ECEN620: Network Theory
Broadband Circuit Design
Fall 2023

Lecture 2: Linear Circuit Analysis Review

Sam Palermo
Analog & Mixed-Signal Center
Texas A&M University
Agenda

- Transfer Functions
- Mason’s Rule
- Second-Order Systems
- Review Material
  - Laplace Transform
  - Passive Circuit s-Domain Models
  - Transfer Functions
  - Sinusoidal Steady-State Response
  - Poles & Zeros
  - Bode Plots
The transfer function $H(s)$ of a network is the ratio of the Laplace transform of the output and input signals when the initial conditions are zero.

$$H(s) = \frac{\mathcal{L}\{v_o(t)\}}{\mathcal{L}\{v_i(t)\}} = \frac{V_o(s)}{V_i(s)}$$

- The transfer function $H(s)$ of a network is the ratio of the Laplace transform of the output and input signals when the initial conditions are zero.
- This is also the Laplace transform of the network’s impulse response.
If input $v_i(t)$ is sinusoidal

$$v_i(t) = A \cos(\omega t + \phi)$$

The steady-state output will be

$$v_{ss}(t) = |H(j\omega)|A \cos(\omega t + \phi + \angle H(j\omega))$$
Mason’s Rule

- Mason’s Rule is useful to find the transfer function of complex networks
- For Mason’s Rule, you need to find the following
  - The direct (forward) path(s) from the input(s) to output
  - The system loops
  - The loops that do not touch the forward path(s)
  - Loops that don’t touch, i.e. share elements or nodes
Mason’s Rule

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum G_i \Delta_i \]

\( G_i = \) path gain of the \( i \)th forward path

\( \Delta = \) the system determinant = 1 – \( \sum \) (all individual loop gains) + \( \sum \) (gain products of all possible two loops that do not touch) – \( \sum \) (gain products of all possible three loops that do not touch) + …

\( \Delta_i = \) \( i \)th forward path determinant = value of \( \Delta \) for that part of the block diagram that does not touch the \( i \)th forward path
Mason’s Rule Example 1

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_i G_i \Delta_i \]

System Determinant: Note, all loops touch

\[ \Delta = 1 - \left( \frac{-a_1}{s} - \frac{-a_2}{s^2} - \frac{-a_3}{s^3} \right) + 0 = 1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3} \]

Forward Path Determinants

Note, all loops touch the forward paths

\[ \Delta_1 = 1 - 0 = 1 \]
\[ \Delta_2 = 1 - 0 = 1 \]
\[ \Delta_3 = 1 - 0 = 1 \]

System Transfer Function

\[ \frac{Y(s)}{U(s)} = \frac{\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \]
Mason’s Rule Example 2

Forward Path Gains
\[ G_1 = 1236 = H_4 \]
\[ G_2 = 12456 = H_1H_2H_3 \]

Loop Gains
\[ l_1 = 242 = H_1H_5 \text{ (does not touch } l_3 \text{)} \]
\[ l_2 = 454 = H_2H_6 \]
\[ l_3 = 565 = H_3H_7 \text{ (does not touch } l_1 \text{)} \]
\[ l_4 = 236542 = H_4H_7H_6H_5 \]

System Determinant: Note, 2 loops don’t touch
\[ \Delta = 1 - (H_1H_5 + H_2H_6 + H_3H_7 + H_4H_7H_6H_5) + (H_1H_5H_3H_7) \]

Forward Path Determinants
Note, \( l_2 \) does not touch \( G_1 \)
\[ \Delta_1 = 1 - H_2H_6 \]
\[ \Delta_2 = 1 - 0 = 1 \]

System Transfer Function
\[
\frac{Y(s)}{U(s)} = \frac{H_4(1 - H_2H_6) + H_1H_2H_3}{1 - (H_1H_5 + H_2H_6 + H_3H_7 + H_4H_7H_6H_5) + H_1H_5H_3H_7}
\]
Second-Order Systems

