomials, and in the calculation of stability margins for control systems.

Many results of stability theory extend far beyond the realm of polynomials. The Hermite-Biehler Theorem, in particular, extends to a vast class of entire functions. In the last section of this chapter some of these extensions are briefly overviewed, with an emphasis on those results which are more directly related to problems in control theory.

## 1.2 THE BOUNDARY CROSSING THEOREM

We begin with the well known Principle of the Argument of complex variable theory. Let $\mathcal{C}$ be a simple closed contour in the complex plane and $w = f(z)$ a function of the complex variable $z$, which is analytic on $\mathcal{C}$. Let $Z$ and $P$ denote the number of zeros and poles, respectively, of $f(z)$ contained in $\mathcal{C}$. Let $\Delta_C \arg[f(z)]$ denote the net change of argument (angle) of $f(z)$ as $z$ transverses the contour $\mathcal{C}$.

**Theorem 1.1 (Principle of the Argument)**

$$\Delta_C \arg[f(z)] = 2\pi(Z - P) \tag{1.1}$$

An important consequence of this result is the well known theorem of Rouché.

**Theorem 1.2 (Rouché's Theorem)**
*Let $f(z)$ and $g(z)$ be two functions which are analytic inside and on a simple closed contour $\mathcal{C}$ in the complex plane. If*

$$|g(z)| < |f(z)| \tag{1.2}$$

*for any $z$ on $\mathcal{C}$, then $f(z)$ and $f(z) + g(z)$ have the same number (multiplicities included) of zeros inside $\mathcal{C}$.*

**Proof.** Since $f(z)$ cannot vanish on $\mathcal{C}$, because of (1.2), we have

$$\begin{aligned} \Delta_C \arg[f(z) + g(z)] &= \Delta_C \arg\left\{ f(z) \left[ 1 + \frac{g(z)}{f(z)} \right] \right\} \\ &= \Delta_C \arg[f(z)] + \Delta_C \arg\left[ 1 + \frac{g(z)}{f(z)} \right]. \end{aligned} \tag{1.3}$$

Moreover, since

$$\left| \frac{g(z)}{f(z)} \right| < 1$$

for all $z \in \mathcal{C}$, the variable point

$$w = 1 + \frac{g(z)}{f(z)}$$

stays in the disc $|w-1|<1$ as $z$ describes the curve $\mathcal{C}$. Therefore $w$ cannot wind around the origin, which means that

$$\Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right]=0. \tag{1.4}$$

Combining (1.3) and (1.4), we find that

$$\Delta_C \arg\left[f(z)+g(z)\right]=\Delta_C \arg[f(z)].$$

Since $f(z)$ and $g(z)$ are analytic in and on $\mathcal{C}$ the theorem now follows as an immediate consequence of the argument principle. ♣

Note that the condition $|g(z)|<|f(z)|$ on $\mathcal{C}$ implies that neither $f(z)$ nor $f(z)+g(z)$ may have a zero on $\mathcal{C}$. Theorem 1.2 is just one formulation of Rouché's Theorem but it is sufficient for our purposes. The next theorem is a simple application of Rouché's Theorem. It is however most useful since it applies to polynomials.

**Theorem 1.3** *Let*

$$P(s)=p_0+p_1s+\cdots+p_ns^n=\prod_{j=1}^{m}(s-s_j)^{t_j},\ p_n\neq 0, \tag{1.5}$$

$$Q(s)=(p_0+\epsilon_0)+(p_1+\epsilon_1)s+\cdots+(p_n+\epsilon_n)s^n, \tag{1.6}$$

*and consider a circle $\mathcal{C}_k$, of radius $r_k$, centered at $s_k$ which is a root of $P(s)$ of multiplicity $t_k$. Let $r_k$ be fixed in such a way that,*

$$0<r_k<\min|s_k-s_j|, \quad \text{for} \quad j=1,2,\cdots,k-1,k+1,\cdots,m. \tag{1.7}$$

*Then, there exists a positive number $\epsilon$, such that $|\epsilon_i|\le\epsilon$, for $i=0,1,\cdots,n$, implies that $Q(s)$ has precisely $t_k$ zeros inside the circle $\mathcal{C}_k$.*

**Proof.** $P(s)$ is non-zero and continuous on the compact set $\mathcal{C}_k$ and therefore it is possible to find $\delta_k>0$ such that

$$|P(s)|\ge\delta_k>0, \quad \text{for all } s\in\mathcal{C}_k. \tag{1.8}$$

On the other hand, consider the polynomial $R(s)$, defined by

$$R(s)=\epsilon_0+\epsilon_1s+\cdots+\epsilon_ns^n. \tag{1.9}$$

If $s$ belongs to the circle $\mathcal{C}_k$, then

$$\begin{aligned}|R(s)| &\le \sum_{j=0}^{n}|\epsilon_j||s^j|\le\sum_{j=0}^{n}|\epsilon_j|\left(|s-s_k|+|s_k|\right)^j\\ &\le \epsilon\underbrace{\sum_{j=0}^{n}\left(r_k+|s_k|\right)^j}_{M_k}.\end{aligned}$$

Thus if $\epsilon$ is chosen so that $\epsilon < \frac{\delta_k}{M_k}$, it is concluded that

$$|R(s)| < |P(s)| \quad \text{for all } s \text{ on } \mathcal{C}_k, \tag{1.10}$$

so that by Rouché's Theorem, $P(s)$ and $Q(s) = P(s) + R(s)$ have the same number of zeros inside $\mathcal{C}_k$. Since the choice of $r_k$ ensures that $P(s)$ has just one zero of multiplicity $t_k$ at $s_k$, we see that $Q(s)$ has precisely $t_k$ zeros in $\mathcal{C}_k$. ♣

**Corollary 1.1** *Fix $m$ circles $\mathcal{C}_1, \cdots, \mathcal{C}_m$, that are pairwise disjoint and centered at $s_1, s_2, \cdots, s_m$ respectively. By repeatedly applying the previous theorem, it is always possible to find an $\epsilon > 0$ such that for any set of numbers $\{\epsilon_0, \cdots, \epsilon_n\}$ satisfying $|\epsilon_i| \le \epsilon$, for $i = 0, 1, \cdots, n$, $Q(s)$ has precisely $t_j$ zeros inside each of the circles $\mathcal{C}_j$.*

Note, that in this case, $Q(s)$ always has $t_1 + t_2 + \cdots + t_m = n$ zeros and must remain therefore of degree $n$, so that necessarily $\epsilon < |p_n|$. The above theorem and corollary lead to our main result, the Boundary Crossing Theorem.

Let us consider the complex plane $C$ and let $\mathcal{S} \subset C$ be any given open set. We know that $\mathcal{S}$, its boundary $\partial\mathcal{S}$ together with the interior $\mathcal{U}^o$ of the closed set $\mathcal{U} = C - \mathcal{S}$ form a partition of the complex plane, that is

$$\mathcal{S} \cup \partial\mathcal{S} \cup \mathcal{U}^o = C, \qquad \mathcal{S} \cap \mathcal{U}^o = \mathcal{S} \cap \partial\mathcal{S} = \partial\mathcal{S} \cap \mathcal{U}^o = \emptyset. \tag{1.11}$$

Assume moreover that each one of these three sets is non-empty. These assumptions are very general. In stability theory one might choose for $\mathcal{S}$ the open left half plane $\mathcal{C}^-$ (for continuous-time systems) or the open unit disc $\mathcal{D}$ (for discrete-time systems) or suitable subsets of these, as illustrated in Figure 1.1.

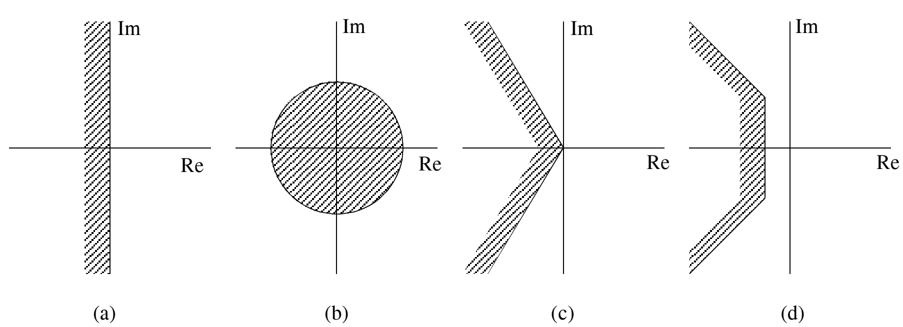


**Figure 1.1.** Some typical stability regions

Consider a family of polynomials $P(\lambda, s)$ satisfying the following assumptions.

**Assumption 1.1.** $P(\lambda, s)$ is a family of polynomials of

1) fixed degree $n$, (invariant degree),

2) continuous with respect to $\lambda$ on a fixed interval $I = [a, b]$.

In other words, a typical element of $P(\lambda, s)$ can be written as

$$P(\lambda, s) = p_0(\lambda) + p_1(\lambda)s + \cdots + p_n(\lambda)s^n, \tag{1.12}$$

where $p_0(\lambda), p_1(\lambda), \cdots, p_n(\lambda)$ are continuous functions of $\lambda$ on $I$ and where $p_n(\lambda) \neq 0$ for all $\lambda \in I$. From the results of Theorem 1.3 and its corollary, it is immediate that in general, for any open set $\mathcal{O}$, the set of polynomials of degree $n$ that have all their roots in $\mathcal{O}$ is itself open. In the case above, if for some $t \in I$, $P(t, s)$ has all its roots in $\mathcal{S}$, then it is always possible to find a positive real number $\alpha$ such that

$$\text{for all } \ t' \in (t - \alpha, t + \alpha) \cap I, P(t', s) \text{ also has all its roots in } \mathcal{S}. \tag{1.13}$$

This leads to the following fundamental result.

**Theorem 1.4 (Boundary Crossing Theorem)**
*Under the Assumptions 1.1, suppose that $P(a, s)$ has all its roots in $\mathcal{S}$ whereas $P(b, s)$ has at least one root in $\mathcal{U}$. Then, there exists at least one $\rho$ in $(a, b]$ such that:*

*a) $P(\rho, s)$ has all its roots in $\mathcal{S} \cup \partial\mathcal{S}$, and*

*b) $P(\rho, s)$ has at least one root in $\partial\mathcal{S}$.*

**Proof.** To prove this result, let us introduce the set $E$ of all real numbers $t$ belonging to $(a, b]$ and satisfying the following property:

$$\mathcal{P}: \qquad \text{for all } \ t' \in (a, t), \quad P(t', s) \text{ has all its roots in } \mathcal{S}. \tag{1.14}$$

By assumption, we know that $P(a, s)$ itself has all its roots in $\mathcal{S}$, and therefore as mentioned above, it is possible to find $\alpha > 0$ such that

$$\text{for all } \ t' \in [a, a + \alpha) \cap I, \quad P(t', s) \text{ also has all its roots in } \mathcal{S}. \tag{1.15}$$

From this it is easy to conclude that $E$ is not empty since, for example, $a + \frac{\alpha}{2}$ belongs to $E$.

Moreover, from the definition of $E$ the following property is obvious:

$$t_2 \in E, \text{ and } a < t_1 < t_2, \text{ implies that } t_1 \text{ itself belongs to } E. \tag{1.16}$$

Given this, it is easy to see that $E$ is an interval and if

$$\rho := \sup_{t \in E} t \tag{1.17}$$

then it is concluded that $E = (a, \rho]$.

A) On the one hand it is impossible that $P(\rho, s)$ has all its roots in $\mathcal{S}$. If this were the case then necessarily $\rho < b$, and it would be possible to find an $\alpha > 0$ such that $\rho + \alpha < b$ and

$$\text{for all } t' \in (\rho - \alpha, \rho + \alpha) \cap I, \quad P(t', s) \text{ also has all its roots in } \mathcal{S}. \tag{1.18}$$

As a result, $\rho + \frac{\alpha}{2}$ would belong to $E$ and this would contradict the definition of $\rho$ in (1.17).

B) On the other hand, it is also impossible that $P(\rho, s)$ has even one root in the interior of $\mathcal{U}$, because a straightforward application of Theorem 1.3 would grant the possibility of finding an $\alpha > 0$ such that

$$\text{for all } t' \in (\rho - \alpha, \rho + \alpha) \cap I, \quad P(t', s) \text{ has at least one root in } \mathcal{U}^0, \tag{1.19}$$

and this would contradict the fact that $\rho - \epsilon$ belongs to $E$ for $\epsilon$ small enough.

From A) and B) it is thus concluded that $P(\rho, s)$ has all its roots in $\mathcal{S} \cup \partial\mathcal{S}$, and at least one root in $\partial\mathcal{S}$. ♣

The above result is in fact very intuitive and just states that in going from one open set to another open set disjoint from the first, the root set of a continuous family of polynomials $P(\lambda, s)$ of fixed degree must intersect at some intermediate stage the boundary of the first open set. If $P(\lambda, s)$ loses degree over the interval $[a, b]$, that is if $p_n(\lambda)$ in (1.12) vanishes for some values of $\lambda$, then the Boundary Crossing Theorem does not hold.

**Example 1.1.** Consider the Hurwitz stability of the polynomial

$$a_1 s + a_0 \quad \text{where} \quad \mathbf{p} := [a_0 \ \ a_1].$$

Referring to Figure 1.2, we see that the polynomial is Hurwitz stable for $\mathbf{p} = \mathbf{p}_0$. Now let the parameters travel along the path $C_1$ and reach the unstable point $\mathbf{p}_1$. Clearly no polynomial on this path has a $j\omega$ root for finite $\omega$ and thus boundary crossing *does not* occur. However, observe that the assumption of constant degree does not hold on this path because the point of intersection between the path $C_1$ and the $a_0$ axis corresponds to a polynomial where loss of degree occurs. On the other hand, if the parameters travel along the path $C_2$ and reach the unstable point $\mathbf{p}_2$, there is no loss of degree along the path $C_2$ and indeed a polynomial on this path has $s = 0$ as a root at $a_0 = 0$ and thus boundary crossing does occur. We illustrate this point in Figure 1.3(a). Along the path $C_2$, where no loss of degree occurs, the root passes through the stability boundary ($j\omega$ axis). However, on the path $C_1$ the polynomial passes from stable to unstable without its root passing through the stability boundary.

