A Simple Class of Non-Linear Subdivision Schemes

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Subdivision

- Set of rules S that recursively act on a shape p^0

$$p^{k+1} = S\left(p^k\right)$$

- Converges to a smooth shape

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$$p^{\infty} = S^{\infty}\left(p^0\right)$$

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Linear Subdivision

- Locally can be written as matrix multiplication

$$p^{k+1} = M\,p^k$$

- Usually reproduce polynomials
- Easy to analyze
  - Sufficient conditions of continuity based on eigen-structure of M [Reif 95]
- Includes Catmull-Clark, Loop, Butterfly, etc…

Non-linear Subdivision

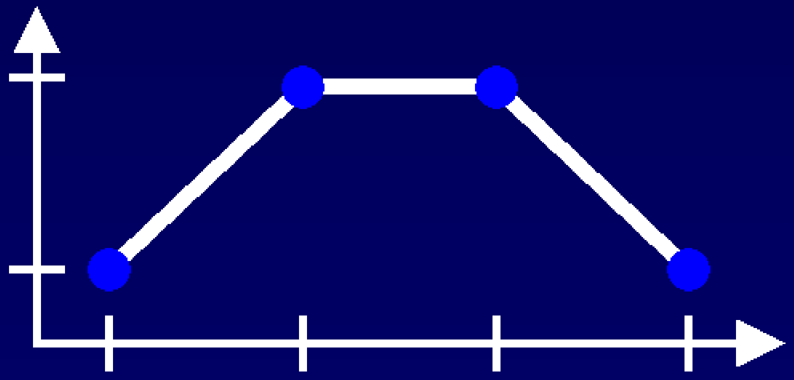
- Greater expression
  - Reproduce non-polynomial functions
    - circles [Sabin et al. 2005]
    - $p(x)e^{l(x)}$ [Micchelli 1996]
  - Preserve convexity [Floater et al. 1998]
  - Subdivision curves on manifolds [Noakes 1998, Wallner et al. 2005]
- Hard to analyze smoothness

Contributions

- Provide a simple class of non-linear subdivision schemes
  - Easy to analyze smoothness
  - Modification of linear subdivision schemes
  - Can reproduce interesting functions: trigonometrics, gaussians
- Applications to intersection calculations

Linear Subdivision Example

- Uniform B-splines [Lane, Reisenfeld 1980]
  - Doubling followed by mid-point averaging
  - Smoothness: C^{n-1} (n = # of averaging steps)
  - Piecewise polynomial



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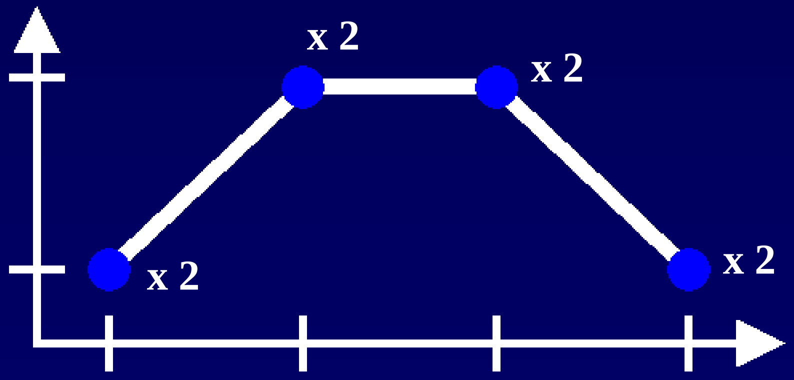

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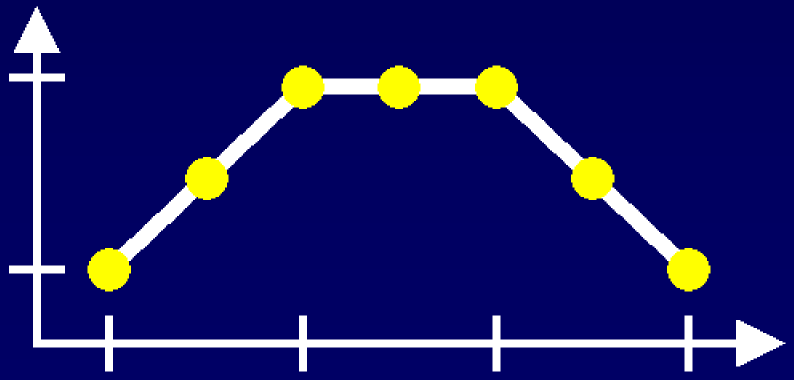

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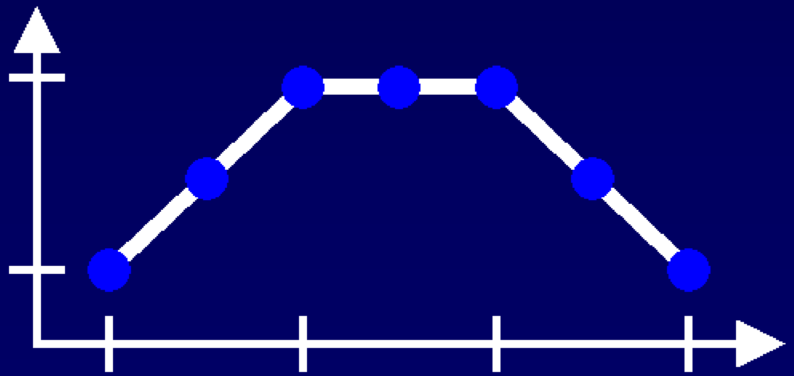

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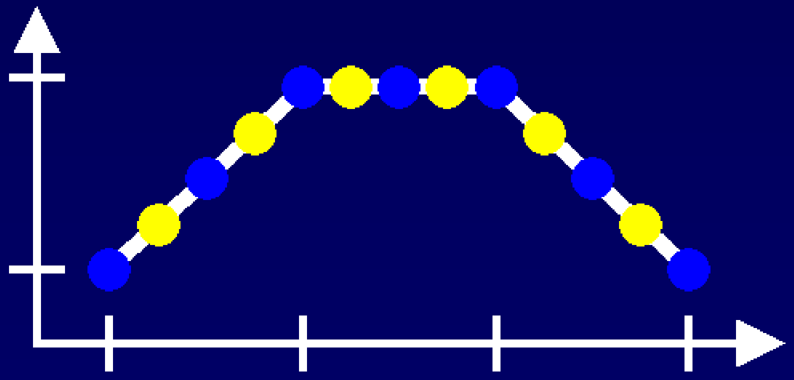

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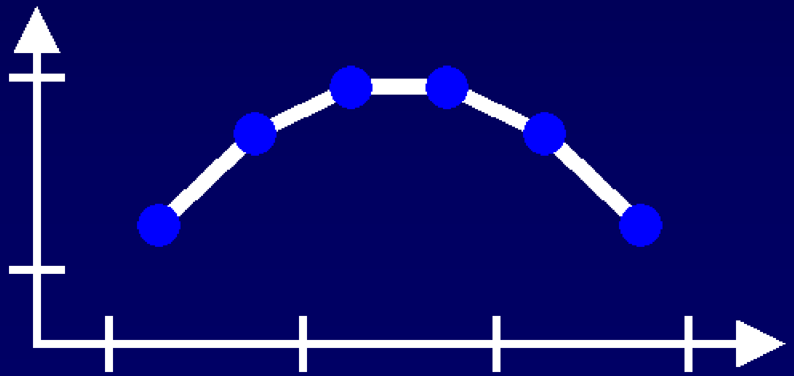

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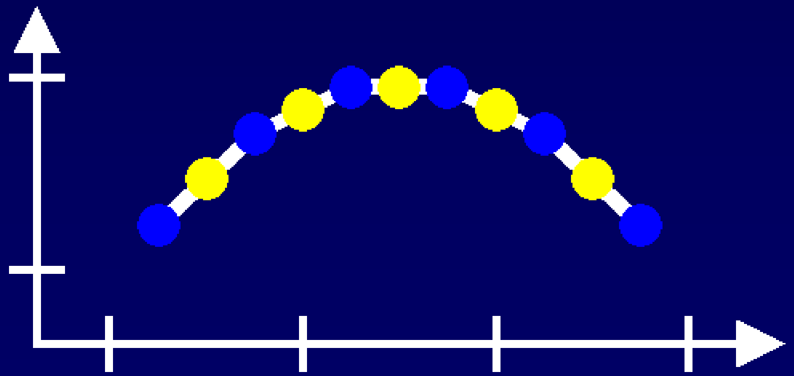

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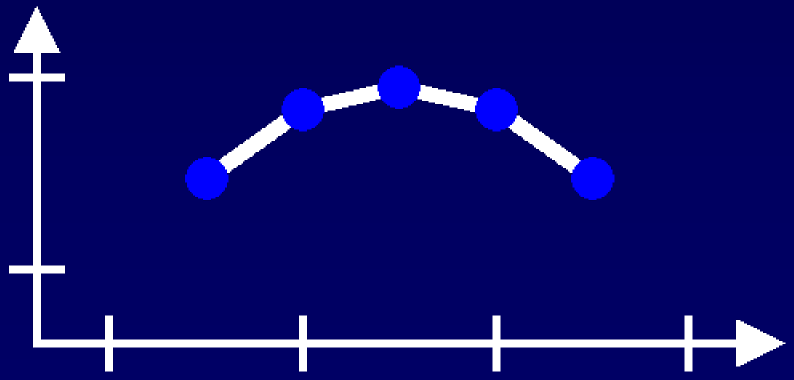

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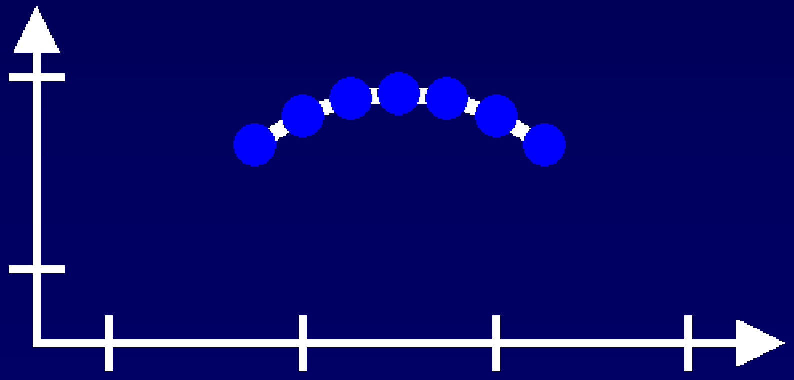

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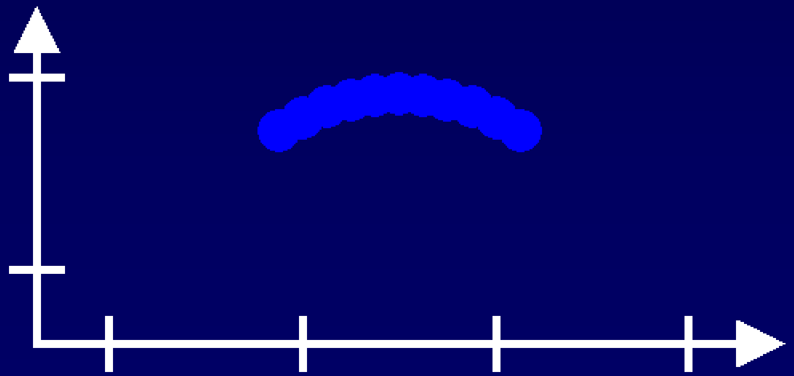

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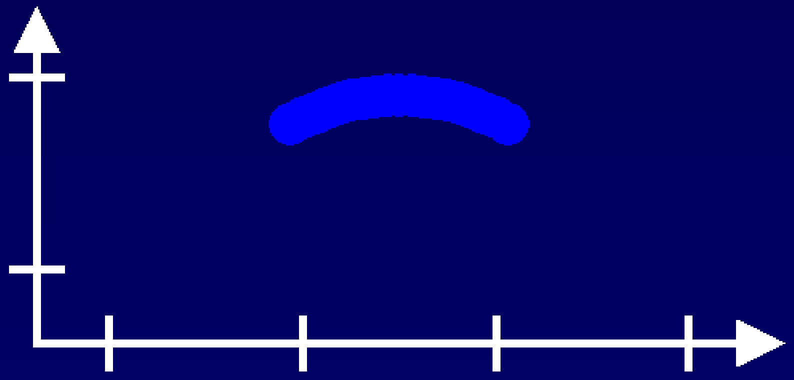

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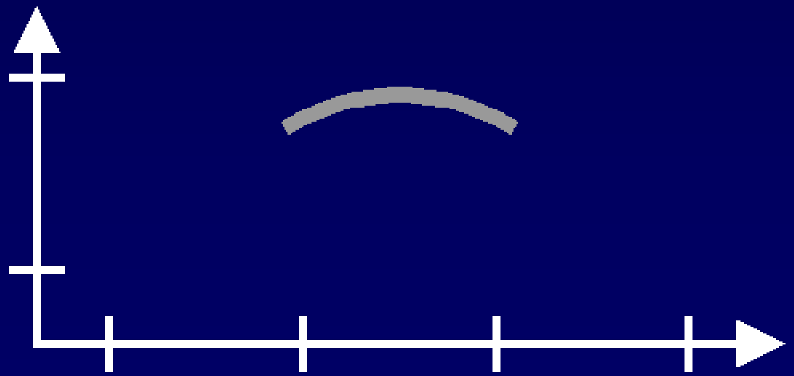

Simple Non-Linear Subdivision

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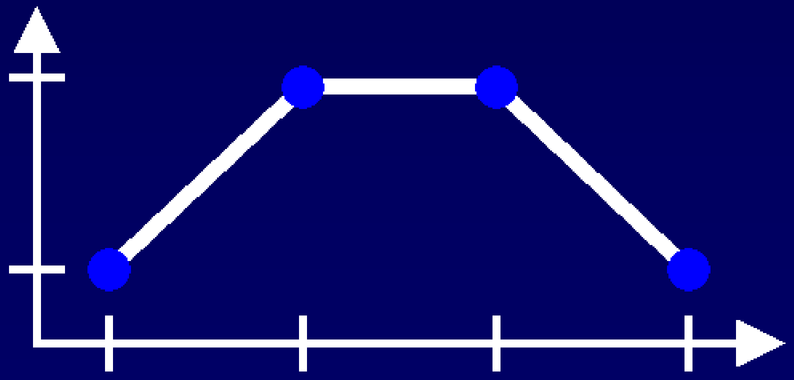
$$\frac{a+b}{2} \rightarrow \sqrt{ab}$$

- Is the curve smooth?
- What functions does this method reproduce?