Forward Path Gain

$$G_1 = k_1 \left( \frac{\omega_0}{s} \right)^2$$

Loop Gains

$$l_1 = -\frac{\omega_0}{sQ}$$

$$l_2 = \left( \frac{\omega_0}{s} \right)^2$$

System Determinant: Note, all loops touch

$$\Delta = 1 - \left( -\frac{\omega_0}{sQ} - \left( \frac{\omega_0}{s} \right)^2 \right) + 0 = 1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2$$

Forward Path Determinant

$$\Delta_1 = 1 - 0 = 1$$

System Transfer Function

$$H(s) = \frac{V_{o2}(s)}{V_i(s)} = \frac{k_1 \left( \frac{\omega_0}{s} \right)^2}{1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2} = \frac{k_1 \omega_0^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_i G_i \Delta_i$$
Second-Order Systems: Real or Complex Poles?

\[ H(s) = \frac{k_1 \omega_0^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

2 poles \( p_1, p_2 = -\frac{\omega_0}{2Q} \pm \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 - \omega_0^2} \)

2 real poles if \( Q \leq 0.5 \)
2 complex conjugate poles if \( Q > 0.5 \)
Bode Plots

- Technique to plot the **Magnitude** (squared) and **Phase** response of a transfer function
  - Magnitude is plotted in Decibels (dB), which is a power ratio unit

\[
|H(j\omega)|^2 \Rightarrow 10\log_{10}(|H(j\omega)|^2)(\text{dB}) = 20\log_{10}(|H(j\omega)|)(\text{dB})
\]

- Phase is typically plotted in degrees

\[
\angle(H(j\omega)) = \tan^{-1}\left(\frac{\text{Im}(H(j\omega))}{\text{Re}(H(j\omega))}\right)
\]
Second-Order Systems – Real Poles (1)

\[ H(s) = \frac{10^4}{s^2 + 1001s + 1000} = \frac{10^4}{(s + 1)(s + 1000)} \]

2 poles: \( p_1 = -1, \ p_2 = -1000 \)

Note, \( Q = 0.032 \)

- If poles are spaced by more than 2 decades, there are 2 distinct regions of \(-45^\circ/\text{dec} \) phase slope
• If poles are spaced by less than 2 decades, there is a region of -90°/dec phase slope
  • Watch out for system stability!

\[ H(s) = \frac{100}{s^2 + 11s + 10} = \frac{100}{(s + 1)(s + 10)} \]

2 poles: \( p_1 = -1, \ p_2 = -10 \)

Note, \( Q = 0.287 \)
Second-Order Systems – Complex Poles

\[ H(s) = \frac{k_i \omega_0^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

What is the low frequency magnitude?

\[ |H(j0)| = k_i \]

What is the high frequency magnitude?

\[ |H(j\omega)| \mid_{\omega=\infty} = \frac{k_i \omega_0^2}{\omega^2} \Rightarrow -40\text{dB/dec. slope at high frequencies} \]

What happens in the middle, particularly near \( \omega_0 \)?

\[ |H(j\omega_0)| = \left| \frac{k_i \omega_0^2}{-\omega_0^2 + j \frac{\omega_0^2}{Q} + \omega_0^2} \right| = k_i Q \]

Note, if \( Q > 1 \) then the magnitude exceeds the low frequency value, i.e. frequency peaking occurs!
Frequency Peaking w/ Complex Poles

Where is the peak frequency?

\[
\frac{d|H(j\omega)|^2}{d\omega} = \frac{d}{d\omega} \left( \frac{k_1^2 \omega_0^4}{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega_0}{Q}\right)^2} \right) = 0
\]

\[
\omega_{pk} = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} \approx \omega_0 \text{ for large } Q
\]

At \(\omega_{pk}\), the peak value is

\[
T_{pk} = \frac{k_1 Q}{\sqrt{1 - \frac{1}{4Q^2}}} \approx k_1 Q \text{ for large } Q
\]

• Note, phase always crosses \(-90^\circ\) at \(\omega_0\)
Next Time

- PLL System Analysis
Review Material

- The following material reviews Laplace transforms, transfer functions, sinusoidal steady-state response, and Bode plots.

