Chapter 0

BACKGROUND AND MOTIVATION

In this chapter we make some introductory and motivational remarks describing the problems of robust stability and control. The first section is written for the reader who is relatively unfamiliar with basic control concepts, terminology and techniques. Next, a brief historical sketch of control theory is included to serve as a background for the contents of the book. Finally these contents and their organization are described in some detail.

0.1 INTRODUCTION TO CONTROL

A **control system** is a mechanism which makes certain physical variables of a system, called a **plant**, behave in a prescribed manner, despite the presence of uncertainties and disturbances.

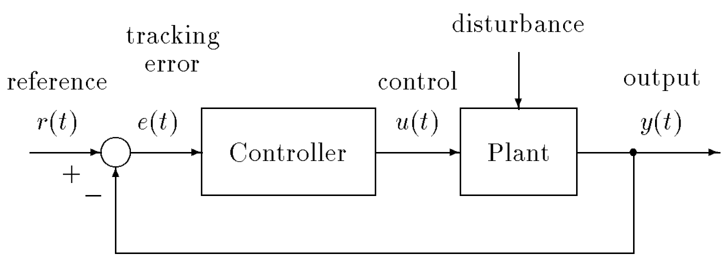

Figure 0.1. Unity feedback control system

The plant or system to be controlled is a dynamic system, such as an aircraft, chemical process, machine tool, electric motor or robot, and the control objective is to make the system output $y(t)$ follow a reference input $r(t)$ as closely as possible despite the disturbances affecting the system. Automatic control is achieved by employing feedback. In a **unity feedback** or **closed loop system** (see Figure 0.1), control action is taken by applying the input $u(t)$ to the plant, and this is based on the difference, at each instant of time t, between the actual value of the plant output to be controlled $y(t)$ and the prescribed reference or desired output $r(t)$. The **controller** is designed to drive this difference, the **tracking error** $e(t)$ to zero. Such control systems are also called **regulators** or **servomechanisms**.

Stability and **performance** are two of the fundamental issues in the design, analysis and evaluation of control systems. Stability means that, in the absence of external excitation, all signals in the system decay to zero. Stability of the closed loop system is an absolute requirement since its absence causes signals to grow without bound, eventually destroying and breaking down the plant. This is what happens when an aircraft crashes, or a satellite spins out of control or a nuclear reactor core heats up uncontrollably and melts down. In many interesting applications the open loop plant is unstable and the job of feedback control is to **stabilize** the system. While feedback is necessary to make the system track the reference input, its presence in control systems causes the potential for instability to be everpresent and very real. We shall make this notion more precise below in the context of servomechanisms. In engineering systems it is of fundamental importance that control systems be designed so that stability is preserved in the face of various classes of **uncertainties**. This property is known as **robust stability**.

The **performance** of a system usually refers to its ability to track reference signals closely and reject disturbances. A well designed control system or servomechanism should be capable of tracking all reference signals belonging to a **class** of signals, without excessive error, despite various types of uncertainties. In other words the worst case performance over the uncertainty set should be acceptable. This is, roughly speaking, referred to as **robust performance**.

The uncertainties encountered in control systems are both in the environment and within the system. In the first place the reference signal to be tracked is usually not known beforehand. Then there are disturbance signals tending to offset the tracking. For example the load torque on the shaft of an electric motor, whose speed is to be maintained constant, can be regarded as a disturbance.

As far as the system is concerned the main source of uncertainty is the behaviour of the plant. These uncertainties can occur, for example, due to changes of operating points, as in an aircraft flying at various altitudes and loadings, or a power system delivering power at differing load levels. Large changes can also occur in an uncontrolled fashion for example, when sensor or actuator failures occur. The complexity of even the simplest plants is such that any mathematical representation of the system must include significant uncertainty.

In analysis and design it is customary to work with a **nominal** mathematical model. This is invariably assumed to be **linear** and **time invariant**, because this is the only class of systems for which there exists any reasonably general design theory. Nevertheless, such models are usually a gross oversimplification and it is therefore necessary to test the validity of any proposed design by testing its performance when the model is significantly different from the nominal.

In summary the requirements of robust stability and performance are meant to ensure that the control system functions **reliably** despite the presence of significant uncertainty regarding the model of the system and the precise description of the external signals to be tracked or rejected. In the next few subsections we discuss these requirements and their impact on control system modelling, design, and evaluation, in greater detail.

0.1.1 Regulators and Servomechanisms

The block diagram of a typical feedback control system is shown in Figure 0.2.

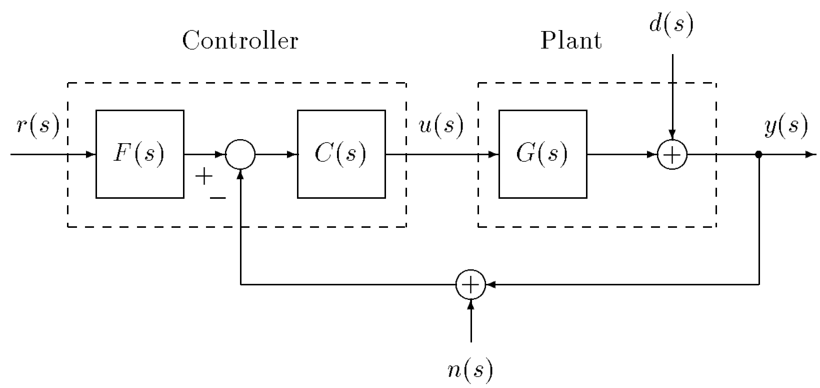

Figure 0.2. General feedback configuration

Here, as in the rest of the book, we will be considering linear time invariant systems which can be represented, after Laplace transformation, in terms of the complex variable s. In Figure 0.2 the vectors u and y represent (the Laplace transforms of) the plant inputs and outputs respectively, d represents disturbance signals reflected to the plant output, n represents measurement noise and r represents the reference signals to be tracked. The plant and the feedback controller are represented by the rational proper transfer matrices $G(s)$, and $C(s)$ respectively, while $F(s)$ represents a feedforward controller or prefilter. The usual problem considered in control theory assumes that $G(s)$ is given while $C(s)$ and $F(s)$ are to be designed. Although every control system has a unique structure and corresponding signal flow representation the standard system represented above is general enough that it captures the essential features of most feedback control systems.

0.1.2 Performance: Tracking and Disturbance Rejection

In the system of Figure 0.2 the plant output $y(t)$ is supposed to follow or track the command reference signal $r(t)$ as closely as possible despite the disturbances

$d(t)$ and the measurement noise $n(t)$. The exogenous signals r, d, n are of course not known exactly as time functions but are known qualitatively. Based on this knowledge the control designer uses certain classes of test signals to evaluate any proposed design. A typical design specification could state that the system is to have zero steady state error whenever the command reference r and disturbance d consist of steps and ramps of arbitrary and unknown magnitude and slope. Often the measurement noise n is known to have most of its energy lying in a frequency band $[\omega_1, \omega_2]$. In addition to steps and ramps the signals r and d would have significant energy in a low frequency band $[0, \omega_0]$. A reasonable requirement to impose is that y track r with "small" error for every signal in this uncertainty class without excessive use of control energy.

There are two approaches to achieving this objective. One approach is to require that the average error over the uncertainty class be small. The other approach is to require that the error response to the worst case exogenous signal from the given class be less than a prespecified value. These correspond to regarding the control system as an operator mapping the exogenous signals to the error and imposing bounds on the norms of these operators or transfer functions. We examine these ideas more precisely by deriving the equations of the closed loop system. These are:

$$\begin{aligned} y(s) &= G(s)u(s) + d(s) \\ u(s) &= C(s)\left[F(s)r(s) - n(s) - y(s)\right] \\ e(s) &:= r(s) - y(s) \qquad \text{(tracking error)}. \end{aligned}$$

Solving the above equations, (we drop the explicit dependence on s whenever it is obvious)

$$\begin{aligned} y &= [I + GC]^{-1} d + [I + GC]^{-1} GCFr - [I + GC]^{-1} GCn \\ e &= -[I + GC]^{-1} d + \left[I - (I + GC)^{-1} GCF\right] r + [I + GC]^{-1} GCn. \end{aligned}$$

Introducing the **sensitivity function** $S(s)$ and the **complementary sensitivity function** $T(s)$:

$$\begin{aligned} S &:= [I + GC]^{-1} \\ T &:= [I + GC]^{-1} GC \end{aligned}$$

we can rewrite the above equations more compactly as

$$\begin{aligned} y &= Sd + TFr - Tn \\ e &= -Sd + (I - TF)r + Tn. \end{aligned}$$

As we have mentioned, in many practical cases the control system is of **unity feedback type** which means that the feedback controller is driven only by the tracking error. In such cases the feedforward element $F(s)$ is the identity and the system equations simplify to

$$\begin{aligned} y &= Sd + Tr - Tn \\ e &= -Sd + Sr + Tn. \end{aligned}$$

Ideally the response of the tracking error e to each of the signals r, d and n should be zero. This translates to the requirement that the transfer functions $S(s)$ and $T(s)$ should both be "small" in a suitable sense. However we see that

$$S(s) + T(s) = I, \qquad \text{for all } s \in \mathbb{C}$$

and thus there is a **built-in trade off** since S and T cannot be small at the same values of s. This trade off can be resolved satisfactorily if the frequency bands in which r and d lie is disjoint from that in which n lies. In this case $S(j\omega)$ should be kept small over the frequency band $[0, \omega_0]$ to provide accurate tracking of low frequency signals and $T(j\omega)$ should be kept small over the frequency band $[\omega_1, \omega_2]$, to attenuate noise.

