

1.1 Propositions and logical operations

Logic is the study of formal reasoning. A statement in a spoken language, such as in English, is often ambiguous in its meaning. By contrast, a statement in logic always has a well defined meaning. Logic is important in mathematics for proving theorems. Logic is also used in computer science in areas such as artificial intelligence for automated reasoning and in designing digital circuits. Logic is useful in any field in which it is important to make precise statements. In law, logic can be used to define the implications of a particular law. In medicine, logic can be used to specify precisely the conditions under which a particular diagnosis would apply.

The most basic element in logic is a proposition. A **proposition** is a statement that is either true or false.

Table 1.1.1: Examples of propositions: Statements that are either true or false.

Proposition	Truth value
There are an infinite number of prime numbers.	True
The Declaration of Independence was signed on July 4, 1812.	False

Propositions are typically declarative sentences. For example, the following are *not* propositions.

Table 1.1.2: English sentences that are not propositions.

Sentence	Comment
What time is it?	A question, not a proposition. A question is neither true nor false.
Have a nice day.	A command, not a proposition. A command is neither true nor false.

A proposition's **truth value** is a value indicating whether the proposition is actually true or false. A proposition is still a proposition whether its truth value is known to be true, known to be false, unknown, or a matter of opinion. The following are all propositions.

Table 1.1.3: Examples of propositions and their truth values.

Proposition	Comment
Two plus two is four.	Truth value is true.
Two plus two is five.	Truth value is false.
Monday will be cloudy.	Truth value is unknown.
The movie was funny.	Truth value is a matter of opinion.
The extinction of the dinosaurs was caused by a meteor.	Truth value is unknown.

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1.1.1: Propositions.

Indicate which statements are propositions.

1) 10 is a prime number.

- Proposition
 Not a proposition

2) Shut the door.

- Proposition
 Not a proposition

3) All politicians are dishonest.

- Proposition
 Not a proposition

4) Would you like some cake?

- Proposition
 Not a proposition

5) Interest rates will rise this year.

- Proposition
 Not a proposition

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The conjunction operation

Propositional variables such as p , q , and r can be used to denote arbitrary propositions, as in:

p: January has 31 days.

q: February has 33 days.

A **compound proposition** is created by connecting individual propositions with logical operations. A **logical operation** combines propositions using a particular composition rule. For example, the conjunction operation is denoted by \wedge . The proposition $p \wedge q$ is read "p and q" and is called the **conjunction** of p and q. $p \wedge q$ is true if both p is true and q is true. $p \wedge q$ is false if p is false, q is false, or both are false.

Using the definitions for $p \wedge q$ given above, the proposition $p \wedge q$ is expressed in English as:

$p \wedge q$: January has 31 days and February has 33 days.

Proposition p's truth value is true – January does have 31 days. Proposition q's truth value is false – February does not have 33 days. The compound proposition $p \wedge q$ is therefore false, because it is not the case that both propositions are true.

A **truth table** shows the truth value of a compound proposition for every possible combination of truth values for the variables contained in the compound proposition. Every row in the truth table shows a particular truth value for each variable, along with the compound proposition's corresponding truth value. Below is the truth table for $p \wedge q$, where **T** represents true and **F** represents false.

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1.1.2: Truth table for the conjunction operation.

Animation captions:

1. $p \wedge q$ is true only when both p and q are true.
2. $p \wedge q$ is false for all other combinations.

Different ways to express a conjunction in English

Define the propositional variables p and h as:

p: Sam is poor.

h: Sam is happy.

There are many ways to express the proposition $p \wedge h$ in English. The sentences below have slightly different meanings in English but correspond to the same logical meaning.

Table 1.1.4: Examples of different ways to express a conjunction in English.

p and h	Sam is poor and he is happy.
p, but h	Sam is poor, but he is happy.

Despite the fact that p, h	Despite the fact that Sam is poor, he is happy.
Although p, h	Although Sam is poor, he is happy.

The disjunction operation

The disjunction operation is denoted by \vee . The proposition $p \vee q$ is read "p or q", and is called the **disjunction** of p and q. $p \vee q$ is true if either one of p or q is true, or if both are true. The proposition $p \vee q$ is false if neither p nor q is true. Using the same p and q from the example above, $p \vee q$ is the statement:

$p \vee q$: January has 31 days or February has 33 days.

The proposition $p \vee q$ is true because January does have 31 days. The truth table for the \vee operation is given below.

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1.1.3: Truth table for the disjunction operation.

Animation captions:

1. $p \vee q$ is true when either of p or q is true.
2. $p \vee q$ is false only when p and q are both false.

Ambiguity of "or" in English

The meaning of the word "or" in common English depends on context. Often when the word "or" is used in English, the intended meaning is that one or the other of two things is true, but not both. One would normally understand the sentence "Lucy is going to the park or the movie" to mean that Lucy is either going to the park, or is going to the movie, but not both. Such an either/or meaning corresponds to the "exclusive or" operation in logic. The **exclusive or** of p and q evaluates to true when p is true and q is false or when q is true and p is false. The **inclusive or** operation is the same as the disjunction (\vee) operation and evaluates to true when one or both of the propositions are true. For example, "Lucy opens the windows or doors when warm" means she opens windows, doors, or possibly both. Since the inclusive or is most common in logic, it is just called "or" for short.

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1.1.4: Truth table for the exclusive or.

The exclusive or operation is usually denoted with the symbol \oplus . The proposition $p \oplus q$ is true if exactly one of the propositions p and q is true but not both. This question asks you to fill in the truth table for $p \oplus q$.

--	--	--	--

p	q	$p \oplus q$
T	T	1?
T	F	2?
F	T	3?
F	F	4?

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1) What is the truth value for the square labeled 1?

- True
 False

2) What is the truth value for the square labeled 2?

- True
 False

3) What is the truth value for the square labeled 3?

- True
 False

4) What is the truth value for the square labeled 4?

- True
 False

The **negation** operation acts on just one proposition and has the effect of reversing the truth value of the proposition. The negation of proposition p is denoted $\neg p$ and is read as "not p ". Since the negation operation only acts on a single proposition, its truth table only has two rows for the proposition's two possible truth values.

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1.1.5: Truth table for the negation operation.

Animation captions:

1. The truth value of $\neg p$ is the opposite of the truth value of p .

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1.1.6: Applying logical operations.



Assume propositions p , q , and r have the following truth values:

p is true
 q is true
 r is false

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What are the truth values for the following compound propositions?

1) $p \wedge q$

- True
 False

2) $\neg r$

- True
 False

3) $p \wedge r$

- True
 False

4) $p \vee r$

- True
 False

5) $p \vee q$

- True
 False

Example 1.1.1: Searching the web.

The language of logic is useful in database searches, such as searching the web. Suppose one is interested in finding web pages related to higher education. A search on the term "college" could potentially miss many pages related to universities. A search on "college OR university" would yield results on both topics. A search on "dogs AND fleas" would yield pages that pertain to both dogs and fleas. A typical web search engine, though, implicitly uses an AND operation for multiple words in queries like "dogs fleas".

Additional exercises

EXERCISE 1.1.1: Identifying propositions.

Determine whether each of the following sentences is a proposition. If the sentence is a proposition, then write its negation.

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(a) Have a nice day.

 **Solution** 

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Ok, got it

(b) The soup is cold.

 **Solution** 

(c) The patient has diabetes.

 **Solution** 

(d) The light is on.

 **Solution** 

(e) It's a beautiful day.

 **Solution** 

(f) Do you like my new shoes?

 **Solution** 

(g) The sky is purple.

 **Solution** 

(h) $2 + 3 = 6$

 **Solution** 

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(i) Every prime number is even.

 **Solution** 

(j) There is a number that is larger than 17.

 **Solution** 

EXERCISE 1.1.2: Expressing English sentences using logical notation.

Express each English statement using logical operations \vee , \wedge , \neg and the propositional variables t , n , and m defined below. The use of the word "or" means inclusive or.

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t : The patient took the medication.

n : The patient had nausea.

m : The patient had migraines.

(a) The patient had nausea and migraines.

 **Solution** 

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Ok, got it

(b) The patient took the medication, but still had migraines.

 **Solution** 

(c) The patient had nausea or migraines.

 **Solution** 

(d) The patient did not have migraines.

 **Solution** 

(e) Despite the fact that the patient took the medication, the patient had nausea.

 **Solution** 

(f) There is no way that the patient took the medication.

 **Solution** 

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EXERCISE 1.1.3: Applying logical operations.

Assume the propositions p , q , r , and s have the following truth values:

p is false

q is true

r is false

s is true

What are the truth values for the following compound propositions?

(a) $\neg p$

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Ok, got it

(b) $p \vee r$

 **Solution** 

(c) $q \wedge s$

 **Solution** 

(d) $q \vee s$

 **Solution** 

(e) $q \oplus s$

 **Solution** 

(f) $q \oplus r$

 **Solution** 

EXERCISE 1.1.4: Truth values for statements with inclusive and exclusive or.

Indicate whether each statement is true or false, assuming that the "or" in the sentence means the inclusive or. Then indicate whether the statement is true or false if the "or" means the exclusive or.

(a) February has 31 days or the number 5 is an integer.

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Ok, got it

(b) The number π is an integer or the sun revolves around the earth.

 **Solution** 

(c) 20 nickels are worth one dollar or whales are mammals.

 **Solution** 

(d) There are eight days in a week or there are seven days in a week.

 **Solution** 

(e) January has exactly 31 days or April has exactly 30 days.

 **Solution** 

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1.2 Evaluating compound propositions

A compound proposition can be created by using more than one operation. For example, the proposition $p \vee \neg q$ evaluates to true if p is true or the negation of q is true.

The order in which the operations are applied in a compound proposition such as $p \vee \neg q \wedge r$ may affect the truth value of the proposition. In the absence of parentheses, the rule is that negation is applied first, then conjunction, then disjunction:

Figure 1.2.1: Order of operations in absence of parentheses.

1. \neg (not)
2. \wedge (and)
3. \vee (or)

For example, the proposition $p \vee q \wedge r$ should be read as $p \vee (q \wedge r)$, instead of $(p \vee q) \wedge r$. However, good practice is to use parentheses to specify the order in which the operations are to be performed, as in $p \vee (q \wedge r)$. One exception to using parentheses in compound propositions is with the negation operation. Parentheses around $\neg p$ are usually omitted to make compound propositions more readable. Because the negation operation is always applied first, the proposition $\neg p \vee q$ is evaluated as $(\neg p) \vee q$ instead of $\neg(p \vee q)$. Also, when there are multiple \vee operations or multiple \wedge operations, such as in the compound proposition $p \vee q \vee r$ or the compound proposition $p \wedge q \wedge r$, parentheses are usually omitted because the order in which the operations are applied does not affect the final truth value.

Animation captions:

1. Compound proposition $p \wedge \neg(p \vee r)$ is evaluated by filling in the given truth values for variables p, q, and r.
2. and then evaluating the operations in the required order.
3. The compound proposition $p \wedge \neg(q \vee r)$ evaluates to false.

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1.2.2: Evaluating complex compound propositions.

Assume the propositions p, q, r have the following truth values:

p is true
q is true
r is false

What are the truth values for the following compound propositions?

1) $p \vee \neg q$

- True
 False

2) $\neg r \wedge (p \vee \neg q)$

- True
 False

3) $\neg(p \wedge \neg r)$

- True
 False

4) $(p \vee r) \wedge \neg p$

- True
 False

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1.2.1: Write proposition using symbols.

Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: The weather is bad or the trip is delayed.

Proposition in symbols:

Navigation bar with 5 steps: 1 (highlighted), 2, 3, 4, 5.

Check

Next

Example 1.2.1: Searching the web -- continued.

Compound propositions can be created with logical operations to conduct refined web searches. Suppose one is interested in studying jaguars (the animal from the cat family). Try searching (e.g., at google.com) for the term "jaguar" — the results may include numerous hits related to the car "Jaguar". To avoid results involving cars, try a second search using the query "jaguar AND -car" — the "-" symbol indicates negation. Notice that results are then mostly about the animal.

Filling in the rows of a truth table

A truth table for a compound statement will have a row for every possible combination of truth assignments for the statement's variables. If there are n variables, there are 2^n rows. In the truth table for the compound proposition $(p \vee r) \wedge \neg q$, there are three variables and $2^3 = 8$ rows.

Table 1.2.1: Truth table with three variables.

p	q	r	$(p \vee r) \wedge \neg q$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F

F	F	T	T
F	F	F	F

Note that the T and F values for each row are unique, and that they are chosen methodically to create all possible combinations. (The column for variable r alternates T F T F..., the column for q alternates T T F F ..., etc.)

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1.2.3: Number of entries in a truth table.

- 1) How many rows are in a truth table for a compound proposition with propositional variables p , q , and r ?

Check

[Show answer](#)

- 2) How many rows are in a truth table for the proposition $(p \wedge q) \vee (\neg r \wedge \neg q) \vee \neg(p \wedge t)$?

Check

[Show answer](#)

When filling out a truth table for a complicated compound proposition, completing intermediate columns for smaller parts of the full compound proposition can be helpful.

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1.2.4: Truth table with intermediate columns.

Animation captions:

1. The truth table for $\neg q \wedge (p \vee r)$ can be computed by first filling in a column for $\neg q$,
2. then filling in a column for $(p \vee r)$,
3. and finally filling in the column for $\neg q \wedge (p \vee r)$ using the intermediate columns.

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1.2.5: Filling in a truth table.

Indicate how the missing items in the truth table below should be filled in:

p	q	r	$p \wedge \neg q$	$(p \wedge \neg q) \vee r$
T	T	T	F	(A)
T	T	F	F	F
T	F	T	T	T
T	F	F	(B)	T
F	T	T	F	T
F	T	F	F	(C)
F	F	T	F	T
F	F	(D)	F	F

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1) What is the correct value for A?

- T
 F



2) What is the correct value for B?

- T
 F



3) What is the correct value for C?

- T
 F



4) What is the correct value for D?

- T
 F



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1.2.2: Truth tables for compound propositions

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Start

Using the pattern above, fill in all combinations of p and q.

p	q

T	T
T	<input type="text"/>
<input type="text"/>	<input type="text"/>
<input type="text"/>	F

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1	2	3	4	5	6	7	8
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Check

Next

Example 1.2.2: Logic in electronic devices.

Most of the devices and appliances used in everyday life are controlled by electronic circuitry that works according to the laws of logic. A designer of an electronically-controlled device must first express the desired behavior in logic, and then test that the device behaves as desired under every set of circumstances.

An electronic fan automatically turns on and off depending on the humidity in a room. Since the humidity detector is not very accurate, the fan will stay on for 20 minutes once triggered in order to ensure that the room is cleared of moisture. There is also a manual off switch that can be used to override the automatic functioning of the control.

Define the following propositions:

M: The fan has been on for twenty minutes.

H: The humidity level in the room is low.

O: The manual "off" button has been pushed.

The fan will turn off if the following proposition evaluates to true:

$$(M \wedge H) \vee O$$

A truth table can be useful in testing the device to make sure it works as intended under every set of circumstances. The following table might be used by a technician testing the electronic fan.

M	H	O	Should be off? (T:yes)	Your observation
T	T	T	T	
T	T	F	T	
T	F	T	T	

T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

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Additional exercises

EXERCISE 1.2.1: Truth values for compound English sentences.

Determine whether the following propositions are true or false:

(a) 5 is an odd number and 3 is a negative number.

 **Solution** ▾

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) 5 is an odd number or 3 is a negative number.

 **Solution** ▾

(c) 8 is an odd number or 4 is not an odd number.

 **Solution** ▾

(d) 6 is an even number and 7 is odd or negative.

 **Solution** ▾

(e) It is not true that 7 is an odd number or 8 is an even number.

 **Solution** ▾

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EXERCISE 1.2.2: Truth values for compound propositions.

The propositional variables, p , q , and s have the following truth assignments: $p = T$, $q = T$, $s = F$. Give the truth value for each proposition.