- Please review this material, as it is fundamental for the analysis of the broadband circuits covered in the class.
References


Laplace Transform

• Laplace transforms are useful for solving differential equations

• One-Sided Laplace Transform

\[ \mathcal{L}\{x(t)\} = X(s) = \int_{0}^{\infty} x(t)e^{-st} dt \]

where \( s \) is a complex variable

\[ s = \sigma + j\omega \]

Note, \( j = \sqrt{-1} \) and \( \omega \) is the angular frequency (rad/s)

• \( s \) has units of inverse seconds (s\(^{-1}\))
# Laplace Transform of Signals

## Laplace Transforms of Signals

<table>
<thead>
<tr>
<th>$X(s)$</th>
<th>$x(t)$</th>
<th>$X(s)$</th>
<th>$x(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^n$</td>
<td>$\delta^{(n)}(t)$</td>
<td>$\frac{\beta}{s^2 + \beta^2}$</td>
<td>$\sin \beta t u(t)$</td>
</tr>
<tr>
<td>$s$</td>
<td>$\delta'(t)$</td>
<td>$\frac{s}{s^2 + \beta^2}$</td>
<td>$\cos \beta t u(t)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\delta(t)$</td>
<td>$\frac{\beta}{s^2 + \beta^2}$</td>
<td>$\sin \beta t u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{s}$</td>
<td>$u(t)$</td>
<td>$\frac{s}{s^2 + \beta^2}$</td>
<td>$\cos \beta t u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{s^2}$</td>
<td>$t u(t)$</td>
<td>$\frac{\beta}{(s + \alpha)^2 + \beta^2}$</td>
<td>$\frac{1}{s + \alpha} u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{s^n}$</td>
<td>$\frac{t^{n-1}}{(n-1)!} u(t)$</td>
<td>$\frac{1}{(s + \alpha)^2 + \beta^2}$</td>
<td>$\frac{1}{(s + \alpha)(s + b)} u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{s + \alpha}$</td>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{(s + \alpha)(s + b)}$</td>
<td>$\frac{1}{b - a} u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{(s + \alpha)^2}$</td>
<td>$t e^{-\alpha t} u(t)$</td>
<td>$\frac{s + c}{(c - a) e^{-\alpha t} - (c - b) e^{-\beta t}}$</td>
<td>$\frac{1}{b - a} u(t)$</td>
</tr>
</tbody>
</table>

[McGillem]
Laplace Transforms of Operations

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 x_1(t) + a_2 x_2(t)$</td>
<td>$a_1 X_1(s) + a_2 X_2(s)$</td>
</tr>
<tr>
<td>$x'(t)$</td>
<td>$sX(s) - x(0^-)$</td>
</tr>
<tr>
<td>$\int_0^t x(\xi),d\xi$</td>
<td>$\frac{1}{s} X(s)$</td>
</tr>
<tr>
<td>$tx(t)$</td>
<td>$- \frac{dX(s)}{ds}$</td>
</tr>
<tr>
<td>$\frac{1}{t} x(t)$</td>
<td>$\int_s^\infty X(\xi),d\xi$</td>
</tr>
<tr>
<td>$u(t - t_0)$</td>
<td>$e^{-st_0}X(s)$</td>
</tr>
<tr>
<td>$e^{-at}x(t)$</td>
<td>$X(s + a)$</td>
</tr>
<tr>
<td>$x(at), \ a &gt; 0$</td>
<td>$\frac{1}{a} X \left( \frac{s}{a} \right)$</td>
</tr>
</tbody>
</table>