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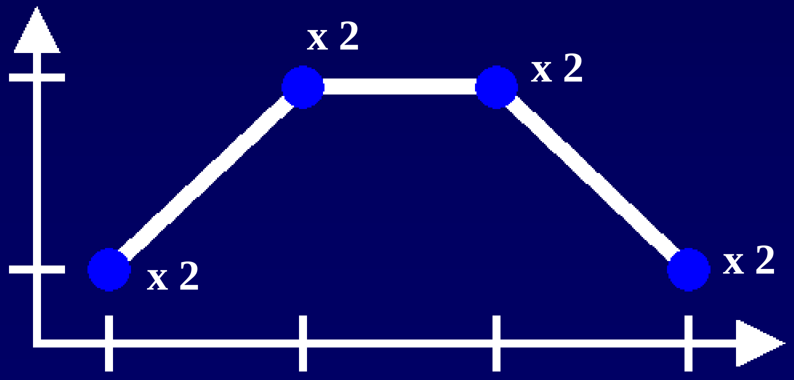

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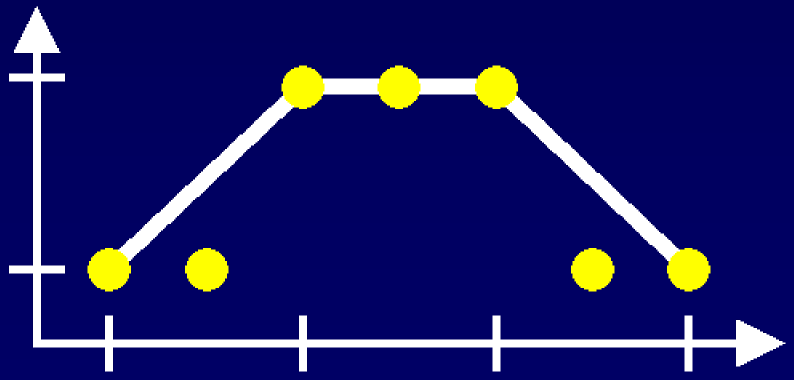

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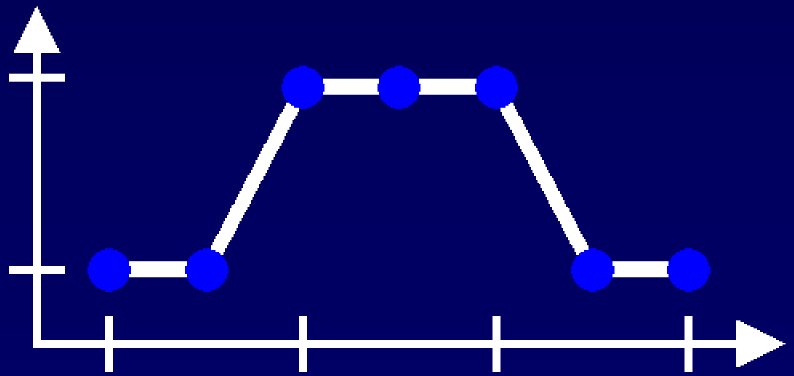

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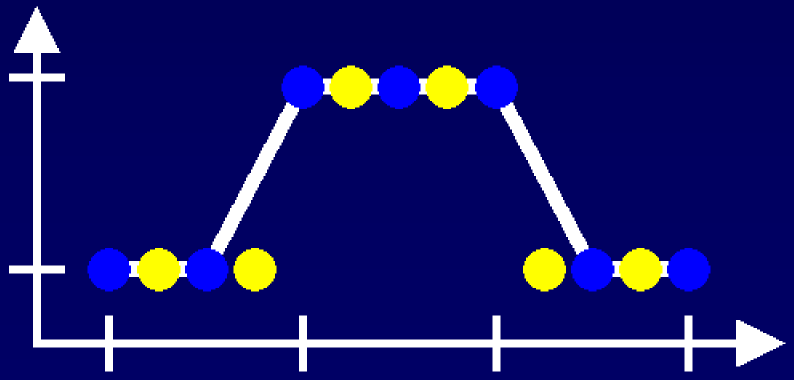

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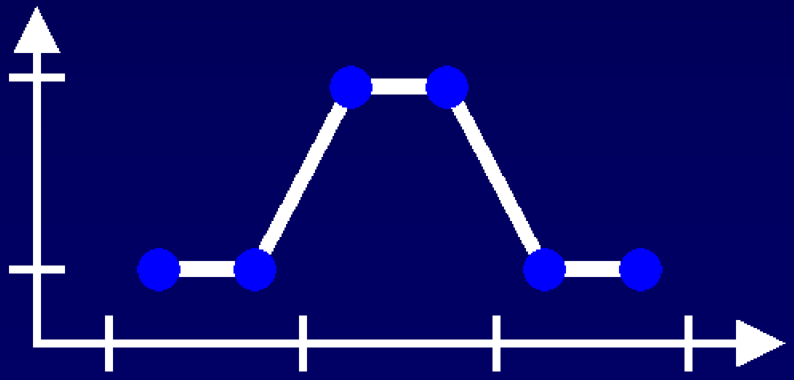

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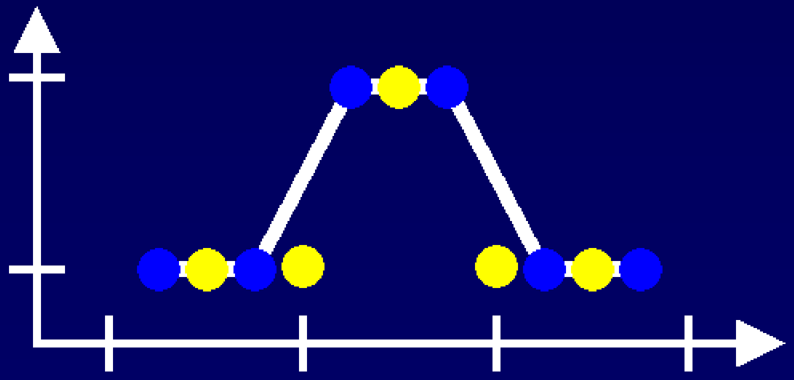

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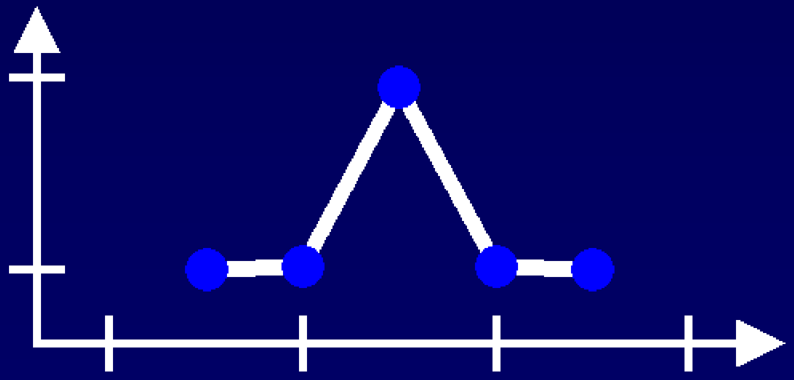

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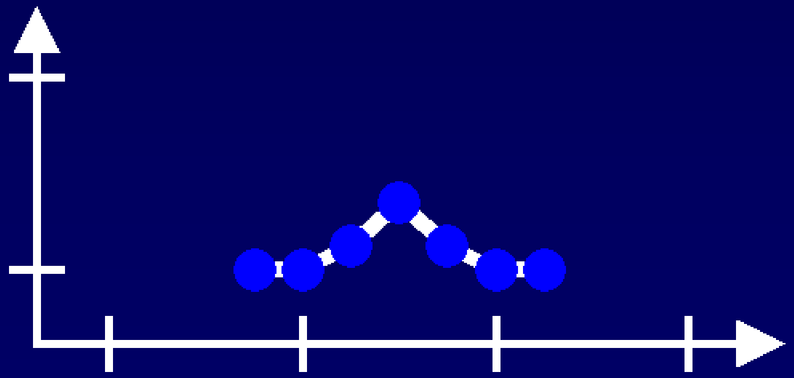

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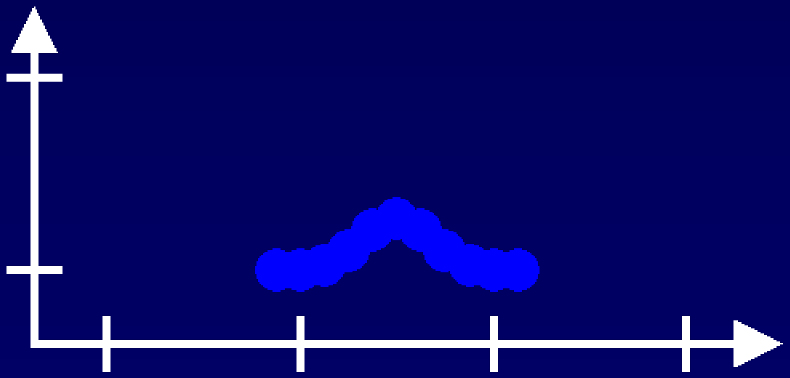

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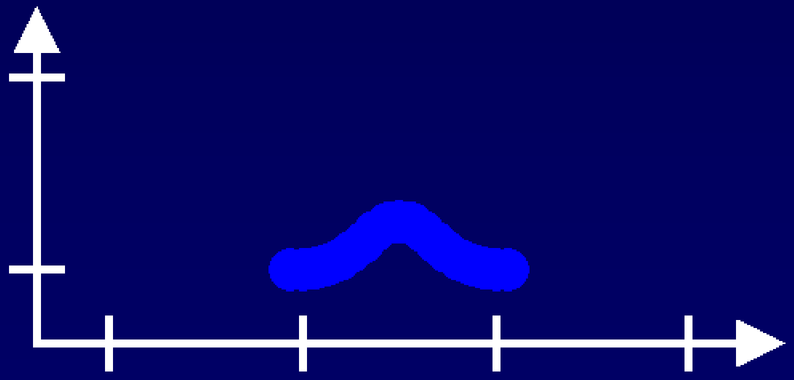

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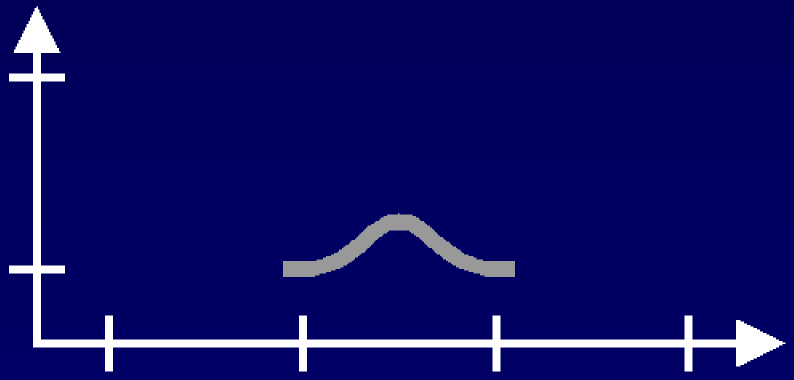

Functional Equations

- Find parametric midpoint of a function *F*

$$F\left(\frac{x_0+x_1}{2}\right)=G(F(x_0),F(x_1))$$

- Example: $L(x) = m\,x + b$

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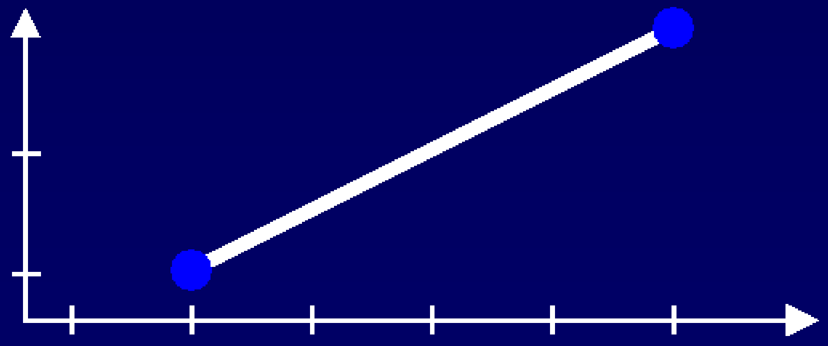

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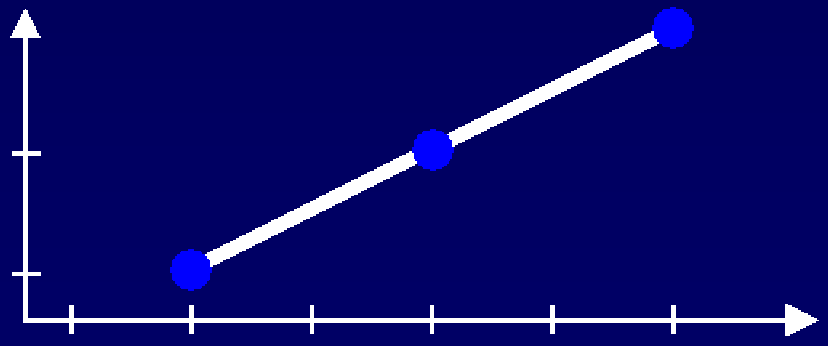

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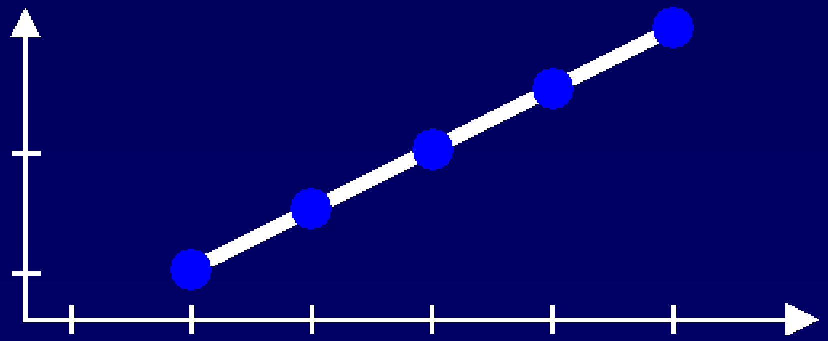