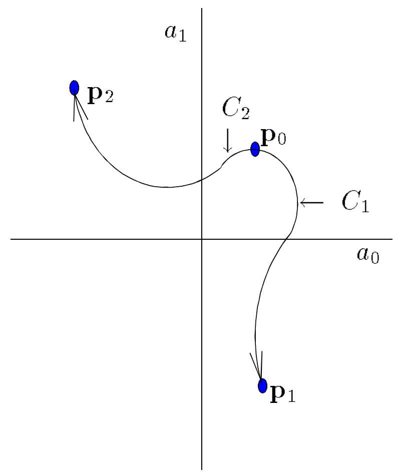


**Figure 1.2.** Degree loss on $C_1$, no loss on $C_2$ (Example 1.1)

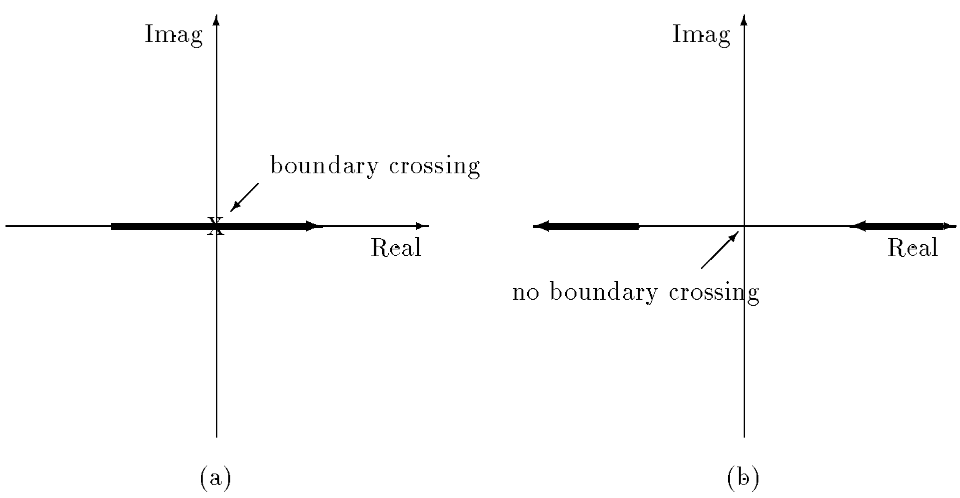


**Figure 1.3.** (a) Root locus corresponding to the path $C_2$ (b) Root locus corresponding to the path $C_1$ (Example 1.1)

The above example shows that the invariant degree assumption is important. Of course we can eliminate the assumption regarding invariant degree and modify the statement of the Boundary Crossing Theorem to require that any path connecting $P_a(s)$ and $P_b(s)$ contains a polynomial which has a root on the boundary or which drops in degree. If degree dropping does occur, it is always possible to apply the result on subintervals over which $p_n(\lambda)$ has a constant sign. In other words if the family of polynomials $P(\lambda, s)$ does not have a constant degree then of course Theorem 1.4 cannot be directly applied but that does not complicate the analysis terribly and similar results can be derived.

The following result gives an example of a situation where the assumption on the degree can be relaxed. As usual let $\mathcal{S}$ be the stability region of interest.

**Theorem 1.5** *Let $\{P_n(s)\}$ be a sequence of stable polynomials of bounded degree and assume that this sequence converges to a polynomial $Q(s)$. Then the roots of $Q(s)$ are contained in $\mathcal{S} \cup \partial\mathcal{S}$.*

In words the above theorem says that the limit of a sequence of stable polynomials of bounded degree can only have unstable roots which are on the boundary of the stability region.

**Proof.** By assumption, there exists an integer $N$ such that degree$[P_n] \leq N$ for all $n \geq 0$. Therefore we can write for all $n$,

$$P_n(s) = p_{0,n} + p_{1,n}s + \cdots + p_{N,n}s^N. \tag{1.20}$$

Since the sequence $\{P_n(s)\}$ converges to $Q(s)$ then $Q(s)$ itself has degree less than or equal to $N$ so that we can also write,

$$Q(s) = q_0 + q_1 s + \cdots + q_N s^N. \tag{1.21}$$

Moreover

$$\lim_{n \longrightarrow +\infty} p_{k,n} = q_k, \quad \text{for } k = 0, 1, \cdots, N. \tag{1.22}$$

Now, suppose that $Q(s)$ has a root $s^*$ which belongs to $\mathcal{U}^o$. We show that this leads to a contradiction. Since $\mathcal{U}^o$ is open, one can find a positive number $r$ such that the disc $\mathcal{C}$ centered at $s^*$ and of radius $r$ is included in $\mathcal{U}^o$. By Theorem 1.3, there exists a positive number $\epsilon$, such that for $|\epsilon_i| \leq \epsilon$, for $i = 0, 1, \cdots, N$, the polynomial

$$(q_0 + \epsilon_0) + (q_1 + \epsilon_1)s + \cdots + (q_N + \epsilon_N)s^N \tag{1.23}$$

has at least one root in $\mathcal{C} \subset \mathcal{U}^o$. Now, according to (1.22) it is possible to find an integer $n_0$ such that

$$n \geq n_0 \implies |p_{k,n} - q_k| < \epsilon, \quad \text{for } k = 0, 1, \cdots, N. \tag{1.24}$$

But then (1.24) implies that for $n \geq n_0$,

$$(q_0 + p_{0,n} - q_0) + (q_1 + p_{1,n} - q_1)s + \cdots + (q_N + p_{N,n} - q_N)s^N = P_n(s) \tag{1.25}$$

has at least one root in $\mathcal{C} \subset \mathcal{U}^o$, and this contradicts the fact that $P_n(s)$ is stable for all $n$. ♣

### 1.2.1 Zero Exclusion Principle

The Boundary Crossing Theorem can be applied to a family of polynomials to detect the presence of unstable polynomials in the family. Suppose that $\delta(s,\mathbf{p})$ denotes a polynomial whose coefficients depend continuously on the parameter vector $\mathbf{p} \in \mathbb{R}^l$ which varies in a set $\Omega \in \mathbb{R}^l$ and thus generates the family of polynomials

$$\mathbf{\Delta}(s) := \{\delta(s,\mathbf{p}) : \mathbf{p} \in \Omega\}. \tag{1.26}$$

We are given a stability region $\mathcal{S}$ and would like to determine if the family $\mathbf{\Delta}(s)$ contains unstable polynomials. Let us assume that there is at least one stable polynomial $\delta(s,\mathbf{p}_a)$ in the family and every polynomial in the family has the same degree. Then if $\delta(s,\mathbf{p}_b)$ is an unstable polynomial, it follows from the Boundary Crossing Theorem that on any continuous path connecting $\mathbf{p}_a$ to $\mathbf{p}_b$ there must exist a point $\mathbf{p}_c$ such that the polynomial $\delta(s,\mathbf{p}_c)$ contains roots on the stability boundary $\partial\mathcal{S}$. If such a path can be constructed entirely inside $\Omega$, that is, if $\Omega$ is pathwise connected, then the point $\mathbf{p}_c$ lies in $\Omega$. In this case the presence of unstable polynomials in the family is equivalent to the presence of polynomials in the family with boundary roots. If $s^*$ is a root of a polynomial in the family it follows that $\delta(s^*,\mathbf{p}) = 0$ for some $\mathbf{p} \in \Omega$ and this implies that $0 \in \mathbf{\Delta}(s^*)$. Therefore the presence of unstable elements in $\mathbf{\Delta}(s)$ can be detected by generating the complex plane image set $\mathbf{\Delta}(s^*)$ of the family at $s^* \in \partial\mathcal{S}$, sweeping $s^*$ along the stability boundary $\partial\mathcal{S}$, and checking if the zero exclusion condition $0 \notin \mathbf{\Delta}(s^*)$ is violated for some $s^* \in \partial\mathcal{S}$. This is stated formally as an alternative version of the Boundary Crossing Theorem.

**Theorem 1.6 (Zero Exclusion Principle)**
*Assume that the family of polynomials (1.26) is of constant degree, contains at least one stable polynomial, and $\Omega$ is pathwise connected. Then the entire family is stable if and only if*

$$0 \notin \mathbf{\Delta}(s^*), \quad \text{for all } s^* \in \partial\mathcal{S}.$$

The Zero Exclusion Principle can be used to derive both theoretical and computational solutions to many robust stability problems. It is systematically exploited in Chapters 2-12 to derive various results on robust parametric stability.

In the rest of this chapter however we restrict attention to the problem of stability determination of a *fixed* polynomial and demonstrate the power of the Boundary Crossing Theorem in tackling some classical stability problems.

## 1.3 THE HERMITE-BIEHLER THEOREM

We first present the Hermite-Biehler Theorem, sometimes referred to as the Interlacing Theorem. For the sake of simplicity we restrict ourselves to the case of polynomials with real coefficients. The corresponding result for complex polynomials will be stated separately. We deal with the Hurwitz case first and then the Schur case.

### 1.3.1 Hurwitz Stability

Consider a polynomial of degree $n$,

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n. \tag{1.27}$$

$P(s)$ is said to be a *Hurwitz* polynomial if and only if all its roots lie in the open left half of the complex plane. We have the two following properties.

**Property 1.1.** If $P(s)$ is a real Hurwitz polynomial then all its coefficients are non zero and have the same sign, either all positive or all negative.

**Proof.** Follows from the fact that $P(s)$ can be factored into a product of first and second degree real Hurwitz polynomials for which the property obviously holds. ♣

**Property 1.2.** If $P(s)$ is a Hurwitz polynomial of degree $n$, then $\arg[P(j\omega)]$, also called the phase of $P(j\omega)$, is a continuous and strictly increasing function of $\omega$ on $(-\infty, +\infty)$. Moreover the net increase in phase from $-\infty$ to $+\infty$ is

$$\arg[P(+j\infty)] - \arg[P(-j\infty)] = n\pi. \tag{1.28}$$

**Proof.** If $P(s)$ is Hurwitz then we can write

$$P(s) = p_n \prod_{i=1}^{n} (s - s_i), \text{ with } s_i = a_i + jb_i, \text{ and } a_i < 0. \tag{1.29}$$

Then we have,

$$\begin{aligned} \arg[P(j\omega)] &= \arg[p_n] + \sum_{i=1}^{n} \arg[j\omega - a_i - jb_i] \\ &= \arg[p_n] + \sum_{i=1}^{n} \arctan\left[\frac{\omega - b_i}{-a_i}\right] \end{aligned} \tag{1.30}$$

and thus $\arg[P(j\omega)]$ is a sum of a constant plus $n$ continuous, strictly increasing functions. Moreover each of these $n$ functions has a net increase of $\pi$ in going from $\omega = -\infty$ to $\omega = +\infty$, as shown in Figure 1.4. ♣

The even and odd parts of a real polynomial $P(s)$ are defined as:

$$\begin{aligned} P^{\text{even}}(s) &:= p_0 + p_2 s^2 + p_4 s^4 + \cdots \\ P^{\text{odd}}(s) &:= p_1 s + p_3 s^3 + p_5 s^5 + \cdots \quad . \end{aligned} \tag{1.31}$$

Define

$$\begin{aligned} P^e(\omega) &:= P^{\text{even}}(j\omega) = p_0 - p_2\omega^2 + p_4\omega^4 - \cdots \\ P^o(\omega) &:= \frac{P^{\text{odd}}(j\omega)}{j\omega} = p_1 - p_3\omega^2 + p_5\omega^4 - \cdots \quad . \end{aligned} \tag{1.32}$$

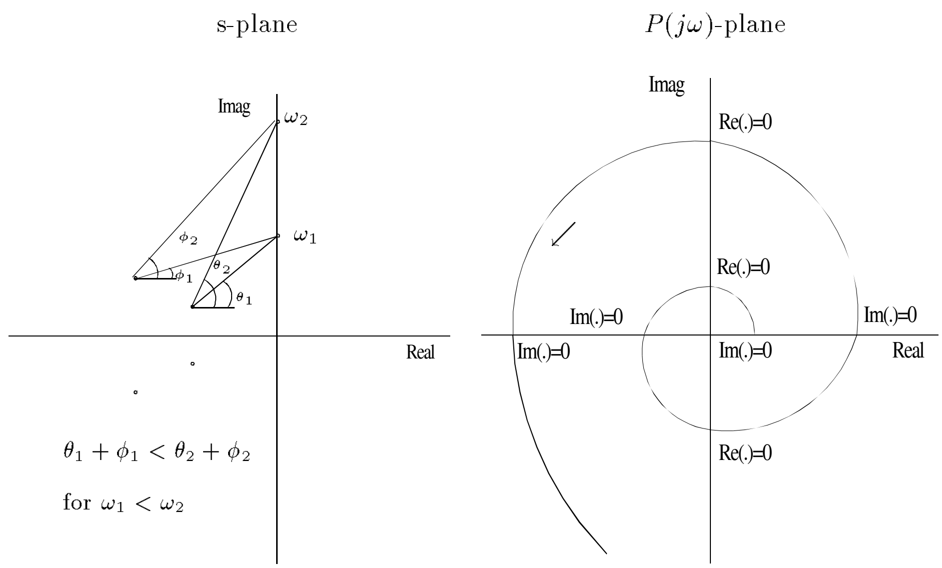


**Figure 1.4.** Monotonic phase increase property for Hurwitz polynomials

$P^e(\omega)$ and $P^o(\omega)$ are both polynomials in $\omega^2$ and as an immediate consequence their root sets will always be symmetric with respect to the origin of the complex plane.