$x_1 \ast x_2 = \int_0^\infty x_1(\lambda)x_2(t - \lambda)\,d\lambda$  

\[
x(0^+) = \lim_{s \to \infty} sx(s) \\
x(\infty) = \lim_{s \to 0} sx(s) \\
x''(t) = s^2 X(s) - sx(0^-) - x'(0^-) \\
x_1(t)x_2(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_1(s - \lambda)X_2(\lambda)\,d\lambda
\]

[McGillem]
Resistor s-Domain Equivalent Circuit

Time-domain Representation:

\[ v(t) = Ri(t) \]

\[ i(t) = \frac{1}{R} v(t) \]

Complex Frequency Representation:

\[ V(s) = RI(s) \]

\[ I(s) = \frac{1}{R} V(s) \]
Capacitor s-Domain Equivalent Circuit

Time-domain Representation:
\[ v(t) = \frac{1}{C} \int_{0}^{t} i(\lambda) d\lambda + v(0) \]
\[ i(t) = C \frac{dv}{dt} \]

Complex Frequency Representation:
\[ V(s) = \frac{1}{sC} I(s) + \frac{1}{s} v(0) \]
\[ I(s) = sCV(s) - Cv(0) \]
Inductor s-Domain Equivalent Circuit

Time-domain Representation:

\[ v(t) = L \frac{di}{dt} \]

\[ i(t) = \frac{1}{L} \int_0^t v(\lambda)d\lambda + i(0) \]

Complex Frequency Representation:

\[ V(s) = sL(s) - Li(0) \]

\[ I(s) = \frac{1}{sL} V(s) + \frac{1}{s} i(0) \]
s-Domain Impedance w/o I.C.

\[ V(s) = I(s)R \]

\[ Z(s) = R \]

\[ V(s) = I(s) \frac{1}{sC} \]

\[ Z(s) = \frac{1}{sC} \]

\[ V(s) = I(s)sL \]

\[ Z(s) = sL \]
The transfer function $H(s)$ of a network is the ratio of the Laplace transform of the output and input signals when the initial conditions are zero,

\[ H(s) = \frac{\mathcal{L}\{v_o(t)\}}{\mathcal{L}\{v_i(t)\}} = \frac{V_o(s)}{V_i(s)} \]

- The transfer function $H(s)$ of a network is the ratio of the Laplace transform of the output and input signals when the initial conditions are zero.
- This is also the Laplace transform of the network’s impulse response.
RC Transfer Function

\[ V_o(s) = \frac{Z_C}{Z_R + Z_C} \quad V_{in}(s) = \frac{1}{sC} \quad V_{in}(s) = \frac{1}{1 + sRC} V_{in}(s) \]

AC Transfer Function, \( H(S) \)

\[ H(S) = \frac{V_o(S)}{V_{in}(S)} = \frac{1}{1 + sRC} \]
Laplace Transform Circuit Example

Given $v_o(0) = 0$

Convert to Laplace Domain

$$\begin{align*}
H(s) &= \frac{V_o(s)}{V_{in}(s)} = \frac{1}{1 + sRC} = \frac{1}{1 + \frac{s}{10^5}} = \frac{10^5}{s + 10^5} \\

V_o(s) &= H(s)V_i(s) = \left(\frac{10^5}{s + 10^5}\right)\left(\frac{10^5}{s^2 + (10^5)^2}\right)
\end{align*}$$

with partial fraction expansion

$$V_o(s) = \frac{1}{2}\left(\frac{10^5}{s + 10^5}\right) - \frac{1}{2}\left(\frac{s}{s^2 + (10^5)^2}\right) + \frac{1}{2}\left(\frac{10^5}{s^2 + (10^5)^2}\right)$$

with inverse Laplace Transform

$$v_o(t) = \frac{1}{2}e^{-10^5t} - \frac{1}{2}\cos 10^5t + \frac{1}{2}\sin 10^5t = \frac{1}{2}e^{-10^5t} + \frac{1}{\sqrt{2}}\sin\left(10^5t - 45^\circ\right)$$
Laplace Transform Circuit Example