(a) $p \vee \neg q$
 **Solution** 



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Ok, got it



(b) $(p \wedge q) \vee s$
 **Solution** 

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(c) $p \wedge (q \vee s)$
 **Solution** 

(d) $p \wedge \neg(q \vee s)$
 **Solution** 

(e) $\neg(q \wedge p \wedge \neg s)$
 **Solution** 

(f) $\neg(p \wedge \neg(q \wedge s))$
 **Solution** 

EXERCISE 1.2.3: Translating compound propositions into English sentences.

Express the following compound propositions in English using the following definitions:

p: I am going to a movie tonight.

q: I am going to the party tonight.



(a) $\neg p$
 **Solution** 

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Ok, got it

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(b) $p \wedge q$
 **Solution** 

(c) $p \wedge \neg q$

 **Solution** 

(d) $\neg p \vee \neg q$

 **Solution** 

(e) $\neg(p \wedge q)$

 **Solution** 

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EXERCISE 1.2.4: Multiple disjunction or conjunction operations.

Suppose that p , q , r , s , and t are all propositional variables.

(a) Describe in words when the expression $p \vee q \vee r \vee s \vee t$ is true and when it is false.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Describe in words when the expression $p \wedge q \wedge r \wedge s \wedge t$ is true and when it is false.

 **Solution** 

EXERCISE 1.2.5: Expressing a set of conditions using logical operations.

Consider the following pieces of identification a person might have in order to apply for a credit card

B: Applicant presents a birth certificate.

D: Applicant presents a driver's license.

M: Applicant presents a marriage license.

Write a logical expression for the requirements under the following conditions:

(a) The applicant must present either a birth certificate, a driver's license or a marriage license.

 **Solution** 

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Ok, got it

(b) The applicant must present at least two of the following forms of identification: birth certificate driver's license, marriage license.

 **Solution** 

(c) Applicant must present either a birth certificate or both a driver's license and a marriage license

 **Solution** 

EXERCISE 1.2.6: Boolean expression to express a condition on the input variables.

(a) Give a logical expression with variables p , q , and r that is true if p and q are false and r is true and is otherwise false.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

1.3 Conditional statements

The **conditional operation** is denoted with the symbol \rightarrow . The proposition $p \rightarrow q$ is read "if p then q ". The proposition $p \rightarrow q$ is false if p is true and q is false; otherwise, $p \rightarrow q$ is true.

A compound proposition that uses a conditional operation is called a **conditional proposition**. A conditional proposition expressed in English is sometimes referred to as a **conditional statement**, as in "If there is a traffic jam today, then I will be late for work."

In $p \rightarrow q$, the proposition p is called the **hypothesis**, and the proposition q is called the **conclusion**. The truth table for $p \rightarrow q$ is given below.

Table 1.3.1: Truth table for the conditional operation.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

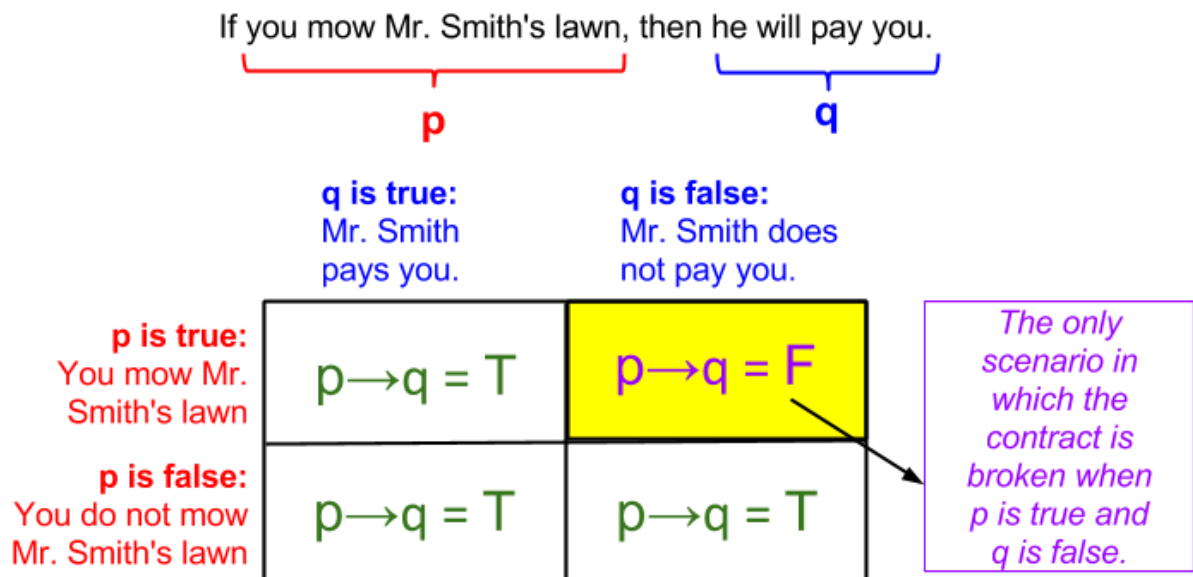
F	F	T
---	---	---

A conditional proposition can be thought of like a contract between two parties, as in:

If you mow Mr. Smith's lawn, then he will pay you.

The only way for the contract between you and Mr. Smith to be broken, is for you to mow Mr. Smith's lawn and for him not to pay you. If you do not mow his lawn, then he can either pay you or not, and the contract is not broken. In the words of logic, the only way for a conditional statement to be false is if the hypothesis is true and the conclusion is false. If the hypothesis is false, then the conditional statement is true regardless of the truth value of the conclusion.

Figure 1.3.1: A conditional statement illustrated.



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1.3.1: Understanding conditional statements.

Each question has a proposition p that is a conditional statement. Truth values are also given for the individual propositions contained in that conditional statement. Indicate whether the conditional statement p is true or false.

- 1) p : If it rains today, I will have my umbrella.
It is raining today.
I do not have my umbrella.

- True
 False

- 2) p : If Sally took too long getting ready,

she missed the bus.
 Sally did not take too long getting ready.
 Sally missed the bus.

- True
 False

- 3) p: If it is sunny out, I ride my bike.
 It is not sunny out.
 I am not riding my bike.

- True
 False

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There are many ways to express the conditional statement $p \rightarrow q$ in English:

Table 1.3.2: English expressions of the conditional operation.

Consider the propositions:

p: You mow Mr. Smith's lawn.

q: Mr. Smith will pay you.

If p, then q.	If you mow Mr. Smith's lawn, then he will pay you.
If p, q.	If you mow Mr. Smith's lawn, he will pay you.
q if p	Mr. Smith will pay you if you mow his lawn.
p implies q.	Mowing Mr. Smith's lawn implies that he will pay you.
p only if q.	You will mow Mr. Smith's lawn only if he pays you.
p is sufficient for q.	Mowing Mr. Smith's lawn is sufficient for him to pay you.
q is necessary for p.	Mr. Smith's paying you is necessary for you to mow his lawn.

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There is sometimes some confusion about the fact that the statement "p only if q" is the same as the proposition $p \rightarrow q$. Both statements mean that the only way for p to be true is if q is also true.

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1.3.2: Conditional proposition from English sentences.



This question uses the following propositions:

p: I will share my cookie with you.
 q: You will share your soda with me.

Select the conditional statement that has the same logical meaning as the English sentence given.

1) If you share your soda with me, then I will share my cookie with you.

- $q \rightarrow p$
 $p \rightarrow q$

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2) My sharing my cookie with you is sufficient for you to share your soda with me.

- $q \rightarrow p$
 $p \rightarrow q$

3) I will share my cookie with you only if you share your soda with me.

- $q \rightarrow p$
 $p \rightarrow q$

**CHALLENGE
 ACTIVITY**

1.3.1: Convert proposition from words to symbols.

Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: If the weather is bad, then the trip will be cancelled.

Proposition in symbols:

1	2	3	4	5
----------	---	---	---	---

Check

Next

The converse, contrapositive, and inverse

Three conditional statements related to proposition $p \rightarrow q$ are so common that they have special names. The **converse** of $p \rightarrow q$ is $q \rightarrow p$. The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

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Table 1.3.3: The converse, contrapositive, and inverse.

Proposition:	$p \rightarrow q$	Ex: If it is raining today, the game will be cancelled.
Converse:	$q \rightarrow p$	If the game is cancelled, it is raining today.
Contrapositive:	$\neg q \rightarrow \neg p$	If the game is not cancelled, then it is not raining today.
Inverse:	$\neg p \rightarrow \neg q$	If it is not raining today, the game will not be cancelled.

PARTICIPATION ACTIVITY

1.3.3: Converse, contrapositive, and inverse of a conditional proposition.



Consider the conditional statement below:

If he studied for the test, then he passed the test.

Match each statement below to the term describing how it is related to the statement above.

Contrapositive

Inverse

Converse

If he did not pass the test, then he did not study for the test.

If he passed the test, then he studied for the test.

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If he did not study for the test, then he did not pass the test.

Reset

The biconditional operation

If p and q are propositions, the proposition " p if and only if q " is expressed with the **biconditional operation** and is denoted $p \leftrightarrow q$. The proposition $p \leftrightarrow q$ is true when p and q have the same truth value and is false when p and q have different truth values.

Alternative ways of expressing $p \leftrightarrow q$ in English include " p is necessary and sufficient for q " or " p then q , and conversely". The term **iff** is an abbreviation of the expression "if and only if", as in " p iff q ". The truth table for $p \leftrightarrow q$ is given below:

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Table 1.3.4: Truth table for the biconditional operation.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound propositions with conditional and biconditional operations

The conditional and biconditional operations can be combined with other logical operations, as in $(p \rightarrow q) \wedge r$. If parentheses are not used to explicitly indicate the order in which the operations should be applied, then \wedge , \vee , and \neg should be applied before \rightarrow or \leftrightarrow . Thus, the proposition $p \rightarrow q \wedge r$ should be evaluated as $p \rightarrow (q \wedge r)$. Good practice, however, is to use parentheses so that the order of operations is clear.

PARTICIPATION ACTIVITY

1.3.4: Example of evaluating a compound proposition with a biconditional.



Animation captions:

1. The compound proposition, $p \vee (q \leftrightarrow r)$ is evaluated by first filling in the given truth values for p , q and r .
2. The biconditional operation is evaluated first,
3. then the negation and then the disjunction, yielding a truth value for the entire compound proposition.

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PARTICIPATION ACTIVITY

1.3.5: Evaluating compound propositions with conditional and biconditional operations.



Assume the propositions p , q , r , and s have the following truth values:

p is true
q is true
r is false
s is false

What are the truth values for the following compound propositions?

1) $s \rightarrow q$

- True
 False

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2) $(r \leftrightarrow s) \wedge q$

- True
 False



3) $q \rightarrow \neg r$

- True
 False



4) $(q \wedge s) \rightarrow p$

- True
 False



5) $(p \leftrightarrow r) \wedge (\neg r \wedge \neg s)$

- True
 False



6) $q \rightarrow \neg(r \vee q)$

- True
 False



**CHALLENGE
ACTIVITY**

1.3.2: Truth tables for conditional propositions.

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Example 1.3.1: Automatic degree requirements check.

Large universities with thousands of students usually have an automated system for checking whether a student has satisfied the requirements for a particular degree before graduation. Degree requirements can be expressed in the language of logic so that they can be checked by a computer program. For example, let X be the proposition that the student has taken course X . For a degree in Computer Science, a student must take one of three project courses, $P1$, $P2$, or $P3$. The student must also take one of two theory courses, $T1$ or $T2$. Furthermore, if the student is an honors student, he or she must take the honors seminar S . Let H be the proposition indicating whether the student is an honors student. We can express these requirements with the following proposition:

$$(P1 \vee P2 \vee P3) \wedge (T1 \vee T2) \wedge (H \rightarrow S)$$

Additional exercises

EXERCISE 1.3.1: Truth values for conditional statements in English.

Which of the following conditional statements are true and why?

(a) If February has 30 days, then 7 is an odd number.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) If January has 31 days, then 7 is an even number.

 **Solution** 

(c) If 7 is an odd number, then February does not have 30 days.

 **Solution** 

(d) If 7 is an even number, then January has exactly 28 days.

 **Solution** 

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EXERCISE

1.3.2: The inverse, converse, and contrapositive of conditional sentences in English.

Give the inverse, converse and contrapositive for each of the following statements:

(a) If she finished her homework, then she went to the party.

 **Solution** 

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Ok, got it

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(b) If he trained for the race, then he finished the race.

 **Solution** 

(c) If the patient took the medicine, then she had side effects.

 **Solution** 

EXERCISE 1.3.3: Expressing conditional statements in English using logic.

Define the following propositions:

c: I will return to college.

j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) If I return to college, then I won't get a job.

 **Solution** 

(c) I am not getting a job, but I am still not returning to college.

 **Solution** 

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(d) I will return to college only if I won't get a job.

 **Solution** 

(e) There's no way I am returning to college.

 **Solution** 

(f) I will get a job and return to college.

 **Solution** 

EXERCISE

1.3.4: Expressing conditional statements in English using logic.

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Define the following propositions:

s: a person is a senior

y: a person is at least 17 years of age

p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.

 **Solution** 

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Ok, got it

(b) A person can park in the school parking lot if they are a senior or at least seventeen years of age.

 **Solution** 

(c) Being 17 years of age is a necessary condition for being able to park in the school parking lot.

 **Solution** 

(d) A person can park in the school parking lot if and only if the person is a senior and at least 17 years of age.

 **Solution** 

(e) Being able to park in the school parking lot implies that the person is either a senior or at least 17 years old.

 **Solution** 

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EXERCISE

1.3.5: Translating logical expressions into English.

Define the following propositions:

w: the roads were wet
 a: there was an accident
 h: traffic was heavy

Express each of the logical expressions as an English sentence:

(a) $w \rightarrow h$

 **Solution** 

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Ok, got it

(b) $w \wedge a$

 **Solution** 

(c) $\neg(a \wedge h)$

 **Solution** 

(d) $h \rightarrow (a \vee w)$

 **Solution** 

(e) $w \wedge \neg h$

 **Solution** 

EXERCISE 1.3.6: Finding logical expressions to match a truth table.

For each table, give a logical expression whose truth table is the same as the one given.

p	q	?
T	T	F
T	F	T
F	T	F
F	F	F

(a)

 **Solution** 

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Ok, got it

(b)

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

 **Solution** 

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1.4 Logical equivalence

A compound proposition is a **tautology** if the proposition is always true, regardless of the truth value of the individual propositions that occur in it. A compound proposition is a **contradiction** if the proposition is always false, regardless of the truth value of the individual propositions that occur in it. $p \vee \neg p$ is a simple example of a tautology since the proposition is always true whether p is true or false. The fact that $p \vee \neg p$ is a tautology can be verified in a truth table, which shows that every truth value in the rightmost column is true.

Table 1.4.1: Truth table for tautology $p \vee \neg p$.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

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Similarly, the proposition $p \wedge \neg p$ is an example of a simple contradiction, because the proposition is false regardless of whether p is true or false. The truth table below shows that $p \wedge \neg p$ is a contradiction because every truth value in the rightmost column is false.

Table 1.4.2: Truth table for contradiction $p \wedge \neg p$.

p	$\neg p$	$p \wedge \neg p$
---	----------	-------------------

T	F	F
F	T	F

Showing that a compound proposition is not a tautology only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to false. For example, the proposition $(p \wedge q) \rightarrow r$ is not a tautology because when $p = q = T$ and $r = F$, then $(p \wedge q) \rightarrow r$ is false. Showing that a compound proposition is not a contradiction only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to true. For example, the proposition $\neg(p \vee q)$ is not a contradiction because when $p = q = F$, then $\neg(p \vee q)$ is true.

**PARTICIPATION
ACTIVITY**

1.4.1: Identifying tautologies and contradictions.



Determine whether the following compound propositions are tautologies, contradictions, or neither.

1) $p \leftrightarrow \neg p$



- Tautology
- Contradiction
- Neither a tautology or a contradiction

2) $p \rightarrow \neg p$



- Tautology
- Contradiction
- Neither a tautology or a contradiction

3) $(p \wedge q) \rightarrow p$



- Tautology
- Contradiction.
- Neither a tautology or a contradiction.