Suppose now that the degree of the polynomial $P(s)$ is even, that is $n = 2m$, $m > 0$. In that case we have

$$\begin{aligned} P^e(\omega) &= p_0 - p_2\omega^2 + p_4\omega^4 - \cdots + (-1)^m p_{2m}\omega^{2m} \\ P^o(\omega) &= p_1 - p_3\omega^2 + p_5\omega^4 - \cdots + (-1)^{m-1} p_{2m-1}\omega^{2m-2}. \end{aligned} \tag{1.33}$$

**Definition 1.1.** A real polynomial $P(s)$ satisfies the interlacing property if

a) $p_{2m}$ and $p_{2m-1}$ have the same sign.

b) All the roots of $P^e(\omega)$ and $P^o(\omega)$ are real and distinct and the $m$ positive roots of $P^e(\omega)$ together with the $m-1$ positive roots of $P^o(\omega)$ interlace in the following manner:

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \cdots < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m}. \tag{1.34}$$

If, on the contrary, the degree of $P(s)$ is odd, then $n = 2m+1$, $m \geq 0$, and

$$\begin{aligned} P^e(\omega) &= p_0 - p_2\omega^2 + p_4\omega^4 - \cdots + (-1)p_{2m}^m\omega^{2m} \\ P^o(\omega) &= p_1 - p_3\omega^2 + p_5\omega^4 - \cdots + (-1)^m p_{2m+1}\omega^{2m} \end{aligned} \tag{1.35}$$

and the definition of the interlacing property, for this case, is then naturally modified to

a) $p_{2m+1}$ and $p_{2m}$ have the same sign.

b) All the roots of $P^e(\omega)$ and $P^o(\omega)$ are real and the $m$ positive roots of $P^e(\omega)$ together with the $m$ positive roots of $P^o(\omega)$ interlace in the following manner:

$$0 < \omega_{e,1} < \omega_{o,1} < \cdots < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m} < \omega_{o,m}. \tag{1.36}$$

An alternative description of the interlacing property is as follows: $P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$ satisfies the interlacing property if and only if

a) the leading coefficients of $P^{\text{even}}(s)$ and $P^{\text{odd}}(s)$ are of the same sign, and

b) all the zeroes of $P^{\text{even}}(s) = 0$ and of $P^{\text{odd}}(s) = 0$ are distinct, lie on the imaginary axis and alternate along it.

We can now enunciate and prove the following theorem.

**Theorem 1.7 (Interlacing or Hermite-Biehler Theorem)**
*A real polynomial $P(s)$ is Hurwitz if and only if it satisfies the interlacing property.*

**Proof.** To prove the necessity of the interlacing property consider a real Hurwitz polynomial of degree $n$,

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n.$$

Since $P(s)$ is Hurwitz it follows from Property 1.1 that all the coefficients $p_i$ have the same sign, thus part a) of the interlacing property is already proven and one can assume without loss of generality that all the coefficients are positive. To prove part b) it is assumed arbitrarily that $P(s)$ is of even degree so that $n = 2m$. Now, we also know from Property 1.2 that the phase of $P(j\omega)$ strictly increases from $-n\pi/2$ to $n\pi/2$ as $\omega$ runs from $-\infty$ to $+\infty$. Due to the fact that the roots of $P(s)$ are symmetric with respect to the real axis it is also true that $\arg(P(j\omega))$ increases from 0 to $+n\pi/2 = m\pi$ as $\omega$ goes from 0 to $+\infty$. Hence as $\omega$ goes from 0 to $+\infty$, $P(j\omega)$ starts on the positive real axis ($P(0) = p_0 > 0$), circles strictly counterclockwise around the origin $m\pi$ radians before going to infinity, and never passes through the origin since $P(j\omega) \neq 0$ for all $\omega$. As a result it is very easy to see that the plot of $P(j\omega)$ has to cut the imaginary axis $m$ times so that the *real* part of $P(j\omega)$ becomes zero $m$ times as $\omega$ increases, at the positive values

$$\omega_{\mathcal{R},1}, \quad \omega_{\mathcal{R},2}, \quad \cdots, \quad \omega_{\mathcal{R},m}. \tag{1.37}$$

Similarly the plot of $P(j\omega)$ starts on the positive real axis and cuts the real axis another $m-1$ times as $\omega$ increases so that the imaginary part of $P(j\omega)$ also becomes zero $m$ times (including $\omega = 0$) at

$$0, \quad \omega_{\mathcal{I},1}, \quad \omega_{\mathcal{I},2}, \quad \cdots, \quad \omega_{\mathcal{I},m-1} \tag{1.38}$$

before growing to infinity as $\omega$ goes to infinity. Moreover since $P(j\omega)$ *circles* around the origin we obviously have

$$0 < \omega_{\mathcal{R},1} < \omega_{\mathcal{I},1} < \omega_{\mathcal{R},2} < \omega_{\mathcal{I},2} < \cdots < \omega_{\mathcal{R},m-1} < \omega_{\mathcal{I},m-1} < \omega_{\mathcal{R},m}. \tag{1.39}$$

Now the proof of necessity is completed by simply noticing that the real part of $P(j\omega)$ is nothing but $P^e(\omega)$, and the imaginary part of $P(j\omega)$ is $\omega P^o(j\omega)$.

For the converse assume that $P(s)$ satisfies the interlacing property and suppose for example that $P(s)$ is of degree $n = 2m$ and that $p_{2m}$, $p_{2m-1}$ are both positive. Consider the roots of $P^e(\omega)$ and $P^o(\omega)$,

$$0 < \omega^p_{e,1} < \omega^p_{o,1} < \cdots < \omega^p_{e,m-1} < \omega^p_{o,m-1} < \omega^p_{e,m}. \tag{1.40}$$

From this, $P^e(\omega)$ and $P^o(\omega)$ can be written as

$$P^e(\omega) = p_{2m} \prod_{i=1}^{m} (\omega^2 - {\omega^p_{e,i}}^2)$$
$$P^o(\omega) = p_{2m-1} \prod_{i=1}^{m-1} (\omega^2 - {\omega^p_{o,i}}^2).$$

Now, consider a polynomial $Q(s)$ that is known to be stable, of the same degree $2m$ and with all its coefficients positive. For example, take $Q(s) = (s+1)^{2m}$. In any event, write

$$Q(s) = q_0 + q_1 s + q_2 s^2 + \cdots + q_{2m} s^{2m}.$$

Since $Q(s)$ is stable, it follows from the first part of the theorem that $Q(s)$ satisfies the interlacing property, so that $Q^e(\omega)$ has $m$ positive roots $\omega^q_{e,1}, \cdots, \omega^q_{e,m}$ and $Q^o(\omega)$ has $m-1$ positive roots $\omega^q_{o,1}, \cdots, \omega^q_{o,m-1}$, and,

$$0 < \omega^q_{e,1} < \omega^q_{o,1} < \cdots < \omega^q_{e,m-1} < \omega^q_{o,m-1} < \omega^q_{e,m}. \tag{1.41}$$

Therefore we can also write:

$$Q^e(\omega) = q_{2m} \prod_{i=1}^{m} (\omega^2 - {\omega^q_{e,i}}^2)$$
$$Q^o(\omega) = q_{2m-1} \prod_{i=1}^{m-1} (\omega^2 - {\omega^q_{o,i}}^2).$$

Consider now the polynomial $P_\lambda(s) := P_\lambda^{even}(s) + sP_\lambda^{odd}(s)$ defined by

$$P^e_\lambda(\omega) := \big((1-\lambda)q_{2m} + \lambda p_{2m}\big) \prod_{i=1}^{m} \left(\omega^2 - \left[(1-\lambda)(\omega^q_{e,i})^2 + \lambda(\omega^p_{e,i})^2\right]\right)$$
$$P^o_\lambda(\omega) := \big((1-\lambda)q_{2m-1} + \lambda p_{2m-1}\big) \prod_{i=1}^{m-1} \left(\omega^2 - \left[(1-\lambda)(\omega^q_{o,i})^2 + \lambda(\omega^p_{o,i})^2\right]\right).$$

Obviously, the coefficients of $P_\lambda(s)$ are polynomial functions in $\lambda$ which are therefore continuous on $[0,1]$. Moreover, the coefficient of the highest degree term in $P_\lambda(s)$ is $(1-\lambda)q_{2m} + \lambda p_{2m}$ and always remains positive as $\lambda$ varies from 0 to 1. For $\lambda = 0$ we have $P_0(s) = Q(s)$ and for $\lambda = 1$, $P_1(s) = P(s)$. Suppose now that $P(s)$ is not Hurwitz. From the Boundary Crossing Theorem it is then clear that there necessarily exists some $\lambda$ in $(0,1]$ such that $P_\lambda(s)$ has a root on the imaginary axis. However, $P_\lambda(s)$ has a root on the imaginary axis if and only if $P_\lambda^e(\omega)$ and $P_\lambda^o(\omega)$ have a common real root. But, obviously, the roots of $P_\lambda^e(\omega)$ satisfy

$$\omega_{e,i}^{\lambda}{}^2 = (1-\lambda)\omega_{e,i}^{q}{}^2 + \lambda\omega_{e,i}^{p}{}^2, \tag{1.42}$$

and those of $P_\lambda^o(\omega)$,

$$\omega_{o,i}^{\lambda}{}^2 = (1-\lambda)\omega_{o,i}^{q}{}^2 + \lambda\omega_{o,i}^{p}{}^2. \tag{1.43}$$

Now, take any two roots of $P_\lambda^e(\omega)$ in (1.42). If $i < j$, from (1.40) $\omega_{e,i}^{p}{}^2 < \omega_{e,j}^{p}{}^2$, and similarly from (1.41), $\omega_{e,i}^{q}{}^2 < \omega_{e,j}^{q}{}^2$, so that

$$\omega_{e,i}^{\lambda}{}^2 < \omega_{e,j}^{\lambda}{}^2.$$

In the same way, it can be seen that the same ordering as in (1.40) and (1.41) is preserved between the roots of $P_\lambda^o(\omega)$, and also between any root of $P_\lambda^e(\omega)$ and any root of $P_\lambda^o(\omega)$. In other words, part b) of the interlacing property is invariant under such convex combinations so that we also have for every $\lambda$ in $[0,1]$:

$$0 < \omega_{e,1}^{\lambda}{}^2 < \omega_{o,1}^{\lambda}{}^2 < \cdots < \omega_{e,m-1}^{\lambda}{}^2 < \omega_{o,m-1}^{\lambda}{}^2 < \omega_{e,m}^{\lambda}{}^2.$$

But this shows that, whatever the value of $\lambda$ in $[0,1]$, $P_\lambda^e(\omega)$ and $P_\lambda^o(\omega)$ can never have a common root, and this therefore leads to a contradiction which completes the proof. ♣

It is clear that the interlacing property is equivalent to the monotonic phase increase property. If the stability region $\mathcal{S}$ is such that a stable polynomial does not have the monotonic phase increase property, the interlacing of the real and imaginary parts will in general fail to hold. However, even in the case of such a region $\mathcal{S}$ the boundary crossing property must hold. This means that the transition from stability to instability can only occur if the real and imaginary parts simultaneously become zero at some boundary point.

### A Frequency Domain Plot for Hurwitz Stability

The interlacing property of a polynomial can be verified by plotting either the graphs of $P^e(\omega)$ and $P^o(\omega)$ or the polar plot of $P(j\omega)$ as shown below.

**Example 1.2.**

$$P(s) = s^9 + 11s^8 + 52s^7 + 145s^6 + 266s^5 + 331s^4 + 280s^3 + 155s^2 + 49s + 6.$$

Then

$$P(j\omega) := P^e(\omega) + j\omega P^o(\omega)$$

with

$$P^e(\omega) = 11\omega^8 - 145\omega^6 + 331\omega^4 - 155\omega^2 + 6$$
$$P^o(\omega) = \omega^8 - 52\omega^6 + 266\omega^4 - 280\omega^2 + 49.$$

The plots of $P^e(\omega)$ and $P^o(\omega)$ are shown in Figure 1.5. They show that the polynomial $P(s)$ is Hurwitz because it satisfies the interlacing property.

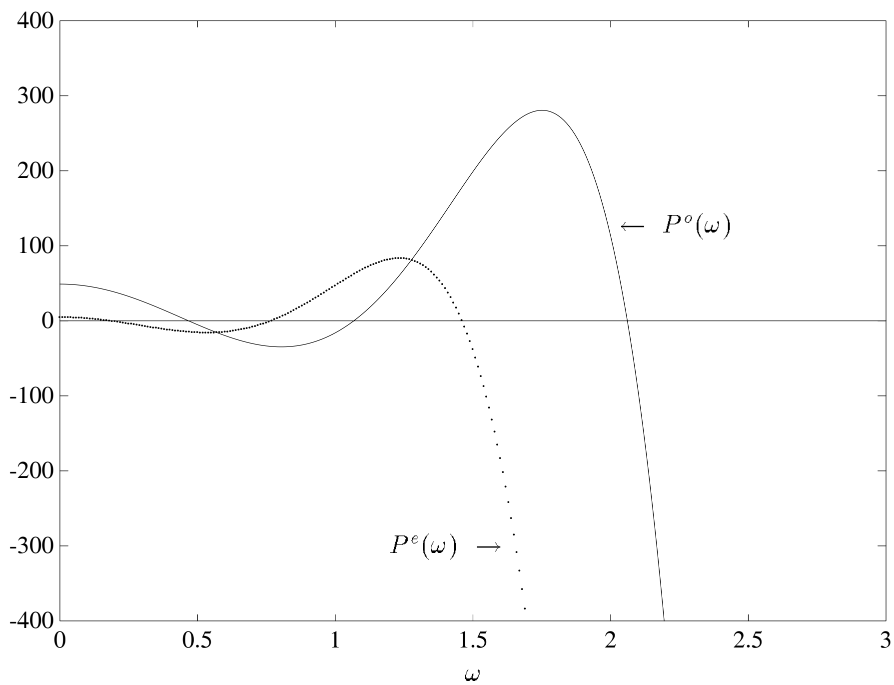


**Figure 1.5.** Interlacing property for Hurwitz polynomials (Example 1.2)

**Example 1.3.**

$$P(s) = s^9 + 21s^8 + 52s^7 + 145s^6 + 266s^5 + 331s^4 + 280s^3 + 155s^2 + 49s + 6.$$

Then

$$P(j\omega) := P^e(\omega) + j\omega P^o(\omega)$$

where

$$P^e(\omega) = 21\omega^8 - 145\omega^6 + 331\omega^4 - 155\omega^2 + 6$$
$$P^o(\omega) = \omega^8 - 52\omega^6 + 266\omega^4 - 280\omega^2 + 49.$$

The plots of $P^e(\omega)$ and $P^o(\omega)$ are shown in Figure 1.6. They show that the polynomial $P(s)$ is not Hurwitz because it fails to satisfy the interlacing property.