We can decompose the output into its transient and steady-state response:

\[ v_o(t) = \frac{1}{2} e^{-10^5 t} + \frac{1}{\sqrt{2}} \sin(10^5 t - 45^\circ) = v_{tr}(t) + v_{ss}(t) \]

\[ v_{tr}(t) = \frac{1}{2} e^{-10^5 t} \]

\[ v_{ss}(t) = \frac{1}{\sqrt{2}} \sin(10^5 t - 45^\circ) \]

- Note that the transient response decays very quickly!
Sinusoidal Steady-State Response

If input $v_i(t)$ is sinusoidal

$$v_i(t) = A \cos(\omega t + \phi)$$

The steady-state output will be

$$v_{ss}(t) = |H(j\omega)|A \cos(\omega t + \phi + \angle H(j\omega))$$
RC Circuit Sinusoidal Steady-State Response

\[ H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{1+sRC} \Rightarrow H(j\omega) = \frac{1}{1+j\omega RC} \]

Output Magnitude

\[ |H(j\omega)| = \sqrt{H(j\omega)H^*(j\omega)} = \sqrt{\left(\frac{1}{1+j\omega RC}\right)\left(\frac{1}{1-j\omega RC}\right)} \]

Output Phase

\[ \angle H(j\omega) = \tan^{-1}\left(\frac{\text{Im}(H(j\omega))}{\text{Re}(H(j\omega))}\right) = \tan^{-1}\left(\frac{\text{Im}(\text{Num})}{\text{Re}(\text{Num})}\right) - \tan^{-1}\left(\frac{\text{Im}(\text{Den})}{\text{Re}(\text{Den})}\right) \]

where Num = Numerator and Den = Denominator of \( H(j\omega) \)

\[ \angle H(j\omega) = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega RC}{1}\right) = -\tan^{-1}(\omega RC) \]
RC Circuit Sinusoidal Steady-State Response Example

\[ H(s) = \frac{1}{1 + \frac{s}{10^5}} \]

with \( s = j\omega = j10^5 \)

\[ H(j10^5) = \frac{1}{1 + j} \]

\[ |H(j10^5)| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \]

\[ \angle H(j10^5) = -\tan^{-1}(1) = -45^\circ \]

\[ v_{ss}(t) = \frac{1}{\sqrt{2}} \sin(10^5 t - 45^\circ) \]
# Complex Numbers Properties

## Function and Evaluation

<table>
<thead>
<tr>
<th>Function</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = R + j \text{Im} )</td>
<td>( f(x) =</td>
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<tr>
<td></td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>( \phi_f = \tan^{-1}(\text{Im}/R) )</td>
</tr>
<tr>
<td>( f(x) \cdot g(x) )</td>
<td>(</td>
</tr>
<tr>
<td>( \frac{f(x)}{g(x)} )</td>
<td>( \left</td>
</tr>
<tr>
<td>( \prod_{j=1}^{n} f_j(x) )</td>
<td>( \left</td>
</tr>
</tbody>
</table>

## Numerical Example

\[
\frac{(1 + j10)(10 + j10)}{(100 + j10)(1000 + j10)}
\]

\[
\left| \frac{(1 + j10)(10 + j10)}{(100 + j10)(1000 + j10)} \right| = \frac{\sqrt{1^2 + 10^2} \cdot \sqrt{10^2 + 10^2}}{\sqrt{100^2 + 10^2} \cdot \sqrt{1000^2 + 10^2}} = 1.41 \times 10^{-3}
\]

\[
\angle \frac{(1 + j10)(10 + j10)}{(100 + j10)(1000 + j10)} = \tan^{-1}\left( \frac{10}{1} \right) + \tan^{-1}\left( \frac{10}{10} \right) - \tan^{-1}\left( \frac{10}{100} \right) - \tan^{-1}\left( \frac{10}{1000} \right) = 123^\circ
\]
Poles & Zeros

\[ H(s) = A \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \]