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Showing logical equivalence using truth tables

Two compound propositions are said to be **logically equivalent** if they have the same truth value regardless of the truth values of their individual propositions. If s and r are two compound propositions, the notation $s \equiv r$ is used to indicate that r and s are logically equivalent. For example, p and $\neg\neg p$ have the same truth value regardless of whether p is true or false, so $p \equiv \neg\neg p$. Propositions s and r are logically equivalent if and only if the proposition $s \leftrightarrow r$ is a tautology.

A truth table can be used to show that two compound propositions are logically equivalent.

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**PARTICIPATION
ACTIVITY**1.4.2: Showing $p \rightarrow \neg p \equiv \neg p$ with a truth table.**Animation captions:**

1. $p \rightarrow \neg p$ can be shown to be equivalent to $\neg p$ by filling in a column for $\neg p$,
2. then filling in a column for $p \rightarrow \neg p$, and verifying that the two columns are the same.

Table 1.4.3: Truth table to show: $\neg p \vee \neg q \equiv \neg(p \wedge q)$.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Showing that two propositions are not logically equivalent only requires showing a particular set of truth values for their individual propositions that cause the two compound proposition to have different truth values. For example, $p \leftrightarrow r$ and $p \rightarrow r$ are not logically equivalent because when $p = F$ and $r = T$, then $p \leftrightarrow r$ is false but $p \rightarrow r$ is true.

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**PARTICIPATION
ACTIVITY**

1.4.3: Logical equivalence by truth table.



The table below shows the truth table for three compound propositions:

- $(p \vee \neg q) \rightarrow r$
- $(p \leftrightarrow q) \rightarrow r$
- $\neg r \rightarrow (\neg p \wedge q)$

p	q	r	$(p \vee \neg q) \rightarrow r$	$(p \leftrightarrow q) \rightarrow r$	$\neg r \rightarrow (\neg p \wedge q)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	F	F

1) Use the truth table to determine which logical equivalence is true.

- $(p \vee \neg q) \rightarrow r \equiv (p \leftrightarrow q) \rightarrow r$
 $(p \leftrightarrow q) \rightarrow r \equiv \neg r \rightarrow (\neg p \wedge q)$
 $(p \vee \neg q) \rightarrow r \equiv \neg r \rightarrow (\neg p \wedge q)$



De Morgan's laws

De Morgan's laws are logical equivalences that show how to correctly distribute a negation operation inside a parenthesized expression. Both versions of De Morgan's laws are particularly useful in logical reasoning. The first De Morgan's law is:

$$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$$

When the negation operation is distributed inside the parentheses, the disjunction operation changes to a conjunction operation. Consider an English example with the following propositions for p and q.

- p: The patient has migraines
 q: The patient has high blood pressure

The use of the English word "or" throughout the example is assumed to be disjunction (i.e., the inclusive or). De Morgan's law says that the following two English statements are logically equivalent:

It is not true that the patient has migraines or high blood pressure.
The patient does not have migraines and does not have high blood pressure.

The logical equivalence $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$ to evaluate to true provides intuition about why the two expressions are logically equivalent.

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**PARTICIPATION
ACTIVITY**

1.4.4: Reasoning about De Morgan's laws.

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1) There is only one truth assignment for p and q that makes the expression $\neg(p \vee q)$ evaluate to true. Which one is it?

- $p = q = T$
- $p = T$ and $q = F$
- $p = q = F$

2) There is only one truth assignment for p and q that makes the expression $\neg p \wedge \neg q$ evaluate to true. Which one is it?

- $p = q = T$
- $p = T$ and $q = F$
- $p = q = F$

The second version of De Morgan's law swaps the role of the disjunction and conjunction:

$$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$$

Continuing with the same example, the following two statements are logically equivalent:

It is not true that the patient has migraines and high blood pressure.
The patient does not have migraines or does not have high blood pressure.

The logical equivalence $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions $\neg(p \wedge q)$ and $(\neg p \vee \neg q)$ to evaluate to false provides intuition about why the two expressions are logically equivalent.

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**PARTICIPATION
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1.4.5: Reasoning about De Morgan's laws.

1) There is only one truth assignment for p and q that makes the expression $\neg(p \wedge$

q) evaluate to false. Which one is it?

- $p = q = T$
- $p = T$ and $q = F$
- $p = q = F$

2) There is only one truth assignment for p and q that makes the expression $\neg p \vee \neg q$ evaluate to false. Which one is it?

- $p = q = T$
- $p = T$ and $q = F$
- $p = q = F$

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**PARTICIPATION
ACTIVITY**

1.4.6: Matching equivalent English expressions using De Morgan's laws.

Select the English sentence that is logically equivalent to the given sentence.

1) It is not true that the child is at least 8 years old and at least 57 inches tall.

- The child is at least 8 years old and at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

2) It is not true that the child is at least 8 years old or at least 57 inches tall.

- The child is at least 8 years old or at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

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Additional exercises

EXERCISE 1.4.1: Proving tautologies and contradictions.

Show whether each logical expression is a tautology, contradiction or neither.

(a) $(p \vee q) \vee (q \rightarrow p)$

 **Solution** 

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Ok, got it

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(b) $(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$

 **Solution** 

(c) $(p \rightarrow q) \leftrightarrow p$

 **Solution** 

(d) $(p \rightarrow q) \vee p$

 **Solution** 

(e) $(\neg p \vee q) \leftrightarrow (p \wedge \neg q)$

 **Solution** 

(f) $(\neg p \vee q) \leftrightarrow (\neg p \wedge q)$

 **Solution** 

EXERCISE 1.4.2: Truth tables to prove logical equivalence.

Use truth tables to show that the following pairs of expressions are logically equivalent.

(a) $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$

 **Solution** 

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Ok, got it

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(b) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$

 **Solution** 

(c) $\neg p \rightarrow q$ and $p \vee q$

 **Solution** 

EXERCISE 1.4.3: Proving two logical expressions are not logically equivalent.

Prove that the following pairs of expressions are not logically equivalent.

(a) $p \rightarrow q$ and $q \rightarrow p$

 **Solution** 

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Ok, got it

(b) $\neg p \rightarrow q$ and $\neg p \vee q$

 **Solution** 

(c) $(p \rightarrow q) \wedge (r \rightarrow q)$ and $(p \wedge r) \rightarrow q$

 **Solution** 

(d) $p \wedge (p \rightarrow q)$ and $p \vee q$

 **Solution** 
EXERCISE 1.4.4: Proving whether two logical expressions are equivalent.

Determine whether the following pairs of expressions are logically equivalent. Prove your answer. If the pair is logically equivalent, then use a truth table to prove your answer.

(a) $\neg(p \vee \neg q)$ and $\neg p \wedge q$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\neg(p \vee \neg q)$ and $\neg p \wedge \neg q$

 **Solution** 

(c) $p \wedge (p \rightarrow q)$ and $p \rightarrow q$

 **Solution** 

(d) $p \wedge (p \rightarrow q)$ and $p \wedge q$

 **Solution** 

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EXERCISE 1.4.5: Applying De Morgan's laws.

Translate each English sentence into a logical expression using the propositional variables defined below. Then negate the entire logical expression using parentheses and the negation operation. Apply De Morgan's law to the resulting expression and translate the final logical expression back into English.

- p : the applicant has written permission from his parents
- e : the applicant is at least 18 years old
- s : the applicant is at least 16 years old

(a) The applicant has written permission from his parents and is at least 16 years old.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) The applicant has written permission from his parents or is at least 18 years old.

 **Solution** 

1.5 Laws of propositional logic

If two propositions are logically equivalent, then one can be substituted for the other within a more complex proposition. The compound proposition after the substitution is logically equivalent to the compound proposition before the substitution.

For example $p \rightarrow q \equiv \neg p \vee q$. Therefore,

$$(p \vee r) \wedge (\neg p \vee q) \equiv (p \vee r) \wedge (p \rightarrow q)$$

In the next example, the logical equivalence $p \rightarrow q \equiv \neg p \vee q$ is applied where the variables p and q represent compound propositions:

$$(\neg t \wedge r) \rightarrow (\neg s \vee t) \equiv \neg(\neg t \wedge r) \vee (\neg s \vee t)$$

**PARTICIPATION
ACTIVITY**

1.5.1: Substituting logically equivalent propositions.

Use the logical equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ to match logically equivalent propositions below:

$$(s \wedge t) \vee \neg(t \vee r) \quad \neg((s \wedge t) \vee (t \vee r)) \quad (\neg s \wedge \neg t) \vee (t \vee r)$$

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$$(s \wedge t) \vee (\neg t \wedge \neg r)$$

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$$\neg(s \wedge t) \wedge \neg(t \vee r)$$

$$\neg(s \vee t) \vee (t \vee r)$$

Reset

Using the laws of propositional logic to show logical equivalence

Substitution gives an alternate way of showing that two propositions are logically equivalent. If one proposition can be obtained from another by a series of substitutions using equivalent expressions, then the two propositions are logically equivalent. The table below shows several laws of propositional logic that are particularly useful for establishing the logical equivalence of compound propositions:

Table 1.5.1: Laws of propositional logic.

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

**PARTICIPATION
ACTIVITY**

1.5.2: The laws of propositional logic can be used to show logical equivalence.

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**Animation content:**

undefined

Animation captions:

1. The proof that $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$ starts with $p \rightarrow q$ and applies the Conditional identity to get $\neg p \vee q$.
2. The Commutative law is applied to get $q \vee \neg p$.
3. The Double negation law is applied to get $\neg \neg q \vee \neg p$.
4. Finally, the Conditional identity is applied again to get $\neg q \rightarrow \neg p$.

**PARTICIPATION
ACTIVITY**

1.5.3: The laws of propositional logic can be used to simplify compound propositions.

**Animation content:**

undefined

Animation captions:

1. The proof that $\neg(p \vee q) \vee (\neg p \wedge q)$ is equivalent to $\neg p$ starts with $\neg(p \vee q) \vee (\neg p \wedge q)$ and applies De Morgan's law to get $\neg p \wedge \neg q) \vee (\neg p \wedge q)$.
2. The Distributive law is applied to get $\neg p \wedge (\neg q \vee q)$.
3. The Commutative law is applied to get $\neg p \wedge (q \vee \neg q)$.
4. The Complement law is applied to get $\neg p \wedge T$.
5. Finally, the Identity law is applied again to get $\neg p$.

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**PARTICIPATION
ACTIVITY**

1.5.4: Using the laws of propositional logic to show logical equivalence.



Put the steps in the correct order to show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$. Each step should follow from the previous step using the given law.

$\neg(\neg p \vee q)$ Conditional identity

$\neg(p \rightarrow q)$

$p \wedge \neg q$ Double negation law

$\neg\neg p \wedge \neg q$ De Morgan's Law

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1

2

3

4

Reset

**CHALLENGE
ACTIVITY**

1.5.1: Reduce the proposition using laws.



**CHALLENGE
ACTIVITY**

1.5.2: Reduce the proposition using laws, including de Morgan's and conditional.



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**CHALLENGE
ACTIVITY**

1.5.3: Expand then reduce the proposition.

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Additional exercises**EXERCISE**

1.5.1: Label the steps in a proof of logical equivalence.

Below are several proofs showing that two logical expressions are logically equivalent. Label the steps in each proof with the law used to obtain each proposition from the previous proposition. The first line in the proof does not have a label.

(a)

$(p \rightarrow q) \wedge (q \vee p)$
$(\neg p \vee q) \wedge (q \vee p)$
$(q \vee \neg p) \wedge (q \vee p)$
$q \vee (\neg p \wedge p)$
$q \vee (p \wedge \neg p)$
$q \vee F$
q

**Solution**

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Ok, got it

$(\neg p \vee q) \rightarrow (p \wedge q)$

(b)	$\neg(\neg p \vee q) \vee (p \wedge q)$
	$(\neg\neg p \wedge \neg q) \vee (p \wedge q)$
	$(p \wedge \neg q) \vee (p \wedge q)$
	$p \wedge (\neg q \vee q)$
	$p \wedge T$
	p

 **Solution** 

(c)	$r \vee (\neg r \rightarrow p)$
	$r \vee (\neg\neg r \vee p)$
	$r \vee (r \vee p)$
	$(r \vee r) \vee p$
	$r \vee p$

 **Solution** 

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EXERCISE 1.5.2: Using the laws of logic to prove logical equivalence.

Use the laws of propositional logic to prove the following:

(a) $\neg p \rightarrow \neg q \equiv q \rightarrow p$

 **Solution** 

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Ok, got it

(b) $p \wedge (\neg p \rightarrow q) \equiv p$

 **Solution** 

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(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

 **Solution** 

(d) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

 **Solution** 

(e) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

 **Solution** 

(f) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

 **Solution** 

(g) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \equiv p \wedge \neg r$

 **Solution** 

(h) $p \leftrightarrow (p \wedge r) \equiv \neg p \vee r$

 **Solution** 

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EXERCISE 1.5.3: Using the laws of logic to prove tautologies.

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $(p \wedge q) \rightarrow (p \vee r)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $p \rightarrow (r \rightarrow p)$

 **Solution** 

(c) $\neg r \vee (\neg r \rightarrow p)$

 **Solution** 

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1.6 Predicates and quantifiers

Many mathematical statements contain variables. The statement "x is an odd number" is not a proposition because its truth value depends on the value of variable x. If $x = 5$, the statement is

true. If $x = 4$, the statement is false. The truth value of the statement can be expressed as a function P of the variable x , as in $P(x)$. The expression $P(x)$ is read "P of x ". A logical statement whose truth value is a function of one or more variables is called a **predicate**. If $P(x)$ is defined to be the statement " x is an odd number", then $P(5)$ corresponds to the statement "5 is an odd number". $P(5)$ is a proposition because it has a well defined truth value.

A predicate can depend on more than one variable. Define the predicates Q and R as:

$$Q(x, y) : x^2 = y$$

$$R(x, y, z) : x + y = z$$

The proposition $Q(5, 25)$ is true because $5^2 = 25$. The proposition $R(2, 3, 6)$ is false because $2 + 3 \neq 6$.

The **domain** of a variable in a predicate is the set of all possible values for the variable. For example, a natural domain for the variable x in the predicate " x is an odd number" would be the set of all integers. If the domain of a variable in a predicate is not clear from context, the domain should be given as part of the definition of the predicate.

PARTICIPATION ACTIVITY

1.6.1: Truth values for predicates on specific inputs.

The domain for all the variables in the following predicates is the set of positive integers:

$P(x)$: x is a prime number

$L(x, y)$: $x < y$

$S(x, y, z)$: $x^2 + y^2 \geq z^2$

1) Is $P(7)$ true?

True

False

2) Is $L(6, 6)$ true?

True

False

3) Is $S(3, 4, 5)$ true?

True

False

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Statements outside the realm of mathematics can also be predicates. For example, consider the statement: "The city has a population over 1,000,000." The "city" is the variable and the domain is defined to be the set of all cities in the United States. When the city is New York, the statement becomes: "New York has a population over 1,000,000" and the statement is true. When the city is

Toledo, the statement becomes: "Toledo has a population over 1,000,000" and the statement is false.

Note that it may happen that a statement $P(x)$ is true for all values in the domain. However, if the statement contains a variable, the statement is still considered to be a predicate and not a proposition. For example, if $P(x)$ is the statement " $x + 1 > 1$ " and the domain is all positive integers, the statement is true for each value in the domain. However, $P(x)$ is considered to be a predicate and not a proposition because it contains a variable.

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PARTICIPATION ACTIVITY

1.6.2: Predicates and propositions.



Which sentences are propositions and which are predicates? The domain is the set of all positive integers.

1) x is odd.

Proposition

Predicate



2) 23 is a prime number.

Proposition

Predicate



3) $\frac{1}{1+x} < 1$.

Proposition

Predicate



4) $16 = x^2$.

Proposition

Predicate



Universal quantifier

If all the variables in a predicate are assigned specific values from their domains, then the predicate becomes a proposition with a well defined truth value. Another way to turn a predicate into a proposition is to use a quantifier. The logical statement $\forall x P(x)$ is read "for all x , $P(x)$ " or "for every x , $P(x)$ ". The statement $\forall x P(x)$ asserts that $P(x)$ is true for every possible value for x in its domain. The symbol \forall is a **universal quantifier** and the statement $\forall x P(x)$ is called a **universally quantified statement**. $\forall x P(x)$ is a proposition because it is either true or false. $\forall x P(x)$ is true if and only if $P(n)$ is true for every n in the domain.