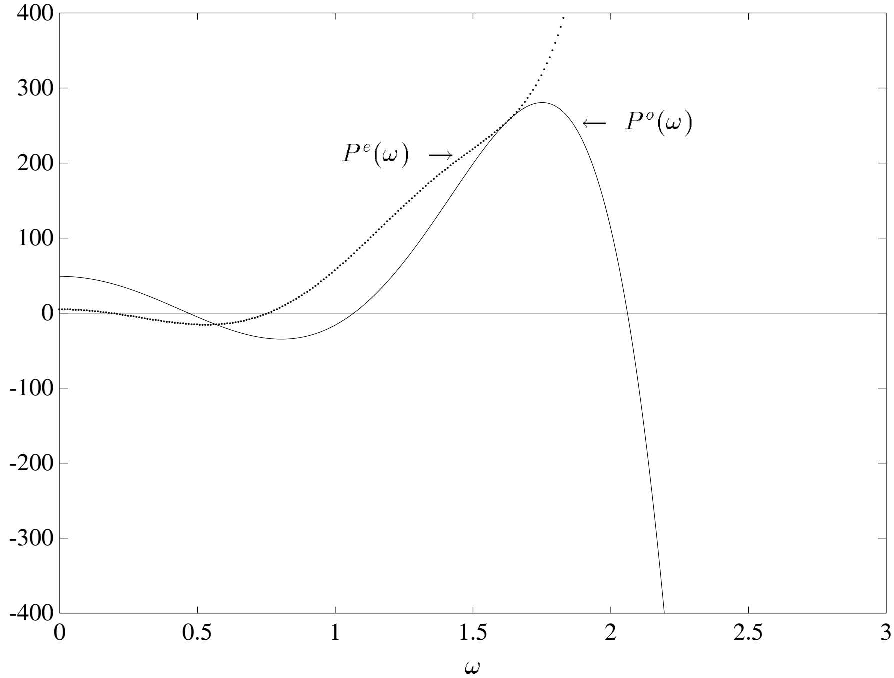


**Figure 1.6.** Interlacing fails for nonHurwitz polynomials (Example 1.3)

Both the plots in the above examples are unbounded as $\omega$ tends to $\infty$. A bounded plot containing the same information can be constructed as follows. For a polynomial

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n, \quad p_n > 0$$

write as usual

$$P(j\omega) = P^e(\omega) + j\omega P^o(\omega)$$

and let $S(\omega)$ and $T(\omega)$ denote arbitrary continuous positive functions on $0 \le \omega < \infty$. Let

$$x(\omega) := \frac{P^e(\omega)}{S(\omega)}, \quad y(\omega) := \frac{P^o(\omega)}{T(\omega)}.$$

**Lemma 1.1** *A real polynomial $P(s)$ is Hurwitz if and only if the frequency plot $z(\omega) := x(\omega) + jy(\omega)$ moves strictly counterclockwise and goes through $n$ quadrants in turn.*

**Proof.** The Hermite-Biehler Theorem and the monotonic phase property of Hurwitz polynomials shows that the plot of $P(j\omega)$ must go through n quadrants if and only if $P(s)$ is Hurwitz. Since the signs of $P^e(\omega)$ and $x(\omega)$, $\omega P^o(\omega)$ and $y(\omega)$ coincide for $\omega > 0$, the lemma is true. ♣

Although the $P(j\omega)$ plot is unbounded, the plot of $z(\omega)$ can always be bounded by choosing the functions $T(\omega)$ and $S(\omega)$ appropriately. For example $T(\omega)$ and $S(\omega)$ can be chosen to be polynomials with degrees equal to that of $P^e(\omega)$ and $P^o(\omega)$ respectively. A similar result can be derived for the complex case. Lemma 1.1 is illustrated with the following example.

**Example 1.4.** Taking the same polynomial as in Example 1.2:

$$P(s) = s^9 + 11s^8 + 52s^7 + 145s^6 + 266s^5 + 331s^4 + 280s^3 + 155s^2 + 49s + 6$$

and writing

$$P(j\omega) := P^e(\omega) + j\omega P^o(\omega)$$

we have

$$\begin{aligned} P^e(\omega) &= 11\omega^8 - 145\omega^6 + 331\omega^4 - 155\omega^2 + 6 \\ P^o(\omega) &= \omega^8 - 52\omega^6 + 266\omega^4 - 280\omega^2 + 49. \end{aligned}$$

We choose

$$\begin{aligned} S(\omega) &= \omega^8 + \omega^6 + \omega^4 + \omega^2 + 1 \\ T(\omega) &= \omega^8 + \omega^6 + \omega^4 + \omega^2 + 1. \end{aligned}$$

The function $z(\omega)$ in Figure 1.7 turns strictly counterclockwise and goes through nine quadrants and this shows that the polynomial $P(s)$ is Hurwitz according to Lemma 1.1.

It will be shown in Chapter 3 that the weighting functions $S(\omega)$ and $T(\omega)$ can be suitably chosen to extend this frequency domain criterion to verify robust Hurwitz stability of an $l_p$ ball in coefficient space.

### 1.3.2 Hurwitz Stability for Complex Polynomials

The Hermite-Biehler Theorem for complex polynomials is given below. Its proof is a straightforward extension of that of the real case and will not be given. Let $P(s)$ be a complex polynomial

$$P(s) = (a_0 + jb_0) + (a_1 + jb_1)s + \cdots + (a_{n-1} + jb_{n-1})s^{n-1} + (a_n + jb_n)s^n. \quad (1.44)$$

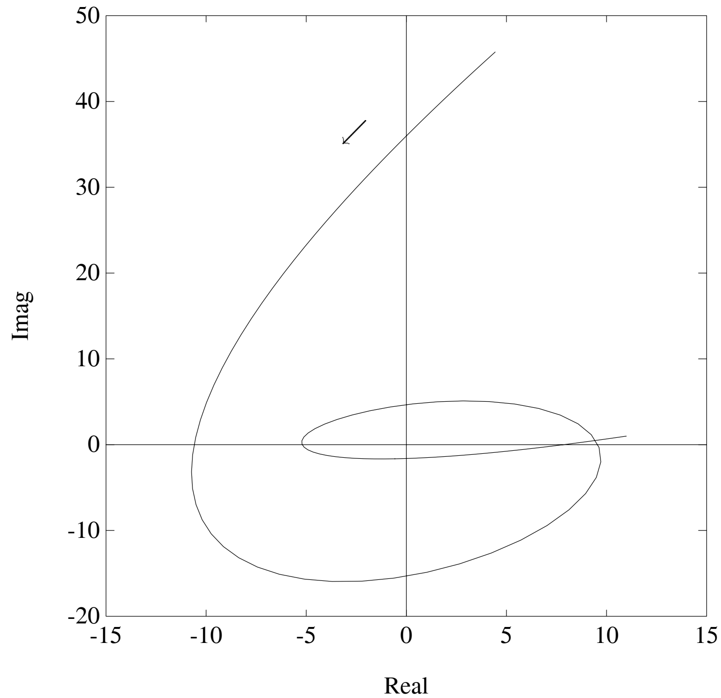


**Figure 1.7.** Frequency plot of $z(\omega)$ (Example 1.4)

Define

$$\begin{aligned} P_R(s) &= a_0 + jb_1 s + a_2 s^2 + jb_3 s^3 + \cdots \\ P_I(s) &= jb_0 + a_1 s + jb_2 s^2 + a_3 s^3 + \cdots \end{aligned} \tag{1.45}$$

and write

$$P(j\omega) = P^r(\omega) + jP^i(\omega),$$

where

$$\begin{aligned} P^r(\omega) &:= P_R(j\omega) = a_0 - b_1\omega - a_2\omega^2 + b_3\omega^3 + \cdots\ , \\ P^i(\omega) &:= \frac{1}{j}P_I(j\omega) = b_0 + a_1\omega - b_2\omega^2 - a_3\omega^3 + \cdots\ . \end{aligned} \tag{1.46}$$

The Hermite-Biehler Theorem for complex polynomials can then be stated as follows.

**Theorem 1.8** *The complex polynomial $P(s)$ in (1.44) is a Hurwitz polynomial if and only if,*

*1)* $a_{n-1}a_n + b_{n-1}b_n > 0$,

*2) The zeros of $P^r(\omega)$ and $P^i(\omega)$ are all simple and real and interlace, as $\omega$ runs from $-\infty$ to $+\infty$.*

Note that condition 1) follows directly from the fact that the sum of the roots of the polynomial $P(s)$ in (1.44) is equal to

$$-\frac{a_{n-1} + jb_{n-1}}{a_n + jb_n} = -\frac{a_{n-1}a_n + b_{n-1}b_n + j(b_{n-1}a_n - a_{n-1}b_n)}{a_n^2 + b_n^2},$$

so that if $P(s)$ is Hurwitz, then the real part of the above complex number must be negative.

### 1.3.3 The Hermite-Biehler Theorem: Schur Case

In fact it is always possible to derive results similar to the interlacing theorem with respect to any stability region $\mathcal{S}$ which has the property that the phase of a stable polynomial evaluated along the boundary of $\mathcal{S}$ increases monotonically. In this case the stability of the polynomial with respect to $\mathcal{S}$ is equivalent to the interlacing of its real and imaginary parts evaluated along the boundary of $\mathcal{S}$. Here we concentrate on the case where $\mathcal{S}$ is the open unit disc. This is the stability region for discrete time systems.

**Definition 1.2.** A polynomial,

$$P(z) = p_n z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0,$$

is said to be a Schur polynomial if all its roots lie in the open unit disc of the complex plane. A necessary condition for Schur stability is $|p_n| > |p_0|$ (see Property 1.3).

#### A frequency plot for Schur stability

$P(z)$ can be written as

$$P(z) = p_n(z - z_1)(z - z_2)\cdots(z - z_n) \tag{1.47}$$

where the $z_i$'s are the $n$ roots of $P(z)$. If $P(z)$ is Schur, all these roots are located inside the unit disc $|z| < 1$, so that when $z$ varies along the unit circle, $z = e^{j\theta}$, the argument of $P(e^{j\theta})$ increases monotonically. For a Schur polynomial of degree $n$, $P(e^{j\theta})$ has a net increase of argument of $2n\pi$, and thus the plot of $P(e^{j\theta})$ encircles the origin $n$ times. This can be used as a frequency domain test for Schur stability.

**Example 1.5.** Consider the stable polynomial

$$\begin{aligned}P(z) &= 2z^4 - 3.2z^3 + 1.24z^2 + 0.192z - 0.1566\\ &= 2(z + 0.3)(z - 0.5 + 0.2j)(z - 0.5 - 0.2j)(z - 0.9)\end{aligned}$$

Let us evaluate $P(z)$ when $z$ varies along the unit circle. The plot obtained in Figure 1.8 encircles the origin four times, which shows that this fourth order polynomial is Schur stable.

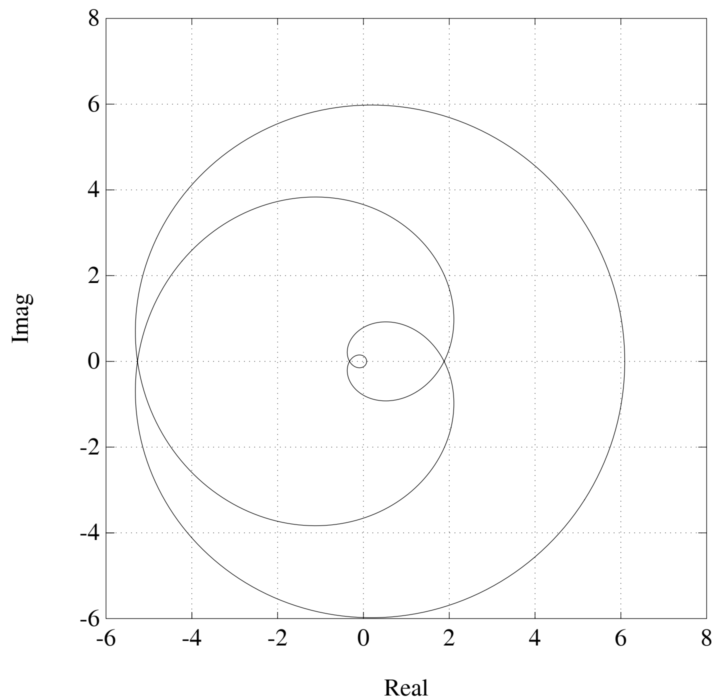


**Figure 1.8.** Plot of $P(e^{j\theta})$ (Example 1.5)

A simplification can be made by considering the *reversed polynomial* $z^n P(z^{-1})$.

$$\begin{aligned} z^n P(z^{-1}) &= p_0 z^n + p_1 z^{n-1} + \cdots + p_n \\ &= p_n(1 - z_1 z)(1 - z_2 z) \cdots (1 - z_n z) \end{aligned} \tag{1.48}$$

$z^n P(z^{-1})$ becomes zero at $z = z_i^{-1}, \quad i = 1, \cdots, n$. If $P(z)$ is Schur the $z_i$'s have modulus less than one, so that the $z_i^{-1}$ are located outside the unit disc. If we let $z = e^{j\theta}$ vary along the unit circle the net increase of argument of $e^{jn\theta} P(e^{-j\theta})$ must therefore be *zero*. This means that for Schur stability of $P(z)$ it is necessary and sufficient that the frequency plot, $e^{jn\theta} P(e^{-j\theta})$ of the reverse polynomial *not encircle the origin*.