- Poles are the roots of the denominator \((p_1, p_2, \ldots p_n)\) where \(H(s) \to \infty\)
- Zeros are the roots of the numerator \((z_1, z_2, \ldots z_m)\) where \(H(s) \to 0\)

Example 1: \( H(s) = \frac{10^5}{s + 10^5} \)
\[ s + 10^5 = 0 \]
\[ p_1 = s = -10^5 \text{ rad} / s \]

Example 2: \( H(s) = \frac{s}{s + 10^5} \)
\[ z_1 = s = 0 \text{ rad} / s \]
\[ s + 10^5 = 0 \]
\[ p_1 = s = -10^5 \text{ rad} / s \]

Example 3: \( H(s) = \frac{100(s+15)}{s^2 + 50s + 1500} \)
\[ s + 15 = 0 \]
\[ z_1 = s = -15 \text{ rad} / s \]
\[ s^2 + 50s + 1500 = 0 \]
\[ p_{1,2} = s_{1,2} = \frac{-50 \pm \sqrt{2500 - 6000}}{2} = -25 \pm j29.6 \text{ rad} / s \]
Bode Plots

- Technique to plot the **Magnitude** (squared) and **Phase** response of a transfer function
  - Magnitude is plotted in Decibels (dB), which is a power ratio unit
    \[
    |H(j\omega)|^2 \overset{dB}{\Rightarrow} 10 \log_{10} \left( |H(j\omega)|^2 \right) \text{ (dB)} = 20 \log_{10} \left( |H(j\omega)| \right) \text{ (dB)}
    \]
  - Phase is typically plotted in degrees
    \[
    \angle(H(j\omega)) = \tan^{-1}\left( \frac{\text{Im}(H(j\omega))}{\text{Re}(H(j\omega))} \right)
    \]
**RC Bode Plot Example**

\[
H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{1 + sRC} = \frac{1}{1 + s10^{-5}} = \frac{1}{1 + j\omega 10^{-5}}
\]

\[
H(s) = \frac{1}{1 + j\omega 10^{-5}} = \frac{1}{\frac{1}{1 - \frac{j\omega}{p_1}}}, \text{ where } p_1 = -10^5 \text{ rad/s}
\]

**Magnitude Squared (dB):**

\[
20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{1}{\sqrt{1+(\omega 10^{-5})^2}}\right| = 20\log_{10}(1) - 20\log_{10}\left(\sqrt{1+(\omega 10^{-5})^2}\right)
\]

**Phase:**

\[
\text{Phase}(H(j\omega)) = -\tan^{-1}(\omega 10^{-5})
\]
RC Bode Plot Example

Magnitude:

\[ 20 \log_{10} |H(j\omega)| = 20 \log_{10} \left| \frac{1}{\sqrt{1 + (\omega 10^{-5})^2}} \right| = 20 \log_{10} (1) - 20 \log_{10} \left( \sqrt{1 + (\omega 10^{-5})^2} \right) \]

Phase:

\[ \text{Phase}(H(j\omega)) = -\tan^{-1}(\omega 10^{-5}) \]

| \(\omega \) (rad/s) | \(|H(j\omega)|\) | \(|H(j\omega)|^2\) | 20\(\log_{10}\) | \(|H(j\omega)|\) (dB) | Phase (\(H(j\omega)\)) (°) |
|---------------------|----------------|----------------|----------------|----------------|----------------|
| 10^3                | 0.9999         | 0.9999         | ~0             | ~0             | ~0             |
| 10^4                | 0.995          | 0.990          | -0.043         | -5.71          |
| 5 \times 10^4       | 0.894          | 0.800          | -0.969         | -26.6          |
| 10^5                | 0.707          | 0.500          | -3.01          | -45.0          |
| 5 \times 10^5       | 0.196          | 0.039          | -14.2          | -78.7          |
| 10^6                | 0.100          | 0.010          | -20.0          | -84.3          |
| 10^7                | 10^-2          | 10^-4          | -40.0          | -89.4          |
| 10^8                | 10^-3          | 10^-6          | -60.0          | -89.9          |