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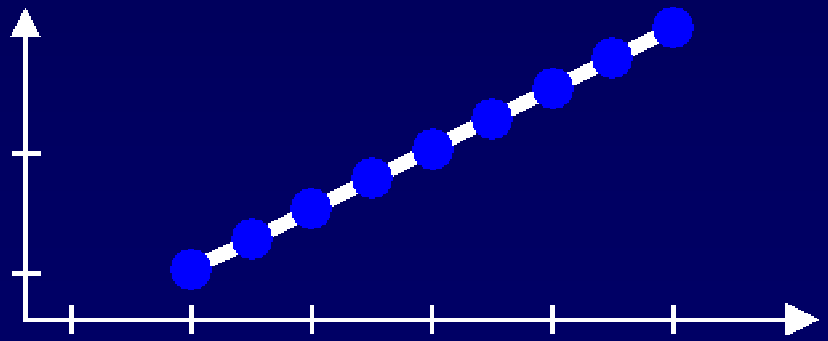

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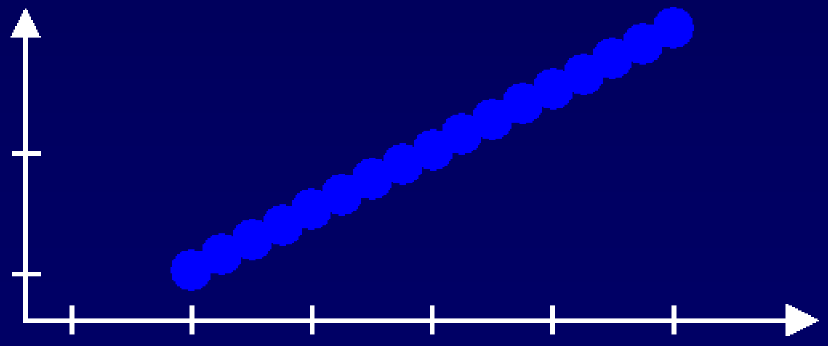

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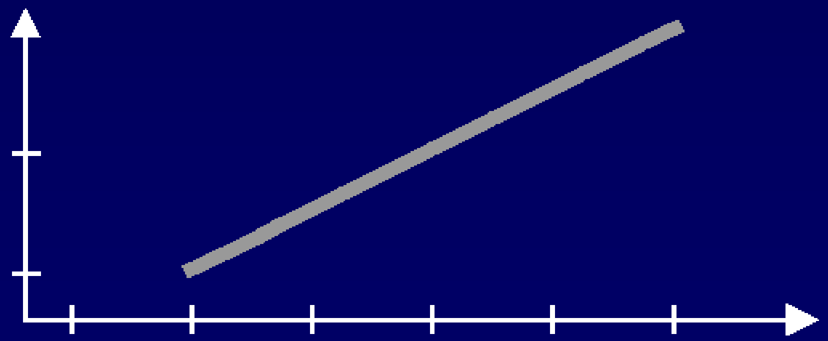

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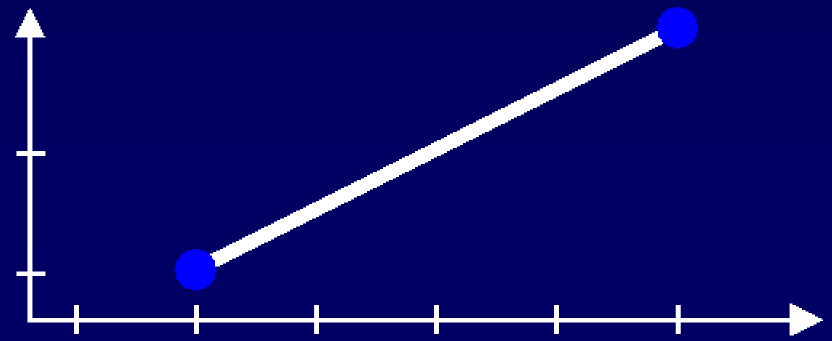

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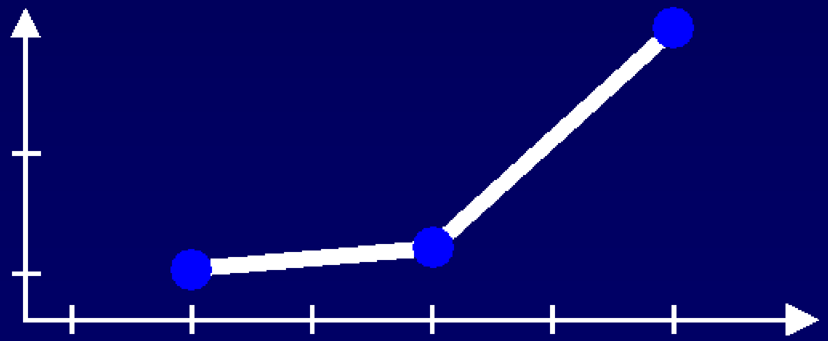

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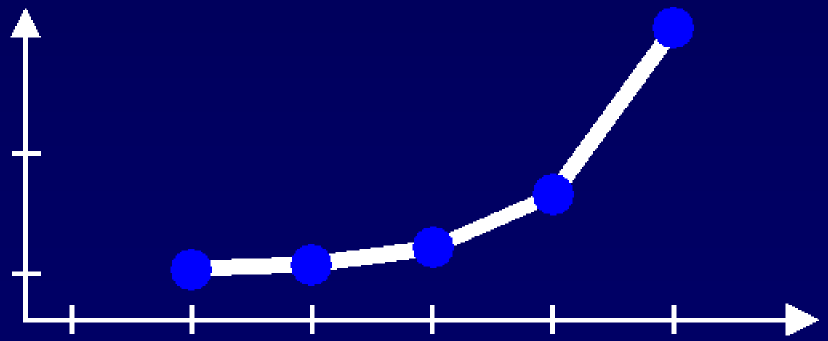

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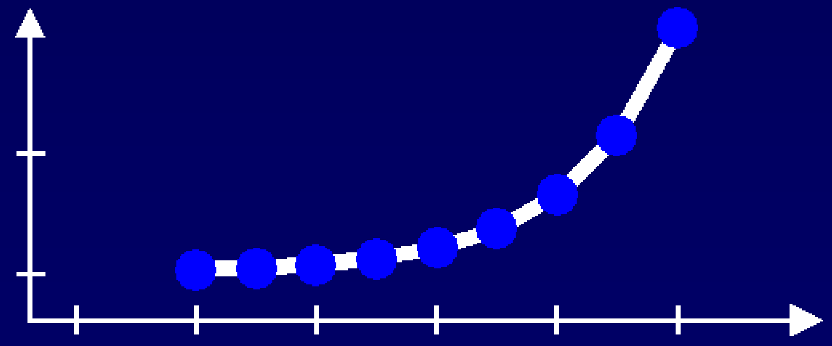

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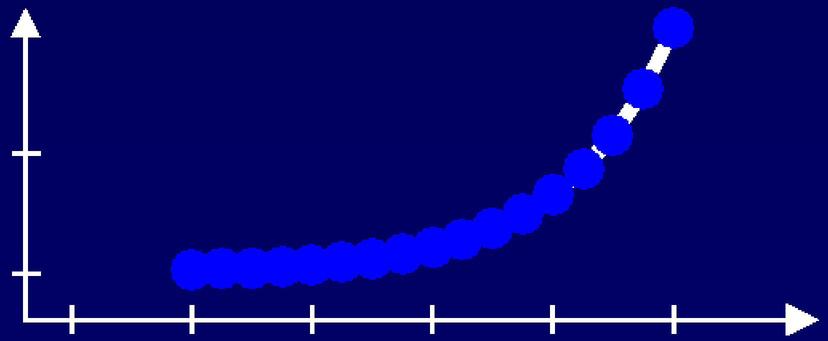

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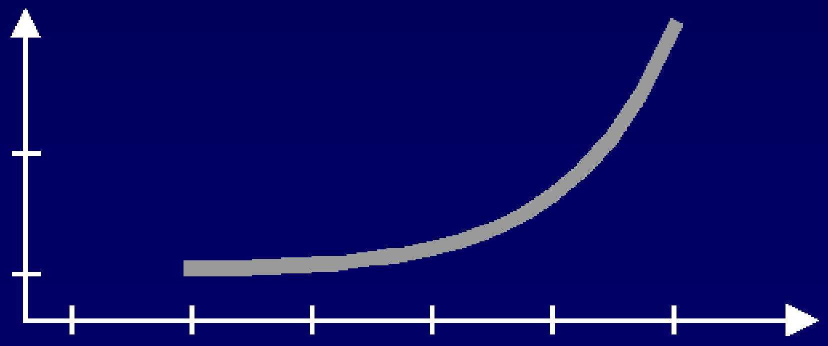

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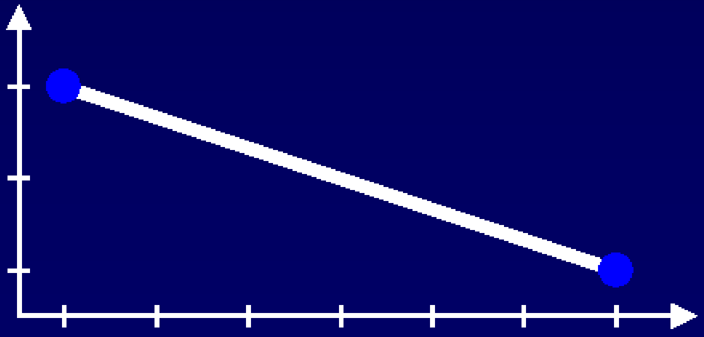

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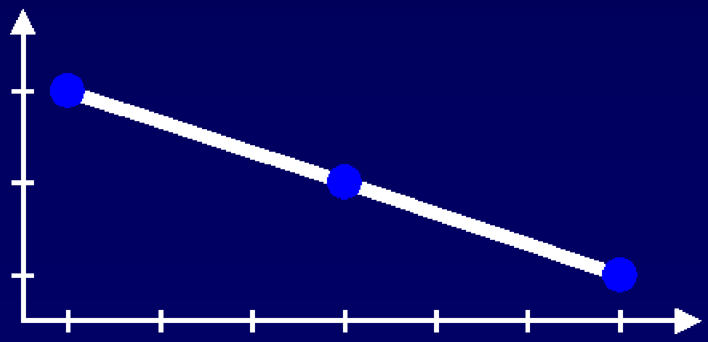

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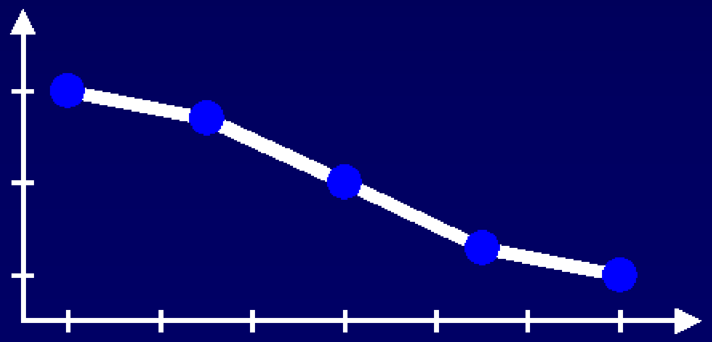

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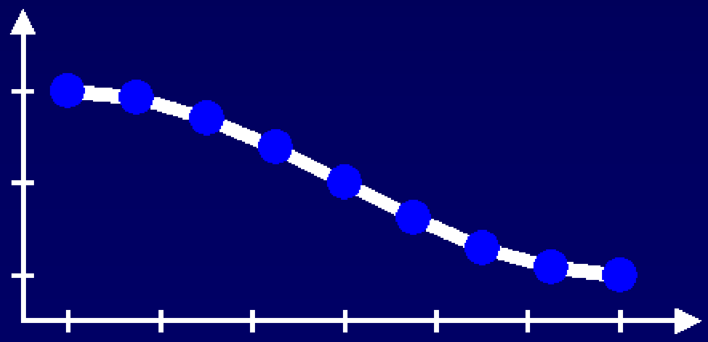

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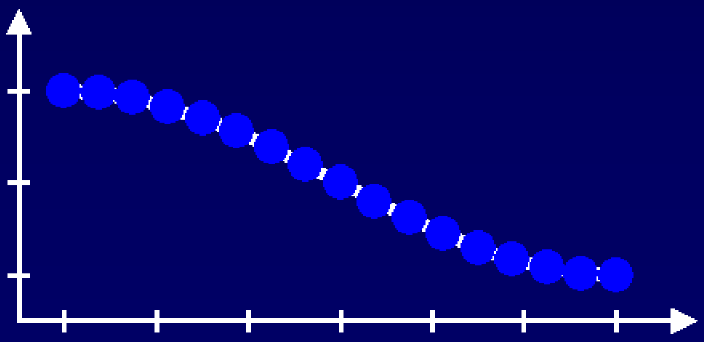

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- Example: $F(x) = \cos(m\,x + b)$

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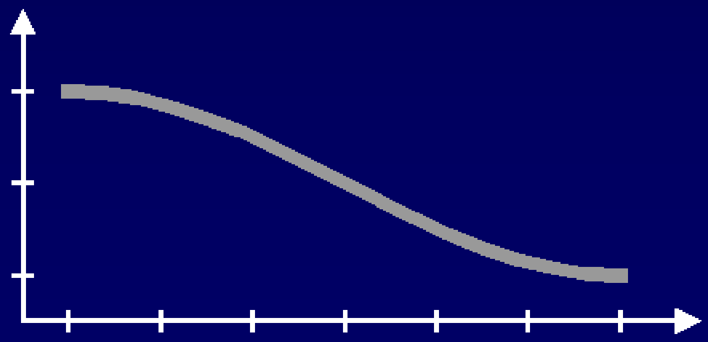

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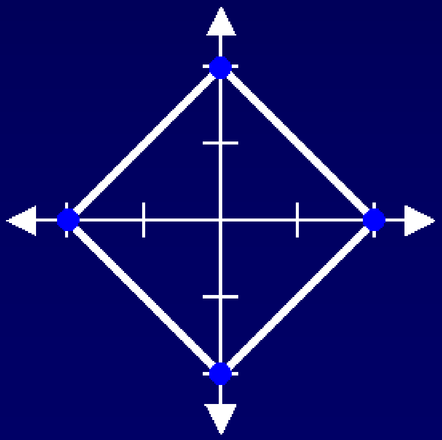