**Example 1.6.** Consider the polynomial in the previous example. The plot of $z^n P(z^{-1})$ when $z$ describes the unit circle is shown in Figure 1.9. As seen, the plot does not encircle the origin and thus we conclude that $P(z)$ is stable.

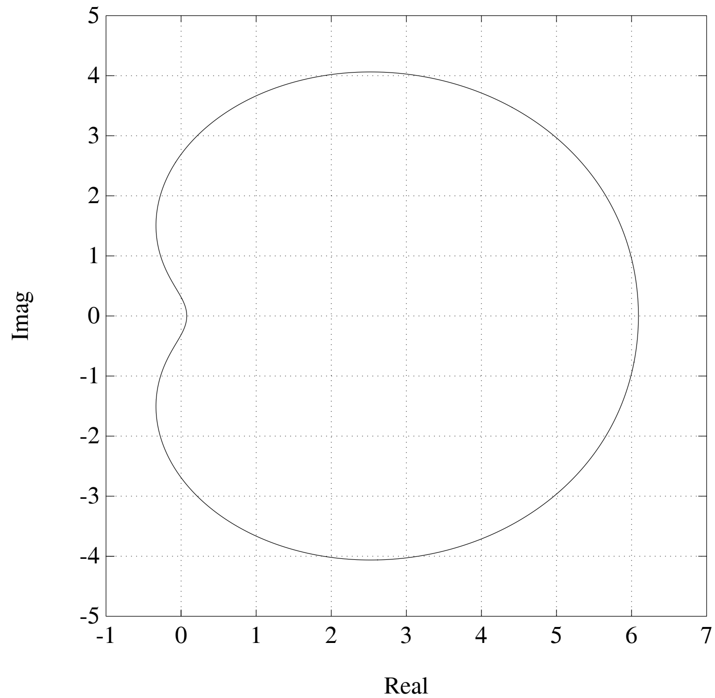


**Figure 1.9.** Plot of $e^{j4\theta}P(e^{-j\theta})$ (Example 1.6)

We see that when using the plot of $P(z)$ we must verify that the plot of $P(e^{j\theta})$ encircles the origin the correct number of times $n$, whereas using the reverse polynomial $R(z) = z^n P(z^{-1})$ we need only check that the plot of $R(e^{j\theta})$ excludes the origin. This result holds for real as well as complex polynomials.

For a real polynomial, it is easy to see from the above that the stability of $P(z)$ is equivalent to the interlacing of the real and imaginary parts of $P(z)$ evaluated along the upper-half of the unit-circle. Writing $P(e^{j\theta}) = R(\theta) + jI(\theta)$ we have:

$$R(\theta) = p_n \cos(n\theta) + \cdots + p_1 \cos(\theta) + p_0$$

and,

$$I(\theta) = p_n \sin(n\theta) + \cdots + p_1 \sin(\theta).$$

**Lemma 1.2** *A real polynomial P(z) is Schur with* $|p_n| > |p_0|$ *if and only if*

*a)* $R(\theta)$ *has exactly* $n$ *zeros in* $[0, \pi]$,

*b)* $I(\theta)$ *has exactly* $n+1$ *zeros in* $[0, \pi]$, *and* *c) the zeros of $R(\theta)$ and $I(\theta)$ interlace.*

**Example 1.7.** Consider the polynomial

$$P(z) = z^5 + 0.2z^4 + 0.3z^3 + 0.4z^2 + 0.03z + 0.02.$$

As seen in Figure 1.10 the polynomial $P(z)$ is Schur since $\mathrm{Re}[P(e^{j\theta})]$ and $\mathrm{Im}[P(e^{j\theta})]$ have respectively 5 and 6 distinct zeros for $\theta \in [0, \pi]$, and the zeros of $\mathrm{Re}[P(e^{j\theta})]$ interlace with the zeros of $\mathrm{Im}[P(e^{j\theta})]$.

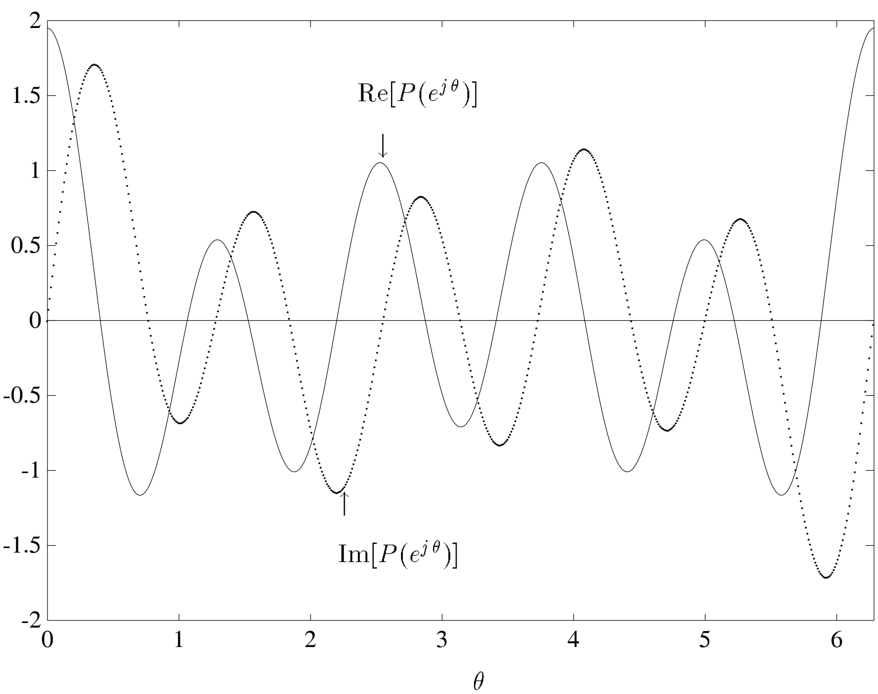


**Figure 1.10.** $\mathrm{Re}[P(e^{j\theta})]$ and $\mathrm{Im}[P(e^{j\theta})]$(Schur case)(Example 1.7)

**Example 1.8.** Consider the polynomial

$$P(z) = z^5 + 2z^4 + 0.3z^3 + 0.4z^2 + 0.03z + 0.02.$$

Since $\mathrm{Re}[P(e^{j\theta})]$ and $\mathrm{Im}[P(e^{j\theta})]$ each do not have $2n = 10$ distinct zeros for $0 \leq \theta < 2\pi$, as shown in Figure 1.11, the polynomial $P(z)$ is not Schur.

These conditions, can in fact be further refined to the interlacing on the unit circle of the two polynomials $P_s(z)$ and $P_a(z)$ which represent the *symmetric* and

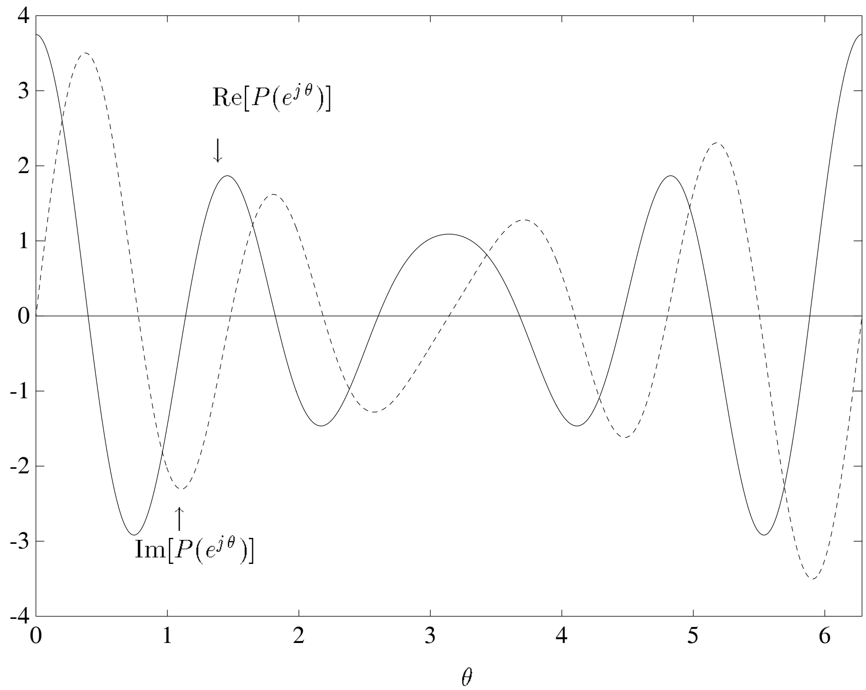


**Figure 1.11.** Re$[P(e^{j\theta})]$ and Im$[P(e^{j\theta})]$ (nonSchur case) (Example 1.8)

*asymmetric* parts of the real polynomial $P(z) = P_s(z) + P_a(z)$:

$$P_s(z) = \frac{1}{2}\left[P(z) + z^n P\left(\frac{1}{z}\right)\right], \quad \text{and} \quad P_a(z) = \frac{1}{2}\left[P(z) - z^n P\left(\frac{1}{z}\right)\right].$$

**Theorem 1.9** *A real polynomial P(z) is Schur if and only if $P_s(z)$ and $P_a(z)$ satisfy the following:*

*a) $P_s(z)$ and $P_a(z)$ are polynomials of degree n with leading coefficients of the same sign.*

*b) $P_s(z)$ and $P_a(z)$ have only simple zeros which belong to the unit circle.*

*c) The zeros of $P_s(z)$ and $P_a(z)$ interlace on the unit circle.*

**Proof.** Let $P(z) = p_0 + p_1 z + p_2 z^2 \cdots + p_n z^n$. The condition a) is equivalent to $p_n^2 - p_0^2 > 0$ which is clearly necessary for Schur stability (see Property 1.3). Now we apply the bilinear transformation of the unit circle into the left half plane and use the Hermite Biehler Theorem for Hurwitz stability. It is known that the bilinear mapping,

$$z = \frac{s+1}{s-1}$$

maps the open unit disc into the open left half plane. It can be used to transform a polynomial $P(z)$ into $\hat{P}(s)$ as follows:

$$(s-1)^n P\left(\frac{s+1}{s-1}\right) = \hat{P}(s).$$

Write

$$\hat{P}(s) = \hat{p}_0 + \hat{p}_1 s + \cdots + \hat{p}_{n-1} s^{n-1} + \hat{p}_n s^n$$

where each $\hat{p}_i$ is a function which depends on the coefficients of $P(z)$. It follows that if the transformation is degree preserving then $P(z)$ is Schur stable if and only if $\hat{P}(s)$ is Hurwitz stable. It is easy to verify that the transformation described above is degree preserving if and only if

$$\hat{p}_n = \sum_{i=0}^{i=n} p_i = P(1) \neq 0$$

and that this holds is implied by condition c).

The transformation of $P(z)$ into $\hat{P}(s)$ is a linear transformation $T$. That is, $\hat{P}(s)$ is the image of $P(z)$ under the mapping $T$. Then $TP(z) = \hat{P}(s)$ may be written explicitly as

$$(s-1)^n P\left(\frac{s+1}{s-1}\right) = TP(z) = \hat{P}(s).$$

For example, for $n = 4$, expressing $P(z)$ and $\hat{P}(s)$ in terms of their coefficient vectors

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 0 & 2 & 4 \\ 6 & 0 & -2 & 0 & 6 \\ -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \end{bmatrix}.$$

Consider the symmetric and anti-symmetric parts of $P(z)$, and their transformed images, given by $TP_s(z)$ and $TP_a(z)$ respectively. A straightforward computation shows that

$$TP_s(z) = \hat{P}^{\text{even}}(s), \quad TP_a(z) = \hat{P}^{\text{odd}}(s), \qquad n \ \text{ even}$$

and

$$TP_s(z) = \hat{P}^{\text{odd}}(s), \quad TP_a(z) = \hat{P}^{\text{even}}(s), \qquad n \ \text{ odd.}$$

The conditions b) and c) now follow immediately from the interlacing property for Hurwitz polynomials applied to $\hat{P}(s)$. ♣

The functions $P_s(z)$ and $P_a(z)$ are easily evaluated as $z$ traverses the unit circle. Interlacing may be verified from a plot of the zeros of these functions as in Figure 1.12.

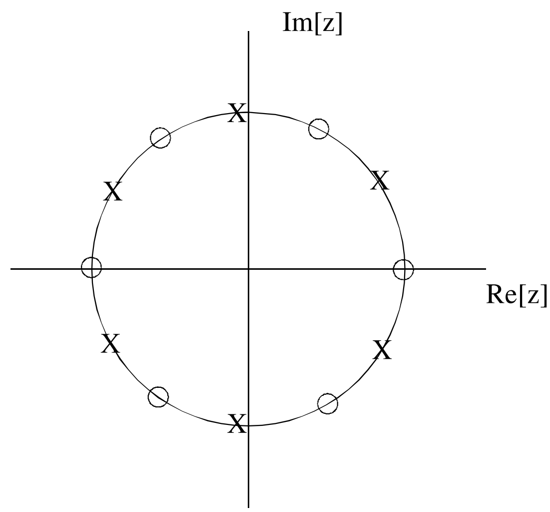


**Figure 1.12.** Interlacing of the symmetric and anti-symmetric parts of a polynomial on the unit circle

### General Stability Regions

The Hermite-Biehler interlacing theorem holds for any stability region $\mathcal{S}$ which has the property that the phase of any polynomial which is stable with respect to $\mathcal{S}$ varies monotonically along the boundary $\partial\mathcal{S}$. The left half plane and the unit circle satisfy this criterion. Obviously there are many other regions of practical interest for which this property holds. For example, the regions shown in Figure 1.1.(c) and (d) also satisfy this condition.