If the domain is a finite set of elements $\{a_1, a_2, \dots, a_k\}$, then:

$$\forall x P(x) \equiv P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_k)$$

The equivalence symbol means that the two expressions always have the same truth value, regardless of the truth values for $P(a_1), \dots, P(a_n)$. If the domain is the set of students in a class and the predicate $A(x)$ means that student x completed the assignment, then the proposition $\forall x A(x)$ means: "Every student completed the assignment." Establishing that $\forall x A(x)$ is true requires showing that each and every student in the class did in fact complete the assignment.

Some universally quantified statements can be shown to be true by showing that the predicate holds for an arbitrary element from the domain. An "arbitrary element" means nothing is assumed about the element other than the fact that it is in the domain. In the following example, the domain is the set of all positive integers:

$$\forall x \left(\frac{1}{x+1} < 1 \right)$$

The statement is true because when x is assigned any arbitrary value from the set of all positive integers, the inequality $\frac{1}{1+x} < 1$ holds.

PARTICIPATION ACTIVITY

1.6.3: Proving $\forall x (1/(x+1) < 1)$ is true for an arbitrary positive real number x .

Animation captions:

1. The proof starts with the fact that $0 < x$ for all positive integers and adds 1 to both sides to get $1 < 1 + x$.
2. Dividing both sides by $(x + 1)$ gives that $\frac{1}{(x+1)} \leq 1$. The inequality is true for all positive integers x .

A **counterexample** for a universally quantified statement is an element in the domain for which the predicate is false. A single counterexample is sufficient to show that a universally quantified statement is false. For example, consider the statement $\forall x (x^2 > x)$, in which the domain is the set of positive integers. When $x = 1$, then $x^2 = x$ and the statement $x^2 > x$ is not true. Therefore $x = 1$ is a counterexample that shows the statement " $\forall x (x^2 > x)$ " is false.

PARTICIPATION ACTIVITY

1.6.4: Truth values for universally quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the universally quantified statement is true or false.

1) $\forall x (x^2 > 0)$.

True

False

2) $\forall x (x - 1 \geq 0)$.

True False3) $\forall x (x - 1 > 0)$. True False

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Existential quantifier

The logical statement $\exists x P(x)$ is read "There exists an x , such that $P(x)$ ". The statement $\exists x P(x)$ asserts that $P(x)$ is true for at least one possible value for x in its domain. The symbol \exists is an **existential quantifier** and the statement $\exists x P(x)$ is called a **existentially quantified statement**. $\exists x P(x)$ is a proposition because it is either true or false. $\exists x P(x)$ is true if and only if $P(n)$ is true for at least one value n in the domain of variable x .

If the domain is a finite set of elements $\{a_1, a_2, \dots, a_k\}$, then:

$$\exists x P(x) \equiv P(a_1) \vee P(a_2) \vee \dots \vee P(a_k)$$

If the domain is the set of students in a class and the predicate $A(x)$ means that student x completed the assignment, then $\exists x A(x)$ is the statement: "There is a student who completed the assignment." Establishing that $\exists x A(x)$ is true only requires finding one particular student who completed the assignment. However, showing that $\exists x A(x)$ is false requires showing that every student in the class did not complete the assignment.

Some existentially quantified statements can be shown to be false by showing that the predicate is false for an arbitrary element from the domain. For example, consider the existentially quantified statement in which the domain of x is the set of all positive integers:

$$\exists x (x + 1 < x)$$

The statement is false because no positive integer satisfies the expression $x + 1 < x$.

PARTICIPATION ACTIVITY

1.6.5: Showing $\exists x (x + 1 < x)$ is false.

Animation captions:

1. The proof starts with the inequality $x + 1 < x$ and subtracts x from both sides to get $1 < 0$.
2. Since $1 < 0$ is false, the inequality $x + 1 < x$ is false for every x and therefore $\exists x (x + 1 < x)$ is false.

PARTICIPATION ACTIVITY

1.6.6: Truth values for existentially quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the existentially quantified statement is true or false.

1) $\exists x(x^2 < 0)$.

True

False



2) $\exists x(x - 1 > 0)$.

True

False



3) $\exists x(x^2 = x)$.

True

False



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**PARTICIPATION
ACTIVITY**

1.6.7: Proving universal and existentially quantified statements.



Match what needs to be done to show that an existential or universally quantified statement is true or false.

Show the following statement is false: $\forall x P(x)$.

Show the following statement is true: $\exists x P(x)$.

Show the following statement is true: $\forall x P(x)$.

Show the following statement is false: $\exists x P(x)$.

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Give a particular element n in the domain for which $P(n)$ is true.

Show that for every element n in the domain, $P(n)$ is false.

Show that for every element n in the

domain, $P(n)$ is true.

Give a counterexample: a particular element n in the domain for which $P(n)$ is false.

Reset

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Additional exercises

EXERCISE 1.6.1: Which expressions with predicates are propositions?

Predicates P , T and E are defined below. The domain of discourse is the set of all positive integers

$P(x) : x$ is even

$T(x, y) : 2^x = y$

$E(x, y, z) : x^y = z$

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) $P(3)$

 **Solution** ▾

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\neg P(3)$

 **Solution** ▾

(c) $T(5, 32)$

 **Solution** ▾

(d) $T(5, x)$

 **Solution** ▾

(e) $E(6, 2, 36)$

 **Solution** ▾

(f) $E(2, y, 7)$

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 **Solution** 

(g) $P(3) \vee T(5, 32)$

 **Solution** 

(h) $T(5, 16) \rightarrow E(6, 3, 36)$

 **Solution** 

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EXERCISE 1.6.2: Truth values for quantified statements about integers.

In this problem, the domain of discourse is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a) $\exists x (x + x = 1)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\exists x (x + 2 = 1)$

 **Solution** 

(c) $\forall x (x^2 - x \neq 1)$

 **Solution** 

(d) $\forall x (x^2 - x \neq 0)$

 **Solution** 

(e) $\forall x (x^2 > 0)$

 **Solution** 

(f) $\exists x (x^2 > 0)$

 **Solution** 

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EXERCISE 1.6.3: Translating mathematical statements in English into logical expressions.

Consider the following statements in English. Write a logical expression with the same meaning. The domain of discourse is the set of all real numbers.

- (a) There is a number whose cube is equal to 2.

 **Solution** 

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Ok, got it

- (b) The square of every number is at least 0.

 **Solution** 

- (c) There is a number that is equal to its square.

 **Solution** 

- (d) Every number is less than or equal to its square.

 **Solution** 

EXERCISE 1.6.4: Truth values for quantified statements for a given set of predicates.

The domain for this problem is a set a, b, c, d . The table below shows the value of three predicates for each of the elements in the domain. For example, $Q(b)$ is false because the truth value in row b , column Q is F .

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

- (a) $\forall x P(x)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\exists x P(x)$  **Solution** (c) $\forall x Q(x)$  **Solution** (d) $\exists x Q(x)$  **Solution** (e) $\forall x R(x)$  **Solution** (f) $\exists x R(x)$  **Solution** 

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1.7 Quantified statements

Universally and existentially quantified statements can also be constructed from logical operations. Consider an example in which the domain is the set of positive integers and define the following predicates:

$P(x)$: x is prime

$O(x)$: x is odd

The proposition $\exists x (P(x) \wedge \neg O(x))$ states that there exists a positive number that is prime and not odd. This proposition is true because of the number $x = 2$.

The proposition $\forall x (P(x) \rightarrow O(x))$ says that for every positive integer x , if x is prime then x is odd. This proposition is false, because of the counterexample $x = 2$. Since 2 is prime and not odd, the conditional statement $P(2) \rightarrow O(2)$ is false.

The universal and existential quantifiers are generically called **quantifiers**. A logical statement that includes a universal or existential quantifier is called a **quantified statement**. The quantifiers \forall and \exists are applied before the logical operations (\wedge , \vee , \rightarrow , and \leftrightarrow) used for propositions. This means that the statement $\forall x P(x) \wedge Q(x)$ is equivalent to $(\forall x P(x)) \wedge Q(x)$ as opposed to $\forall x (P(x) \wedge Q(x))$.

1.7.1: Evaluating quantified statements.

ACTIVITY

For the following questions, the domain for the variable x is the set of all positive integers. The first three questions use predicates O and M which are defined as follows:

$O(x)$: x is odd

$M(x)$: x is an integer multiple of 4 (e.g., 4, 8, 12,...)

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Indicate whether each quantified statement is true or false:

1) $\exists x (O(x) \wedge M(x))$ □

True

False

2) $\exists x (\neg O(x) \wedge \neg M(x))$ □

True

False

3) $\forall x (M(x) \rightarrow \neg O(x))$ □

True

False

4) $\forall x ((x = 1) \vee (x^2 \neq x))$ □

True

False

A variable x in the predicate $P(x)$ is called a **free variable** because the variable is free to take on any value in the domain. The variable x in the statement $\forall x P(x)$ is a **bound variable** because the variable is bound to a quantifier. A statement with no free variables is a proposition because the statement's truth value can be determined.

In the statement $(\forall x P(x)) \wedge Q(x)$, the variable x in $P(x)$ is bound by the universal quantifier, but the variable x in $Q(x)$ is not bound by the universal quantifier. Therefore the statement $(\forall x P(x)) \wedge Q(x)$ is not a proposition. In contrast, the universal quantifier in the statement $\forall x (P(x) \wedge Q(x))$ binds both occurrences of the variable x . Therefore $\forall x (P(x) \wedge Q(x))$ is a proposition.

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PARTICIPATION
ACTIVITY

1.7.2: Free and bound variables in quantified statements.

1) The expression $\exists x P(x)$ is a proposition. □

True

False

2) The expression $(\exists x S(x)) \vee R(x)$ is a proposition.

True

False

3) The expression $\exists x (S(x) \vee R(x))$ has a free variable.

True

False

4) The expression $\forall x P(x) \vee \exists x Q(x)$ is a proposition.

True

False

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Logical equivalence with quantified statements

Two quantified statements (whether they are expressed in English or the language of logic) have the same logical meaning if they have the same truth value regardless of value of the predicates for the elements in the domain. Consider as an example a domain consisting of a set of people invited to a party. Define the predicates:

$P(x)$: x came to the party

$S(x)$: x was sick

The statement "Everyone was not sick" is logically equivalent to " $\forall x \neg S(x)$ " because the two statements have the same truth value regardless of who was invited to the party and whether they were sick.

The table below gives an example of a set of people who could have been invited to the party and the value of the predicate $S(x)$ and $P(x)$ for each person. For example, Gertrude came to the party (i.e., $P(\text{Gertrude}) = T$) because the truth value in the row labeled Gertrude and column labeled $P(x)$ is true.

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Table 1.7.1: Values for predicates $S(x)$ and $P(x)$ for a particular group.

Name	$S(x)$	$P(x)$
Joe	F	T

Theodore	T	F
Gertrude	F	T
Samuel	F	F

For the group of people in the domain, the statement "Someone was sick and came to the party" is false because there is no individual for whom $S(x)$ and $P(x)$ are true. However, $\exists x (S(x) \vee P(x))$ is true because, for example, $S(\text{Joe}) \vee P(\text{Joe})$ is true. Therefore the two statements "Someone was sick and came to the party" and " $\exists x (S(x) \vee P(x))$ " are not logically equivalent.

**PARTICIPATION
ACTIVITY**

1.7.3: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate $N(x)$ says that x is a new employee. The predicate $D(x)$ says that x met his deadline. Consider the group defined in the table below:

Name	$N(x)$	$D(x)$
Happy	F	F
Sleepy	F	T
Grumpy	T	T
Bashful	T	T

- Is the statement "Every new employee met his deadline" true or false for the group defined in the table?
 - True
 - False
- Is the statement " $\forall x (N(x) \wedge D(x))$ " true or false for the group defined in the table?
 - True
 - False
- Select the logical expression that is equivalent to the statement: "Every new employee met his deadline".
 - $\forall x (N(x) \wedge D(x))$

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$\forall x (N(x) \rightarrow D(x))$

**PARTICIPATION
ACTIVITY**

1.7.4: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate $N(x)$ says that x is a new employee. The predicate $D(x)$ says that x met his deadline. Consider the group defined in the table below:

Name	$N(x)$	$D(x)$
Happy	F	F
Sleepy	F	T
Grumpy	T	F
Bashful	T	F

1) Is the statement "There is a new employee who met his deadline" true or false for the group defined in the table?

- True
 False

2) Is the statement " $\exists x (N(x) \rightarrow D(x))$ " true or false for the group defined in the table?

- True
 False

3) Select the logical expression that is equivalent to the statement: "There is a new employee who met his deadline".

- $\exists x (N(x) \wedge D(x))$
 $\exists x (N(x) \rightarrow D(x))$

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Example 1.7.1: Translating quantified statements from English to logic.

The domain in this problem are the members of a board of directors for a company who are considering a proposal for that company. Define the following predicates:

$R(x)$: person x read the proposal.

$V(x)$: person x voted in favor of the proposal.

Translate each of the following sentences into an equivalent logical expression:

1. Everyone who read the proposal voted in favor of it.

Solution: $\forall x (R(x) \rightarrow V(x))$. [Video explanation of the solution \(2:01\)](#)

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2. Someone who did not read the proposal, voted in favor of it.

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Solution: $\exists x (\neg R(x) \wedge V(x))$. [Video explanation of the solution \(1:09\)](#)

3. Someone did not read the proposal and someone voted in favor of it.

Solution: $\exists x \neg R(x) \wedge \exists x V(x)$. [Video explanation of the solution \(2:57\)](#)

**PARTICIPATION
ACTIVITY**

1.7.5: Quantified statements expressed in English.

In the following question, the domain is the set of fourth graders at Lee Elementary School. The predicates P and Q are defined as follows:

$P(x)$: x took the math test

$Q(x)$: x is present today

Match the English sentence with the corresponding logical proposition.

$\forall x (Q(x) \rightarrow P(x))$

$\exists x (Q(x) \wedge \neg P(x))$

$\forall x (Q(x) \wedge P(x))$

$\exists x \neg P(x)$

Every student was present and took the math test.

There is a student who did not take the math test.

Every student who is present took the math test.

There is a student who is present and did not take the math test.

Reset

**CHALLENGE
ACTIVITY**

1.7.1: Give an example or counterexample of quantified statement.



Mark the statement as true or false.

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**Additional exercises****EXERCISE**

1.7.1: Translating quantified statements in English into logic.

In the following question, the domain of discourse is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

$T(x)$: x is a member of the executive team

$B(x)$: x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(a) Someone did not get a large bonus.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Everyone got a large bonus.

 **Solution** 

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(c) Sam did not get a large bonus even though he is a member of the executive team.

 **Solution** 

(d) Someone who is not on the executive team received a large bonus.

 **Solution** 

(e) Every executive team member got a large bonus.

 **Solution** 

EXERCISE 1.7.2: Determining whether a quantified statement is a proposition.

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Predicates P and Q are defined below. The domain of discourse is the set of all positive integers.

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P(x): x is prime

Q(x): x is a perfect square (i.e., $x = y^2$, for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) $\exists x Q(x)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\forall x Q(x) \wedge \neg P(x)$

 **Solution** 

(c) $\forall x Q(x) \vee P(3)$

 **Solution** 

(d) $\exists x (Q(x) \wedge P(x))$

 **Solution** 

(e) $\forall x (\neg Q(x) \vee P(x))$

 **Solution** 

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EXERCISE 1.7.3: Translating quantified statements from English to logic.

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In the following question, the domain of discourse is a set of students at a university. Define the following predicates:

E(x): x is enrolled in the class

T(x): x took the test

Translate the following English statements into a logical expression with the same meaning.

- (a) Someone took the test who is enrolled in the class.



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Ok, got it

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- (b) All students enrolled in the class took the test.



- (c) Everyone who took the test is enrolled in the class.



- (d) At least one student who is enrolled in the class did not take the test.