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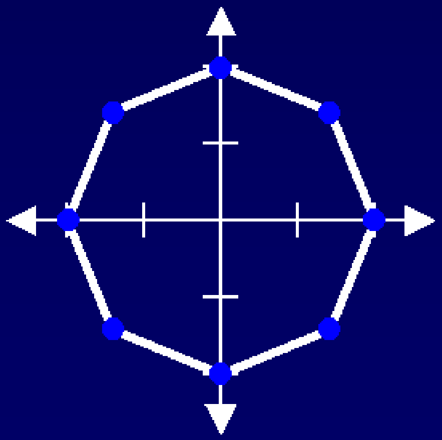

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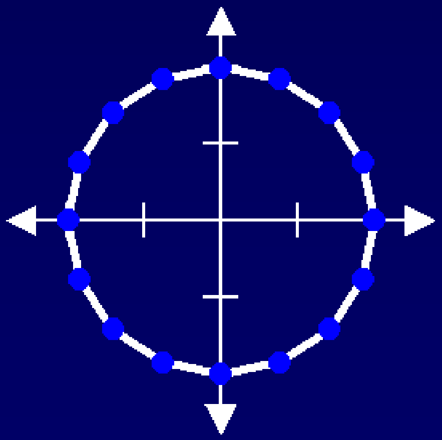

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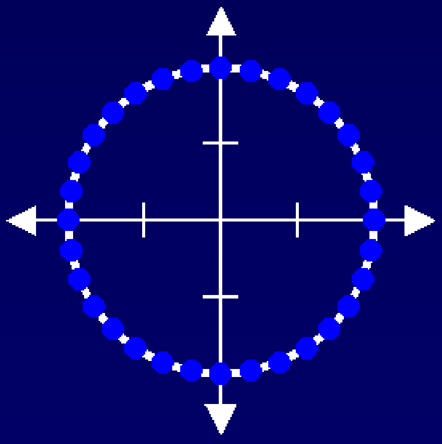

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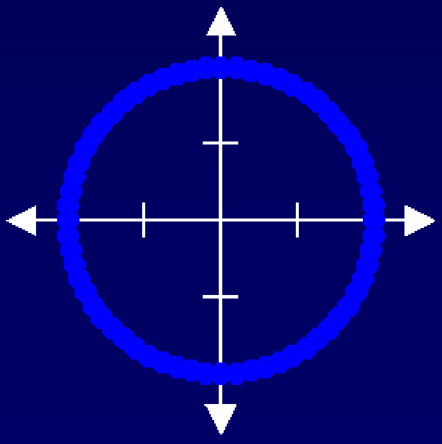

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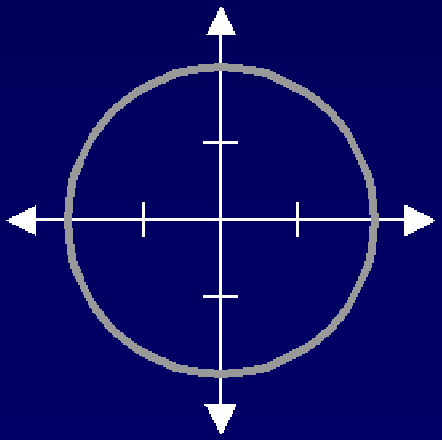


Other Averaging Rules

Function	**Averaging Rule**
$F(x)=\sqrt{x}$	$F(\frac{x_0+x_1}{2})=\sqrt{\frac{F(x_0)^2+F(x_1)^2}{2}}$
$F(x)=x^2$	$F(\frac{x_0+x_1}{2})=\frac{((F(x_0)+F(x_1))/2+\sqrt{F(x_0)F(x_1)}}{2}$
$F(x)=\frac{1}{x}$	$F(\frac{x_0+x_1}{2})=\frac{F(x_0)F(x_1)}{(F(x_0)+F(x_1))/2}$
$F(x)=\frac{1}{x^2}$	$F(\frac{x_0+x_1}{2})=\frac{F(x_0)F(x_1)}{\sqrt{(F(x_0)^2+F(x_1)^2)/2}}$
$F(x)=\cosh(x)$	$F(\frac{x_0+x_1}{2})=\frac{\sqrt{(F(x_0)+1)(F(x_1)+1)}+\sqrt{(F(x_0)-1)(F(x_1)-1)}}{2}$

Non-linear Maps

- Given
  - F: 1-1 function on $\Omega \subseteq R^n$
  - S: subdivision scheme
  - $\hat{S} = F \circ S \circ F^{-1}$
- Then
  - $\hat{S}^\infty = F \circ S^\infty \circ F^{-1}$
  - $S^\infty\left(p^0\right) = p^\infty \quad \Rightarrow \quad \hat{S}^\infty\left(F\left(p^0\right)\right) = F\left(p^\infty\right)$

Non-linear Maps

- Given
  - F: 1-1 function on $\Omega \subseteq R^n$
  - $S = S_d \circ \ldots \circ S_2 \circ S_1$: subdivision scheme
  - $\hat{S} = F \circ S \circ F^{-1}$
- Then
  - $\hat{S} = \left(F \circ S_d \circ F^{-1}\right) \circ \ldots \circ \left(F \circ S_2 \circ F^{-1}\right) \circ \left(F \circ S_1 \circ F^{-1}\right)$

Non-linear Maps Example

$$\hat{S} = \left(F \circ S_d \circ F^{-1}\right) \circ \ldots \circ \left(F \circ S_2 \circ F^{-1}\right) \circ \left(F \circ S_1 \circ F^{-1}\right)$$

Lane-Reisenfeld	$F(x)$
$S_1(p)_j = p_{\lfloor j/2 \rfloor}$	$\hat{S}_1(p)_j = F(F^{-1}(p_{\lfloor j/2 \rfloor}))$
$S_{i \neq 1}(p)_j = \frac{p_j + p_{j+1}}{2}$	$\hat{S}_{i \neq 1}(p)_j = F(\frac{F^{-1}(p_j) + F^{-1}(p_{j+1})}{2})$

Non-linear Maps Example

$$\hat{S} = \left(F \circ S_d \circ F^{-1}\right) \circ \ldots \circ \left(F \circ S_2 \circ F^{-1}\right) \circ \left(F \circ S_1 \circ F^{-1}\right)$$

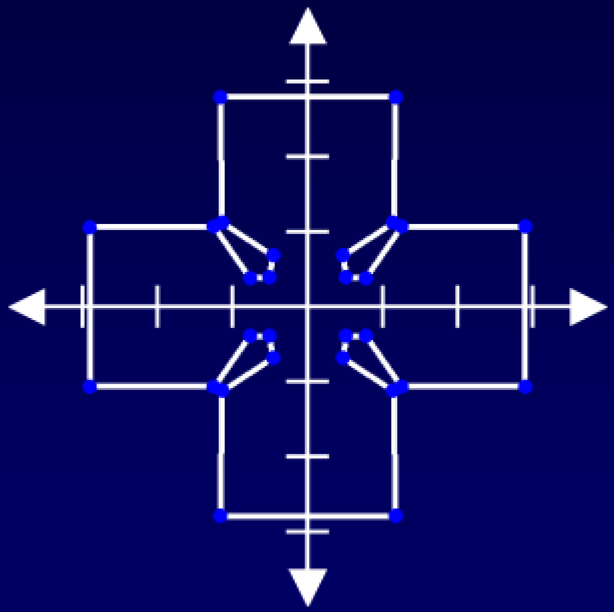
Lane-Reisenfeld	$F(x) = e^x$
$S_1(p)_j = p_{\lfloor j/2 \rfloor}$	$\hat{S}_1(p)_j = p_{\lfloor j/2 \rfloor}$
$S_{i \neq 1}(p)_j = \frac{p_j + p_{j+1}}{2}$	$\hat{S}_{i \neq 1}(p)_j = \sqrt{p_j p_{j+1}}$

Smoothness and Interpolation

- Given
  - F: 1-1 function on $\Omega \subseteq R^n$
  - S: subdivision scheme
  - $\hat{S} = F \circ S \circ F^{-1}$
- Then
  - $S^{\infty}(p^0) : C^k \quad \& \quad F : C^n \quad \Rightarrow \quad \hat{S}^{\infty}(\hat{p}^0) : C^{\min(k,n)}$
  - S:interpolatory $\Rightarrow$ $\hat{S}$:interpolatory

Example

Four-Point [Dyn et al. 1987]



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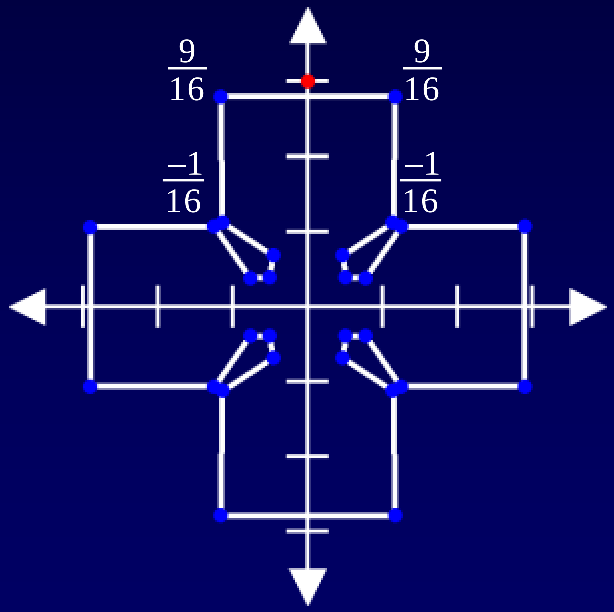

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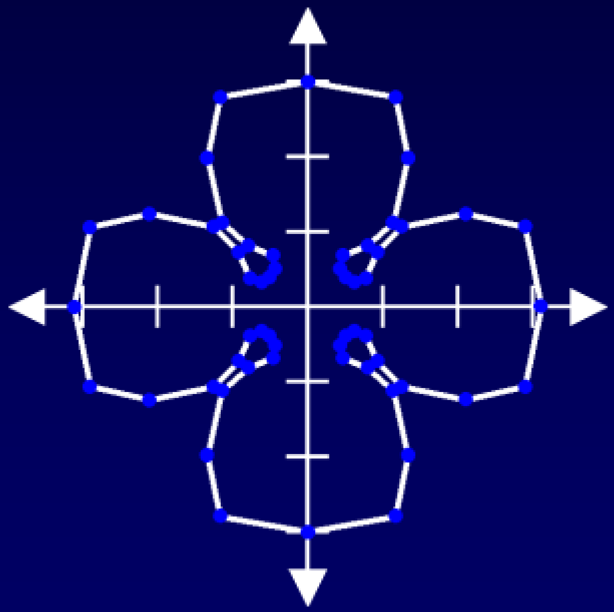

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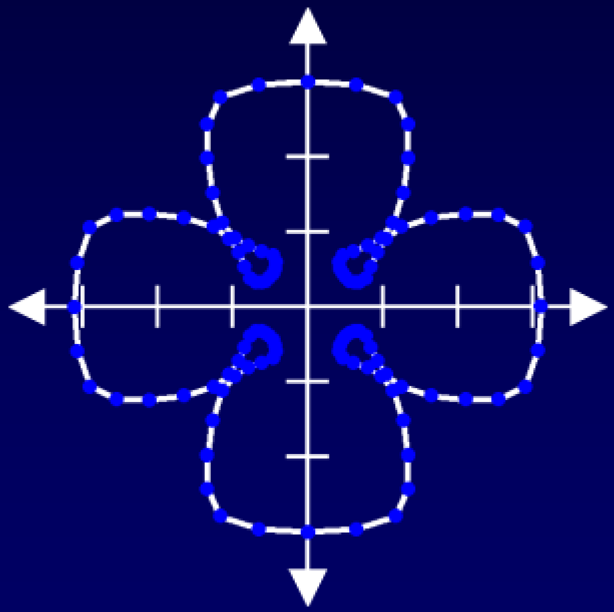

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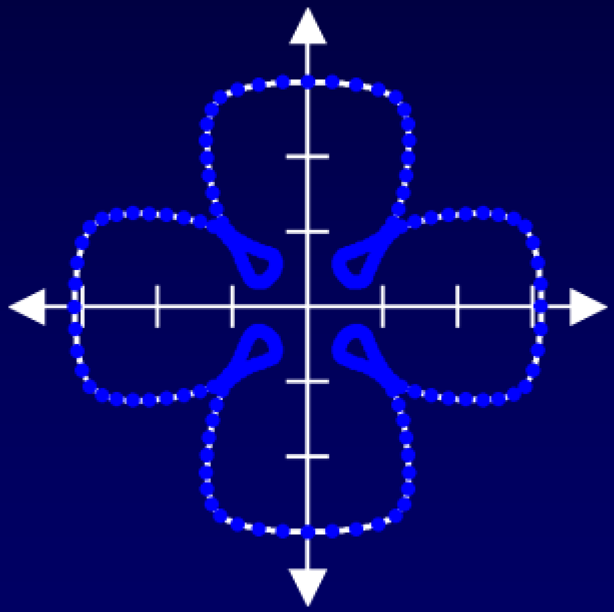

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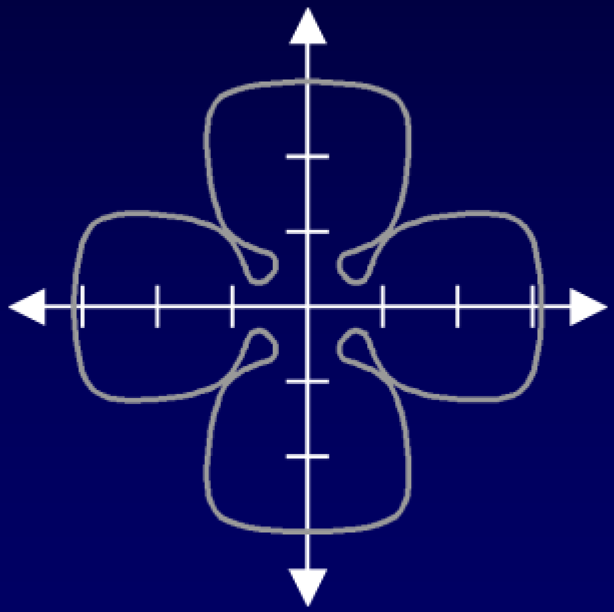