Suppose for the moment, that the plant is a single-input, single-output (SISO) system. Then G, C, F, S and T are scalar transfer functions. In this case, we have, for the unity feedback case,

$$u = \frac{T}{G}r - \frac{T}{G}n - \frac{T}{G}d.$$

From Parseval's Theorem we have

$$\int_0^\infty u^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |u(j\omega)|^2 d\omega$$

and so we see that the control signal energy can be kept small by keeping the magnitude of $T(j\omega)$ as small as possible over the frequency bands of r, d and n. This obviously conflicts with the requirement of making $S(j\omega)$ small over the band $[0, \omega_0]$. Thus for good tracking, $T(j\omega)$ must necessarily be large over $[0, \omega_0]$ but then should rapidly decrease with increasing frequency to "conserve" control energy.

The above arguments also extend to multiinput-multioutput (MIMO) systems. We know that the "gain" of a SISO linear system, with transfer function $M(s)$, for a sinusoidal input of radian frequency ω is given by $|M(j\omega)|$. When $M(s)$ is a proper transfer function with poles in the open left half plane we can define the maximum gain, which is also the H_∞ norm of M:

$$\sup_{0 \leq \omega \leq \infty} |M(j\omega)| := \|M\|_\infty.$$

For a MIMO system, $M(j\omega)$ is a *matrix* and the corresponding H_∞ norm of M is defined in terms of the maximum singular value $\bar{\sigma}_M(\omega)$ of $M(j\omega)$:

$$\sup_{0 \leq \omega \leq \infty} \bar{\sigma}_M(\omega) := \|M\|_\infty.$$

The above requirements on S and T can be stated in terms of norms. Let the stable transfer matrix $W(s)$ denote a low pass filter transfer function matrix with

$$\begin{aligned} W(j\omega) &\simeq I, \qquad \text{for } \omega \in [0, \omega_0] \\ W(j\omega) &\simeq 0, \qquad \text{for } \omega \geq \omega_0. \end{aligned}$$

Similarly let the stable transfer function matrix $V(s)$ denote a bandpass filter with

$$\begin{aligned} V(j\omega) &\simeq I, \quad &&\text{for } \omega \in B := [\omega_1, \omega_2] \\ V(j\omega) &\simeq 0, \quad &&\text{for } \omega \notin B. \end{aligned}$$

Then the requirement that S be small over the band $[0, \omega_0]$ and that T be small over the band B can be stated in a **worst case** form:

$$\begin{aligned} \|WS\|_\infty &\leq \epsilon_1 \\ \|VT\|_\infty &\leq \epsilon_2. \end{aligned}$$

0.1.3 Quantitative feedback theory

In classical control theory, the objective is to find a feedback compensator to satisfy the above or similar types of design objectives for the nominal system. This type of approach was extended to the domain of uncertain systems by the **Quantitative Feedback Theory (QFT)** approach pioneered by Horowitz [120] in the early 1960's. In the QFT approach, which we briefly outline here, one considers the plant model $G(s, \mathbf{p})$ with the uncertain parameter $\mathbf{p}$ lying in a set Ω and a control system configuration including feedback *as well as* feedforward elements. The quality of tracking is measured by the closeness to unity of the transfer function relating y to r, the sensitivity of the feedback loop by the transfer function S, and the noise rejection properties of the loop by the transfer function T. Typically the tracking performance and sensitivity reduction specifications are to be robustly attained over a low frequency range $[0, \omega_0]$ and the noise rejection specifications are to be achieved over a high frequency range $[\omega_1, \infty)$. For SISO systems the specifications assume the forms

$$m_1(\omega) \leq \left| F(j\omega) \frac{C(j\omega)G(j\omega, \mathbf{p})}{1 + C(j\omega)G(j\omega, \mathbf{p})} \right| < m_2(\omega), \quad \text{for } \omega \in [0, \omega_0]$$

$$\begin{aligned} |S(j\omega, \mathbf{p})| &< l_1(\omega), \quad &&\text{for } \omega \in [0, \omega_0] \\ |T(j\omega, \mathbf{p})| &< l_2(\omega), \quad &&\text{for } \omega \in [\omega_1, \infty) \end{aligned}$$

for suitably chosen frequency dependent functions $m_1(\omega), m_2(\omega), l_1(\omega)$ and $l_2(\omega)$. Robust design means that the above performance specifications are met for all $\mathbf{p} \in \Omega$ by a suitable choice of feedback and feedforward compensators which stabilize the feedback system for all $\mathbf{p} \in \Omega$. An additional requirement in QFT design is that the bandwidth of the feedback controller $C(s)$ be as small as possible.

The advantage of the feedforward compensator is roughly, that it frees up the feedback controller $C(s)$ to carry out certain closely related tasks such as robust stabilization, sensitivity and noise reduction, while the feedforward controller can subsequently attempt to satisfy the tracking requirement. This freedom allows for a better chance of finding a solution and also reducing the bandwidth of the feedback controller, often referred to in QFT theory as the "cost of feedback". QFT design is typically carried out by loopshaping the nominal loop gain $L_0 = G_0 C$, on the Nichols chart, to satisfy a set of bounds at each frequency, with each bound reflecting a performance specification. Once such an L_0 is found, the controller $C(s)$ can be found by "dividing" out the plant and the feedforward filter $F(s)$ can be computed from the tracking specification. The QFT approach thus has the flavor of classical frequency domain based design, however with the added ingredient of robustness. QFT techniques have been extended to multivariable and even nonlinear systems.

0.1.4 Perfect Steady State Tracking

Control systems are often evaluated on the basis of their ability to track certain test signals such as steps and ramps accurately. To make the system track arbitrary steps with zero steady state error it is enough to use unity feedback ($F = I$) and to place one integrator or pole at the origin in each input channel of the error driven controller $C(s)$. This causes each element of $S(s)$ to have a zero at the origin, as required, to cancel the $s = 0$ poles of r and d, to ensure perfect steady state tracking. Likewise, if steps and ramps are to be tracked then *two* integrators, or poles at $s = 0$ must be included.

These types of controllers have been known from the early days of automatic control in the last century when it was known that steady state tracking and disturbance rejection of steps, ramps etc. could be achieved by incorporating enough **integral action** in a unity feedback control loop (see Figure 0.1). When the closed loop is stable such a controller *automatically converges to the correct control input* which is required to maintain zero error in the steady state. A precise model of the plant is not needed nor is it necessary to know the plant parameters; all that is required is that the closed loop be robustly stable. The generalization of this principle, known as the **Internal Model Principle**, states that in order to track with zero steady state error the control system should contain, internal to the feedback loop, a model signal generator of the unstable external reference and disturbances and the controller should be driven by the tracking error.

Although a controller incorporating integral action, or an unstable internal model, allows the system to achieve reliable tracking, it potentially makes the system **more prone to instability** because the forward path is **rendered unstable**. The designer therefore faces the task of setting the rest of the controller parameters, the part that stabilizes the closed loop system, so that closed loop stability is preserved under various contingencies. In practice this is usually done by using a fixed linear time invariant model of the plant, called the **nominal model**, stabilizing this model, and ensuring that adequate **stability margins** are attained about this nominal in order to preserve closed loop stability under all the possible system uncertainties.

0.1.5 The Characteristic Polynomial and Nyquist Criterion

For linear time invariant control systems, stability is characterized by the root locations of the characteristic polynomial. Consider the standard feedback control system shown in Figure 0.3. If the plant and controller are linear, time invariant

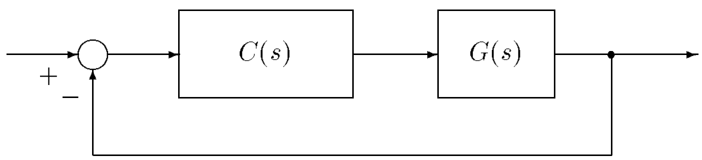

Figure 0.3. Standard feedback system

dynamic systems they can be described by their respective real rational transfer function matrices $G(s)$ and $C(s)$. Suppose that $\mathbf{p}$ is a vector of physical parameters contained in the plant and that $\mathbf{x}$ denotes a vector of adjustable design parameters contained in the controller. For continuous time systems we therefore write

$$C(s) = N_c(s, \mathbf{x})D_c^{-1}(s, \mathbf{x}) \quad \text{and} \quad G(s) = D_p^{-1}(s, \mathbf{p})N_p(s, \mathbf{p})$$

where N_c, D_c, N_p and D_p are polynomial matrices in the complex variable s. The **characteristic polynomial** of the closed loop control system is given by

$$\delta(s, \mathbf{x}, \mathbf{p}) = \det\left[D_c(s, \mathbf{x})D_p(s, \mathbf{p}) + N_c(s, \mathbf{x})N_p(s, \mathbf{p})\right].$$

System stability is equivalent to the condition that the characteristic polynomial have all its roots in a certain region $\mathcal{S}$ of the complex plane. For continuous time systems the stability region $\mathcal{S}$ is the open left half, $\mathbb{C}^-$, of the complex plane and for discrete time systems it is the open unit disc, $\mathbb{D}^1$, centered at the origin.