In the next two sections we display another application of the Boundary Crossing Theorem by using it to derive the Jury and Routh tables for Schur and Hurwitz stability respectively.

## 1.4 SCHUR STABILITY TEST

The problem of checking the stability of a discrete time system reduces to the determination of whether or not the roots of the characteristic polynomial of the system lie strictly within the unit disc, that is whether or not the characteristic polynomial is a Schur polynomial. In this section we develop a simple test procedure for this problem based on the Boundary Crossing Theorem. The procedure turns out to be equivalent to Jury's test for Schur stability.

The development here is given generally for complex polynomials and of course it applies to real polynomials as well. Now, let

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n,$$

be a polynomial of degree $n$. The following is a simple necessary condition.

**Property 1.3.** A necessary condition for $P(z)$ to be a Schur polynomial is that

$$|p_n| > |p_0|.$$

Indeed, if $P(z)$ has all its roots $z_1, \cdots, z_n$ inside the unit-circle then the product of these roots is given by

$$(-1)^n \prod_{i=1}^{n} z_i = \frac{p_0}{p_n},$$

$$\text{hence} \qquad \left|\frac{p_0}{p_n}\right| = \prod_{i=1}^{n} |z_i| < 1.$$

Now, consider a polynomial $P(z)$ of degree $n$,

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n.$$

Let $\overline{z}$ denote the conjugate of $z$ and define

$$\begin{aligned} Q(z) &= z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = \overline{p}_0 z^n + \overline{p}_1 z^{n-1} + \cdots + \overline{p}_{n-1} z + \overline{p}_n, \\ R(z) &= \frac{1}{z}\left[P(z) - \frac{p_0}{\overline{p}_n} Q(z)\right]. \end{aligned} \tag{1.49}$$

It is easy to see that $R(z)$ is always of degree less than or equal to $n-1$. The following key lemma allows the degree of the test polynomial to be reduced without losing stability information.

**Lemma 1.3** *If $P(z)$ satisfies $|p_n| > |p_0|$, we have the following equivalence*

$$P(z) \textit{ is a Schur polynomial} \Longleftrightarrow R(z) \textit{ is a Schur polynomial.}$$

**Proof.** First notice that obviously,

$$R(z) \text{ is a Schur polynomial} \Longleftrightarrow zR(z) \text{ is a Schur polynomial.}$$

Now consider the family of polynomials

$$P_\lambda(z) = P(z) - \lambda \frac{p_0}{\overline{p}_n} Q(z), \text{ where } \lambda \in [0, 1].$$

It can be seen that $P_0(z) = P(z)$, and $P_1(z) = zR(z)$. Moreover the coefficient of degree $n$ of $P_\lambda(z)$ is

$$p_n - \lambda \frac{|p_0|^2}{\overline{p}_n},$$

and satisfies

$$\left| p_n - \lambda \frac{|p_0|^2}{\overline{p}_n} \right| > |p_n| - \lambda \left| \frac{p_0}{p_n} \right| |p_0| > |p_n| - |p_0| > 0,$$

so that $P_\lambda(z)$ remains of fixed degree $n$.

Assume now by way of contradiction that one of these two polynomials $P_0(z)$ or $P_1(z)$ is stable whereas the other one is not. Then, from the Boundary Crossing Theorem it can be concluded that there must exist a $\lambda$ in $[0,1]$ such that $P_\lambda(z)$ has a root on the unit-circle at the point $z_0 = e^{j\theta}$, $\theta \in [0, 2\pi)$, that is

$$P_\lambda(z_0) = P(z_0) - \lambda \frac{p_0}{\overline{p}_n} z_0^n \overline{P\left[\frac{1}{\overline{z}_0}\right]} = 0. \tag{1.50}$$

But for any complex number $z$ on the unit circle, $\overline{z} = \frac{1}{z}$, and therefore (1.50) implies that,

$$P_\lambda(z_0) = P(z_0) - \lambda \frac{p_0}{\overline{p}_n} z_0^n \overline{P(z_0)} = 0. \tag{1.51}$$

Taking the complex conjugate of this last expression it is deduced that

$$\overline{P(z_0)} - \lambda \frac{\overline{p}_0}{p_n} \overline{z}_0^n P(z_0) = 0. \tag{1.52}$$

Therefore, from (1.51) and (1.52) after using the fact that $|z_0| = 1$,

$$P(z_0) \left(1 - \lambda^2 \frac{|p_0|^2}{|p_n|^2}\right) = 0. \tag{1.53}$$

By assumption $\lambda^2 \dfrac{|p_0|^2}{|p_n|^2} < 1$, and therefore (1.53) implies that,

$$P(z_0) = 0. \tag{1.54}$$

But then this implies that

$$\overline{P(z_0)} = \overline{P\left(\frac{1}{\overline{z}_0}\right)} = 0,$$

and therefore (see (1.49))

$$R(z_0) = 0. \tag{1.55}$$

But (1.54) and (1.55) contradict the assumption that one of the two polynomials $P(z)$ and $zR(z)$ is stable, and this concludes the proof of the lemma. ♣

The above lemma leads to the following procedure for successively reducing the degree and testing for stability.

**Algorithm 1.1. (Schur stability for real or complex polynomials)**

1) Set $P^{(0)}(z) = P(z)$,

2) Verify $|p_n^{(i)}| > |p_0^{(i)}|$,

3) Construct $P^{(i+1)}(z) = \frac{1}{z}\left[P^{(i)}(z) - \frac{p_0^{(i)}}{\overline{p}_n^{(i)}} z^n \overline{P(\frac{1}{\overline{z}})}\right]$,

4) Go back to 2) until you either find that 2) is violated ($P(z)$ is not Schur) or until you reach $P^{(n-1)}(z)$ (which is of degree 1) in which case condition 2) is also sufficient and $P(z)$ is a Schur polynomial.

It can be verified by the reader that this procedure leads precisely to the Jury stability test.

**Example 1.9.** Consider a real polynomial of degree 3 in the variable $z$,

$$P(z) = z^3 + az^2 + bz + c.$$

According to the algorithm, we form the following polynomial

$$\begin{aligned} P^{(1)}(z) &= \frac{1}{z}\left[P(z) - cz^3 \overline{P\left(\frac{1}{\overline{z}}\right)}\right] \\ &= (1-c^2)z^2 + (a-bc)z + b - ac, \end{aligned}$$

and then,

$$\begin{aligned} P^{(2)}(z) &= \frac{1}{z}\left[P^{(1)}(z) - \left(\frac{b-ac}{1-c^2}\right) z^2 \overline{P^{(1)}\left(\frac{1}{\overline{z}}\right)}\right] \\ &= \frac{(1-c^2)^2 - (b-ac)^2}{1-c^2} z + (a-bc)\left(1 - \frac{b-ac}{1-c^2}\right). \end{aligned}$$

On the other hand, the Jury table is given by,

$$\begin{bmatrix} c & b & a & 1 \\ 1 & a & b & c \\ c^2-1 & cb-a & ca-b & \\ ca-b & cb-a & c^2-1 & \\ (c^2-1)^2 - (ca-b)^2 & (cb-a)[(c^2-1)-(ca-b)] & & \end{bmatrix}$$

Here, the first two lines of this table correspond to the coefficients of $P(z)$, the third and fourth lines to those of $P^{(1)}(z)$ and the last one to a constant times $P^{(2)}(z)$, and the tests to be carried out are exactly similar.

## 1.5 HURWITZ STABILITY TEST

We now turn to the problem of left half plane or Hurwitz stability for real polynomials and develop an elementary test procedure for it based on the Interlacing Theorem and therefore on the Boundary Crossing Theorem. This procedure turns out to be equivalent to Routh's well known test.

Let $P(s)$ be a real polynomial of degree $n$, and assume that all the coefficients of $P(s)$ are positive,

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n, \qquad p_i > 0, \quad \text{for} \quad i = 0, \cdots, n \ .$$

Remember that $P(s)$ can be decomposed into its odd and even parts as

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s).$$

Now, define the polynomial $Q(s)$ of degree $n-1$ by:

$$\begin{aligned} \text{If } n = 2m: \quad Q(s) &= \left[P^{\text{even}}(s) - \frac{p_{2m}}{p_{2m-1}} s P^{\text{odd}}(s)\right] + P^{\text{odd}}(s), \\ \text{If } n = 2m+1: \quad Q(s) &= \left[P^{\text{odd}}(s) - \frac{p_{2m+1}}{p_{2m}} s P^{\text{even}}(s)\right] + P^{\text{even}}(s) \end{aligned} \qquad (1.56)$$

that is in general, with $\mu = \frac{p_n}{p_{n-1}}$,

$$Q(s) = p_{n-1}s^{n-1} + (p_{n-2} - \mu p_{n-3})s^{n-2} + p_{n-3}s^{n-3} + (p_{n-4} - \mu p_{n-5})s^{n-4} + \cdots . \qquad (1.57)$$

Then the following key result on degree reduction is obtained.

**Lemma 1.4** *If $P(s)$ has all its coefficients positive,*

$$P(s) \textit{ is stable } \iff Q(s) \textit{ is stable.}$$

**Proof.** Assume, for example that, $n = 2m$, and use the interlacing theorem.

(a) Assume that $P(s) = p_0 + \cdots + p_{2m}s^{2m}$ is stable and therefore satisfies the interlacing theorem. Let

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \omega_{o,2} < \cdots < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m},$$

be the interlacing roots of $P^e(\omega)$ and $P^o(\omega)$. One can easily check that (1.56) implies that $Q^e(\omega)$ and $Q^o(\omega)$ are given by

$$\begin{aligned} Q^e(\omega) &= P^e(\omega) + \mu\omega^2 P^o(\omega), \quad \mu = \frac{p_{2m}}{p_{2m-1}}, \\ Q^o(\omega) &= P^o(\omega). \end{aligned}$$

From this it is already concluded that $Q^o(\omega)$ has the required number of positive roots, namely the $m-1$ roots of $P^o(\omega)$:

$$\omega_{o,1}, \quad \omega_{o,2}, \quad \cdots, \quad \omega_{o,m-1}.$$

Moreover, due to the form of $Q^e(\omega)$, it can be deduced that,

$$
\begin{aligned}
Q^e(0) &= P^e(0) > 0, \\
Q^e(\omega_{o,1}) &= P^e(\omega_{o,1}) < 0, \\
&\vdots \\
Q^e(\omega_{o,m-2}) &= P^e(\omega_{o,m-2}), \text{ has the sign of } (-1)^{m-2}, \\
Q^e(\omega_{o,m-1}) &= P^e(\omega_{o,m-1}), \text{ has the sign of } (-1)^{m-1}.
\end{aligned}
$$

Hence, it is already established that $Q^e(\omega)$ has $m-1$ positive roots $\omega'_{e,1}$, $\omega'_{e,2}$, $\cdots$, $\omega'_{e,m-1}$, that do interlace with the roots of $Q^o(\omega)$. Since moreover $Q^e(\omega)$ is of degree $m-1$ in $\omega^2$, these are the only positive roots it can have. Finally, it has been seen that the sign of $Q^e(\omega)$ at the last root $\omega_{o,m-1}$ of $Q^o(\omega)$ is that of $(-1)^{m-1}$. But the highest coefficient of $Q^e(\omega)$ is nothing but

$$q_{2m-2}(-1)^{m-1}.$$

From this $q_{2m-2}$ must be strictly positive, as $q_{2m-1} = p_{2m-1}$ is, otherwise $Q^e(\omega)$ would again have a change of sign between $\omega_{o,m-1}$ and $+\infty$, which would result in the contradiction of $Q^e(\omega)$ having $m$ positive roots (whereas it is a polynomial of degree only $m-1$ in $\omega^2$). Therefore $Q(s)$ satisfies the interlacing property and is stable if $P(s)$ is.

(b) Conversely assume that $Q(s)$ is stable. Write

$$P(s) = [Q^{\text{even}}(s) + \mu s Q^{\text{odd}}(s)] + Q^{\text{odd}}(s) \quad \mu = \frac{p_{2m}}{p_{2m-1}}.$$

By the same reasoning as in a) it can be seen that $P^o(\omega)$ already has the required number $m-1$ of positive roots, and that $P^e(\omega)$ already has $m-1$ roots in the interval $(0, \omega_{o,m-1})$ that interlace with the roots of $P^o(\omega)$. Moreover the sign of $P^e(\omega)$ at $\omega_{o,m-1}$ is the same as $(-1)^{m-1}$ whereas the term $p_{2m}s^{2m}$ in $P(s)$, makes the sign of $P^e(\omega)$ at $+\infty$ that of $(-1)^m$. Thus $P^e(\omega)$ has an $m^{th}$ positive root,

$$\omega_{e,m} > \omega_{o,m-1},$$

so that $P(s)$ satisfies the interlacing property and is therefore stable.

♣

The above lemma shows how the stability of a polynomial $P(s)$ can be checked by successively reducing its degree as follows.

**Algorithm 1.2. (Hurwitz stability for real polynomials)**

1) Set $P^{(0)}(s) = P(s)$,

2) Verify that all the coefficients of $P^{(i)}(s)$ are positive,

3) Construct $P^{(i+1)}(s) = Q(s)$ according to (1.57),

4) Go back to 2) until you either find that any 2) is violated ($P(s)$ is not Hurwitz) or until you reach $P^{(n-2)}(s)$ (which is of degree 2) in which case condition 2) is also sufficient ($P(s)$ is Hurwitz).

The reader may verify that this procedure is identical to Routh's test since it generates the Routh table. The proof also shows the known property that for a stable polynomial not only the first column but the entire Routh table must consist only of positive numbers. However the procedure described here does not allow to count the stable and unstable zeroes of the polynomial as can be done with Routh's Theorem.