EXERCISE 1.7.4: Translating quantified statements from English to logic.

In the following question, the domain of discourse is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

$S(x)$: x was sick yesterday

$W(x)$: x went to work yesterday

$V(x)$: x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

- (a) At least one person was sick yesterday.



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Ok, got it

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- (b) Everyone was well and went to work yesterday.



- (c) Everyone who was sick yesterday did not go to work.



(d) Yesterday someone was sick and went to work.

 **Solution** 

(e) Everyone who did not get to work yesterday was sick.

 **Solution** 

(f) Everyone who missed work was sick or on vacation (or both).

 **Solution** 

(g) Someone who missed work was neither sick nor on vacation.

 **Solution** 

(h) Each person missed work only if they were sick or on vacation (or both).

 **Solution** 

(i) Ingrid was sick yesterday but she went to work anyway.

 **Solution** 

Someone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression ($x \neq$ Ingrid).)

 **Solution** 

Everyone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression ($x \neq$ Ingrid).)

 **Solution** 

EXERCISE 1.7.5: Translating quantified statements from logic to English.

In the following question, the domain of discourse is the set of employees of a company. Define the following predicates:

$A(x)$: x is on the board of directors

$E(x)$: x earns more than \$100,000

$W(x)$: x works more than 60 hours per week

Translate the following logical expressions into English:

(a) $\forall x (A(x) \rightarrow E(x))$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\exists x (E(x) \wedge \neg W(x))$

 **Solution** 

(c) $\forall x (W(x) \rightarrow E(x))$

 **Solution** 

(d) $\exists x (\neg A(x) \wedge E(x))$

 **Solution** 

(e) $\forall x (E(x) \rightarrow (A(x) \vee W(x)))$

 **Solution** 

(f) $\exists x (A(x) \wedge \neg E(x) \wedge W(x))$

 **Solution** 

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EXERCISE 1.7.6: Determining whether a quantified logical statement is true.

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

$P(x)$: x was given the placebo

$D(x)$: x was given the medication

$A(x)$: x had fainting spells

$M(x)$: x had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

	$P(x)$	$D(x)$	$A(x)$	$M(x)$
Frodo	T	F	F	T
Gandalf	F	T	F	F
Gimli	F	T	T	F

Aragorn	T	F	T	T
Bilbo	T	T	F	F

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

(a) $\exists x (M(x) \wedge D(x))$

 **Solution** 

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Ok, got it

(b) $\exists x M(x) \wedge \exists x D(x)$

 **Solution** 

(c) $\exists x M(x) \wedge D(x)$

 **Solution** 

(d) $\forall x (A(x) \vee M(x))$

 **Solution** 

(e) $\forall x (M(x) \leftrightarrow A(x))$

 **Solution** 

(f) $\forall x ((M(x) \wedge A(x)) \rightarrow \neg D(x))$

 **Solution** 

(g) $\exists x (D(x) \wedge \neg A(x) \wedge \neg M(x))$

 **Solution** 

(h) $\forall x (D(x) \rightarrow (A(x) \vee M(x)))$

 **Solution** 

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1.8 De Morgan's law for quantified statements

De Morgan's law for quantified statements

The negation operation can be applied to a quantified statement, such as $\neg\forall x F(x)$. If the domain for the variable x is the set of all birds and the predicate $F(x)$ is "x can fly", then the statement $\neg\forall x F(x)$ is equivalent to:

"Not every bird can fly."

which is logically equivalent to the statement:

"There exists a bird that cannot fly."

The equivalence of the previous two statements is an example of De Morgan's law for quantified statements, which is formally stated as $\neg\forall x F(x) \equiv \exists x \neg F(x)$. The diagram below illustrates that for a finite domain, De Morgan's law for universally quantified statements is the same as De Morgan's law for propositions:

Figure 1.8.1: De Morgan's law for universally quantified statements.

$$\begin{array}{ccc} \text{Domain of discourse} = \{a_1, a_2, \dots, a_n\} & & \\ \neg \forall x P(x) & \equiv & \exists x \neg P(x) \\ \text{III} & & \text{III} \\ \neg(P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n)) & \equiv & \neg P(a_1) \vee \neg P(a_2) \vee \dots \vee \neg P(a_n) \end{array}$$

Similarly, consider the statement $\neg\exists x A(x)$ in which the domain is the set of children enrolled in a class and $A(x)$ is the predicate "x is absent today". The statement is expressed in English as:

"It is not true that there is a child in the class who is absent today."

which is logically equivalent to:

"Every child in the class is not absent today."

The logical equivalence of the last two statements is an example of the second of De Morgan's laws for quantified statements: $\neg\exists x P(x) \equiv \forall x \neg P(x)$. The diagram below illustrates that for a finite domain, De Morgan's law for existentially quantified statements is the same as De Morgan's law for propositions:

Figure 1.8.2: De Morgan's law for existentially quantified statements.

Domain of discourse = $\{a_1, a_2, \dots, a_n\}$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

III

III

$$\neg(P(a_1) \vee P(a_2) \vee \dots \vee P(a_n)) \equiv \neg P(a_1) \wedge \neg P(a_2) \wedge \dots \wedge \neg P(a_n)$$

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Table 1.8.1: Summary of De Morgan's laws for quantified statements.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

**PARTICIPATION
ACTIVITY**

1.8.1: De Morgan's law can be used to simplify an existentially quantified statement.

Animation captions:

1. Start with $\neg \exists x (P(x) \rightarrow \neg Q(x))$ and apply De Morgan's law to get $\forall x \neg (P(x) \rightarrow \neg Q(x))$
2. Then apply the Conditional Identity to get $\forall x \neg (\neg P(x) \vee \neg Q(x))$.
3. Then apply De Morgan's law to get $\forall x (\neg \neg P(x) \wedge \neg \neg Q(x))$.
4. Finally, apply the Double Negation law to get $\forall x (P(x) \wedge Q(x))$.

**PARTICIPATION
ACTIVITY**

1.8.2: De Morgan's law can be used to simplify a universally quantified statement.

Animation captions:

1. Start with $\neg \forall x (P(x) \wedge \neg Q(x))$ and apply De Morgan's law to get $\exists x \neg (P(x) \wedge \neg Q(x))$.
2. Then apply De Morgan's law to get $\exists x (\neg P(x) \vee \neg \neg Q(x))$.
3. Finally, apply the Double Negation law to get $\exists x (\neg P(x) \vee Q(x))$.

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PARTICIPATION

1.8.3: Applying De Morgan's laws for quantified statements.

ACTIVITY



Match the logically equivalent propositions.

$$\neg \forall x \neg P(x)$$

$$\neg \exists x (\neg P(x) \wedge Q(x))$$

$$\neg \forall x (P(x) \wedge \neg Q(x))$$

$$\neg \exists x (P(x) \vee Q(x))$$

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$$\exists x P(x)$$

$$\exists x (\neg P(x) \vee Q(x))$$

$$\forall x (\neg P(x) \wedge \neg Q(x))$$

$$\forall x (P(x) \vee \neg Q(x))$$

Reset

Additional exercises

EXERCISE 1.8.1: Applying De Morgan's law for quantified statements to logical expressions.

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negator sign applies directly to a predicate. (i.e., $\exists x (\neg P(x) \vee \neg Q(x))$ is an acceptable final answer, but not $\neg \exists x P(x)$ or $\exists x \neg(P(x) \wedge Q(x))$).

(a) $\neg \exists x P(x)$

Solution

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\neg \exists x (P(x) \vee Q(x))$

Solution

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(c) $\neg \forall x (P(x) \wedge Q(x))$

Solution

(d) $\neg \forall x (P(x) \wedge (Q(x) \vee R(x)))$


EXERCISE 1.8.2: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

$P(x)$: x was given the placebo

$D(x)$: x was given the medication

$M(x)$: x had migraines

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Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Some patient was given the placebo and the medication.

- $\exists x (P(x) \wedge D(x))$
- Negation: $\neg \exists x (P(x) \wedge D(x))$
- Applying De Morgan's law: $\forall x (\neg P(x) \vee \neg D(x))$
- English: Every patient was either not given the placebo or not given the medication (or both).

(a) Every patient was given the medication.



Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Every patient was given the medication or the placebo or both.



(c) There is a patient who took the medication and had migraines.



(d) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditional identity, $p \rightarrow q \equiv \neg p \vee q$.)



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EXERCISE 1.8.3: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain of discourse is a set of students who show up for a test. Define the following predicates:

$P(x)$: x showed up with a pencil

$C(x)$: x showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Every student showed up with a calculator.

- $\forall x C(x)$
- Negation: $\neg \forall x C(x)$
- Applying De Morgan's law: $\exists x \neg C(x)$
- English: Some student showed up without a calculator.

(a) One of the students showed up with a pencil.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Every student showed up with a pencil or a calculator (or both).

 **Solution** 

(c) Every student who showed up with a calculator also had a pencil.

 **Solution** 

(d) There is a student who showed up with both a pencil and a calculator.

 **Solution** 

EXERCISE

1.8.4: Using De Morgan's law for quantified statements to prove logical equivalence.

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Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a) $\neg \forall x (P(x) \wedge \neg Q(x)) \equiv \exists x (\neg P(x) \vee Q(x))$

 **Solution** 

Click the eye icon to toggle solution visibility for

Ok, got it

students

(b) $\neg \forall x (\neg P(x) \rightarrow Q(x)) \equiv \exists x (\neg P(x) \wedge \neg Q(x))$

 **Solution** 

(c) $\neg \exists x (\neg P(x) \vee (Q(x) \wedge \neg R(x))) \equiv \forall x (P(x) \wedge (\neg Q(x) \vee R(x)))$

 **Solution** 

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1.9 Nested quantifiers

If a predicate has more than one variable, each variable must be bound by a separate quantifier. A logical expression with more than one quantifier that bind different variables in the same predicate is said to have **nested quantifiers**. The examples below show several logical expressions and which variables are bound in each. The logical expression is a proposition if all the variables are bound.

Figure 1.9.1: Nested quantifiers and bound variables.

$\forall x \exists y P(x, y)$ x and y are both bound.

$\forall x P(x, y)$ x is bound and y is free.

$\exists y \exists z T(x, y, z)$ y and z are bound. x is free.

**PARTICIPATION
ACTIVITY**

1.9.1: Free and bound variables with nested quantifiers.

1) Is the variable y bound in the expression $\forall x Q(x, y)$?

- Yes
 No

2) Is the following logical expression a proposition: $\forall z \exists y Q(x, y, z)$?

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Yes

No

Nested quantifiers of the same type

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M to be:

$M(x, y)$: x sent an email to y

and consider the proposition: $\forall x \forall y M(x, y)$. The proposition can be expressed in English as:

$\forall x \forall y M(x, y) \leftrightarrow$ "Everyone sent an email to everyone."

The statement $\forall x \forall y M(x, y)$ is true if for every pair, x and y , $M(x, y)$ is true. The universal quantifiers include the case that $x = y$, so if $\forall x \forall y M(x, y)$ is true, then everyone sent an email to everyone else and everyone sent an email to himself or herself. The statement $\forall x \forall y M(x, y)$ is false if there is any pair, x and y , that causes $M(x, y)$ to be false. In particular, $\forall x \forall y M(x, y)$ is false even if there is a single individual who did not send himself or herself an email.

Now consider the proposition: $\exists x \exists y M(x, y)$. The proposition can be expressed in English as:

$\exists x \exists y M(x, y) \leftrightarrow$ "There is a person who sent an email to someone."

The statement $\exists x \exists y M(x, y)$ is true if there is a pair, x and y , in the domain that causes $M(x, y)$ to evaluate to true. In particular, $\exists x \exists y M(x, y)$ is true even in the situation that there is a single individual who sent an email to himself or herself. The statement $\exists x \exists y M(x, y)$ is false if all pairs, x and y , cause $M(x, y)$ to evaluate to false.

PARTICIPATION ACTIVITY

1.9.2: Nested quantifiers of the same type.



In the following question, the domain is the set of all non-negative integers. xy means x times y .

1) $\forall x \forall y (xy = 1)$

True

False



2) $\exists x \exists y (xy = 1)$

True

False

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3) $\exists x \exists y ((x+y = x) \wedge (y \neq 0))$.

True

False





$$4) \forall x \forall y ((x+y \neq x) \vee (y = 0)).$$

- True
- False

Alternating nested quantifiers

A quantified expression can contain both types of quantifiers as in: $\exists x \forall y M(x, y)$. The quantifiers are applied from left to right, so the statement $\exists x \forall y M(x, y)$ translates into English as:

$$\exists x \forall y M(x, y) \leftrightarrow \text{"There is a person who sent an email to everyone."}$$

Switching the quantifiers changes the meaning of the proposition:

$$\forall x \exists y M(x, y) \leftrightarrow \text{"Every person sent an email to someone."}$$

In reasoning whether a quantified statement is true or false, it is useful to think of the statement as a **two player game** in which two players compete to set the statement's truth value. One of the players is the "existential player" and the other player is the "universal player". The variables are set from left to right in the expression. The table below summarizes which variables are set by which player and the goal of each player:

Table 1.9.1: Nested quantifiers as a two-person game.

Player	Action	Goal
Existential player	Selects values for existentially bound variables	Tries to make the expression true
Universal player	Selects values for universally bound variables	Tries to make the expression false

If the predicate is true after all the variables are set, then the quantified statement is true. If the predicate is false after all the variables are set, then the quantified statement is false. Consider as an example the following quantified statement in which the domain is the set of all integers:

$$\forall x \exists y (x + y = 0)$$

The universal player first selects the value of x . Regardless of which value the universal player selects for x , the existential player can select y to be $-x$, which will cause the sum $x + y$ to be 0. Because the existential player can always succeed in causing the predicate to be true, the statement $\forall x \exists y (x + y = 0)$ is true.

Switching the order of the quantifiers gives the following statement:

$$\exists x \forall y (x + y = 0)$$

Now, the existential player goes first and selects a value for x . Regardless of the value chosen for x , the universal player can select some value for y that causes the predicate to be false. For example, if x is an integer, then $y = -x + 1$ is also an integer and $x + y = 1 \neq 0$. Thus, the universal player can always win and the statement $\exists x \forall y (x + y = 0)$ is false.

**PARTICIPATION
ACTIVITY**

1.9.3: Examples showing reasoning about nested quantifiers using two player games.

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Animation captions:

1. Is $\forall x \exists y (y^2 = x)$ true? The domain is the set of integers. If the universal player selects $x = \dots$, then the existential player cannot find an integer y such that $x = y^2$.
2. The universal player wins, so $\forall x \exists y (y^2 = x)$ is false.
3. Is $\exists x \forall y (x + y = y)$ true? If the existential player selects $x = 0$, then for any y that the universal player selects, $0 + y = y$.
4. The existential player wins, so $\exists x \forall y (x + y = y)$ is true.



**PARTICIPATION
ACTIVITY**

1.9.4: Truth values for statements with nested quantifiers.

In the following question, the domain is the set of all real numbers. xy means x times y .

1) $\forall x \exists y (xy = 1)$

True

False



2) $\exists x \forall y (xy = 1)$

True

False



3) $\exists x \forall y (xy = y)$

True

False



4) $\forall x \exists y (x^2 = y)$

True

False



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De Morgan's law with nested quantifiers

De Morgan's law can be applied to logical statements with more than one quantifier. Each time the negation sign moves past a quantifier, the quantifier changes type from universal to existential or from existential to universal:

Table 1.9.2: De Morgan's laws for nested quantified statements.

$\neg \forall x \forall y P(x, y) \equiv \exists x \exists y \neg P(x, y)$
$\neg \forall x \exists y P(x, y) \equiv \exists x \forall y \neg P(x, y)$
$\neg \exists x \forall y P(x, y) \equiv \forall x \exists y \neg P(x, y)$
$\neg \exists x \exists y P(x, y) \equiv \forall x \forall y \neg P(x, y)$

Consider a scenario in which the domain is the set of all students in a school. The predicate $L(x, y)$ indicates that x likes y . The statement $\exists x \forall y L(x, y)$ is read as:

$\exists x \forall y L(x, y) \leftrightarrow$ There is a student who likes everyone in the school.