As an example, suppose the controller is a PID controller and has transfer function

$$C(s) = K_P + \frac{K_I}{s} + K_D s$$

and the plant has transfer function $G(s)$ which we write in two alternate forms

$$G(s) = \frac{\mu(s-\alpha)}{(s-\beta)(s-\gamma)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}. \tag{0.1}$$

The characteristic polynomial is given by

$$\delta(s) = s(s-\beta)(s-\gamma) + \mu(s-\alpha)(K_P s + K_I + K_D s^2)$$

and also by

$$\delta(s) = s(b_2 s^2 + b_1 s + b_0) + (a_1 s + a_0)(K_P s + K_I + K_D s^2).$$

If a state space model of the strictly proper plant and proper controller are employed, we have, setting the external input to zero, the following set of differential equations describing the system in the time domain:

$$\text{Plant}: \quad \begin{aligned} \dot{x}_p &= A_p x_p + B_p u \\ y &= C_p x_p \end{aligned}$$

$$\text{Controller}: \quad \begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}$$

$$\text{Closed loop}: \quad \begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$

and the closed loop characteristic polynomial is given by

$$\delta(s) = \det\left[sI - A_{cl}\right].$$

Similar equations hold in the discrete time case.

If one fixes the parameters at the nominal values $\mathbf{p} = \mathbf{p}^0$ and $\mathbf{x} = \mathbf{x}^0$ the root locations of the characteristic polynomial indicate whether the system is stable or not. In control theory it is known that the nominal plant transfer function can always be stabilized by some fixed controller $\mathbf{x} = \mathbf{x}^0$ unless it contains unstable cancellations. For state space systems the corresponding requirement is that the unstable system modes of A be controllable from u and observable from y.

An alternative and sometimes more powerful method of stability verification is the **Nyquist criterion**. Here one seeks to determine conditions on the open loop system $G(s)C(s)$ that guarantee that the closed loop will be stable. The answer is provided by the Nyquist criterion. Consider the SISO case and introduce the Nyquist contour consisting of the imaginary axis along with a semicircle of infinite radius which together enclose the right half of the complex plane (RHP). The directed plot of $G(s)C(s)$, evaluated as s traverses this contour in the clockwise direction is called the **Nyquist plot**. The Nyquist criterion states that the closed loop is stable if and only if the Nyquist plot encircles the $-1 + j0$ point P times in the counterclockwise direction, where P is the number of RHP poles of $G(s)C(s)$. A similar condition can be stated for multivariable systems. The power of the Nyquist criterion is due to the fact that the Nyquist plot is just the frequency response of the open loop system and can often be measured experimentally, thus eliminating the need for having a detailed mathematical model of the system.

The design of a stabilizing controller for the nominal plant can be accomplished in a variety of ways such as classical control, Linear Quadratic Optimal state feedback implemented by observers, and pole placement controllers. The more difficult and unsolved problem is achieving stability in the presence of uncertainty, namely robust stability.

0.1.6 Uncertainty Models and Robustness

The linear time invariant models that are usually employed are **approximations** which are made to render the design and analysis of complex systems tractable. In reality most systems are nonlinear and a linear time invariant model is obtained by fixing the operating point and linearizing the system equations about it. As the operating point changes so do the parameters of the corresponding linear approximation. Thus there is **significant uncertainty** regarding the "true" plant model, and it is necessary that a controller that stabilizes the system do so for the entire range of expected variations in the plant parameters. In addition other reasonable, but less structured perturbations of the plant model must also be tolerated without disrupting closed loop stability. These unstructured perturbations arise typically from truncating a complex model by retaining only some of the dominant modes, which usually lie in the low frequency range. Therefore unstructured uncertainty is usually operational in a high frequency range. The tolerance of both these types of uncertainty is,qualitatively speaking, the problem of robust stability.

In classical control design the above problem is dealt with by means of the Bode or Nyquist plots and the notions of **gain or phase margin**. It is customary to fix the controller structure based on existing hardware and software constraints and to optimize the design over the numerical values of the fixed number of adjustable controller parameters and any adjustable plant parameters. Thus most often the controller to be designed is constrained to be of the proportional, integral, or PID (proportional, integral and derivative) lag, lead or lead-lag types. Robustness was interpreted to mean that the closed loop system remained stable, despite an adequate amount of uncertainty about the nominal Bode and Nyquist plots, because these represent the measured data from the physical system. This requirement in turn was quantified roughly as the gain and phase margin of the closed loop system.

Gain and Phase Margin

In the gain margin calculation one considers the loop breaking m (see Figure 0.4) and inserts a gain at that point. Thus the open loop plant transfer function $G(s)$ is replaced by its perturbed version $kG(s)$ and one determines the range of excursion of k about its nominal value of 1 for which stability is preserved (Figure 0.5). Likewise in the phase margin calculation one replaces $G(s)$ by $e^{-j\theta}G(s)$ (Figure 0.6) and determines the range of θ for which closed loop stability is preserved.

The importance of these concepts lie in the fact that they provide measures of the robustness of a given designed controller since they indicate the **distance from instability** or **stability margin** of the closed-loop system under gain and phase perturbations of the plant's frequency response. They can be readily determined from the Nyquist or Bode plots and are thus useful in design. However it is to be emphasized that this notion of distance to instability does not extend in an obvious way to the realistic case where *several independent parameters* are simultaneously subject to perturbation. We also remark here that simultaneous gain and phase perturbations constitute a mixed real-complex perturbation problem which has only

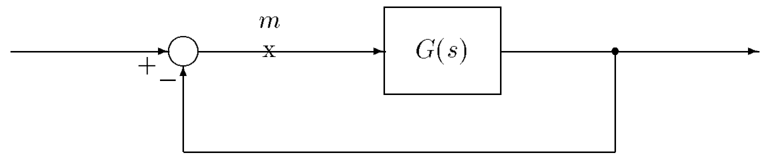

Figure 0.4. Unity feedback system

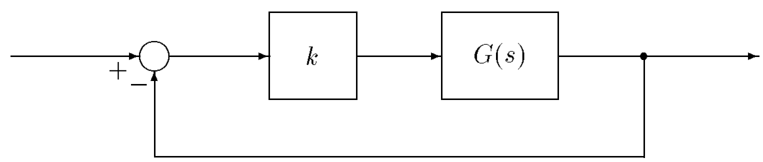

Figure 0.5. Gain margin calculation

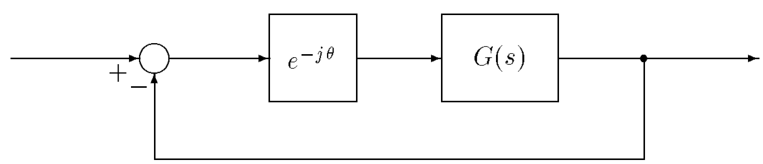

Figure 0.6. Phase margin calculation

recently been effectively analyzed by the techniques described in this book. Finally the analysis of robust performance can be reduced to robust stability problems involving frequency dependent perturbations which cannot be captured by gain and phase margin alone.

Parametric Uncertainty

Consider equation (0.1) in the example treated above. The parametric uncertainty in the plant model may be expressed in terms of the gain μ and the pole and zero locations α, β, γ. Alternatively it may be expressed in terms of the transfer function coefficients a_0, a_1, b_0, b_1, b_2. Each of these sets of plant parameters are subject to variation. The PID gains are, in this example, the adjustable controller parameters. Thus, in the first case, the uncertain plant parameter vector is

$$\mathbf{p}_1 = \left[\begin{array}{cccc} \mu & \alpha & \beta & \gamma \end{array} \right],$$

the controller parameter vector is

$$\mathbf{x} = \left[\begin{array}{ccc} K_P & K_I & K_D \end{array} \right]$$

and the characteristic polynomial is

$$\delta(s, \mathbf{p}_1, \mathbf{x}) = s(s-\beta)(s-\gamma) + \mu(s-\alpha)(K_P s + K_I + K_D s^2).$$

In the second case the uncertain plant parameter vector is

$$\mathbf{p}_2 = \left[\begin{array}{ccccc} a_0 & a_1 & b_0 & b_1 & b_2 \end{array} \right]$$

and the characteristic polynomial is

$$\delta(s, \mathbf{p}_2, \mathbf{x}) = s(b_2 s^2 + b_1 s + b_0) + (a_1 s + a_0)(K_P s + K_I + K_D s^2).$$

In most control systems the controller remains fixed during operation while the plant parameter varies over a wide range about a nominal value $\mathbf{p}^0$. The term **robust parametric stability** refers to the ability of a control system to maintain stability despite such large variations. Robustness with respect to parameter variations is necessary because of inherent uncertainties in the modelling process and also because of actual parameter variations that occur during the operation of the system. In the design process, the parameters $\mathbf{x}$ of a controller are regarded as adjustable variables and robust stability with respect to these parameters is also desirable in order to allow for adjustments to a nominal design to accommodate other design constraints. Additionally, it is observed that if *insensitivity* with respect to plant parameters is obtained, it is generally obtained at the expense of *heightened sensitivity* with respect to controller parameters. Thus, in many cases, it may be reasonable to lump the plant and controller parameters into a global parameter vector $\mathbf{p}$, with respect to which the system performance must be evaluated.