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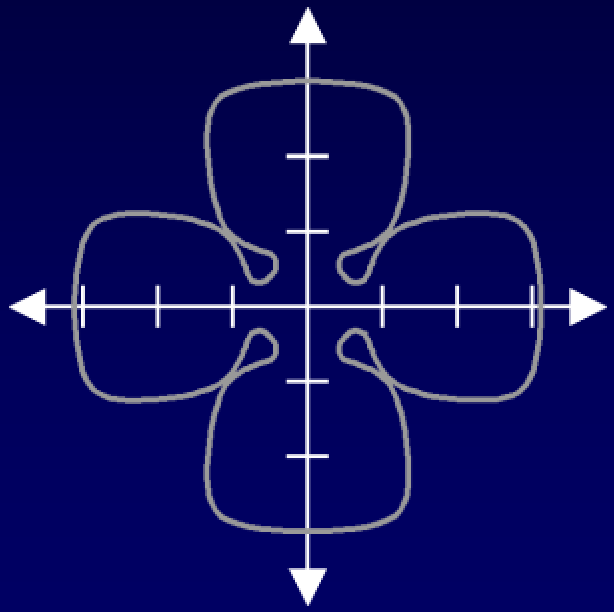
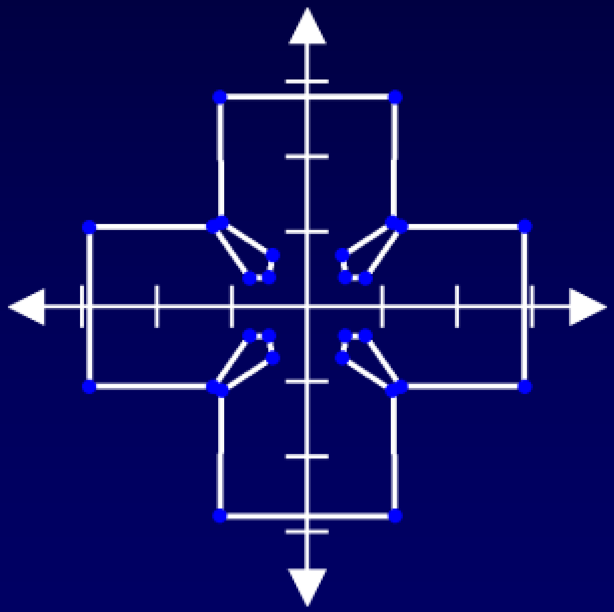


Mobius Transform



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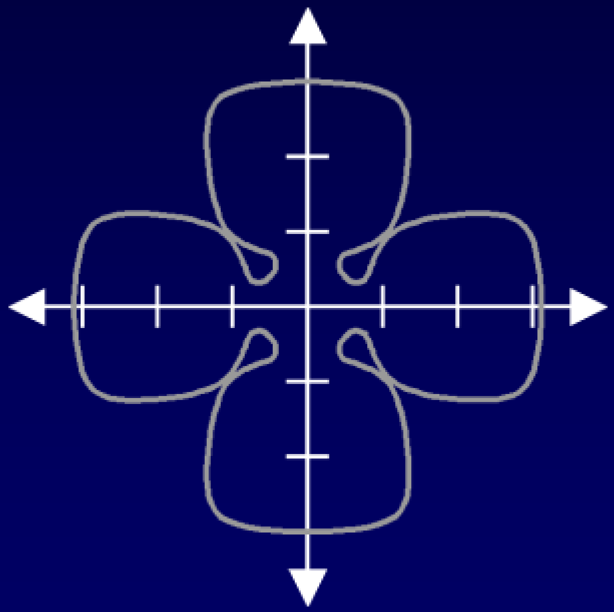

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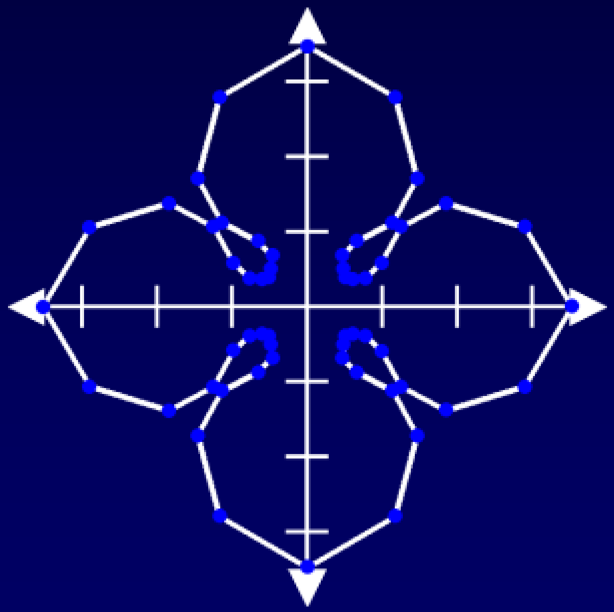

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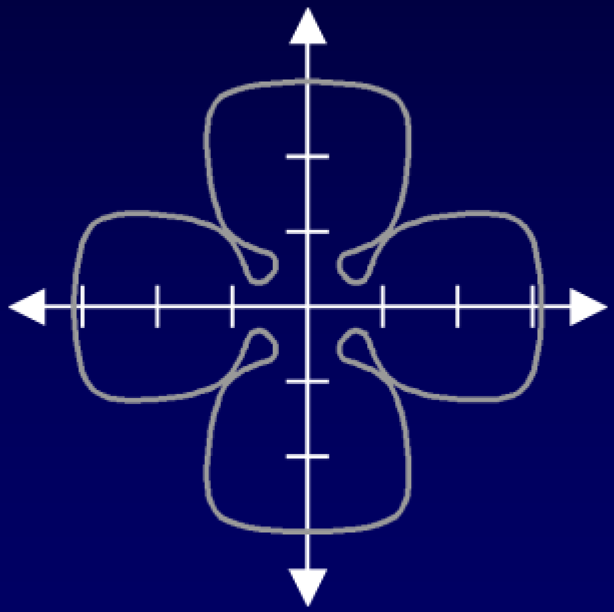

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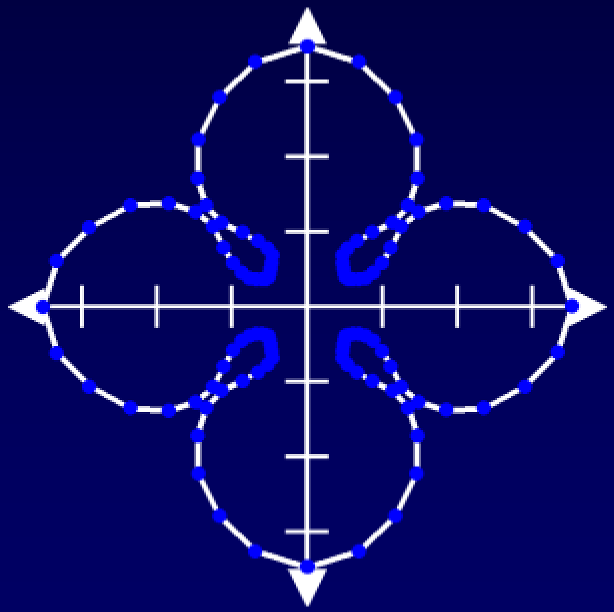

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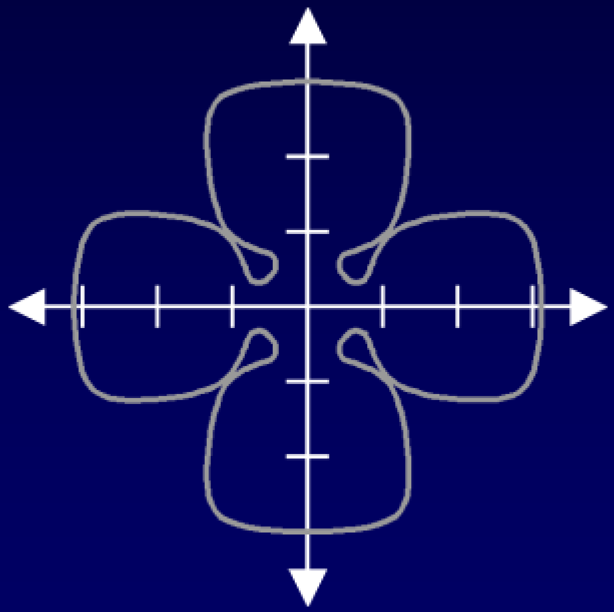

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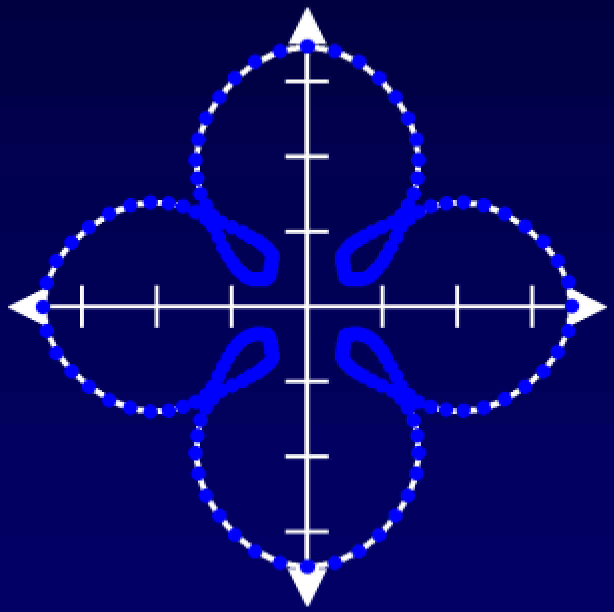

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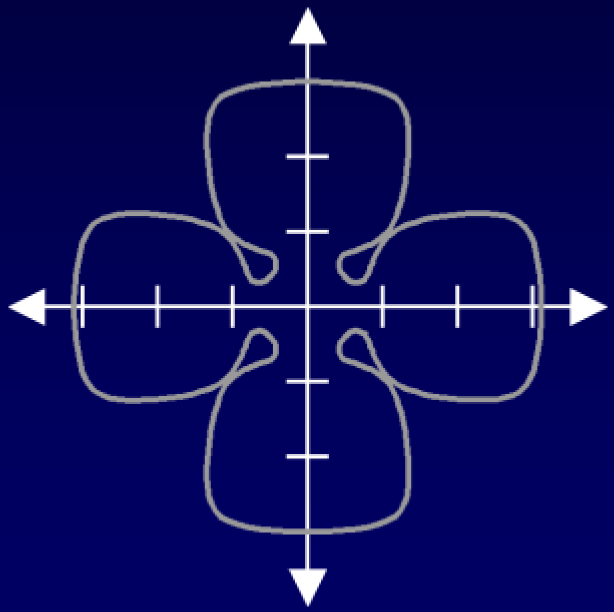

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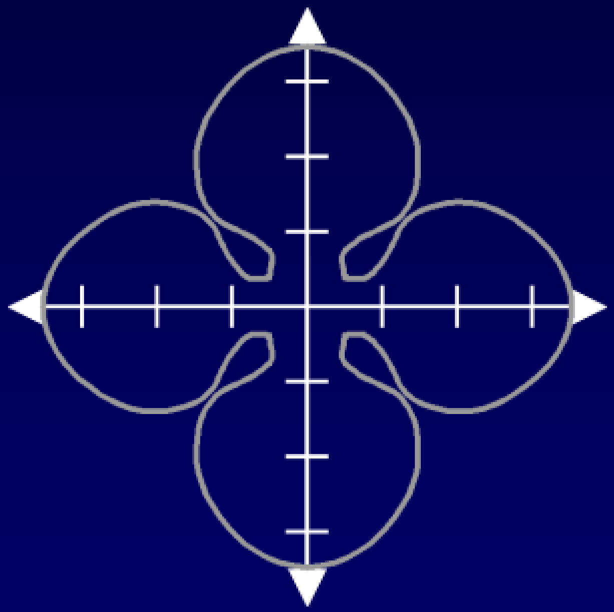
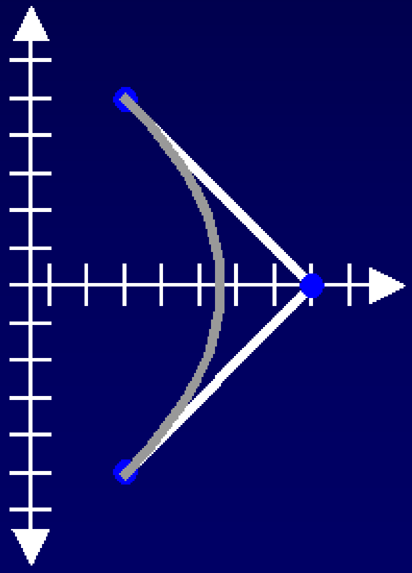