The negation of the statement is:

$\neg \exists x \forall y L(x, y) \leftrightarrow$ There is no student who likes everyone in the school.

Applying De Morgan's laws yields: $\neg \exists x \forall y L(x, y) \equiv \forall x \exists y \neg L(x, y)$ which is translated into:

$\forall x \exists y \neg L(x, y) \leftrightarrow$ Every student in the school has someone that they do not like.

PARTICIPATION ACTIVITY

1.9.5: De Morgan's law for nested quantifiers.

Animation captions:

1. If De Morgan's law is applied once to $\neg \forall x \forall y P(x, y)$, the result is $\exists x \neg \forall y P(x, y)$.
2. Applying De Morgan's law again gives $\exists x \exists y \neg P(x, y)$.
3. Applying De Morgan's law once to $\neg \forall x \exists y P(x, y)$ gives $\exists x \neg \exists y P(x, y)$. The second application gives $\exists x \forall y \neg P(x, y)$.
4. Applying De Morgan's law once to $\neg \exists x \forall y P(x, y)$ gives $\forall x \neg \forall y P(x, y)$. The second application gives $\forall x \exists y \neg P(x, y)$.
5. Applying De Morgan's law once to $\neg \exists x \exists y P(x, y)$ gives $\forall x \neg \exists y P(x, y)$. The second application gives $\forall x \forall y \neg P(x, y)$.

PARTICIPATION ACTIVITY

1.9.6: De Morgan's law for nested quantifiers.

Match logically equivalent propositions.

$$\neg \exists x \exists y (\neg P(x) \wedge Q(y))$$

$$\neg \exists x \forall y (P(x) \vee \neg Q(y))$$

$$\neg \forall x \forall y (P(x) \wedge Q(y))$$

$$\neg \forall x \exists y (P(x) \vee Q(y))$$

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$$\exists x \exists y (\neg P(x) \vee \neg Q(y))$$

$$\exists x \forall y (\neg P(x) \wedge \neg Q(y))$$

$$\forall x \exists y (\neg P(x) \wedge Q(y))$$

$$\forall x \forall y (P(x) \vee \neg Q(y))$$

Reset

Additional exercises

EXERCISE 1.9.1: Which logical expressions with nested quantifiers are propositions?

The table below shows the value of a predicate $M(x, y)$ for every possible combination of values of x and y . The domain for x and y is $\{1, 2, 3\}$. The row number indicates the value for x and the column number indicates the value for y . For example $M(1, 2) = F$ because the value in row 1, column 2, is F.

M	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Indicate whether each of the logical expressions is a proposition. If so, indicate whether the proposition is true or false.

(a) $M(1, 1)$



Solution

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\forall y M(x, y)$

 **Solution** 

(c) $\exists x M(x, 3)$

 **Solution** 

(d) $\exists x \exists y M(x, y)$

 **Solution** 

(e) $\exists x \forall y M(x, y)$

 **Solution** 

(f) $M(x, 2)$

 **Solution** 

(g) $\exists y \forall x M(x, y)$

 **Solution** 

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EXERCISE 1.9.2: Truth values for statements with nested quantifiers - small finite domain.

The tables below show the values of predicates $P(x, y)$, $Q(x, y)$, and $S(x, y)$ for every possible combination of values of the variables x and y . The domain for x and y is $\{1, 2, 3\}$.

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a) $\exists x \forall y P(x, y)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\exists x \forall y Q(x, y)$

 **Solution** 

(c) $\exists x \forall y P(y, x)$

 **Solution** 

(d) $\exists x \exists y S(x, y)$

 **Solution** 

(e) $\forall x \exists y Q(x, y)$

 **Solution** 

(f) $\forall x \exists y P(x, y)$

 **Solution** 

(g) $\forall x \forall y P(x, y)$

 **Solution** 

(h) $\exists x \exists y Q(x, y)$

 **Solution** 

(i) $\forall x \forall y \neg S(x, y)$

 **Solution** 

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EXERCISE 1.9.3: Truth values for mathematical expressions with nested quantifiers.

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a) $\forall x \exists y (xy > 0)$

 **Solution** 

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Ok, got it

(b) $\exists x \forall y (xy = 0)$

 **Solution** 

(c) $\forall x \forall y \exists z (z = (x - y)/3)$

 **Solution** 

(d) $\forall x \exists y \forall z (z = (x - y)/3)$

 **Solution** 

(e) $\forall x \forall y (xy = yx)$

 **Solution** 

(f) $\exists x \exists y \exists z (x^2 + y^2 = z^2)$

 **Solution** 

(g) $\forall x \exists y y^2 = x$

 **Solution** 

(h) $\forall x \exists y (x < 0 \vee y^2 = x)$

 **Solution** 

(i) $\exists x \exists y (x^2 = y^2 \wedge x \neq y)$

 **Solution** 

(j) $\exists x \exists y (x^2 = y^2 \wedge |x| \neq |y|)$

 **Solution** 

(k) $\forall x \forall y (x^2 \neq y^2 \vee |x| = |y|)$

 **Solution** 

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EXERCISE 1.9.4: De Morgan's law and nested quantifiers.

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

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(a) $\forall x \exists y \exists z P(y, x, z)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) $\forall x \exists y (P(x, y) \wedge Q(x, y))$

 **Solution** 

(c) $\exists x \forall y (P(x, y) \rightarrow Q(x, y))$

 **Solution** 

(d) $\exists x \forall y (P(x, y) \leftrightarrow P(y, x))$

 **Solution** 

(e) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

 **Solution** 

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EXERCISE 1.9.5: Applying De Morgan's law to English statements with nested quantifiers.

The domain for variables x and y is a group of people. The predicate $F(x, y)$ is true if and only if x is a friend of y . For the purposes of this problem, assume that for any person x and person y , either x is a friend of y or x is an enemy of y . Therefore, $\neg F(x, y)$ means that x is an enemy of y .

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

(a) Everyone is a friend of everyone.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Someone is a friend of someone.

 **Solution** 

(c) Someone is a friend of everyone.

 **Solution** 

(d) Everyone is a friend of someone.

 **Solution** 

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1.10 More nested quantified statements

Using logic to express "everyone else"

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate $M(x, y)$ that indicates whether x sent an email to y . The statement $\forall x \forall y M(x, y)$ asserts that every person sent an email to every other person and every person sent an email to himself or herself. How could we use logic to express that everyone sent an email to everyone else without including the case that everyone sent an email to himself or herself? The idea is to use the conditional operation: $(x \neq y) \rightarrow M(x, y)$.

The table below shows a group of four people and the truth value of $M(x, y)$ for each pair. For example, Agnes sent an email to Fred (i.e., $M(\text{Agnes}, \text{Fred}) = \text{T}$) because the truth value in the row labeled Agnes and the column labeled Fred is T.

Figure 1.10.1: Email predicate truth values.

		y			
		Agnes	Fred	Sue	Marge
x	Agnes	T	T	T	T
	Fred	T	F	T	T
	Sue	T	T	T	T
	Marge	T	T	T	F

The statement $\forall x \forall y M(x, y)$ is false because $M(\text{Fred}, \text{Fred})$ and $M(\text{Marge}, \text{Marge})$ are both false. However, the statement

$$\forall x \forall y ((x \neq y) \rightarrow M(x, y))$$

is true. The statement says that for every pair, x and y , if x and y are different people then x sent an email to y . That is, everyone sent an email to everyone else. The statement is true for the table above because for every pair that is not on the diagonal of the table (i.e., for every pair such that $x \neq y$), $M(x, y)$ is true.

1.10.1: Expressing 'someone else' in logic.

ACTIVITY

The diagrams below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

		y			
		Agnes	Fred	Sue	Marge
x	Agnes	F	F	T	F
	Fred	F	T	F	F
	Sue	F	F	F	T
	Marge	F	T	F	F

- 1) Indicate whether the following statement is true for the group in the table:

"Everyone sent an email to someone else"

- True
 False

- 2) Is the statement " $\forall x \exists y M(x, y)$ " true for the group in the table?

- True.
 False

- 3) Consider the statement $\forall x \exists y ((x \neq y) \wedge M(x, y))$. In the two player game for the group in the table, if the universal player selects $x = \text{Sue}$, who will the existential player select for y ?

- Fred
 Sue
 Marge

- 4) Is the statement " $\forall x \exists y ((x \neq y) \wedge M(x, y))$ " true for the

group in the table?

- True
 False

**PARTICIPATION
ACTIVITY**

1.10.2: Expressing 'someone else' in logic, cont.

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The diagrams below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

		y			
		Sandy	Jen	Frank	Gary
x	Sandy	F	F	T	F
	Jen	F	F	T	F
	Frank	F	F	F	T
	Gary	F	T	F	F

- 1) Indicate whether the following statement is true for the group in the table:
 "Everyone sent an email to someone else"

- True
 False

- 2) Is the statement
 $\forall x \exists y ((x \neq y) \wedge M(x, y))$ true for the group in the table?

- True
 False

- 3) Are the two statements below logically equivalent?
 "Everyone sent an email to someone else"

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and

$\forall x \exists y ((x \neq y) \wedge M(x, y))$

- True
- False

Expressing uniqueness in quantified statements

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An existentially quantified statement evaluates to true even if there is more than one element in the domain that causes the predicate to evaluate to true. If the domain is a set of people who attend a meeting and the predicate $L(x)$ indicates whether or not x came late to the meeting, then the statement $\exists x L(x)$ is true if there are one, two or more people who came late.

PARTICIPATION ACTIVITY

1.10.3: Using logic to express that exactly one person came late to the meeting.



Animation captions:

- How to express: "Exactly one person was late to the meeting." $L(x)$ means x was late to the meeting. $\exists x L(x)$ means that someone was late to the meeting.
- A way is needed to express that x is the only person who came late to the meeting.
- Add that for every y , if $y \neq x$, then y was not late to the meeting:

$$\exists x(L(x) \wedge \forall y((x \neq y) \rightarrow \neg L(y))).$$

PARTICIPATION ACTIVITY

1.10.4: Expressing uniqueness in quantified statements.



Consider a domain consisting of a set of people attending a meeting. The predicate $L(x)$ indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	$L(x)$
Shirley	F
Rob	T
Ted	F
Mindy	T

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- Indicate whether the following statement is true for the group in the table:



"Exactly one person was late for the meeting."

- True
 False

2) If $x = \text{Shirley}$ is the following statement true:

$L(\text{Shirley}) \wedge \forall y ((\text{Shirley} \neq y) \rightarrow \neg L(y))$

- True
 False

3) If $x = \text{Rob}$, is the following statement true: $L(\text{Rob}) \wedge \forall y ((\text{Rob} \neq y) \rightarrow \neg L(y))$

- True
 False

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**PARTICIPATION
ACTIVITY**

1.10.5: Expressing uniqueness in quantified statements, cont.

Consider a domain consisting of a set of people attending a meeting. The predicate $L(x)$ indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	$L(x)$
Shirley	F
Rob	T
Ted	F
Mindy	F

1) Indicate whether the following statement is true for the group in the table: "Exactly one person was late for the meeting."

- True
 False

2) If $x = \text{Rob}$, is the following statement true: $L(\text{Rob}) \wedge \forall y ((\text{Rob} \neq y) \rightarrow \neg L(y))$



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True

False

Moving quantifiers in logical statements

Now consider a set of people at a party as the domain. We would like to find a logical expression that is equivalent to the statement: "Every adult is married to someone at the party." There are two predicates:

$M(x, y)$: x is married to y .

$A(x)$: x is an adult.

Here is an equivalent statement that is closer in form to a logical expression: "For every person x , if x is an adult, then there is a person y to whom x is married." The logic is expressed as:

$$\forall x (A(x) \rightarrow \exists y M(x, y))$$

Since y does not appear in the predicate $A(x)$, " $\exists y$ " can be moved to the left so that it appears just after the $\forall x$ resulting in the following equivalent expression:

$$\forall x \exists y (A(x) \rightarrow M(x, y))$$

PARTICIPATION ACTIVITY

1.10.6: Example of moving quantifiers in logical statements.

Animation captions:

1. How to express: "Every adult is married to exactly one adult." $A(x)$ means x is an adult. $M(x, y)$ means x is married to y .
2. The first step is to express: "Every adult is married to someone." For every x , if x is an adult, there is a y such that x is married to y : $\forall x (A(x) \rightarrow \exists y M(x, y))$.
3. The next step is to express that y is the only person x is married to: for every z , if z is not y , then x is not married to z : $\forall x (A(x) \rightarrow (\exists y M(x, y) \wedge \forall z ((z \neq y) \rightarrow \neg M(x, z))))$.
4. The quantifiers for y and z can be moved to the front just after $\forall x$ because $\exists y$ does not pass references to y and $\forall z$ does not pass references to z .

PARTICIPATION ACTIVITY

1.10.7: Nested quantifiers expressed in English.

Match each proposition to the corresponding English sentence. The domain is the set of all students in a math class. The two predicates are defined as:

$P(x, y)$: x knows y 's phone number.

$H(x)$: x has the homework assignment.

$\forall x \forall y P(x, y)$ $\forall x \exists y (P(x, y) \wedge H(y))$ $\exists x \forall y P(x, y)$ $\exists x \forall y (H(x) \wedge P(x, y))$

Every student knows every student's phone number.

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Some student knows every student's phone number.

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Every student knows the phone number of another student who has the homework assignment.

There is a student who has the homework assignment and knows every student's phone number.

Reset

Additional exercises

EXERCISE 1.10.1: Truth values for expressions with nested quantifiers.

The domain of discourse for this problem is a group of three people who are working on a project. To make notation easier, the people are numbered 1, 2, 3. The predicate $M(x, y)$ indicates whether x has sent an email to y , so $M(2, 3)$ is read "Person 2 has sent an email to person 3." The table below shows the value of the predicate $M(x, y)$ for each (x, y) pair. The truth value in row x and column y gives the truth value for $M(x, y)$.

M	1	2	3
1	T	T	T
2	T	F	T
3	T	T	F

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Indicate whether the quantified statement is true or false. Justify your answer.

(a) $\forall x \forall y M(x, y)$



Solution

Click the eye icon to toggle solution visibility for

Ok, got it

students

(b) $\forall x \forall y ((x \neq y) \rightarrow M(x,y))$

 Solution 

(c) $\exists x \exists y \neg M(x,y)$

 Solution 

(d) $\exists x \exists y ((x \neq y) \wedge \neg M(x,y))$

 Solution 

(e) $\forall x \exists y \neg M(x,y)$

 Solution 

(f) $\exists x \forall y M(x,y)$

 Solution 

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EXERCISE 1.10.2: Showing non-equivalence for expressions with nested quantifiers.

Show that the two quantified statements in each problem are not logically equivalent by filling in a table so that, for the domain of discourse $\{a, b, c\}$, the values of the predicate P you select for the table causes one of the statements to be true and the other to be false. For example, the table below shows that $\forall x \forall y P(x, y)$ and $\exists x \exists y P(x, y)$ are not logically equivalent because for the given values of the predicate P , $\forall x \forall y P(x, y)$ is false and $\exists x \exists y P(x, y)$ is true.

P	a	b	c
a	T	T	T
b	T	F	T
c	T	T	F

(a) $\forall x \exists y P(x, y)$ and $\exists x \forall y P(x, y)$

 Solution 

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Ok, got it

$\forall x \exists y ((x \neq y) \wedge P(x, y))$ and $\forall x \exists y P(x, y)$

(b)

 **Solution** (c) $\exists x \exists y (P(x, y) \wedge P(y, x))$ and $\exists x \exists y P(x, y)$  **Solution** 

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EXERCISE

1.10.3: Statements with nested quantifiers: English to logic. IST230LupoliSummer2018

The domain of discourse is the members of a chess club. The predicate $B(x, y)$ means that person x has beaten person y at some point in time. Give a logical expression equivalent to the following English statements. You can assume that it is possible for a person to beat himself or herself.

(a) Sam has been beaten by someone.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Everyone has been beaten before.

 **Solution** 

(c) No one has ever beaten Nancy.

 **Solution** 

(d) Everyone has won at least one game.

 **Solution** 

(e) No one has beaten both Ingrid and Dominic.