The **maximal** range of variation of the parameter $\mathbf{p}$, measured in a suitable norm, for which closed loop stability is preserved is the **parametric stability margin**, and is a measure of the performance of the controller $\mathbf{x}$. In other words

$$\rho_x := \sup\left\{\alpha : \delta(s, \mathbf{x}, \mathbf{p}) \ \text{ stable}, \quad \|\mathbf{p} - \mathbf{p}^0\| < \alpha\right\}$$

is the parametric stability margin of the system with the controller $\mathbf{x}$. Since ρ represents the *maximal perturbation*, it is indeed a bona fide performance measure to compare the robustness of two proposed controller designs $\mathbf{x}_1$ and $\mathbf{x}_2$. This calculation is an important aid in analysis and design.

Consider a family of polynomials $\delta(s, \mathbf{p})$ of degree n, where the real parameter $\mathbf{p}$ ranges over a connected set Ω. If it is known that one member of the family is stable, a useful technique of verifying robust stability of the family is to ascertain that

$$\delta(j\omega, \mathbf{p}) \neq 0, \quad \text{for all } \mathbf{p} \in \Omega, \quad \omega \in \mathbb{R}.$$

This can also be written as the **zero exclusion condition**

$$0 \notin \delta(j\omega, \Omega), \quad \text{for all } \omega \in \mathbb{R}.$$

The zero exclusion condition is exploited in this text to derive various types of robust stability and performance margins.

We note that the closed loop characteristic polynomial coefficients in the above examples, are linear functions of the controller parameter $\mathbf{x}$. On the other hand, in the first case, the characteristic polynomial coefficients are multilinear functions of the uncertain parameter $\mathbf{p}_1$ whereas in the second representation they are linear functions of the uncertain parameter $\mathbf{p}_2$. In these cases the zero exclusion condition can be verified easily and so can stability margins. Motivated by such examples, the majority of robust parametric stability results developed in this book are directed towards the *linear* and *multilinear* dependency cases, which fortunately, fit many practical applications and also turn out to be mathematically tractable.

The problem of determining $\mathbf{x}$ to achieve stability and a prescribed level of parametric stability margin ρ is the *synthesis* problem, and in a mathematical sense is unsolved. However in an engineering sense, many effective techniques exist for robust parametric controller design. In particular the exact calculation of ρ_x can itself be used in an iterative loop to adjust $\mathbf{x}$ to robustify the system.

Nonparametric and Mixed Uncertainty

Nonparametric uncertainty refers to that aspect of system uncertainty associated with unmodelled dynamics, truncation of high frequency modes, nonlinearities and the effects of linearization and even time-variation and randomness in the system. It is often accounted for by, for instance, replacing the transfer function of the plant $G(s)$ by the perturbed version $G(s) + \Delta G(s)$ (additive unstructured uncertainty), and letting $\Delta G(s)$ range over a ball of H_∞ functions of prescribed radius. The problem then is to ensure that the closed loop remains stable under all such perturbations, and the worst case performance is acceptable in some precise sense.

If the plant transfer function $G(s)$ is not fixed but a function of the parameter $\mathbf{p}$ we have the **mixed uncertainty** problem where the plant transfer function is $G(s, \mathbf{p}) + \Delta G(s)$ and stabilization must be achieved while $\mathbf{p}$ ranges over a ball in parameter space and $\Delta G(s)$ ranges over an H_∞ ball.

Another model of nonparametric uncertainty that is in use in control theory is in the **Absolute Stability problem** where a fixed system is perturbed by a family of **nonlinear feedback gains** that are known to lie in a prescribed sector. By replacing the fixed system with a parameter dependent one we again obtain the more realistic mixed uncertainty problem.

There are now some powerful methods available for quantifying the different amounts of parametric and nonparametric uncertainty that can be tolerated in the above situations. These results in fact extend to the calculation of stability and performance margins for control systems containing several physically distinct interconnected subsystems subjected to mixed uncertainties. Problems of this type are treated in Chapters 9 - 14 of this book. To complete this overview we briefly describe in the next two subsections the H_∞ and μ approaches to control design.

0.1.7 H_∞ Optimal Control

In this section we attempt to give a very brief sketch of H_∞ optimal control theory and its connection to robust control. A good starting point is the sensitivity minimization problem where a controller is sought so that the weighted sensitivity function or error transfer function, with the nominal plant, is small in the H_∞ norm. In other words we want to solve the problem

$$\inf_C \|W(s)(I + G_0(s)C(s))^{-1}\|_\infty$$

where the infimum is sought over all stabilizing controllers $C(s)$. A crucial step in the solution is the so-called YJBK parametrization of all rational, proper, stabilizing controllers. Write

$$G_0(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$$

where $N(s), D(s), \tilde{D}(s), \tilde{N}(s)$ are real, rational, stable, proper (RRSP) matrices with $N(s), D(s)$ being right coprime and $\tilde{D}(s), \tilde{N}(s)$ being left coprime over the ring of RRSP matrices. Let $\tilde{A}(s), \tilde{B}(s)$ be any RRSP matrices satisfying

$$\tilde{N}(s)A(s) + \tilde{D}(s)B(s) = I.$$

The YJBK parametrization states that every stabilizing controller for $G_0(s)$ can be generated by the formula

$$C(s) = (A(s) + D(s)X(s))(B(s) - N(s)X(s))^{-1}$$

by letting $X(s)$ range over all RRSP matrices for which $\det[B(s) - N(s)X(s)] \neq 0$. The weighted sensitivity function minimization problem can now be rewritten as

$$\inf_C \|W(s)(I + G_0(s)C(s))^{-1}\|_\infty = \inf_X \|W(s)(B(s) - N(s)X(s))\tilde{D}(s)\|_\infty$$

where the optimization is now over the free parameter $X(s)$ which is only required to be RRSP.

The form of the optimization problem obtained in the above example is not unique to the sensitivity minimization problem. In fact all closed loop transfer functions of interest can be expressed as affine functions of the free parameter $X(s)$. Thus the generic H_∞ optimization problem that is solved in the literature, known as the model matching problem takes the form

$$\inf_X \|Y_1(s) - Y_2(s)X(s)Y_3(s)\|_\infty$$

where all matrices are RRSP. Various techniques have been developed for the solution of this optimization problem. The most important of these are based on inner outer factorization of transfer matrices and matrix Nevanlinna-Pick interpolation theory, and its state space counterpart involving the solution of two Riccati equations.

Suppose that $C(s)$ is a compensator stabilizing the nominal plant $G_0(s)$. We ask whether it stabilizes a family of plants $G_0(s) + \Delta G(s)$ around $G_0(s)$. The family in question can be specified in terms of a frequency dependent bound on the admissible, **additive** perturbations of the frequency response $G_0(j\omega)$

$$||\Delta G(j\omega)|| = ||G(j\omega) - G_0(j\omega)|| \leq |r(j\omega)|$$

where $r(s)$ is a RRSP transfer function. Under the assumption that every admissible $G(s)$ and $G_0(s)$ have the same number of unstable poles it can be easily shown, based on the Nyquist criterion that, $C(s)$ stabilizes the entire family of plants if and only if

$$||C(j\omega)(I + G_0(j\omega)C(j\omega))^{-1}|| \cdot |r(j\omega)| < 1, \quad \text{for all } \omega \in \mathbb{R}. \tag{0.2}$$