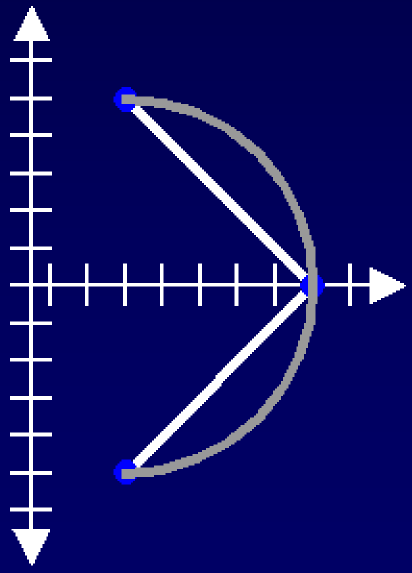
Geometric Properties

- Properties: convex-hull, variation diminishing

Linear



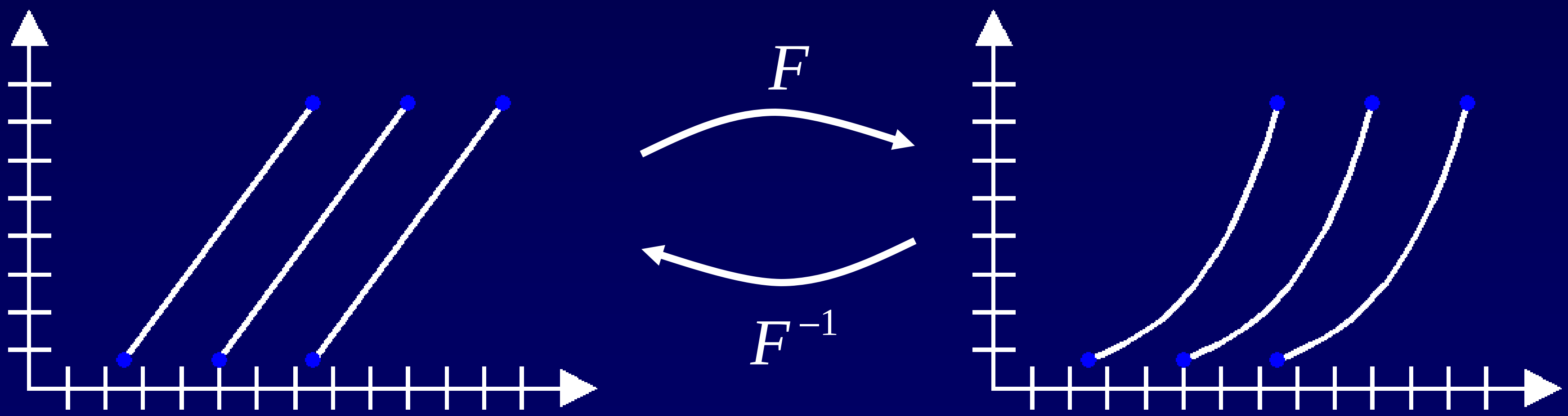
$F(z) = e^z$



Geometric Interpretation

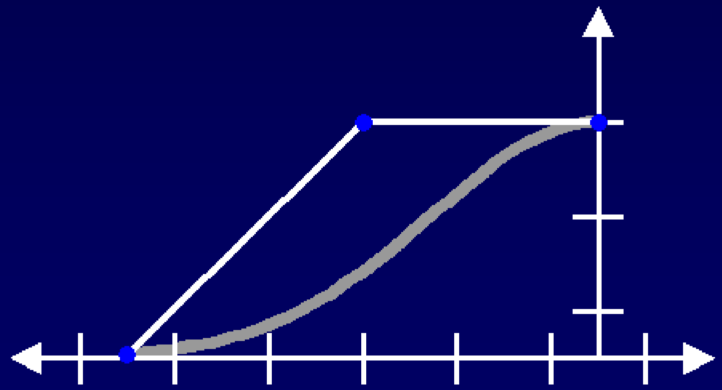
- Modify geodesics so that the properties hold

$$D(\hat{P},\hat{Q}) = Dist_{Euclidean}(F^{-1}(\hat{P}), F^{-1}(\hat{Q}))$$



Geometric Interpretation

- A set *C* is convex w.r.t. the geodesics *G* if the geodesic connecting any two points in *C* lies completely within *C*



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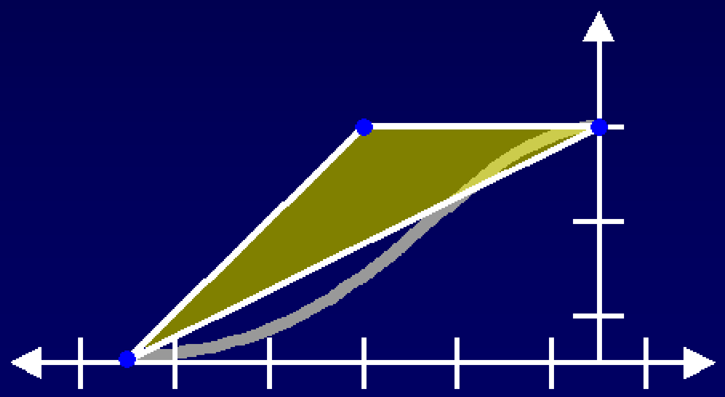

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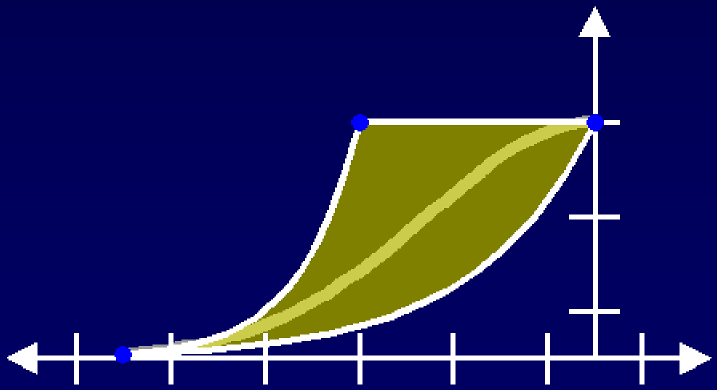
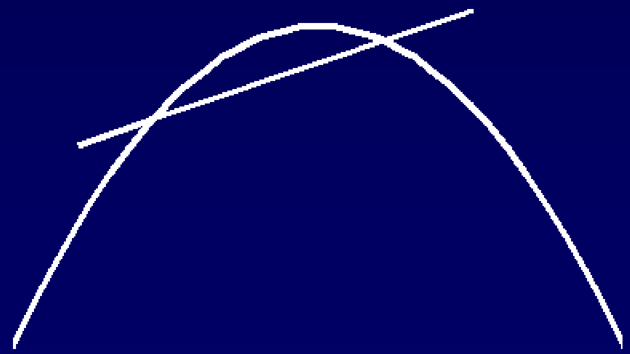


Intersection

1) If convex hulls of the control points do not intersect, then the curves do not intersect
2) If each curve is approximately a straight line, intersect those lines; else subdivide



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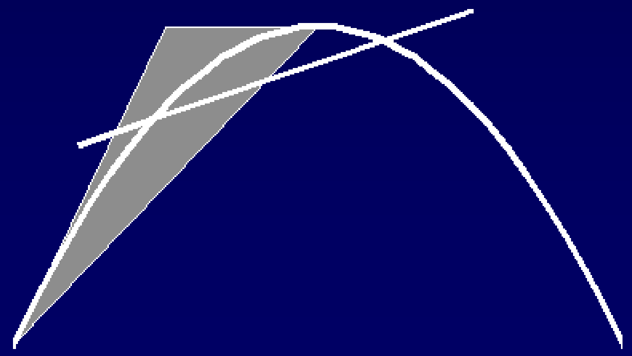

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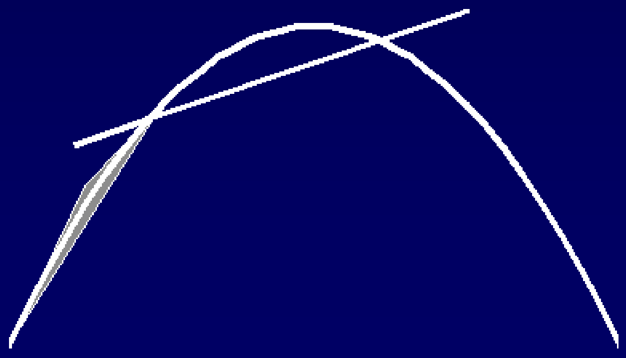

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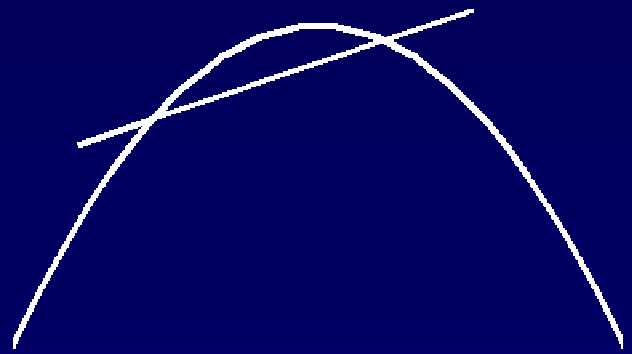

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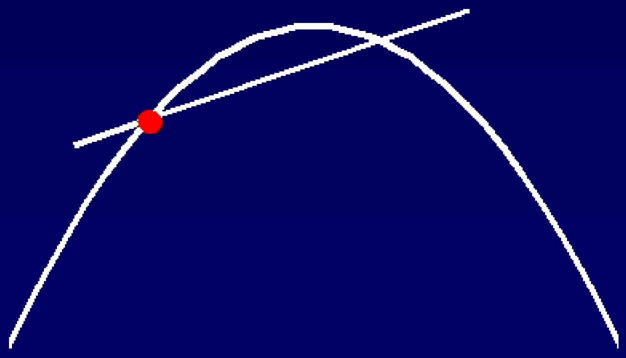

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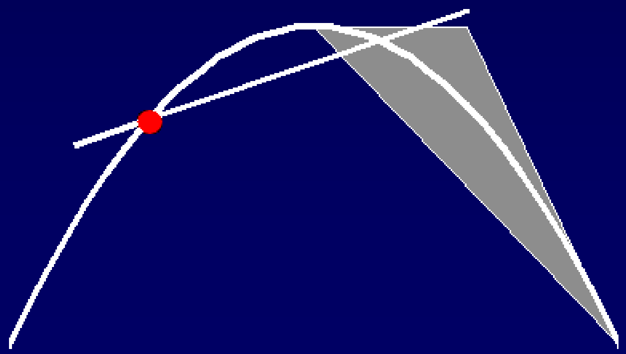

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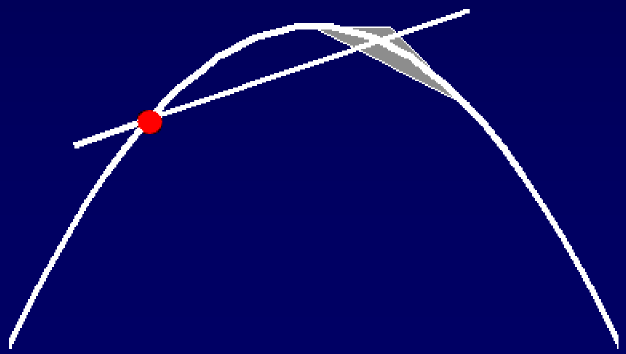

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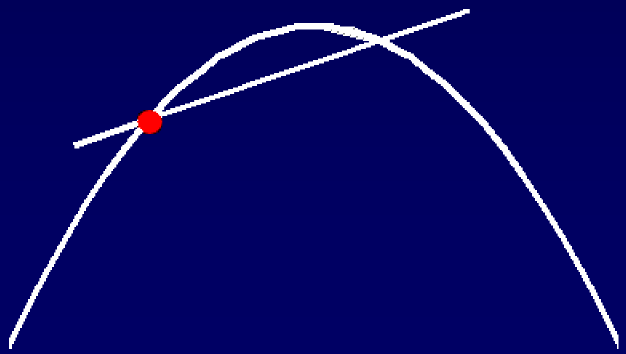

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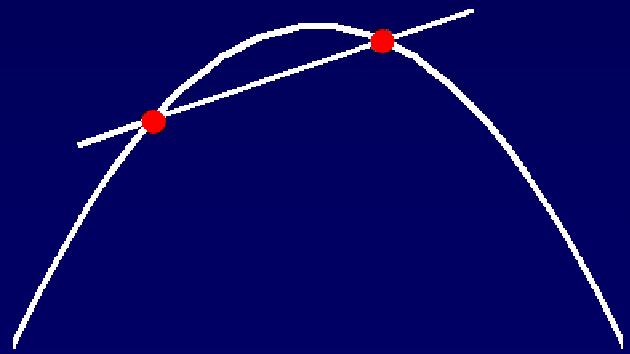

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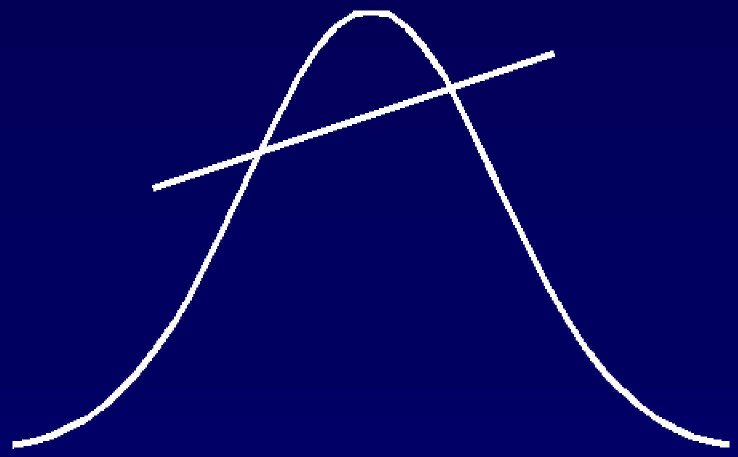

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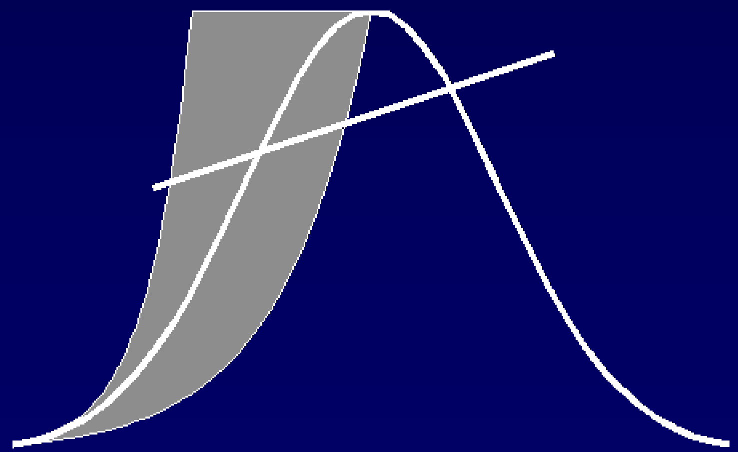

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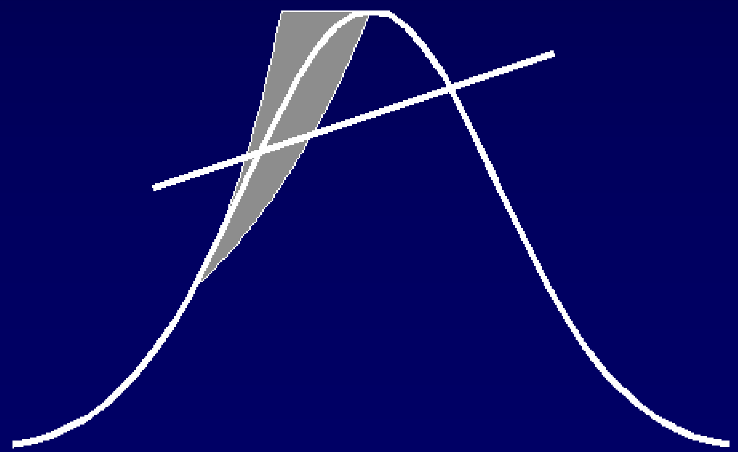