~20\(\log_{10}\) (1) = 0 dB

~20\(\log_{10}\) (\(\omega 10^{-5}\)) = -20 dB/dec

-45°/dec
RC Bode Plot Example

Max Error = 3.01dB

Max Error = 5.71°
Transient Response

\( \omega = 10^3 \text{ rad/s} = -p1/100 \)

Input & Output Signal

\[ |v_o(t)| \approx 1 \]

Phase Shift \( \approx 0^\circ \)

\( \omega = 10^5 \text{ rad/s} = -p1 \)

Input & Output Signal

\[ |v_o(t)| = \frac{1}{\sqrt{2}} \]

Phase Shift = \(-45^\circ\)

\( \omega = 10^6 \text{ rad/s} = 10*p1 \)

Input & Output Signal

\[ |v_o(t)| \approx 0.1 \]

Phase Shift = \(-84.3^\circ\)
Bode Plot Algorithm - Magnitude

1. Where is a good starting point?
   a. Calculate DC value of $|H(j\omega)|$
   b. If not a reasonable value, I like to calculate $|H(j\omega)|$ at $\omega$ equal to the lowest value of $p1/10$ or $z1/10$

2. Where to end?
   a. Calculate $|H(j\omega)|$ as $\omega \rightarrow \infty$

3. Where are the poles and zeros?
   a. Beginning at each pole frequency, the magnitude will decrease with a slope of $-20$dB/dec
   b. Beginning at each zero frequency, the magnitude will increase with a slope of $+20$dB/dec

4. Note, the above algorithm is only valid for real poles and zeros. We will discuss complex poles later.
**Bode Plot Algorithm - Magnitude**

\[
H(s) = \frac{-10^4(s+1)}{(s+10)(s+100)} = \frac{-10(1+s)}{\left(1 + \frac{s}{10}\right) \left(1 + \frac{s}{100}\right)}
\]

DC Magnitude = 10 = 20dB

HF Magnitude = 0 = \(-\infty\)dB

\[z_1 = -1, \quad p_1 = -10, \quad p_2 = -100\]

\[20 \log_{10}|H(j\omega)| = 20 \log_{10}\left|\frac{10\sqrt{1+\omega^2}}{\sqrt{1+(\omega 10^{-1})^2} \sqrt{1+(\omega 10^{-2})^2}}\right| =
\]

\[20 \log_{10}(10) - 20 \log_{10}\left(\sqrt{1+\omega^2}\right) - 20 \log_{10}\left(\sqrt{1+(\omega 10^{-1})^2}\right) - 20 \log_{10}\left(\sqrt{1+(\omega 10^{-2})^2}\right)\]
Bode Plot Algorithm - Phase

1. Calculate low frequency value of Phase(H(jω))
   a. An negative sign introduces -180° phase shift
   b. A DC pole introduces -90° phase shift
   c. A DC zero introduces +90° phase shift

2. Where are the poles and zeros?
   a. For negative poles: 1 dec. before the pole freq., the phase will decrease with a slope of -45°/dec. until 1 dec. after the pole freq., for a total phase shift of -90°
   b. For zeros poles: 1 dec. before the zero freq., the phase will increase with a slope of +45°/dec. until 1 dec. after the zero freq., for a total phase shift of +90°
   c. Note, if you have positive poles or zeros, the phase change polarity is inverted

3. Note, the above algorithm is only valid for real poles and zeros. We will discuss complex poles later.
Bode Plot Algorithm - Phase

\[ H(s) = -\frac{10^4(s+1)}{(s+10)(s+100)} = -\frac{10(1+s)}{(1+\frac{s}{10})(1+\frac{s}{100})} \]

LF Phase = \(-180^\circ\)

\[ z_1 = -1, \ p_1 = -10, \ p_2 = -100 \]

\[ \angle H(j\omega) = -180^\circ + \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{100}\right) \]