The robust stability question posed above can also be formulated in terms of perturbations which are H_∞ functions and lie in a prescribed ball

$$\mathcal{B} := \{\Delta G(s) : ||\Delta G||_\infty \leq \alpha\}$$

in which case the corresponding condition for robust stability is

$$||C(s)(I + G_0(s)C(s))^{-1}||_\infty < \frac{1}{\alpha}. \tag{0.3}$$

If the perturbation of $G_0(s)$ is specified in the **multiplicative** form $G(s) = G_0(s)(I + \Delta G(s))$ where $\Delta G(s)$ is constrained to lie in the H_∞ ball of radius α, and the number of unstable poles of $G(s)$ remains unchanged, we have the robust stability condition

$$||(I + C(s)G_0(s))^{-1}C(s)G_0(s)||_\infty < \frac{1}{\alpha}. \tag{0.4}$$

The conditions (0.2), (0.3), (0.4) can all be derived from the Nyquist criterion by imposing the requirement that the number of encirclements required for stability of the nominal system remain invariant under perturbations. This amounts to verifying that

$$|I + G(j\omega)C(j\omega)| \neq 0, \quad \text{for all } \omega \in \mathbb{R}$$

for deriving (0.2), (0.3), and

$$|I + C(j\omega)G(j\omega)| \neq 0, \quad \text{for all } \omega \in \mathbb{R}$$

for deriving (0.4). The conditions (0.3), (0.4) are examples of the so-called Small Gain Condition and can be derived from a general result called the **Small Gain Theorem**. This theorem states that a feedback connection of stable operators remains stable if and only if the product of the gains is strictly bounded by 1. Thus a feedback loop consisting of a stable system $G_0(s)$ perturbed by stable feedback perturbations $\Delta \in H_\infty$ remains stable, if and only if

$$\bar{\sigma}(\Delta) = \|\Delta\|_\infty < \frac{1}{\|G_0\|_\infty}.$$

Now consider the question of existence of a robustly stabilizing compensator from the set of compensators stabilizing the nominal plant. From the relations derived above we see that we need to determine in each case if a compensator $C(s)$ satisfying (0.3) or (0.4) respectively exists. By using the YJBK parametrization we can rewrite (0.3) as

$$\inf_X \|(A(s) + D(s)X(s))\tilde{D}(s)\|_\infty < \frac{1}{\alpha}$$

and (0.4) as

$$\inf_X \|(A(s) + D(s)X(s))\tilde{N}(s)\|_\infty < \frac{1}{\alpha}.$$

However these problems are precisely of the form of the H_∞ model matching problem. Therefore the standard machinery can be used to determine whether a robustly stabilizing compensator exists within the family of compensators that stabilize $G_0(s)$. From the above discussion it is clear that in the norm-bounded formulation robust stability and robust performance problems are closely related.

0.1.8 μ Theory

The objective of μ theory is to refine the small gain condition derived in H_∞ optimal control, by imposing constraints on the perturbation structure allowed and thus derive robustness results for a more realistic problem. By introducing fictitious signals and perturbation blocks if necessary it is always possible to formulate robust stability and performance problems in a unified framework as robust stability problems with structured feedback perturbations.

The starting point in this approach is the construction of an artificial stable system $M(s)$, which is scaled and defined so that the uncertain elements, which can be real or complex can be pulled out and arranged as a block diagonal feedback perturbation

$$\Delta = diag[\Delta_1, \Delta_2, \cdots, \Delta_m]$$

with each Δ_i being of size $k_i \times k_i$ repeated m_i times and with $\bar{\sigma}(\Delta_i) \leq 1$. Both performance and robust stability requirements can be reduced in this setting to the condition

$$\det[I + M(j\omega)\Delta(j\omega)] \neq 0, \quad \text{for all } \omega \in \mathbb{R}. \tag{0.5}$$

In the above problem formulation the matrices representing the perturbations Δ_i can be of arbitrary size and can be real or complex and can have repeated parameters. The condition (0.5) is similar to the small gain condition derived earlier with the important difference that the allowed perturbation matrices Δ must respect the block diagonal structure imposed. In this general setting the verification of the inequality (0.5) is a N-P complete problem and is therefore hard for problems of significant dimension. In the special case where the Δ_i consist only of complex blocks a sufficient condition (necessary and sufficient condition when $n \leq 3$) for the the inequality (0.5) to hold is

$$\inf_{D} \bar{\sigma}(DM(j\omega)D^{-1}) < 1, \quad \text{for all } \omega \in \mathbb{R} \tag{0.6}$$

where the matrix D is a real block diagonal matrix which possesses the same structure as Δ. The attractive feature about the second problem is that it is convex in D. Techniques of verifying this inequality constitutes the μ analysis machinery.

To proceed to the question of synthesis one can now write out the stable system $M(s)$ in terms of the compensator $C(s)$ or its YJBK representation. This leads to the problem

$$\inf_{X} \inf_{D} \|(D[Y_1(s) - Y_2(s)X(s)Y_3(s)]D^{-1})\|_{\infty} < 1. \tag{0.7}$$

In μ synthesis one attempts to find a solution to the above problem by alternating between D and $X(s)$. This is known as D-K iteration. Whenever D is fixed we have a standard model matching problem which provides the optimum $X(s)$. However the structured matrix D must be chosen to guarantee that the condition (0.7) has a solution and this is difficult. In the case where real parameter uncertainty exists the Δ_i containing real numbers would have to be replaced, in this approach, by complex parameters, which inherently introduces conservatism.

In the next section we give a rapid sketch of the history of control theory tracing the important conceptual developments. This is done partly to make the reader appreciate more fully the significance of the present developments.

0.2 HISTORICAL BACKGROUND

Control theory had its beginnings in the middle of the last century when Maxwell published his paper "On Governors" [178] in 1868. Maxwell's paper was motivated by the need to understand and correct the observed unstable (oscillatory) behaviour of many locomotive steam engines in operation at the time. Maxwell showed that the behaviour of a dynamic system could be approximated in the vicinity of an equilibrium point by a *linear* differential equation. Consequently the stability or instability of such a system could be determined from the location of the roots of the *characteristic equation* of this linear differential equation. The speed of locomotives was controlled by centrifugal governors and so the problem was to determine the design parameters of the controller (flyball mass and inertia, spring tension etc.) to ensure stability of the closed loop system. Maxwell posed this in general terms: Determine the constraints on the coefficients of a polynomial that ensure that the roots are confined to the left half plane, the stability region for continuous time systems.

This problem had actually been already solved for the first time [109] by the French mathematician Hermite in 1856! In his proof, Hermite related the location of the zeros of a polynomial with respect to the real line to the signature of a particular quadratic form. In 1877, the English physicist E.J. Routh, using the theory of Cauchy indices and of Sturm sequences, gave his now famous algorithm [198] to compute the number k of roots which lie in the right half of the complex plane $\mathrm{Re}(s) \geq 0$. This algorithm thus gave a necessary and sufficient condition for stability in the particular case when $k = 0$. In 1895, A. Hurwitz drawing his inspiration from Hermite's work, gave another criterion [121] for the stability of a polynomial. This new set of necessary and sufficient conditions took the form of n determinantal inequalities, where n is the degree of the polynomial to be tested. Equivalent results were discovered at the beginning of the century by I. Schur [202, 203] and A. Cohn [69] for the discrete-time case, where the stability region is the interior of the unit disc in the complex plane.

One of the main concerns of control engineers had always been the analysis and design of systems that are subjected to various type of uncertainties or perturbations. These may take the form of noise or of some particular external disturbance signals. Perturbations may also arise within the system, in its physical parameters. This latter type of perturbations, termed parametric perturbations, may be the result of actual variations in the physical parameters of the system, due to aging or changes in the operating conditions. For example in aircraft design, the coefficients of the models that are used depends heavily on the flight altitude. It may also be the consequence of uncertainties or errors in the model itself; for example the mass of an aircraft varies between an upper limit and a lower limit depending on the passenger and baggage loading. From a design standpoint, this type of parameter variation problem is also encountered when the *controller structure* is fixed but its parameters are adjustable. The choice of controller structure is usually dictated by physical, engineering, hardware and other constraints such as simplicity and economics. In this situation, the designer is left with a restricted number of controller or design parameters that have to be adjusted so as to obtain a satisfactory behavior for the closed-loop system; for example, PID controllers have only three parameters that can be adjusted.

The characteristic polynomial of a closed-loop control system containing a plant with uncertain parameters will depend on those parameters. In this context, it is necessary to analyse the stability of a *family* of characteristic polynomials. It turns out that the Routh-Hurwitz conditions which are so easy to check for a single polynomial, are almost useless for families of polynomials because they lead to conditions that are highly nonlinear in the unknown parameters. Thus, in spite of the fundamental need for dealing with systems affected by parametric perturbations, engineers were faced from the outset with a major stumbling block in the form of the nonlinear character of the Routh-Hurwitz conditions, which moreover was the only tool available to deal with this problem.

One of the most important and earliest contributions to stability analysis under parameter perturbations was made by Nyquist in his classic paper [184] of 1932 on feedback amplifier stability. This problem arose directly from his work on the problems of long-distance telephony. This was soon followed by the work of Bode [42] which eventually led to the introduction of the notions of **gain** and **phase margins** for feedback systems. Nyquist's criterion and the concepts of gain and phase margin form the basis for much of the classical control system design methodology and are widely used by practicing control engineers.