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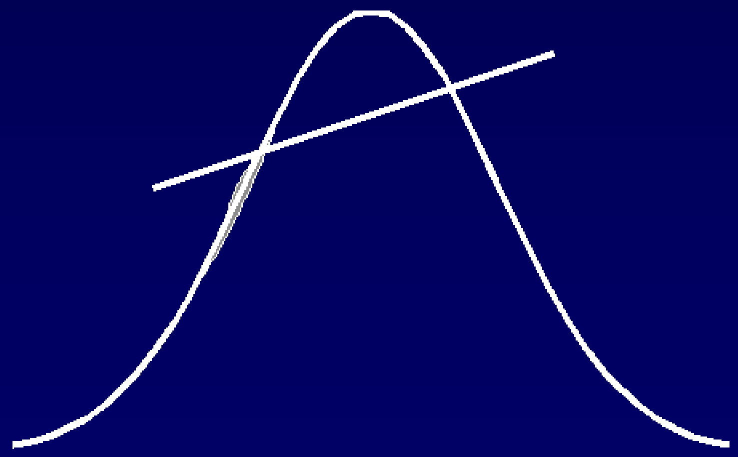

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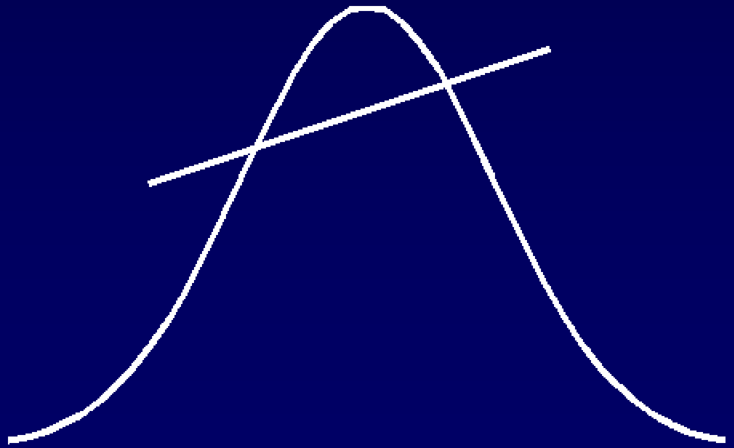

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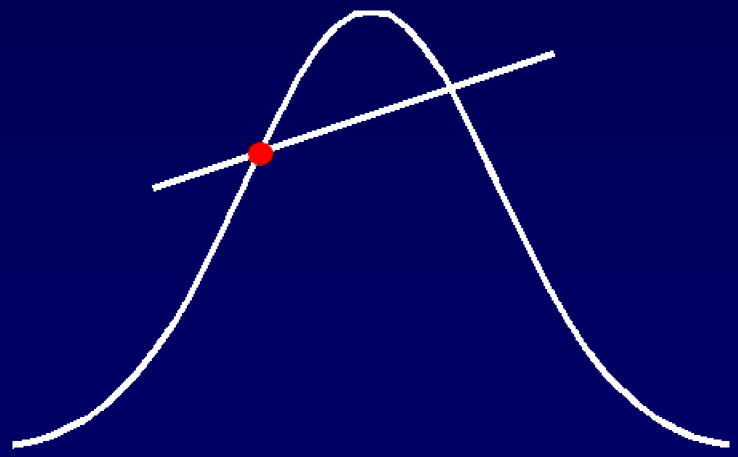

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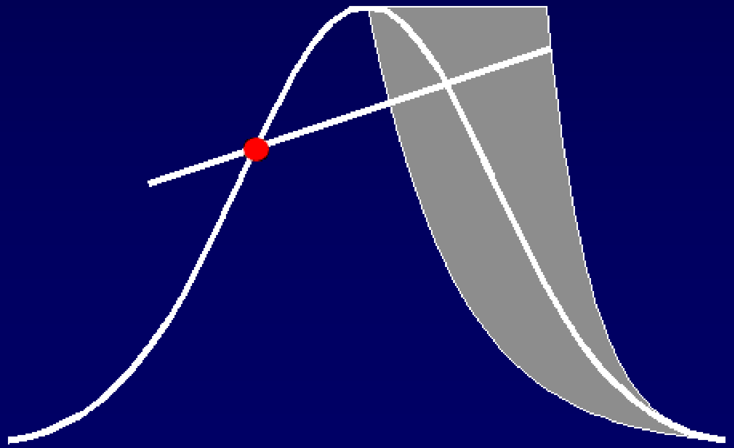

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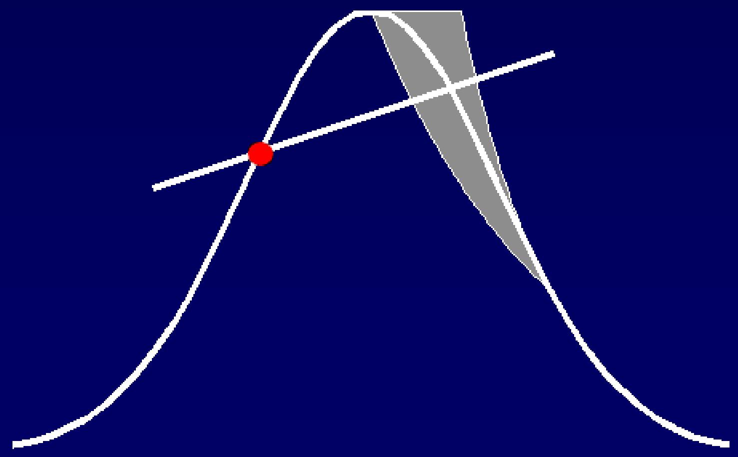

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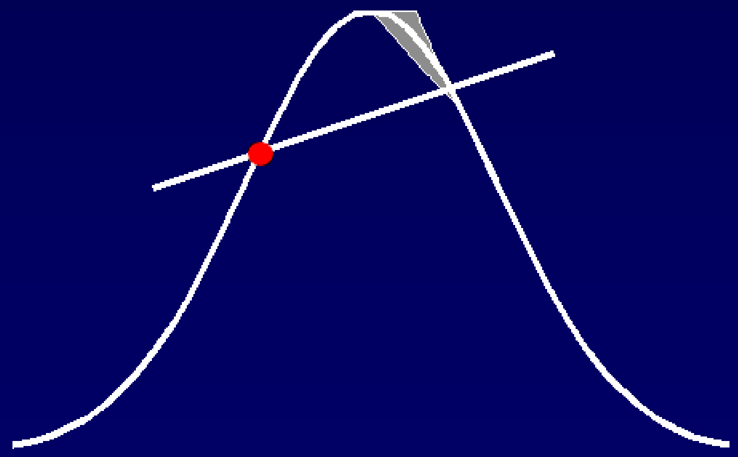

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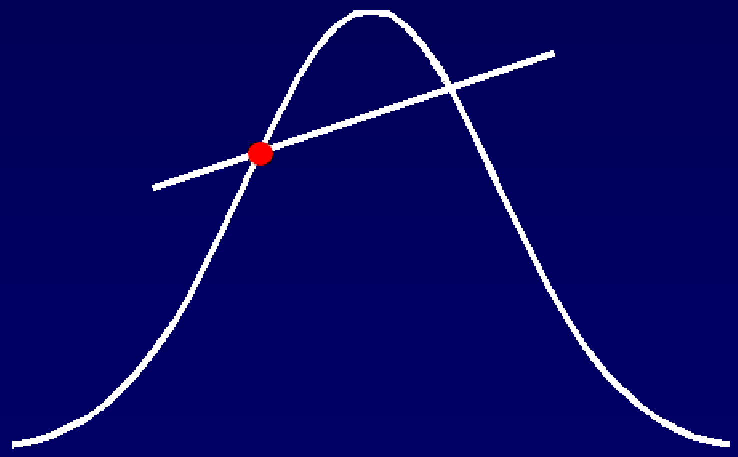

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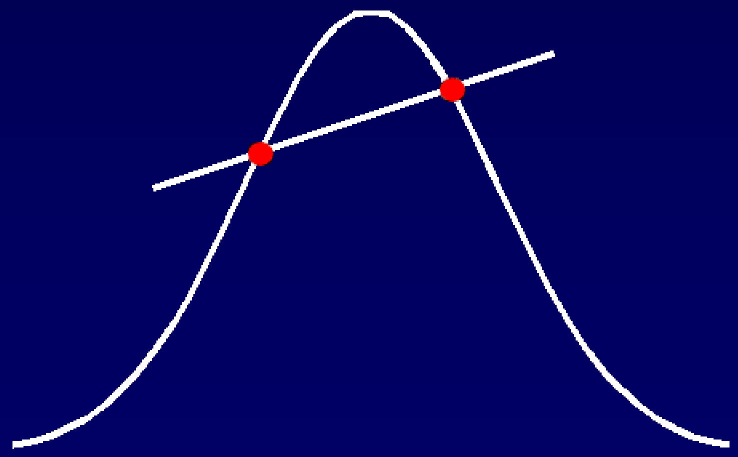

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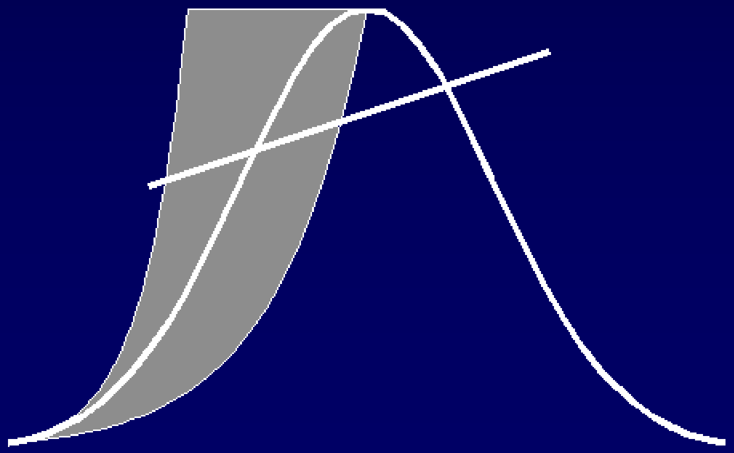

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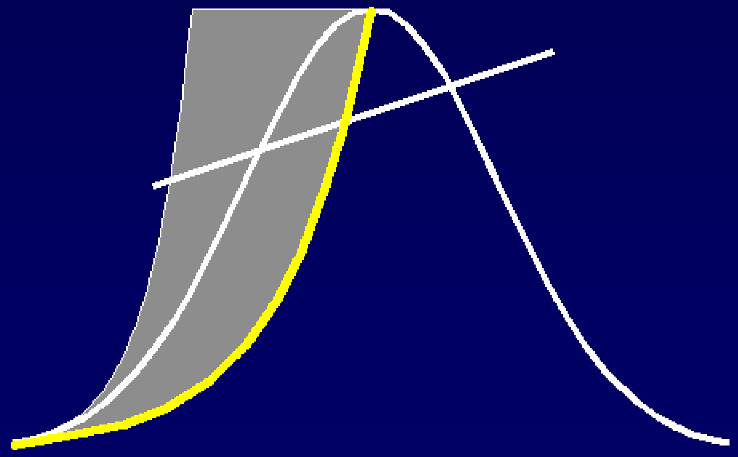

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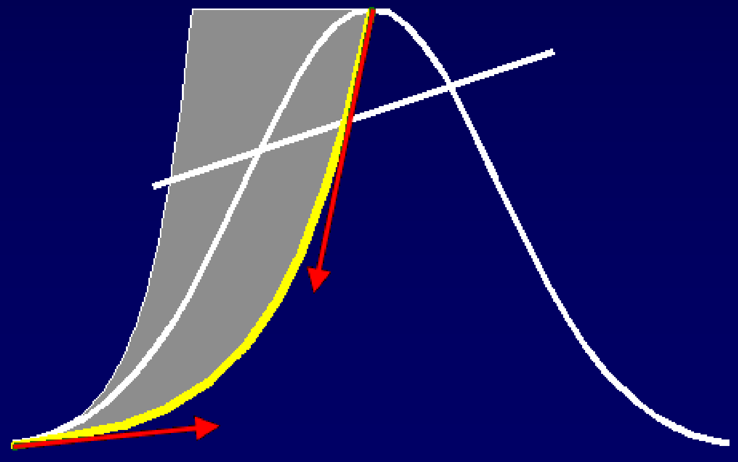

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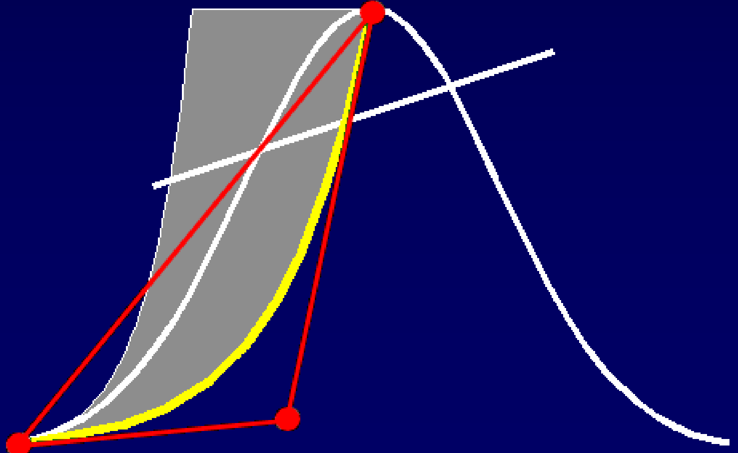