 **Solution** 

(f) Josephine has beaten everyone else.

 **Solution** 

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(g) Nancy has beaten exactly one person.

 **Solution** 

(h) There are at least two members who have never been beaten.

 **Solution** 

EXERCISE 1.10.4: Statements with nested quantifiers: English to logic.

The domain for the variables x and y are the set of musicians in an orchestra. The predicates S , B , and P are defined as:

$S(x)$: x plays a string instrument

$B(x)$: x plays a brass instrument

$P(x, y)$: x practices more than y

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Give a quantified expression that is equivalent to the following English statements:

(a) There are no brass players in the orchestra.

 **Solution** ▾

Click the eye icon to toggle solution visibility for students

Ok, got it

(b) Someone in the orchestra plays a string instrument and a brass instrument.

 **Solution** ▾

(c) There is a brass player who practices more than all the string players.

 **Solution** ▾

(d) All the string players practice more than all the brass players.

 **Solution** ▾

(e) Exactly one person practices more than Sam.

 **Solution** ▾

(f) Sam practices more than anyone else in the orchestra.

 **Solution** ▾

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EXERCISE 1.10.5: Statements with nested quantifiers: variables with different domains.

The domain for the first input variable to predicate T is a set of students at a university. The domain for the second input variable to predicate T is the set of Math classes offered at that university. The predicate $T(x, y)$ indicates that student x has taken class y . Sam is a student at the university and Math 101 is one of the courses offered at the university. Give a logical expression for each sentence

(a) Sam has taken Math 101.

 **Solution** 

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Ok, got it

(b) Every student has taken at least one math class.

 **Solution** 

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(c) Every student has taken at least one class besides Math 101.

 **Solution** 

(d) There is a student who has taken every math class besides Math 101.

 **Solution** 

(e) Everyone besides Sam has taken at least two different math classes.

 **Solution** 

(f) Sam has taken exactly two math classes.

 **Solution** 

EXERCISE 1.10.6: Mathematical statements into logical statements with nested quantifiers.

Translate each of the following English statements into logical expressions. The domain of discourse is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1.

 **Solution** 

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Ok, got it

(b) The reciprocal of every positive number is also positive.

 **Solution** 

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(c) There are two numbers whose sum is equal to their product.

 **Solution** 

(d) The ratio of every two positive numbers is also positive.

 **Solution** 

(e) The reciprocal of every positive number less than one is greater than one.

 **Solution** 

(f) There is no smallest number.

 **Solution** 

(g) Every number besides 0 has a multiplicative inverse.

 **Solution** 

(h) Every number besides 0 has a unique multiplicative inverse.

 **Solution** 

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1.11 Logical reasoning

The language of logic allows us to formally establish the truth of logical statements, assuming that a set of hypotheses is true. An **argument** is a sequence of propositions, called **hypotheses**, followed by a final proposition, called the **conclusion**. An argument is **valid** if the conclusion is true whenever the hypotheses are all true, otherwise the argument is **invalid**. An argument will be denoted as:

p_1
 p_2
....
 p_n
 $\therefore c$

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$p_1 \dots p_n$ are the hypotheses and c is the conclusion. The symbol \therefore reads "therefore". The argument is valid whenever the proposition $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow c$ is a tautology. According to the commutative law, reordering the hypotheses does not change whether an argument is valid or not. Therefore two arguments are considered to be the same even if the hypotheses appear in a different order. For example, the following two arguments are considered to be the same:

p
 $p \rightarrow q$
 $\therefore q$

$p \rightarrow q$
 p
 $\therefore q$

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p and $p \rightarrow q$ are the hypotheses. q is the conclusion.

**PARTICIPATION
 ACTIVITY**

1.11.1: The components of a logical argument.

Consider the following argument:

$\neg q$
 $p \rightarrow q$
 $\therefore \neg p$

1) The conclusion is:

- $p \rightarrow q$
- $\neg p$
- $\neg q$

2) The proposition $\neg q$ is:

- a hypothesis
- the conclusion
- an argument

3) The argument is valid if which proposition is a tautology:

- $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
- $\neg p \rightarrow ((p \rightarrow q) \wedge \neg q)$
- $\neg p \leftrightarrow ((p \rightarrow q) \wedge \neg q)$

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One way to establish the validity of an argument is to use a truth table.

**PARTICIPATION
 ACTIVITY**

1.11.2: Using a truth table to establish the validity of an argument.

**Animation captions:**

1. To show that the argument with hypotheses p and $p \rightarrow q$ and conclusion q is valid, fill in the truth table for p , q , and $p \rightarrow q$.
2. There is only one row in which both hypotheses are true.
3. Since the conclusion is also true in that row, the argument is valid.

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In order to use a truth table to establish the validity of an argument, a truth table is constructed for all the hypotheses and the conclusion. Each row in which all the hypotheses are true is examined. If the conclusion is true in each of the examined rows, then the argument is valid. If there is any row in which all the hypotheses are true but the conclusion is false, then the argument is invalid.

**PARTICIPATION
ACTIVITY**

1.11.3: Validity of an argument from truth tables.



Consider the argument below:

$$p \rightarrow q$$

$$p \vee q$$

$$\therefore q$$

The truth table below shows the truth values for the two hypotheses and the conclusion for every possible truth assignment to p and q .

p	q	$p \rightarrow q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

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1) In which rows are both the hypotheses true?

- Row 1
- Rows 1 and 3
- Rows 1, 3, and 4

2) Is the conclusion true in all of the rows



where both the hypotheses are true?

- Yes
 No

3) Is the argument valid?

- Yes
 No

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An argument can be shown to be invalid by showing an assignment of truth values to its variables that makes all the hypotheses true and the conclusion false. For example, the argument:

$$\begin{array}{l} \neg p \\ p \rightarrow q \\ \therefore \neg q \end{array}$$

is invalid, because when $p = F$ and $q = T$, the hypotheses $p \rightarrow q$ and $\neg p$ are both true, but the conclusion $\neg q$ is false. In some cases, it may be necessary to build the whole truth table in order to actually find a truth assignment that shows an argument is invalid. However, the final proof of invalidity only requires a single truth assignment for which all the hypotheses are true and the conclusion is false.

PARTICIPATION ACTIVITY

1.11.4: Proving an argument is invalid.

Consider the argument below:

$$\begin{array}{l} p \vee q \\ p \\ \therefore \neg q \end{array}$$

The truth table below shows the truth values for the two hypotheses and the conclusion for every possible truth assignment to p and q .

p	q	$p \vee q$	$\neg q$
T	T	T	F
T	F	T	T
F	T	T	F

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F	F	F	T
---	---	---	---

1) In which rows are both the hypotheses true?

- Row 1
 Rows 1 and 2
 Rows 1 and 3

2) Is the conclusion true in all of the rows where both the hypotheses are true?

- Yes
 No

3) Is the argument valid or invalid?

- Valid
 Invalid

4) Which truth assignment is a proof that the argument is invalid?

- $p = q = T$
 $p = T$ and $q = F$
 $p = F$ and $q = T$

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The form of an argument

The hypotheses and conclusion in a logical argument can also be expressed in English, as in:

It is raining today.

If it is raining today, I will not ride my bike to school.

∴ I will not ride my bike to school.

The **form** of an argument expressed in English is obtained by replacing each individual proposition with a variable. While it is common to express a logical argument in English, the validity of an argument is established by analyzing its form. Define propositional variables p and q to be:

p : It is raining today.

q : I will not ride my bike to school.

The argument's form, given below, was already shown to be valid by using a truth table:

p
 $p \rightarrow q$
 $\therefore q$

**PARTICIPATION
ACTIVITY**

1.11.5: Determining the form of an argument expressed in English.

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Use the following variable definitions to match each argument to its form:

p : Sam studied for his test.

q : Sam passed his test.

$p \vee q$

$\neg q$

$\therefore p$

p

q

$\therefore p \wedge q$

$p \rightarrow q$

$\neg q$

$\therefore \neg p$

If Sam studied for his test, then Sam passed his test.

Sam did not pass his test.

\therefore Sam did not study for his test.

Sam studied for his test or Sam passed his test.

Sam did not pass his test.

\therefore Sam studied for his test.

Sam studied for his test.

Sam passed his test.

\therefore Sam studied for his test and Sam passed his test.

Reset

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When arguments are expressed in English, the propositions sometimes have known truth values. In the argument below the hypotheses and conclusion are all true:

5 is not an even number.

If 5 is an even number, then 7 is an even number.

\therefore 7 is not an even number.

The argument is, nonetheless, invalid because its form

$$\begin{aligned} &\neg p \\ &p \rightarrow q \\ &\therefore \neg q \end{aligned}$$

was shown to be invalid.

In a valid argument, the conclusion must follow from the hypotheses for every possible combination of truth values for the individual propositions. The invalid argument form shown above can be used with a different set of propositions to reach a false conclusion even when the hypotheses are true, as in:

$$\begin{aligned} &5 \text{ is not an even number.} \\ &\text{If } 5 \text{ is an even number, then } 6 \text{ is an even number.} \\ &\therefore 6 \text{ is not an even number.} \end{aligned}$$

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**PARTICIPATION
ACTIVITY**

1.11.6: Valid and invalid arguments in English.

In the examples below, all of the hypotheses and the conclusions are true. Indicate which arguments are valid. You can use the fact that the argument form A given below is valid and argument form B is invalid:

(Valid) Argument form A:

$$\begin{aligned} &\neg q \\ &p \rightarrow q \\ &\therefore \neg p \end{aligned}$$

(Invalid) Argument form B:

$$\begin{aligned} &\neg p \\ &p \rightarrow q \\ &\therefore \neg q \end{aligned}$$

- 1) 6 is not a prime number.
If 6 is a prime number, then 4 is a prime number.
 \therefore 4 is not a prime number.

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Valid

Invalid

- 2) 4 is not a prime number.
 If 6 is a prime number, then 4 is a prime number.
 \therefore 6 is not a prime number.



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Valid

Invalid

- 3) π is not a rational number.
 If π is a rational number, then 2π is a rational number.
 \therefore 2π is not a rational number.



Valid

Invalid

Additional exercises

EXERCISE 1.11.1: Valid and invalid arguments expressed in logical notation.

Indicate whether the argument is valid or invalid. For valid arguments, prove that the argument is valid using a truth table. For invalid arguments give truth values for the variables showing that the argument is not valid.

- (a) $p \vee q$
 p
 $\therefore q$

 **Solution** 

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Ok, got it

- (b) $p \leftrightarrow q$
 $p \vee q$
 $\therefore p$

 **Solution** 

- (c) p
 q
 $\therefore p \leftrightarrow q$

 **Solution** 

- (d) $p \vee q$
 $\neg q$
 $\therefore p \leftrightarrow q$

 **Solution** 

- (e) $(p \wedge q) \rightarrow r$
 $\therefore (p \vee q) \rightarrow r$

 **Solution** 

- (f) $(p \vee q) \rightarrow r$
 $\therefore (p \wedge q) \rightarrow r$

 **Solution** 

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EXERCISE 1.11.2: Valid and invalid arguments in English.

Which of the following arguments are invalid and which are valid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid.

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- The patient has high blood pressure or diabetes or both.
- (a) The patient has diabetes or high cholesterol or both.
 \therefore The patient has high blood pressure or high cholesterol.

 **Solution** 

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students

He studied for the test or he failed the test or both.

- (b) He passed the test.
 \therefore He studied for the test.

 **Solution** 

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If $\sqrt{2}$ is an irrational number, then $2\sqrt{2}$ is an irrational number.

- (c) $2\sqrt{2}$ is an irrational number.
 $\therefore \sqrt{2}$ is an irrational number.

 **Solution** 

1.12 Rules of inference with propositions

Using truth tables to establish the validity of an argument can become tedious, especially if an argument uses a large number of variables. Fortunately, some arguments can be shown to be valid by applying rules that are themselves arguments that have already been shown to be valid. The laws of propositional logic can also be used in establishing the validity of an argument.

Table 1.12.1: Rules of inference known to be valid arguments.

Rule of inference	Name
p $p \rightarrow q$ $\therefore q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	Modus tollens
p $\therefore p \vee q$	Addition

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$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

The validity of an argument can be established by applying the rules of inference and laws of propositional logic in a **logical proof**. A logical proof of an argument is a sequence of steps, each of which consists of a proposition and a justification. If the proposition in a step is a hypothesis, the justification is "Hypothesis". Otherwise, the proposition must follow from previous steps by applying one law of logic or rule of inference. The justification indicates which rule or law is used and the previous steps to which it is applied. For example, having justification "Modus ponens, 2, 3" on line 4 means that the Modus ponens rule is applied to the propositions on lines 2 and 3, resulting in the proposition on line 4. The proposition in the last step in the proof must be the conclusion of the argument being proven. For example, in the animation below, the last proposition "t" in step 6 is the same as the conclusion of the argument being proven.

PARTICIPATION ACTIVITY

1.12.1: An example of a logical proof to establish the validity of an argument.

Animation captions:

1. An argument has hypotheses $(p \vee r) \rightarrow q$, $q \rightarrow t$, and r . The conclusion is t . The first line of the proof is r , a hypothesis.

2. Line 2 of the proof is $p \vee r$, by Addition applied to line 1.
3. Line 3 of the proof is $(p \vee r) \rightarrow q$, a hypothesis.
4. Line 4 of the proof is q , by Modus Ponens applied to lines 2 and 3.
5. Line 5 of the proof is $q \rightarrow t$, a hypothesis.
6. Line 6 of the proof is t , by Modus Ponens applied to lines 4 and 5, which is the conclusion of the argument.

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Here is an argument expressed in English:

If it is raining or windy or both, the game will be cancelled.

The game will not be cancelled.

It is not windy.

The first step in proving the validity of the argument is to assign variable names to each of the individual propositions:

w: It is windy

r: It is raining

c: The game will be cancelled

Replacing English phrases with variable names results in the following argument form:

$$(r \vee w) \rightarrow c$$

$$\neg c$$

$$\neg w$$

The final step is to prove that the argument is valid using a logical proof.

Table 1.12.2: An example of a logical proof to establish the validity of an argument.

1.	$(r \vee w) \rightarrow c$	Hypothesis
2.	$\neg c$	Hypothesis
3.	$\neg(r \vee w)$	Modus tollens, 1, 2
4.	$\neg r \wedge \neg w$	De Morgan's law, 3
5.	$\neg w \wedge \neg r$	Commutative law, 4
6.	$\neg w$	Simplification, 5

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**PARTICIPATION
ACTIVITY**

1.12.2: Applying the laws of inference.



Establish the validity of the following argument.

$$\neg(p \wedge \neg q)$$

$$p$$

$$q \vee r$$

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Complete the proof that establishes the validity of the argument above by indicating the rule used to justify each step. The number after the Double negation law indicates the line number to which the rule is applied. The proof starts with the following line:

Hypothesis	1. $\neg(p \wedge \neg q)$
------------	----------------------------

Hypothesis**De Morgan's law****Disjunctive syllogism****Addition****Double negation law, 4****Double negation law, 2**

$$2. \neg p \vee \neg \neg q$$

$$3. \neg p \vee q$$

$$4. p$$

$$5. \neg \neg p$$

$$6. q$$

$$7. q \vee r$$

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Additional exercises**EXERCISE**

1.12.1: Proving arguments are valid using rules of inference.

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

- (a) $p \rightarrow q$
 $q \rightarrow r$
 $\neg r$
 $\therefore \neg p$

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 **Solution** 

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Ok, got it

- (b) $p \rightarrow (q \wedge r)$
 $\neg q$
 $\therefore \neg p$

 **Solution** 

- (c) $(p \wedge q) \rightarrow r$
 $\neg r$
 q
 $\therefore \neg p$

 **Solution** 

- (d) $(p \vee q) \rightarrow r$
 p
 $\therefore r$

 **Solution** 

- (e) $p \vee q$
 $\neg p \vee r$
 $\neg q$
 $\therefore r$

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 **Solution** 

- (f) $p \rightarrow q$
 $r \rightarrow u$
 $p \wedge r$
 $\therefore q \wedge u$

 **Solution** 

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EXERCISE 1.12.2: Proving the rules of inference using other rules.