The next major period in the evolution of control theory was the period between 1960 and 1975 when the state-variable approach and the ideas of optimal control in the time-domain were introduced. This phase in the theory of automatic control systems arose out of the important new technological problems that were encountered at that time: the launching, maneuvering, guidance and tracking of space vehicles. A lot of effort was expended and rapid developments in both theory and practice took place. Optimal control theory was developed under the influence of many great researchers such as Pontryagin [192], Bellman [25, 26], Kalman [125] and Kalman and Bucy [128]. Kalman [126] introduced a number of key state-variable concepts. Among these were controllability, observability, optimal linear-quadratic regulator (LQR), state-feedback and optimal state estimation (Kalman filtering).

In the LQR problem the dynamics of the system to be controlled are represented by the state space model which is a set of linear first order differential equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t)$ represents the n dimensional state vector at time t. The objective of control is to keep x close to zero without excessive control effort. This objective is to be achieved by minimizing the quadratic cost function

$$I = \int_0^\infty (x'(t)Qx(t) + u'(t)Ru(t)dt.$$

The solution is provided by the **optimal state feedback** control

$$u(t) = Fx(t)$$

where the state feedback matrix F is calculated from the algebraic Riccati equation:

$$A'P + PA + Q - PBR^{-1}B'P = 0$$

$$F = -R^{-1}B'P.$$

The optimal state feedback control produced by the LQR problem was guaranteed to be stabilizing for any performance index of the above form, provided only that the pair (Q, A) is detectable and (A, B) is stabilizable. This means that the characteristic roots of the closed loop system which equal the eigenvalues $\sigma(A + BF)$ of $A + BF$ lie in the left half of the complex plane.

At this point it is important to emphasize that, essentially, the focus was on optimality of the nominal system and the problem of plant uncertainty was largely ignored during this period. One notable exception was to be found in a 1963 paper [244] by Zames introducing the concept of the "small gain" principle, which plays such a key role in robust stability criteria today. Another was in a 1964 paper by Kalman [127] which demonstrated that for SISO (single input-single output) systems the optimal LQR state-feedback control laws had some very strong guaranteed robustness properties, namely an infinite gain margin and a 60 degree phase margin, which in addition were independent of the particular quadratic index chosen. This is illustrated in Figure 0.7 where the state feedback system designed via LQR optimal control has the above guaranteed stability margins at the loop breaking point m.

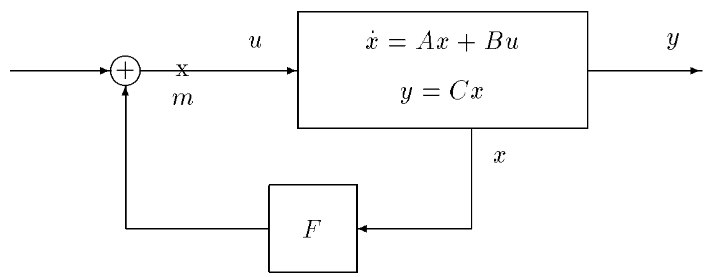

Figure 0.7. State feedback configuration

In implementations, the state variables, which are generally unavailable for direct measurement, would be substituted by their "estimates" generated by an observer or Kalman filter. This takes the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

where $\hat{x}(t)$ is the estimate of the state $x(t)$ at time t. From the above equations it follows that

$$(\dot{x} - \dot{\hat{x}})(t) = (A - LC)(x - \hat{x})(t)$$

so that the estimation error converges to zero, regardless of initial conditions and the input $u(t)$, provided that L is chosen so that $A - LC$ has stable eigenvalues.

To close the feedback loop the optimal feedback control $u = Fx$ would be replaced by the **suboptimal** observed state feedback control $\hat{u} = F\hat{x}$. It is easily shown that the resulting closed loop system has characteristic roots which are precisely the eigenvalues $A + BF$ and those of $A - LC$. This means that the "optimal" eigenvalues were preserved in an output feedback implementation and suggested that the design of the state estimator could be decoupled from that of the optimal controller. This and some related facts came to be known as the **separation principle**.

Invoking this separation principle, control scientists were generally led to believe that the extraordinary robustness properties of the LQR state feedback design were preserved when the control was implemented as an output feedback. We depict this in Figure 0.8 where the stability margin at the point m continues to equal that obtained in the state feedback system.

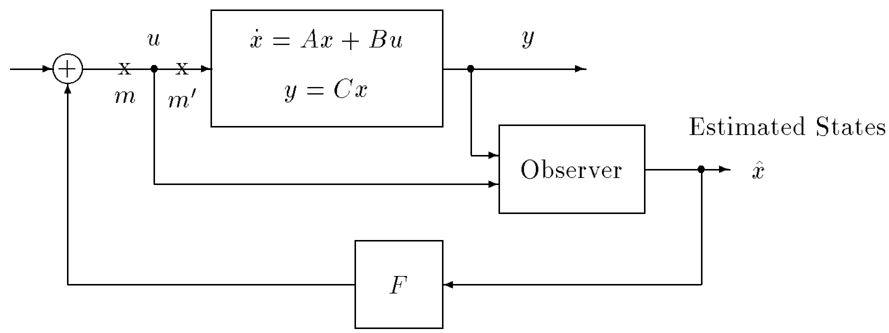

Figure 0.8. Observed state feedback configuration

In 1968 Pearson [187] proposed a scheme whereby an output feedback compensator could be designed to optimize the closed loop dynamics in the LQR sense, by including the requisite number of integrators in the optimal control problem formulation *from the outset*. The philosophy underlying this approach is that the system that is *implemented should also be optimal* and not suboptimal as in the previous observed state feedback case. In [52] Brasch and Pearson showed that arbitrary pole placement could be achieved in the closed loop system by a controller of order no higher than the controllability index or the observability index.

In the late 1960's and early 1970's the interest of control theorists turned to the servomechanism problem. Tracking and disturbance rejection problems with persistent signals such as steps, ramps and sinusoids could not be solved in an obvious way by the existing methods of optimal control. The reason is that unless the proper signals are included in the performance index the cost function usually turns out to be unbounded. In [35] and [36] Bhattacharyya and Pearson gave a solution to the multivariable servomechanism problem. The solution technique in [36] was based on an **output regulator problem** formulated and solved by Bhattacharyya, Pearson and Wonham [37] using the elegant geometric theory of linear systems [237] being developed at that time by Wonham. It clarified the conditions under which the servo problem could be solved by relating it to the stabilization of a suitably defined error system. The robust servomechanism problem, based on error feedback, was treated by Davison [79], and later Francis and Wonham [96] developed the **Internal Model Principle** which established the *necessity* of using error driven controllers, and consequently internal models with suitable redundancy, for reliable tracking and disturbance rejection.

In Ferreira [93] and Ferreira and Bhattacharyya [94] the multivariable servomechanism problem was interpreted as a problem wherein the exogenous signal poles are to be assigned as the zeroes of the error transfer functions. The notion of **blocking zeros** was introduced in [94] and it was shown that robust servo performance, which is equivalent to robust assignment of blocking zeros to the error transfer function, could be achieved by employing error driven controllers containing as poles, in each error channel, the poles of the exogenous signal generators.

It was not realized until the late 1970's that the separation principle of state feedback and LQR control theory really applies to the *nominal system* and is *not* valid for the perturbed system. This fact was dramatically exposed in a paper by Doyle in 1978 [87] who showed by a counterexample that all the guaranteed robustness properties (gain and phase margins) of the LQR design vanished in an output feedback implementation. Referring to the Figure 0.8 we emphasize that the stability margins that are relevant are those that hold at the point m' and *not* those that hold at m. In other words injecting uncertainties at m implies that the observer is aware of the perturbations in the system, which is unrealistic. On the other hand stability margins at m' reflect the realistic situation, that uncertainty in the plant model is unknown to the observer. Doyle's observation really brought back to the attention of the research community the importance of designing feedback controllers which can be *assured* to have desirable robustness properties, and thus a renewed interest appeared in the problem of plant uncertainty. It was realized that the state space based approach via LQR optimal control was inadequate for robust stability. The geometric approach of Wonham [237], which had proved successful in solving many control synthesis problems, was also inadequate for handling robustness issues because it assumed exact knowledge of the plant.

At about this time, significant results were being reported on the analysis of multivariable systems in the frequency domain. In particular, the concept of coprime matrix fraction description of multivariable systems was introduced as a design tool by Youla, Jabr and Bongiorno [241, 242] and Desoer, Liu, Murray and Saeks [81]. In [242] and Kucera [156] a parameterization of all stabilizing controllers was given and this parameterization (which is commonly referred to as the *YJBK parameterization*) has come to play a fundamental role in the theory of robust stabilization of multivariable systems. Also, the Nyquist stability criterion was generalized to multivariable systems by Rosenbrock [197], and Mac Farlane and Postlethwaite [166].

This confluence of interests naturally led to a frequency domain formulation for the robust control problem. More precisely, uncertainty or perturbations in a system with transfer function $G(s)$ were modelled as,

i) $G(s) \longrightarrow G(s)\,(I + \Delta G(s))$, or

ii) $G(s) \longrightarrow G(s) + \Delta G(s)$ with $\|\Delta G(s)\| < \alpha$.