**Example 1.10.** Consider a real polynomial of degree 4,

$$P(s) = s^4 + as^3 + bs^2 + cs + d.$$

Following the algorithm above we form the polynomials,

$$\mu = \frac{1}{a}, \text{ and } P^{(1)}(s) = as^3 + \left(b - \frac{c}{a}\right)s^2 + cs + d,$$

and then,

$$\mu = \frac{a^2}{ab-c}, \text{ and } P^{(2)} = \left(b - \frac{c}{a}\right)s^2 + \left(c - \frac{a^2 d}{ab-c}\right)s + d.$$

Considering that at each step only the even or the odd part of the polynomial is modified, it is needed to verify the positiveness of the following set of coefficients,

$$\begin{bmatrix} 1 & b & d \\ a & c & \\ b - \dfrac{c}{a} & d & \\ & & \\ c - \dfrac{a^2 d}{ab-c} & & \end{bmatrix}$$

But this is just the Routh table for this polynomial.

Note that a lemma similar to Lemma 1.4 could be derived where the assumption that all the coefficients of $P(s)$ are positive is replaced by the assumption that only the two highest degree coefficients $p_{n-1}$ and $p_n$ are positive. The corresponding algorithm would then exactly lead to checking that the first column of the Routh table is positive. However since the algorithm requires that the entire table be constructed, it is more efficient to check that every new coefficient is positive.

### Complex polynomials

A similar algorithm can be derived for checking the Hurwitz stability of complex polynomials. The proof which is very similar to the real case is omitted and a precise description of the algorithm is given below.

Let $P(s)$ be a complex polynomial of degree $n$,

$$P(s) = (a_0+jb_0)+(a_1+jb_1)s+\cdots+(a_{n-1}+jb_{n-1})s^{n-1}+(a_n+jb_n)s^n, \quad a_n+jb_n \neq 0.$$

Let,

$$T(s) = \frac{1}{a_n + jb_n} P(s).$$

Thus $T(s)$ can be written as,

$$T(s) = (c_0 + jd_0) + (c_1 + jd_1)s + \cdots + (c_{n-1} + jd_{n-1})s^{n-1} + s^n,$$

and notice that,

$$c_{n-1} = \frac{a_{n-1}a_n + b_{n-1}b_n}{a_n^2 + b_n^2}.$$

Assume that $c_{n-1} > 0$, which is a necessary condition for $P(s)$ to be Hurwitz (see Theorem 1.8). As usual write, $T(s) = T_R(s) + T_I(s)$, where

$$\begin{aligned} T_R(s) &= c_0 + jd_1 s + c_2 s^2 + jd_3 s^3 + \cdots, \\ T_I(s) &= jd_0 + c_1 s + jd_2 s^2 + c_3 s^3 + \cdots. \end{aligned}$$

Now define the polynomial $Q(s)$ of degree $n-1$ by:

$$\begin{aligned} \text{If } n = 2m: \qquad & Q(s) = \left[T_R(s) - \frac{1}{c_{2m-1}} sT_I(s)\right] + T_I(s), \\ \text{If } n = 2m+1: \qquad & Q(s) = \left[T_I(s) - \frac{1}{c_{2m}} sT_R(s)\right] + T_R(s), \end{aligned}$$

that is in general, with $\mu = \frac{1}{c_{n-1}}$,

$$\begin{aligned} Q(s) = {} & [c_{n-1} + j(d_{n-1} - \mu d_{n-2})]s^{n-1} + [(c_{n-2} - \mu c_{n-3}) + jd_{n-2}]s^{n-2} \\ & + [c_{n-3} + j(d_{n-3} - \mu d_{n-4})]s^{n-3} + \cdots. \end{aligned}$$

Now, exactly as in the real case, the following lemma can be proved.

**Lemma 1.5** *If $P(s)$ satisfies $a_{n-1}a_n + b_{n-1}b_n > 0$, then*

$$P(s) \textit{ is Hurwitz stable } \Longleftrightarrow Q(s) \textit{ is Hurwitz stable.}$$

The corresponding algorithm is as follows.

**Algorithm 1.3. (Hurwitz stability for complex polynomials)**

1) Set $P^{(0)}(s) = P(s)$,

2) Verify that the last two coefficients of $P^{(i)}(s)$ satisfy $a_{n-1}^{(i)}a_n^{(i)} + b_{n-1}^{(i)}b_n^{(i)} > 0$,

3) Construct $T^{(i)}(s) = \dfrac{1}{a_n^{(i)} + jb_n^{(i)}}P^{(i)}(s)$,

4) Construct $P^{(i+1)}(s) = Q(s)$ as above,

5) Go back to 2) until you either find that any 2) is violated ($P(s)$ is not Hurwitz) or until you reach $P^{(n-1)}(s)$ (which is of degree 1) in which case condition 2) is also sufficient ($P(s)$ is Hurwitz).

## 1.6 A FEW COMPLEMENTS

Polynomial functions are analytic in the entire complex plane and thus belong to the so-called class of *entire functions*. It is not straightforward however, to obtain a general version of the Boundary Crossing Theorem that would apply to any family of entire functions. The main reason for this is the fact that a general entire function may have a finite or infinite number of zeros, and the concept of a *degree* is not defined except for polynomials. Similar to Theorem 1.3 for the polynomial case, the following theorem can be considered as a basic result for the analysis of continuous families of entire functions.

**Theorem 1.10** *Let $A$ be an open subset of the complex plane $C$, $F$ a metric space, and $f$ be a complex-valued function continuous in $A \times F$ such that for each $\alpha$ in $F$, $z \longrightarrow f(z,\alpha)$ is analytic in $A$. Let also $B$ be an open subset of $A$ whose closure $\overline{B}$ in $C$ is compact and included in $A$, and let $\alpha_0 \in F$ be such that no zeros of $f(z,\alpha_0)$ belong to the boundary $\partial B$ of $B$. Then, there exists a neighborhood $W$ of $\alpha_0$ in $F$ such that,*

*1) For all $\alpha \in W$, $f(z,\alpha)$ has no zeros on $\partial B$.*

*2) For all $\alpha \in W$, the number (multiplicities included) of zeros of $f(z,\alpha)$ which belong to $B$ is independent of $\alpha$.*

The proof of this theorem is not difficult and is very similar to that of Theorem 1.3. It uses Rouché's Theorem together with the compactness assumption on $B$. Loosely speaking, the above result states the continuity of the zeros of a parametrized family of analytic functions with respect to the parameter values. However, it only applies to the zeros which are contained in a given compact subset of the complex plane and therefore, in terms of stability, a more detailed analysis is required. In the polynomial case, Theorem 1.3 is used together with a knowledge of the degree of the family to arrive at the Boundary Crossing Theorem. The degree of a polynomial indicates not only the number of its zeros in the complex plane, but also its rate of growth at infinity. For more general families of entire functions it is precisely by using this additional notion of growth at infinity that results which are similar to the Boundary Crossing Theorem may sometimes be derived. Clearly, each particular case requires its own analysis. Much deeper results however may be achieved and some of these are presented in the remainder of this section. These results demonstrate the possibility of extending the Hermite-Biehler Theorem to more general classes of entire functions, and some extensions that seem to be of particular interest for Control Theory are selected here.

Let $P(s)$ be a real or complex polynomial. Write,

$$P(j\omega) = P^r(\omega) + jP^i(\omega) \tag{1.58}$$

where $P^r(\omega)$ and $P^i(\omega)$ are real polynomials in $\omega$ which represent respectively the real and imaginary part of $P(j\omega)$. The Hermite-Biehler Theorem for real or complex polynomials can be stated in the following way.

**Theorem 1.11 (Hermite-Biehler)**
*$P(s)$ has all its roots in the open left-half of the complex plane if and only if,*

*a) $P^r(\omega)$ and $P^i(\omega)$ have only simple roots and these roots interlace.*

*b) For all $\omega$ in $\mathbb{R}$, $\quad P^{i\prime}(\omega)P^r(\omega) - P^i(\omega)P^{r\prime}(\omega) > 0$.*

*In condition b) the symbol $'$ indicates derivation with respect to $\omega$. Moreover it can be shown that condition b) can be replaced by the following condition,*

*b') $P^{i\prime}(\omega_o)P^r(\omega_o) - P^i(\omega_o)P^{r\prime}(\omega_o) > 0$, $\quad$ for some $\omega_o$ in $\mathbb{R}$.*

*In other words it is enough that condition b) be satisfied at only one point on the real line.*

It is quite easy to see that the Hermite-Biehler Theorem does not carry over to arbitrary entire functions $f(s)$ of the complex variable $s$. In fact counterexamples can be found which show that conditions a) and b) above are neither necessary nor sufficient for an entire function to have all its zeros in the open left-half plane. The theorem however holds for a large class of entire functions and two such theorems are given below without proofs.

One of the earliest attempt at generalizing the Hermite-Biehler Theorem was made by L. S. Pontryagin. In his paper he studied entire functions of the form $P(s, e^s)$, where $P(s,t)$ is a polynomial in two variables. Before stating his result, some preliminary definitions are needed. Let $P(s,t)$ be a polynomial in two variables with real or complex coefficients,

$$P(s,t) = \sum_k \sum_i a_{ik} s^i t^k. \tag{1.59}$$

Let $r$ be the highest degree in $s$ and $p$ be the highest degree in $t$. $P(s,t)$ is said to have a principal term if and only if $a_{rp} \neq 0$. For example

$$P(s,t) = s + t,$$

does not have a principal term but the following polynomial does

$$P(s,t) = s + t + st.$$

The first result of Pontryagin can be stated as follows.

**Theorem 1.12** *If the polynomial $P(s,t)$ does not have a principal term then the function,*

$$f(s) = P(s, e^s),$$

*has an unbounded number of zeros with arbitrarily large positive real parts.*

In the case where $P(s,t)$ does have a principal term, the main result of Pontryagin's paper is then to show that the Hermite-Biehler Theorem extends to this class of entire functions. More precisely we have the following theorem.

**Theorem 1.13** *Let $f(s) = P(s, e^s)$, where $P(s,t)$ is a polynomial with a principal term, and write*

$$f(j\omega) = f^r(\omega) + jf^i(\omega),$$

*where $f^r(\omega)$ and $f^i(\omega)$ represent respectively the real and imaginary parts of $f(j\omega)$. Then in order that $f(s)$ have all its zeros in the open left-half plane it is necessary and sufficient that the following two conditions hold,*

*a) $f^r(\omega)$ and $f^i(\omega)$ have only simple roots and these roots interlace,*

*b) For all $\omega$ in $\mathbb{R}$, $f^{i\prime}(\omega)f^r(\omega) - f^i(\omega)f^{r\prime}(\omega) > 0$.*

Another interesting extension of the Hermite-Biehler Theorem is now given. Let $f(s)$ be an entire function of the form,

$$f(s) = \sum_{k=1}^{n} e^{\lambda_k s} P_k(s), \tag{1.60}$$

where $P_k(s)$ for $k = 1, \cdots, n$ is an arbitrary polynomial with real or complex coefficients, and the $\lambda_k$'s are real number which satisfy,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n, \quad |\lambda_1| < \lambda_n \ . \tag{1.61}$$

The Hermite-Biehler Theorem also holds for this class of entire functions. Write as usual $f(j\omega) = f^r(\omega) + jf^i(\omega)$.

**Theorem 1.14** *Under the above assumptions, the entire function $f(s)$ in (1.60) has all its zeros in the open left-half plane if and only if*

*a) $f^r(\omega)$ and $f^i(\omega)$ have only simple roots and these roots interlace.*

*b) For all $\omega$ in $\mathbb{R}$, $f^{i\prime}(\omega)f^r(\omega) - f^i(\omega)f^{r\prime}(\omega) > 0$.*

Moreover in both Theorems 1.13 and 1.14, condition b) may be replaced by the weaker condition,

*b′)* $f^{i\prime}(\omega_0)f^r(\omega_0) - f^i(\omega_0)f^{r\prime}(\omega_0) > 0$, for some $\omega_0$ in $\mathbb{R}$.

### Time-delay systems

In control problems involving time delays, we often deal with characteristic equations of the form

$$\delta(s) = d(s) + e^{-sT_1}n_1(s) + e^{-sT_2}n_2(s) + \cdots + e^{-sT_m}n_m(s). \tag{1.62}$$

which are also known as quasipolynomials. Under the assumption that $\deg[d(s)] = n$ and $\deg[n_i(s)] < n$ and

$$0 < T_1 < T_2 < \cdots < T_m,$$

it is easy to show using Theorems 1.13 and 1.14 that the stability of $\delta(s)$ can be checked using the interlacing condition. More precisely, we have

**Theorem 1.15** *Under the above assumptions, $\delta(s)$ in (1.62) is Hurwitz stable if and only if*

*a) $\delta^r(\omega)$ and $\delta^i(\omega)$ have only simple roots and these roots interlace.*

*b) ${\delta^i}'(\omega_0)\delta^r(\omega_0) - \delta^i(\omega_0){\delta^r}'(\omega_0) > 0$, for some $\omega_0$ in $\mathbb{R}$.*

We note that the interlacing condition a) needs to be checked only up to a finite frequency. This follows from the fact that the phasors of $n_i(j\omega)/d(j\omega)$ tend to zero as $\omega$ tend to $\infty$. This ensures that the quasipolynomial $\delta(s)$ has the monotonic phase property for a sufficiently large $\omega$. Therefore, the interlacing condition needs to be verified only for low frequency range.

An immediate consequence of the above result is the following Boundary Crossing Theorem for quasipolynomials. Consider the quasipolynomial family

$$Q(s,\lambda) = d(s,\lambda) + e^{-sT_1}n_1(s,\lambda) + e^{-sT_2}n_2(s,\lambda) + \cdots + e^{-sT_m}n_m(s,\lambda). \tag{1.63}$$

where $\lambda \in [a,b]$ and we assume that 1) $Q(s,a)$ is Hurwitz, 2) $Q(s,b)$ is not Hurwitz, 3) $\deg[d(s,\lambda)] = n, \ \forall \ \lambda \in [a,b]$ and 4) $\deg[n_i(s,\lambda)] < n, \ \forall \ \lambda \in [a,b]$.