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

One of the rules of inference is Modus tollens:

- $p \rightarrow q$
 $\neg q$
 (a) $\therefore \neg p$

Prove that Modus tollens is valid using the laws of propositional logic and any of the other rules of inference besides Modus tollens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

One of the rules of inference is Modus ponens:

- $p \rightarrow q$
 p
 (b) $\therefore q$

Prove that Modus ponens is valid using the laws of propositional logic and any of the other rule of inference besides Modus ponens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

 **Solution** 

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One of the rules of inference is Disjunctive syllogism :

- $p \vee q$
 $\neg p$
 $\therefore q$
 (c)

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

 **Solution** 

One of the rules of inference is Resolution:

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline (d) \quad \therefore q \vee r \end{array}$$

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Prove that Resolution is valid using the laws of propositional logic and any of the other rules of inference besides Resolution. (Hint: you will need one of the conditional identities from the laws of propositional logic).

 **Solution** 

EXERCISE 1.12.3: Proving arguments in English are valid using rules of inference.

Prove that each argument is valid by replacing each proposition with a variable to obtain the form of the argument. Then use the rules of inference to prove that the form is valid.

- (a) If I drive on the freeway, I will see the fire.
I will drive on the freeway or take surface streets (or both).
I am not going to take surface streets.
 \therefore I will see the fire.

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

- (b) If it was not foggy or it didn't rain (or both), then the race was held and there was a trophy ceremony.
The trophy ceremony was not held.
 \therefore It rained.

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 **Solution** 

- (c) If I work out hard, then I am sore.
 If I am sore, I take an aspirin.
 I did not take an aspirin.
 \therefore I did not work out hard.



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1.13 Rules of inference with quantifiers

In order to apply the rules of inference to quantified expressions, such as $\forall x \neg(P(x) \wedge Q(x))$, we need to remove the quantifier by plugging in a value from the domain to replace the variable x . A value that can be plugged in for variable x is called an **element** of the domain of x . For example, if the domain is the set of all integers, $\neg(P(3) \wedge Q(3))$ is a proposition to which De Morgan's law can be applied. Elements of the domain can be defined in a hypothesis of an argument. In the example below, the domain is a set of employees at a company. The second line defines Linda as an element of the domain.

Every employee who received a large bonus works hard.
 Linda is an employee at the company.
 Linda received a large bonus.
 \therefore Some employee works hard.

Elements of the domain can also be introduced within a proof in which case they are given generic names such as "c" or "d". There are two types of named elements used in logical proofs. An **arbitrary** element of a domain has no special properties other than those shared by all the elements of the domain. A **particular** element of the domain may have properties that are not shared by all the elements of the domain. For example, if the domain is the set of all integers, 3 is a particular element of the domain. The number 3 is odd which is not a property that is shared by all integers. Every domain element referenced in a proof must be defined on a separate line of the proof. If the element is defined in a hypothesis, it is always a particular element and the definition of that element in the proof is labeled "Hypothesis". If an element is introduced for the first time in the proof, the definition is labeled "Element definition" and must specify whether the element is arbitrary or particular.

PARTICIPATION ACTIVITY

1.13.1: Definitions of arbitrary and particular elements of a domain.

For each definition of an element in the domain, indicate whether the element defined is particular or arbitrary.

1) The domain is the set of all integers.

3 is an integer. Hypothesis.

Particular

Arbitrary

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2) The domain is the set of all employees at a company.

c is an arbitrary employee of the company. Element definition.

Particular

Arbitrary

3) The domain is the set of all integers.

c is a particular integer. Element definition.

Particular

Arbitrary

4) The domain is the set of students enrolled for a class.

Larry is enrolled in the class.
Hypothesis.

Particular

Arbitrary

The rules **existential instantiation** and **universal instantiation** replace a quantified variable with an element of the domain. The rules **existential generalization** and **universal generalization** replace an element of the domain with a quantified variable.

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Table 1.13.1: Rules of inference for quantified statements.

Rule of Inference	Name	Example
c is an element (arbitrary or particular) $\forall x P(x)$	Universal instantiation	Sam is a student in the class.

$\therefore P(c)$		Every student in the class completed the assignment. Therefore, Sam completed his assignment.
c is an arbitrary element $P(c)$ _____ $\therefore \forall x P(x)$	Universal generalization	Let c be an arbitrary integer. $c \leq c^2$ Therefore, every integer is less than or equal to its square.
$\exists x P(x)$ $\therefore (c \text{ is a particular element}) \wedge P(c)$	Existential instantiation*	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c.
c is an element (arbitrary or particular) $P(c)$ _____ $\therefore \exists x P(x)$	Existential generalization	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

*Note: each use of Existential instantiation must define a new element with its own name (e.g., "c" or "d").

PARTICIPATION ACTIVITY

1.13.2: A logical proof that uses the laws of inference for quantified statements.

Animation captions:

1. An argument has hypotheses $\forall x (P(x) \vee Q(x))$, "3 is an integer", and $\neg P(3)$. The conclusion is $Q(3)$. The first two lines in the proof are the first two hypotheses.
2. Line 3 of the proof is $(P(3) \vee Q(3))$, by Universal instantiation applied to lines 1 and 2.
3. Line 4 of the proof is $\neg P(3)$, a hypothesis.
4. Line 5 of the proof is $Q(3)$, by Disjunctive syllogism applied to lines 3 and 4.

PARTICIPATION ACTIVITY

1.13.3: Correct and incorrect use of generalization and instantiation.

Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

1)

1.	c is an element	Hypothesis
2.	$P(c)$	Hypothesis
3.	$\forall x P(x)$	Universal generalization, 1, 2

- Correct
 Incorrect

2)

1.	c is an element	Hypothesis
2.	$\forall x P(x)$	Hypothesis
3.	$P(c)$	Universal instantiation, 1, 2

- Correct
 Incorrect

3)

1.	c is an element	Hypothesis
2.	$P(c)$	Hypothesis
3.	d is an element	Hypothesis
4.	$Q(d)$	Hypothesis
5.	$P(c) \wedge Q(d)$	Conjunction, 2, 3
6.	$\exists x (P(x) \wedge Q(x))$	Existential generalization, 1, 3, 5

- Correct
 Incorrect

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The proof below establishes the validity of the following argument:

$$\exists x P(x)$$

$$\forall x Q(x)$$

$$\exists x (P(x) \wedge Q(x))$$

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Match each rule to the correct location in the proof.

1. $\exists x P(x)$	Hypothesis
2. (c is a particular element) $\wedge P(c)$	Reason A
3. $P(c) \wedge$ (c is a particular element)	Commutative law, 2
4. $\forall x Q(x)$	Hypothesis
5. c is a particular element	Simplification, 2
6. $Q(c)$	Reason B
7. $P(c)$	Reason C
8. $P(c) \wedge Q(c)$	Reason D
9. $\exists x(P(x) \wedge Q(x))$	Reason E

Simplification, 3

Existential instantiation, 1

Universal instantiation, 4, 5

Conjunction, 6, 7

Existential generalization, 5, 8

Reason A

Reason B

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Reason C

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Reason D

Reason E

Reset

Here is an argument expressed in English. The domain is the set of students enrolled in a class:

Every student who stayed up too late missed the test.

Juan is enrolled in the class.

Juan did not miss the test.

\therefore Some student did not stay up too late.

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The first step in proving that the argument is valid is to determine the form of the argument.

Define the following two predicates:

$S(x)$: x stayed up too late

$M(x)$: x missed the test

Here is the form of the argument:

$\forall x (S(x) \rightarrow M(x))$

Juan, a student in the class

$\neg M(\text{Juan})$

$\therefore \exists x \neg S(x)$

**PARTICIPATION
ACTIVITY**

1.13.5: Completing a proof of validity with quantified statements.

Complete the proof of the argument given above by putting the rule used to justify each step next to the step. The proof starts with the following two lines:

Hypothesis	1. $\forall x (S(x) \rightarrow M(x))$
Hypothesis	2. Juan, a student in the class

Universal instantiation, 1, 2

Hypothesis

Existential generalization, 2, 5

Modus tollens, 3, 4

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3. $S(\text{Juan}) \rightarrow M(\text{Juan})$

4. $\neg M(\text{Juan})$

5. $\neg S(\text{Juan})$ 6. $\exists x \neg S(x)$

Reset

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**PARTICIPATION
ACTIVITY**

1.13.6: Ordering the steps of a proof of validity.

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The proof below establishes the validity of the following argument:

 $\forall x (Q(x) \vee P(x))$ $\forall x \neg Q(x)$ $\forall x P(x)$

Complete the proof of the argument above by putting each step on the correct line. The proof starts with the following line:

 $\forall x (Q(x) \vee P(x))$

1. Hypothesis

 $P(c)$ $\forall x P(x)$ $\forall x \neg Q(x)$ $Q(c) \vee P(c)$ **c is an arbitrary element
of the domain** $\neg Q(c)$

2. Element definition.

3. Universal instantiation, 1, 2

4. Hypothesis

5. Universal instantiation, 2, 4

6. Disjunctive syllogism, 3, 5

7. Universal generalization, 2, 6

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Reset

Multiple uses of existential instantiation: a common mistake

It is important to define a new particular element with a new name for each use of existential instantiation within the same logical proof in order to avoid a faulty proof that an invalid argument is valid.

The mistake in the proof below is in the assumption that the value c (introduced in Step 2), which causes $P(x)$ to be true, is the same value c (introduced in Step 5) that causes $Q(x)$ to be true. A correct use of existential instantiation in line 4 would first introduce a new particular element, d that is not necessarily equal to c , and assert that $Q(d)$ is true. For example, if $P(x)$ means that x owns a cat and $Q(x)$ means that x owns a dog, then the two hypotheses say that there is someone who owns a cat and there is someone who owns a dog. However, the two hypotheses together do not imply that there is someone who owns a cat and a dog.

Table 1.13.2: Incorrect use of existential instantiation leading to an erroneous proof of an invalid argument.

$\exists x P(x)$
 $\exists x Q(x)$
 $\exists x (P(x) \wedge Q(x))$

1.	$\exists x P(x)$	Hypothesis
2.	$(c \text{ is a particular element}) \wedge P(c)$	Existential instantiation, 1
3.	$P(c) \wedge (c \text{ is a particular element})$	Commutative law, 2
4.	$\exists x Q(x)$	Hypothesis
5.	$(c \text{ is a particular element}) \wedge Q(c)$	Existential instantiation, 4
6.	$Q(c) \wedge (c \text{ is a particular element})$	Commutative law, 5
7.	$P(c)$	Simplification, 3
8.	$Q(c)$	Simplification, 6
9.	$P(c) \wedge Q(c)$	Conjunction, 7, 8
10.	$c \text{ is a particular element}$	Simplification, 2
11.	$\exists x (P(x) \wedge Q(x))$	Existential generalization, 9, 10

**PARTICIPATION
ACTIVITY**

1.13.7: Correct and incorrect use of generalization and instantiation.

Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

1)

1.	$\exists x P(x)$	Hypothesis
2.	(c is a particular element) $\wedge P(c)$	Existential instantiation, 1
3.	$\exists x Q(x)$	Hypothesis
4.	(c is a particular element) $\wedge Q(c)$	Existential instantiation, 3

- Correct
- Incorrect

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Showing an argument with quantified statements is invalid

An argument with quantified statements can be shown to be invalid by defining a domain and predicates for which the hypotheses are all true but the conclusion is false. For example, consider the invalid argument:

$\exists x P(x)$
 $\exists x Q(x)$
 $\exists x (P(x) \wedge Q(x))$

Suppose that domain is the set $\{c, d\}$. The two hypotheses, $\exists x P(x)$ and $\exists x Q(x)$, are both true for the values for P and Q on elements c and d given in the table. However, the conclusion $\exists x (P(x) \wedge Q(x))$ is false. Therefore, the argument is invalid.

	P	Q
c	T	F
d	F	T

PARTICIPATION ACTIVITY

1.13.8: Showing an argument with quantified statements is invalid: finite domain.

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- 1) The argument below is invalid. Suppose that the domain of x is the set $\{a, b\}$. Select the table that proves the argument is invalid.

$$\forall x P(x) \vee \forall x Q(x)$$

$$\exists x (P(x) \wedge Q(x))$$

	P	Q
c	T	F
d	T	F

	P	Q
c	T	F
d	F	T

	P	Q
c	T	F
d	T	T

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**PARTICIPATION
ACTIVITY**

1.13.9: Showing an argument with quantified statements is invalid: integer domain.

The following argument is invalid:

$$\exists x P(x)$$

$$\exists x Q(x)$$

$$\exists x (P(x) \wedge Q(x))$$

Which definitions for predicates P and Q show that the argument is invalid? In each question the domain of x is the set of positive integers.

1) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime

Q(x): x is even

Do the definitions for P and Q show that the argument is invalid?

Yes

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No

- 2) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime

Q(x): x is multiple of 4

Do the definitions for P and Q show that the argument is invalid?

Yes

No

- 3) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime

Q(x): $x^2 < x$

Do the definitions for P and Q show that the argument is invalid?

Yes

No

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Additional exercises

EXERCISE 1.13.1: Proving the validity of arguments with quantified statements.

Prove that the given argument is valid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. Then use the rules of inference to prove that the form is valid.

The domain of discourse is the set of musicians in an orchestra.

- (a) Everyone practices hard or plays badly (or both).
Someone does not practice hard.
 \therefore Someone plays badly.

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 **Solution** 

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Ok, got it

The domain of discourse is the set of people who live in a city. Linda lives in the city.

- (b) Linda lives in the city.
Linda owns a Ferrari.
Everyone who owns a Ferrari has gotten a speeding ticket.
 \therefore Linda has gotten a speeding ticket.

 **Solution** 

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The domain of discourse is the set of all paintings.

- (c) All of the paintings by Matisse are beautiful.
The museum has a painting by Matisse.
 \therefore The museum has a beautiful painting.

 **Solution** 

The domain is the set of students at an elementary school.

- (d) Every student who has a permission slip can go on the field trip.
Every student has a permission slip.
 \therefore Every student can go on the field trip.

 **Solution** 

The domain of discourse is the set of students at a university.

- (e) Larry is a student at the university.
Hubert is a student at the university.
Larry and Hubert are taking Boolean Logic.
Any student who takes Boolean Logic can take Algorithms.
 \therefore Larry and Hubert can take Algorithms.

 **Solution** 

EXERCISE 1.13.2: Which arguments are valid?

Which of the following arguments are valid? Explain your reasoning.

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- (a) I have a student in my class who is getting an A. Therefore, John, a student in my class is getting an A.

 **Solution** 

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Ok, got it

- (b) Every girl scout who sells at least 50 boxes of cookies will get a prize. Suzy, a girl scout, got a prize. Therefore Suzy sold at least 50 boxes of cookies.

 **Solution** 

EXERCISE 1.13.3: Show an argument with quantified statements is invalid.

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Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

- $\forall x (P(x) \rightarrow Q(x))$
(a) $\exists x \neg P(x)$
 $\therefore \exists x \neg Q(x)$

 **Solution** 

Click the eye icon to toggle solution visibility for students

Ok, got it

- $\exists x (P(x) \vee Q(x))$
(b) $\exists x \neg Q(x)$
 $\therefore \exists x P(x)$

 **Solution** 

EXERCISE 1.13.4: Determine and prove whether an argument is valid or invalid.

Determine whether each argument is valid. If the argument is valid, give a proof using the laws of logic. If the argument is invalid, give values for the predicates P and Q over the domain {a, b} that demonstrate the argument is invalid.

- (a) $\exists x (P(x) \wedge Q(x))$
 $\therefore \exists x Q(x) \wedge \exists x P(x)$

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 **Solution** 

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Ok, got it

(b) $\exists x Q(x) \wedge \exists x P(x)$
 $\therefore \exists x (P(x) \wedge Q(x))$

 **Solution** 

(c) $\forall x (P(x) \wedge Q(x))$
 $\therefore \forall x Q(x) \wedge \forall x P(x)$

 **Solution** 

(d) $\forall x (P(x) \vee Q(x))$
 $\therefore \forall x Q(x) \vee \forall x P(x)$

 **Solution** 

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