Here $\|\Delta G(s)\|$ denotes a suitable norm in the space of stable transfer function matrices, and case i) represents a *multiplicative* perturbation whereas case ii) represents an *additive* perturbation. From our present perspective, this type of perturbation in a plant model will be referred to as *unstructured* or *norm bounded perturbations.* One reason for this designation is that the norms which are commonly employed in

i) and ii) completely discard all the phase information regarding the perturbation. Also, there is no obvious connection between bounds on the coefficients or other parameters of $G(s)$ and the transfer function norm bounds.

The solution to the robust stabilization problem, namely the problem of determining a controller for a prescribed level of unstructured perturbations, was given in 1984 by Kimura [148] for the case of SISO systems. The multivariable robust stabilization problem was solved in 1986 by Vidyasagar and Kimura [234] and Glover [103].

These results were in fact a by-product of an important line of research initiated by Zames [245] concerning the optimal disturbance rejection problem which can be summarized as the problem of designing a feedback controller which minimizes the worst case effect over a class of disturbances on the system outputs. This problem is mathematically equivalent to the so-called sensitivity minimization problem or to the problem of minimizing the norm of the error transfer function. Note that the idea of designing a feedback control so as to reduce the sensitivity of the closed-loop is a very classical idea which goes back to the work of Bode [42].

In his seminal paper [245] of 1981, Zames proposed the idea of designing feedback controllers which do not merely reduce the sensitivity of the closed-loop system but actually optimize the sensitivity in an appropriate sense. The crucial idea was to consider the sensitivity function as a map between spaces of bounded energy signals and to minimize its induced operator norm. This translates to the physical assumption that the disturbance signal that hits each system is precisely the worst disturbance for *that* system.

The induced operator norm for a convolution operator between spaces of finite energy signals is the so-called H_∞ norm which derives its name from the theory of Hardy spaces in functional analysis. Zames' fundamental paper thus introduced for the first time the H_∞ approach to control system design. In this same paper the solution of the H_∞ sensitivity minimization problem was given for the special case of a system with a single right half plane zero. This paper led to a flurry of results concerning the solution of the H_∞ optimal sensitivity minimization problem or of the H_∞ optimal disturbance rejection problem. Francis and Zames [97] gave a solution for the SISO case, and the multivariable problem was solved by Doyle [89], Chang and Pearson [53] and Safonov and Verma [201]. In [90, 88] Doyle showed the equivalence between robust performance and sensitivity in H_∞ problems. A state space solution to the H_∞ disturbance rejection problem was given by Doyle, Glover, Khargonekar and Francis [91].

Following the same norm optimization design philosophy, Vidyasagar proposed in [233] the L_1 optimal disturbance rejection problem. In this problem, the disturbances that affect the system are no longer bounded energy (L_2) signals but bounded amplitude (L_∞) signals. The L_1 optimal disturbance rejection problem was solved by Dahleh and Pearson in 1987 [73].

0.3 THE PARAMETRIC THEORY

It is important to point out that during the 1960's and 70's the problem of stability under large parameter uncertainty was almost completely ignored by controls researchers. The notable exceptions are Horowitz [120] who in 1963 proposed a quantitative approach to feedback theory, and Šiljak [211] who advocated the use of parameter plane and optimization techniques for control system analysis and design under parameter uncertainty. This was mainly due to the perception that the real parametric robust stability problem was an impossibly difficult one, precluding neat results and effective computational techniques. Supposedly, the only way to deal with it would be to complexify real parameters and to overbound the uncertainty sets induced in the complex plane by real parameter uncertainty with complex plane discs.

The situation changed dramatically with the advent of a remarkable theorem [143] due to the Russian control theorist V.L. Kharitonov. Kharitonov's Theorem showed that the left half plane (Hurwitz) stability of a family of polynomials of fixed but arbitrary degree corresponding to an entire box in the coefficient space could be verified by checking the stability of four prescribed vertex polynomials. The result is extremely appealing because the apparently impossible task of verifying the stability of a continuum of polynomials could now be carried out by simply using the Routh-Hurwitz conditions on 4 fixed polynomials. The number 4 is also surprising in view of the fact that a) for polynomials of degree n the box of polynomials in question has 2^{n+1} vertices in general and b) the Hurwitz region is not convex or even connected. From a computational point of view the fixed number 4 of test polynomials independent of the dimension of the parameter space is very attractive and dramatically counterbalances the usual arguments regarding increase in complexity with dimension.

Kharitonov's work was published in 1978 in the Russian technical literature but his results remained largely unknown for several years partly due to the fact that Kharitonov's original proofs were written in a very condensed form.

The appearance of Kharitonov's Theorem led to a tremendous resurgence of interest in the study of robust stability under **real parametric uncertainty**. For the first time since Routh's and Hurwitz's results, researchers started to believe that the robust control problem for real parametric uncertainties could be approached without conservatism and overbounding, and with computational efficiency built right into the theory. Moreover, it has also revealed the *effectiveness* and *transparency* of methods which exploit the algebraic and geometric properties of the stability region in parameter space, as opposed to blind formulation of an optimization problem. This has led to an outpouring of results in this field over the last few years.

The first notable result following Kharitonov's Theorem is due to Soh, Berger and Dabke [214] in 1985 who, in a sense, adopted a point of view opposite to Kharitonov's. Starting with an already stable polynomial $\delta(s)$, they gave a way to compute the radius of the largest stability ball in the space of polynomial *coefficients* around δ. In their setting, the vector space of all polynomials of degree less than or equal to n is identified with $\mathbb{R}^{n+1}$ equipped with its Euclidean norm, and the largest stability hypersphere is defined by the fact that every polynomial within the sphere is stable whereas at least one polynomial on the sphere itself is unstable.

The next significant development in this field was a paper by Bartlett, Hollot and Lin [21] which considered a family of polynomials whose coefficients vary in an arbitrary polytope of $\mathbb{R}^{n+1}$, with its edges not necessarily parallel to the coordinate axes as in Kharitonov's problem. They proved that the root space of the entire family is bounded by the root loci of the exposed edges. In particular the entire family is stable if and only if all the edges are proved to be stable. Moreover this result applies to general stability regions and is not restricted to Hurwitz stability.

In 1987 Biernacki, Hwang and Bhattacharyya [40] extended the results of [214] and calculated the largest stability ball in the space of *parameters* of the plant transfer function itself. This work was done for the case of a plant with a single input and multiple outputs, or the dual case of a plant with multiple inputs and a single output, and a numerical procedure was given to compute the stability radius. These and other results on parametric robust stability were reported in a monograph [29] by Bhattacharyya.

In 1989, Chapellat and Bhattacharyya [58] proved the Generalized Kharitonov Theorem (GKT) which gave necessary and sufficient conditions for robust stability of a closed loop system containing an interval plant in the forward path. An interval plant is a plant where each transfer function coefficient can vary in a prescribed interval. The solution provided by GKT reveals an extremal set of line segments from which all the important parameters of the behaviour of the entire set of uncertain systems can be extracted. The number of extremal line segments is also *independent* of the number of uncertain parameters.

The extremal set characterizes robust stability under parametric uncertainty, worst case parametric stability margin, frequency domain plots such as Bode and Nyquist envelopes, worst case H_∞ stability margins and mixed uncertainty stability and performance margins. The characterization enjoys the *built-in computational efficiency* of the theory and provides closed form or almost closed form solutions to entire classes of control design problems that could previously only be approached as optimization problems on a case by case basis. In a series of recent papers the Generalized Kharitonov Theorem has been established as a fundamental and unifying approach to control system design under uncertainty. In combination with the recently developed highly efficient computational techniques of determining stability margins, such as the Tsypkin-Polyak locus [225, 226] and those based on the Mapping Theorem of Zadeh and Desoer [243], the GKT provides the control engineer with a powerful set of techniques for analysis and design.

The above fundamental results have laid a solid foundation. The future development of the theory of robust stability and control under parametric and mixed perturbations rests on these. The objective of our book is to describe these results for the use of students of control theory, practicing engineers and researchers.

0.4 DISCUSSION OF CONTENTS

We begin Chapter 1 with a new look at classical stability criteria for a single polynomial. We consider a family of polynomials where the coefficients depend continuously on a set of parameters, and introduce the **Boundary Crossing Theorem** which establishes, roughly, that along any continuous path in parameter space connecting a stable polynomial to an unstable one, the first encounter with instability must be with a polynomial which has unstable roots only on the stability boundary. This is a straightforward consequence of the continuity of the roots of a polynomial with respect to its coefficients. Nevertheless, this simple theorem serves as the unifying idea for the entire subject of robust parametric stability as presented in this book. In Chapter 1 we give simple derivations of the **Routh** and **Jury** stability tests as well the **Hermite-Biehler Theorem** based on this result. The results developed in the rest of the chapters depend directly or indirectly on the Boundary Crossing Theorem.