**Theorem 1.16 (Boundary Crossing Theorem for Quasipolynomials)** *Under the above assumptions there exists at least one $\rho$ in $(a,b]$ such that $Q(0,\rho) = 0$ or $Q(j\omega,\rho) = 0$ for some $\omega \in [-\infty, +\infty]$.*

These results are particularly useful for the stability analysis of control systems which contain time-delays. We illustrate this in the example below.

**Example 1.11.** Consider the quasipolynomial:

$$\delta(s) = d(s) + e^{-sT_1}n_1(s) + e^{-sT_2}n_2(s)$$

where

$$\begin{aligned}
d(s) &= s^9 + 5s^8 + 20s^7 + 100s^6 + 200s^5 + 100s^4 + 100s^3 + 50s^2 + 15s + 1 \\
n_1(s) &= 3s^8 + 10s^7 + 10s^6 + 15s^5 + 100s^4 + 50s^3 + 50s^2 + 10s + 2 \\
n_2(s) &= 2s^8 + 22s^7 + 35s^6 + 51s^5 + 131s^4 + 130s^3 + 55s^2 + 24s + 3
\end{aligned}$$

with $T_1 = 0.1$ sec and $T_2 = 0.3$ sec. We write

$$\begin{aligned} d(j\omega) &= d^{e}(\omega) + j\omega d^{o}(\omega) \\ n_1(j\omega) &= n_1^{e}(\omega) + j\omega n_1^{o}(\omega) \\ n_2(j\omega) &= n_2^{e}(\omega) + j\omega n_2^{o}(\omega) \end{aligned}$$

and $\delta(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$. We have

$$\begin{aligned} \delta_r(\omega) = {} & d^{e}(\omega) + \cos(\omega T_1) n_1^{e}(\omega) - \omega \sin(\omega T_1) n_1^{o}(\omega) \\ & + \cos(\omega T_2) n_2^{e}(\omega) - \omega \sin(\omega T_2) n_2^{o}(\omega) \\ \delta_i(\omega) = {} & \omega d^{o}(\omega) + \omega \cos(\omega T_1) n_1^{o}(\omega) - \sin(\omega T_1) n_1^{e}(\omega) \\ & + \omega \cos(\omega T_2) n_2^{o}(\omega) - \sin(\omega T_2) n_2^{e}(\omega). \end{aligned}$$

Figure 1.13 shows that $\delta_r(\omega)$ and $\delta_i(\omega)$ are interlacing. Thus, we conclude that the quasipolynomial $\delta(s)$ is Hurwitz.

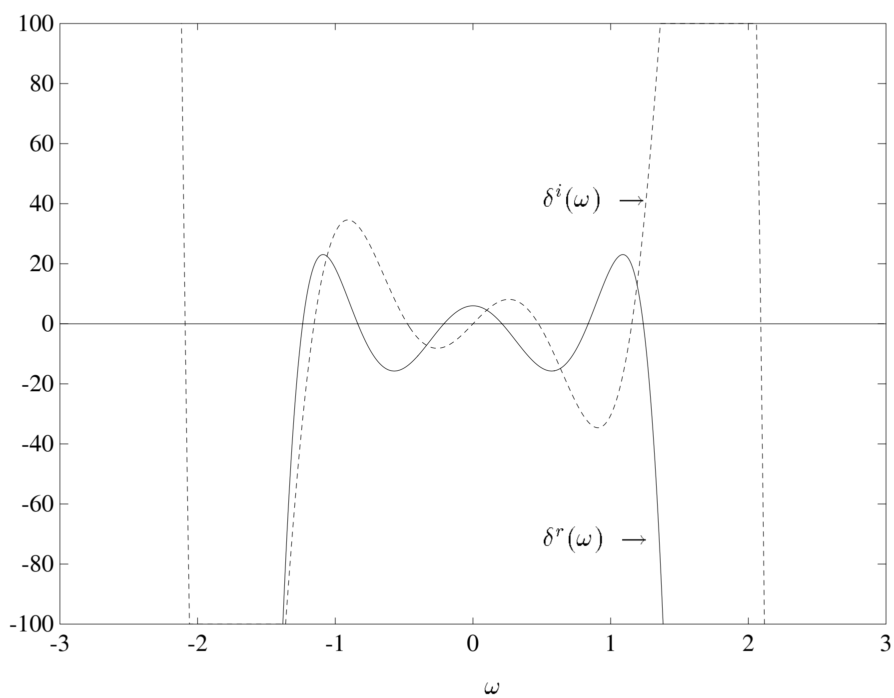


**Figure 1.13.** Interlacing property for a quasipolynomial (Example 1.11)

## 1.7 EXERCISES

The purpose of Problems 1-5 is to illustrate some possible uses of the Boundary Crossing Theorem and of the Interlacing (Hermite-Biehler) Theorem. In all the problems the word stable means *Hurwitz-stable*, and all stable polynomials are assumed to have *positive* coefficients. The following standard notation is also used: Let $P(s)$ be any polynomial. Denote by $P^{\text{even}}(s)$ the even part of $P(s)$, and by $P^{\text{odd}}(s)$ its odd part, that is

$$P(s) = \underbrace{(p_0 + p_2 s^2 + p_4 s^4 + \cdots)}_{P^{\text{even}}(s)} + \underbrace{(p_1 s + p_3 s^3 + p_5 s^5 + \cdots)}_{P^{\text{odd}}(s)}.$$

Also denote,

$$P^e(\omega) := P^{\text{even}}(j\omega) = p_0 - p_2\omega^2 + p_4\omega^4 + \cdots$$
$$P^o(\omega) := \frac{P^{\text{odd}}(j\omega)}{j\omega} = p_1 - p_3\omega^2 + p_5\omega^4 + \cdots.$$

Also for any polynomial $Q(t)$ of the variable $t$, the notation $\left[Q\right]'(t)$ designates the derivative of $Q(t)$ with respect to $t$.

**1.1** Suppose that the polynomial $P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$ is stable. Prove that the following two polynomials are also stable:

$$Q(s) = P^{\text{even}}(s) + [P^{\text{even}}]'(s) = p_0 + 2p_2 s + p_2 s^2 + 4p_4 s^3 + \cdots,$$

$$R(s) = \left[P^{\text{odd}}\right]'(s) + P^{\text{odd}}(s) = p_1 + p_1 s + 3p_3 s^2 + p_3 s^3 + \cdots .$$

**Hint**: In both cases use the Hermite-Biehler (i.e. Interlacing) Theorem. First, check that part a) of the theorem is trivially satisfied. To prove part b) of the theorem for $Q(s)$, show that $-\omega Q^o(\omega) = \left[P^e\right]'(\omega)$. To conclude use the fact that for any continuous function $f(t)$, if $f(a) = f(b) = 0$ for some real numbers $a < b$, and if $f$ is differentiable on the interval $[a, b]$, then there exists a real number $c$ such that:

$$a < c < b \;\text{ and }\; f'(c) = 0. \;\text{(Rolle's Theorem).}$$

Proceed similarly to prove part b) of the theorem for $R(s)$.

**1.2** Suppose that:

$$P_1(s) = P^{\text{even}}(s) + P_1^{\text{odd}}(s)$$
$$P_2(s) = P^{\text{even}}(s) + P_2^{\text{odd}}(s)$$

are two stable polynomials with the same 'even' part. Show that the polynomial $Q_{\lambda,\mu}(s)$ defined by:

$$Q_{\lambda,\mu}(s) = P^{\text{even}}(s) + \lambda P_1^{\text{odd}}(s) + \mu P_2^{\text{odd}}(s),$$

is stable for all $\lambda > 0$ and $\mu > 0$.
**Hint**: You can use directly the Boundary Crossing Theorem. In doing so, check that

$$Q^o_{\lambda,\mu}(\omega) = \lambda P^o_1(\omega) + \mu P^o_2(\omega),$$

and use the fact that the sign of $P^o_i(\omega)$ alternates at the positive roots of $P^e(\omega)$, and does not depend on $i$.

**1.3** Suppose that $P(s)$ is a stable polynomial:

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n, \; n \geq 1.$$

Write as usual: $P(j\omega) = P^e(\omega) + j\omega P^o(\omega)$.

a) Show that the polynomial $Q_\lambda(s)$ associated with:

$$Q^e_\lambda(\omega) = P^e(\omega) - \lambda P^o(\omega),$$

and,

$$Q^o_\lambda(\omega) = P^o(\omega),$$

is stable for all $\lambda$ satisfying $0 \leq \lambda < \frac{p_0}{p_1}$.

b) Deduce from a) that

$$\frac{p_{2k}}{p_{2k+1}} \geq \frac{p_0}{p_1}, \quad \text{for all} \quad k \geq 1.$$

**Hint**: In part a) use the Boundary Crossing Theorem. In doing so, check carefully that the highest coefficient of $Q_\lambda(s)$ equals $p_n$ for all values of $\lambda$ and therefore the degree of $Q_\lambda(s)$ is $n$ no matter what $\lambda$ is.

To prove b) use the fact that since $Q_\lambda(s)$ is stable for all $\lambda$ in the range $[0, \frac{p_0}{p_1})$, then in particular the coefficients of $Q_\lambda(s)$ must remain positive for all values of $\lambda$.

**1.4** Prove that if $P(s)$ is a stable polynomial then:

$$\frac{p_0}{2p_2} \leq \frac{p_{2k}}{(2k+2)p_{2k+2}}, \quad \text{for all } k \geq 1.$$

**Hint**: Use Exercise 1.1 and apply part (b) of Exercise 1.3.

**1.5** Let $P(s)$ be an arbitrary polynomial of degree $n > 0$. Prove that if $P(s)$ satisfies part b) of the interlacing condition, but violates part a) in the sense that $p_n p_{n-1} < 0$, then $P(s)$ is completely unstable, that is $P(s)$ has all its roots in the *open right half plane*. Give the Schur counterpart of this result.
**Hint**: Consider the polynomial $Q(s) = P(-s)$.

**1.6** Show, by using the Boundary Crossing Theorem that the set $\mathcal{H}_n{}^+$ consisting of $n^{th}$ degree Hurwitz polynomials with positive coefficients is connected. A similar result holds for Schur stable polynomials and in fact for any stability region $\mathcal{S}$.

**1.7** Write $s = \sigma + j\omega$ and let the stability region be defined by

$$\mathcal{S} := \{s : \sigma < \omega^2 - 1\}.$$

Consider the parametrized family of polynomials

$$p(s,\lambda) = s^3 + (10 - 14\lambda)s^2 + (65\lambda^2 - 94\lambda + 34)s$$
$$+(224\lambda^2 - 102\lambda^3 - 164\lambda + 40), \quad \lambda \in [0,1].$$

Verify that $p(s,0)$ is stable and $p(s,1)$ is unstable with respect to $\mathcal{S}$. Use the Boundary Crossing Theorem to determine the range of values of $\lambda$ for which the family is stable and the points on the stability boundary through which the roots cross over from stability to instability.
**Hint**: Consider a point $(\sigma, \omega)$ on the stability boundary and impose the condition for this point to be a root of $p(s,\lambda)$ in the form of two polynomial equations in $\omega$ with coefficients which are polynomial functions of $\lambda$. Now use the eliminant to obtain a polynomial equation in $\lambda$ the roots of which determine the possible values of $\lambda$ for which boundary crossing may occur.

**1.8** Use Algorithm 1.1 to check that the following complex polynomial is a Schur polynomial,

$$P(z) = 32z^4 + (8 + 32j)z^3 + (-16 + 4j)z^2 - (2 + 8j)z + 2 - j.$$

Use Algorithm 1.3 to check that the following complex polynomial is a Hurwitz polynomial,

$$P(s) = s^4 + 6s^3 + (14 + j)s^2 + (15 + 3j)s + 2j + 6.$$

**1.9** Show using the Hermite Biehler Theorem that the polynomial $P(s) + jQ(s)$ with $P(s)$ and $Q(s)$ being real polynomials has no zeroes in the lower half plane $Im\ s \leq 0$ if and only if

i) $P(s)$ and $Q(s)$ have only simple real zeroes which interlace and

ii) $Q'(s_0)P(s_0) - P'(s_0)Q(s_0) > 0$ for some point $s_0$ on the real axis.

**Hint:** Use the monotonic phase property.

**1.10** Rework Example 1.11 with $d(s)$, $n_1(s)$, $n_2(s)$ and $T_1$ as before. Determine the maximal value of $T_2$ for which the system is stable by using the interlacing property.

## 1.8 NOTES AND REFERENCES

The material of section 1.2 is mainly based on Marden [175] and Dieudonné [82]. In particular the statement of Theorem 1.2 (Rouché's Theorem) follows Marden's book very closely. The Boundary Crossing Theorem and the unified proof of the Hermite-Biehler Theorem, Routh and Jury tests based on the Boundary Crossing Theorem were developed by Chapellat, Mansour and Bhattacharyya [68] and the treatment given here closely follows this reference. The stability theory for a single polynomial bears countless references, going back to the last century. For a modern exposition of stability theory, the best reference remains Gantmacher [101] and to a lesser extent Marden [175]. The Hermite-Biehler Theorem for Hurwitz polynomials can be found in the book of Guillemin [105]. The corresponding theorem for the discrete time case is stated in Bose [43] where earlier references are also given. The complex case was treated in Bose and Shi [49]. Jury's test is described in [124]. Vaidyanathan and Mitra [228] have given a unified network interpretation of classical stability results. It is to be noted that the type of proof given here of Jury's criterion for Schur stability can also be found in the Signal Processing literature in conjunction with lattice filters (see, for example, the book by Haykin [108]). A considerable amount of research has been done on the distribution of the zeros of entire functions and numerous papers can be found in the literature. For a particularly enlightening summary of this research the reader can consult the book of B. Ja. Levin [160]. Theorems 1.13 and 1.14 are due to Pontryagin [192]. The stability theory of time-delay systems is treated in the book [106] by Hale. A unified approach to the proofs of various stability criteria based on the monotonicity of the argument and the Boundary Crossing Theorem has been described by Mansour [169].