In Chapter 2 we study the problem of determining the stability of a **line segment** joining a pair of polynomials. The pair is said to be **strongly stable** if the entire segment is stable. This is the simplest case of robust stability of a parametrized family of polynomials. A first characterization is obtained in terms of the **Bounded Phase Lemma** which is subsequently generalized to general polytopic families in Chapter 4. We then develop necessary and sufficient conditions for strong stability in the form of the **Segment Lemma** treating both the Hurwitz and Schur cases. We then establish the **Vertex Lemma** which gives some useful sufficient conditions for strong stability of a pair based on certain standard forms for the difference polynomial. We also discuss the related notion of **Convex Directions**. The Segment and Vertex Lemmas are used in proving the Generalized Kharitonov Theorem in Chapter 7.

In Chapter 3 we consider the problem of determining the robust stability of a parametrized family of polynomials where the parameter is just the set of polynomial coefficients. This is the problem treated by Soh, Berger and Dabke in 1985. Using orthogonal projection we derive quasi-closed form expressions for the **real stability radius in coefficient space** in the Euclidean norm. We then describe the **Tsypkin-Polyak locus** which determines the stability radius in the ℓ_p norm for arbitrary p in a convenient graphical form. Then we deal with a family of complex polynomials where each coefficient is allowed to vary in a **disc** in the complex plane and give a constructive solution to the problem of robust stability determination in terms of the H_∞ norms of two transfer functions. The proof relies on a lemma which gives a robust Hurwitz characterization of the H_∞ norm which is useful in its own right.

In Chapter 4 we extend these results to the parameter space concentrating on the case of linear parametrization, where the polynomial coefficients are affine linear functions of the real parameter vector $\mathbf{p}$. We develop the procedure for calculating the **real parametric stability margin** measured in the ℓ_1, ℓ_2 and ℓ_∞ norms. The main conceptual tool is once again the Boundary Crossing Theorem and its computational version the **Zero Exclusion Principle**. We consider the special case in which **p** varies in a box. For linearly parametrized systems this case gives rise to a **polytope of polynomials** in coefficient space. For such families we establish the important fact that stability is determined by the **exposed edges** and in special cases by the vertices. It turns out that this result on exposed edges also follows from a more powerful result, namely the Edge Theorem, which is established in Chapter 6. Here we show that this stability testing property of the exposed edges carries over to complex polynomials as well as to quasipolynomials which arise in control systems containing time-delay. A computationally efficient solution to testing the stability of general polytopic families is given by the **Bounded Phase Lemma**, which reduces the problem to checking the maximal phase difference over the vertex set, evaluated along the boundary of the stability region. The **Tsypkin-Polyak locus** for stability margin determination is also described for such polytopic systems. We close the chapter by giving an extension of the theory of disc polynomials developed in Chapter 3 to the case of **linear disc polynomials** where the characteristic polynomial is a linear combination with polynomial coefficients of complex coefficients which can vary in prescribed discs in the complex plane.

In Chapter 5 we turn our attention to the robust stability of **interval polynomial** families. We state and prove **Kharitonov's Theorem** which deals with the Hurwitz stability of such families, treating both the real and the complex cases. This theorem is interpreted as a generalization of the Hermite-Biehler Interlacing Theorem and a simple derivation is also given using the Vertex Lemma of Chapter 2. An important **extremal property** of the Kharitonov polynomials is established, namely that the worst case real stability margin in the coefficient space over an interval family occurs *precisely* on the Kharitonov vertices. This fact is used to give an application of Kharitonov polynomials to **robust state feedback stabilization**. Finally the problem of **Schur stability of interval polynomials** is studied. Here it is established that a subset of the exposed edges of the underlying interval box suffices to determine the stability of the entire family.

In Chapter 6 we state and prove the **Edge Theorem**. This important result shows that the **root space boundary** of a polytope of polynomials is exactly determined by the root loci of the exposed edges. Since each exposed edge is a one parameter family of polynomials, this result allows us to constructively determine the root space of a family of linearly parametrized systems. This is an effective tool in control system analysis and design.

In Chapter 7 we generalize Kharitonov's problem by considering the robust Hurwitz stability of a linear combination, with polynomial coefficients, of interval polynomials. This formulation is motivated by the problem of robust stability of a feedback control system containing a fixed compensator and an interval plant in its forward path and we refer to such systems as linear interval systems. The solution is provided by the **Generalized Kharitonov Theorem** which shows that for a compensator to robustly stabilize the system it is sufficient that it stabilizes a prescribed set of **line segments** in the plant parameter space. Under special conditions on the compensator it suffices to stabilize the **Kharitonov vertices**. These line segments, labeled **extremal segments**, play a fundamental characterizing role in later chapters.

In Chapter 8 we develop some **extremal frequency domain properties** of linear interval control systems. The extremal segments are shown to possess boundary properties that are useful for generating the **frequency domain templates** and the **Nyquist, Bode and Nichols envelopes** of linear interval systems. The **extremal gain and phase margins** of these systems occur on these segments. We show how these concepts are useful in extending classical design techniques to linear interval systems by giving some examples of **robust parametric classical control design**.

In Chapter 9 we consider mixed uncertainty problems, namely the **robust stability and performance** of control systems subjected to parametric uncertainty as well as unstructured perturbations. The parameter uncertainty is modeled through a linear interval system whereas two types of unstructured uncertainty are considered, namely H^∞ norm bounded uncertainty and nonlinear sector bounded perturbations. The latter class of problems is known as the Absolute Stability problem. We present **robust versions** of the **Small Gain Theorem** and the **Absolute Stability Problem** which allow us to quantify the worst case parametric or unstructured stability margins that the closed loop system can tolerate. This results in the robust versions of the well-known Lur'e criterion, Popov criterion and the Circle criterion of nonlinear control theory.

Chapters 10 and 11 deal with the robust stability of polynomials containing uncertain interval parameters which appear affine multilinearly in the coefficients. The main tool to solve this problem is the **Mapping Theorem** described in the 1963 book of Zadeh and Desoer. We state and prove this theorem and apply it to the robust stability problem. In Chapter 11 we continue to develop results on **multilinear interval systems** extending the Generalized Kharitonov Theorem and the frequency domain properties of Chapters 7, 8 and 9 to the multilinear case.

In Chapter 12 we deal with parameter perturbations in **state space models**. The same mapping theorem is used to give an effective solution to the robust stability of state space systems under real parametric interval uncertainty. This is followed by some techniques for calculating **robust parametric stability regions using Lyapunov theory**. We also include results on the calculation of the **real and complex stability radius** defined in terms of the operator (induced) norm of a feedback matrix as well as some results on the Schur stability of nonnegative interval matrices.

In Chapter 13 we describe some **synthesis techniques**. To begin with we demonstrate a **direct synthesis** procedure whereby any minimum phase interval plant of order n, with m zeros can be robustly stabilized by a fixed stable minimum phase compensator of order $n - m - 1$. Then we show, by examples, how standard results from H^∞ theory such as the Small Gain Theorem, Nevanlinna-Pick interpolation and the two Riccati equation approach can be exploited to deal with parametric perturbations using the extremal properties developed earlier.

In Chapter 14 some examples of **interval identification and design** applied to two experimental space structures are described as an application demonstrating the practical use of the theory described in the book.

The contents of the book have been used in a one-semester graduate course on Robust Control. The numerical exercises given at the end of the chapters should be worked out for a better understanding of the theory. They can be solved by using a MATLAB based Parametric Robust Control Toolbox available separately. A demonstration diskette based on this ToolBox is included in this book, and includes solutions to some of the examples.

0.5 NOTES AND REFERENCES

Most of the important papers on robust stability under norm bounded perturbations are contained in the survey volume [83] edited by Dorato in 1987. Dorato and Yedavalli [86] have also edited in 1990, a volume of papers collected from IEEE Conferences which contains a large collection of papers on parametric stability. Šiljak [212] gave a (1989) survey of parametric robust stability. Many of the results presented in this book were obtained in the M.S. (1987) and Ph.D. (1990) theses [54, 55] of H. Chapellat and the Ph.D (1986) thesis [131] of L. H. Keel. The 1987 book [29] of Bhattacharyya, and the recent books [2] and [14] by Ackermann and Barmish respectively concentrate on the parametric approach. The monograph [85] by Dorato, Fortuna and Muscato, and the 1985 book [232] of Vidyasagar all deal with H_∞ optimization and the book [84] by Dorato, Abdallah and Cerone deals with Linear Quadratic Optimal Control. The book [167] by Maciejowski is a design oriented textbook on control systems concentrating on robustness issues. In the book of Boyd and Barratt [51] several control design problems are treated as convex optimization problems. The proceedings [180, 32, 172] of several International Workshops on Robust Control held since 1988, and edited respectively by Milanese, Tempo and Vicino [180], Hinrichsen and Mårtensson [110], Bhattacharyya and Keel [32] and Mansour, Balemi and Truöl [172], are also useful references.