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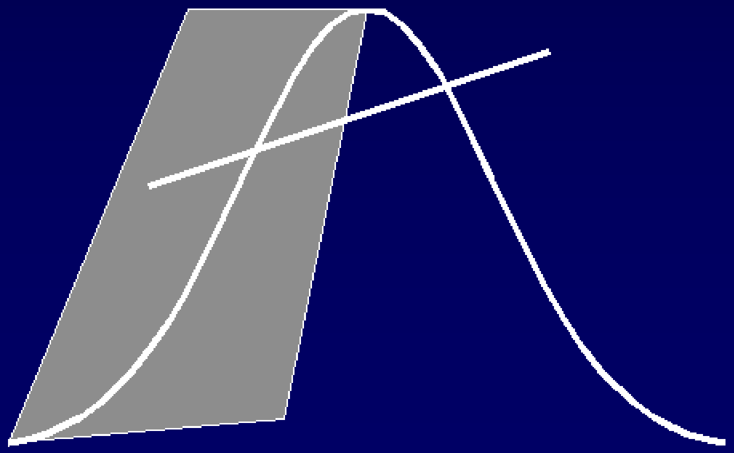

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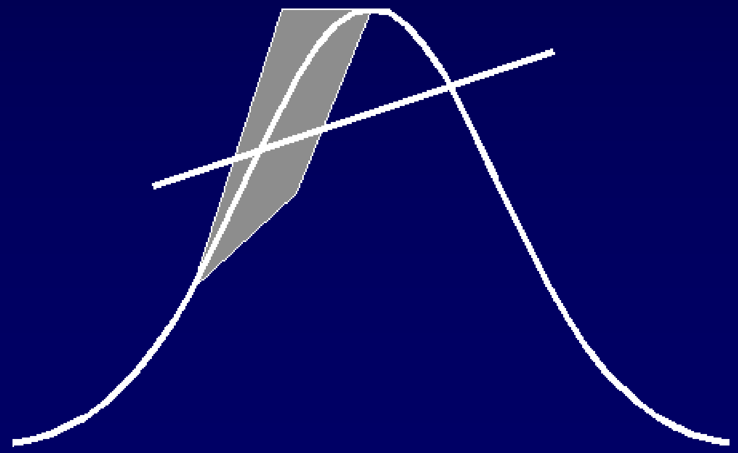

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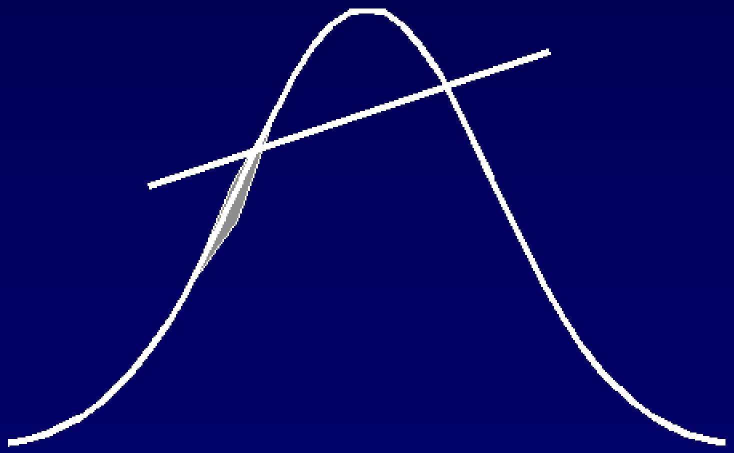

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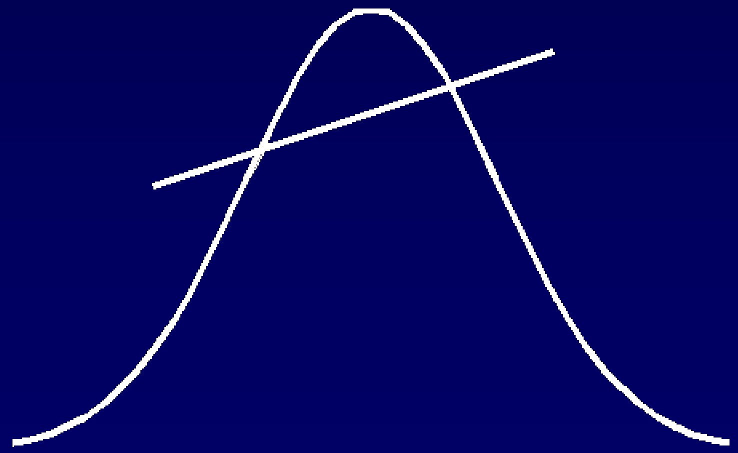

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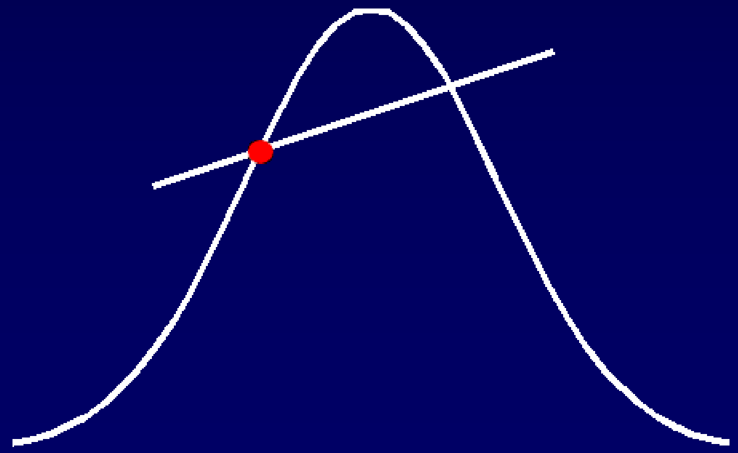

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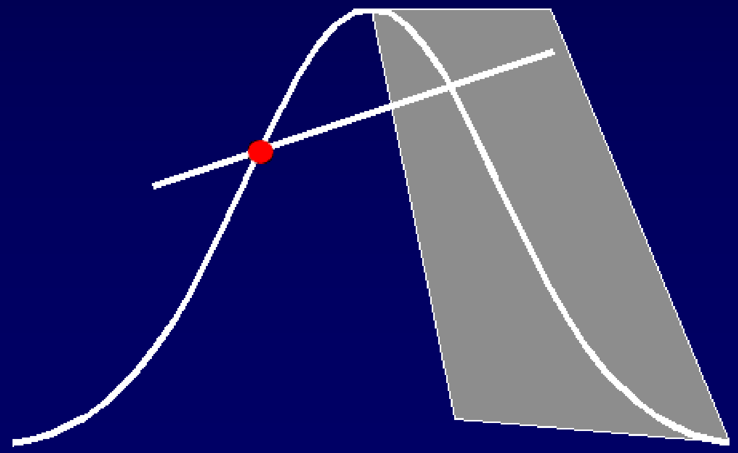

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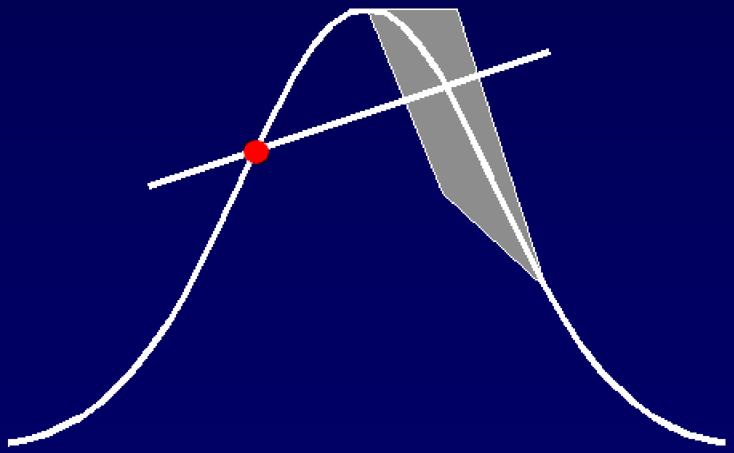

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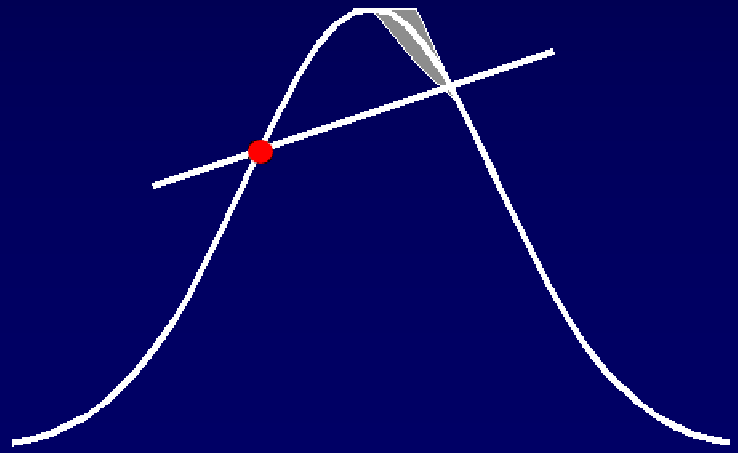

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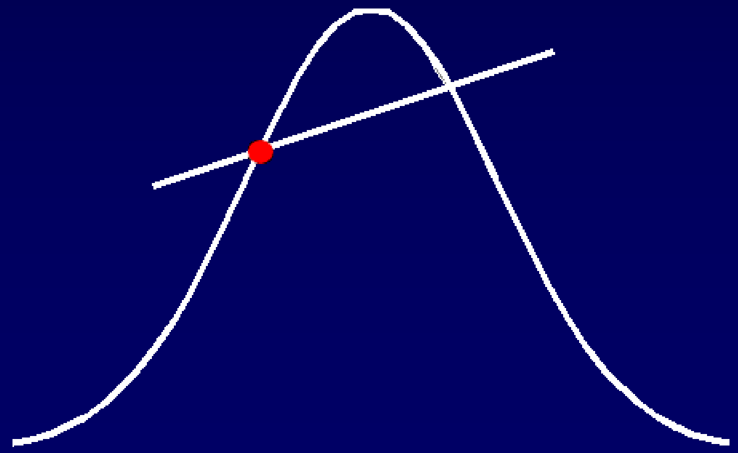

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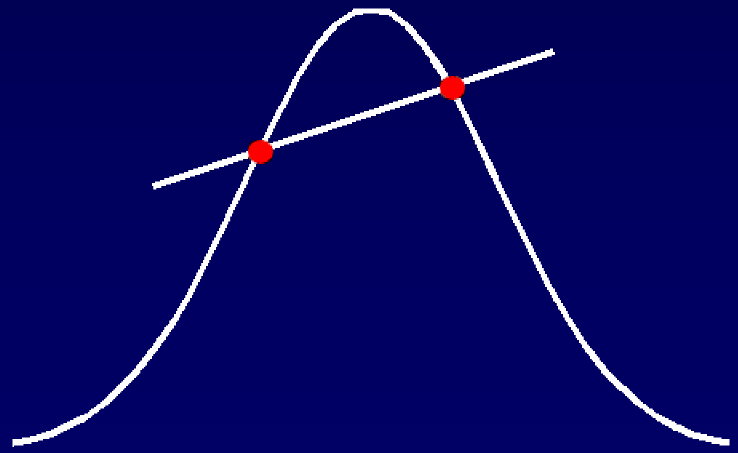

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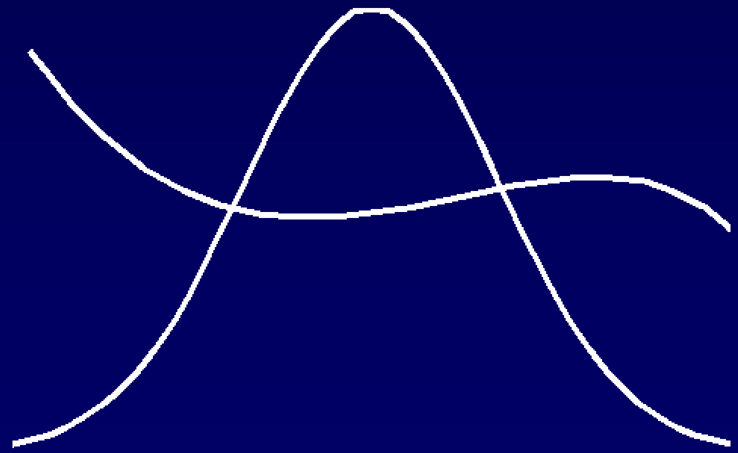

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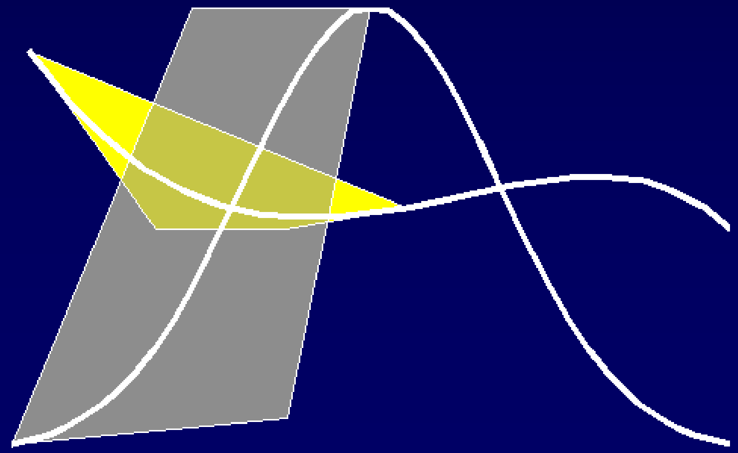

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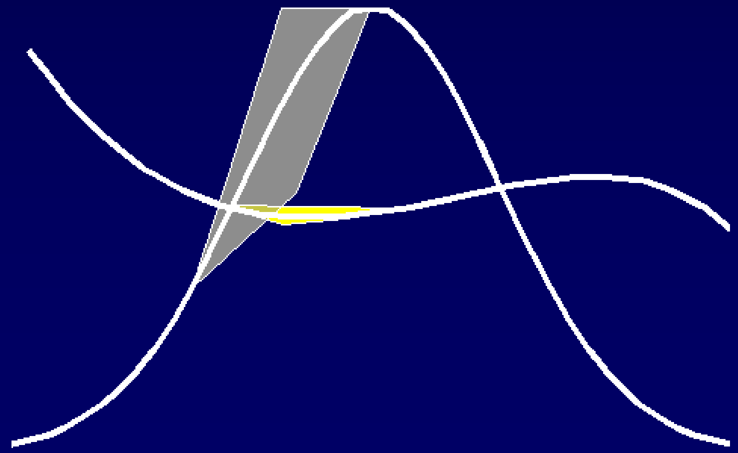

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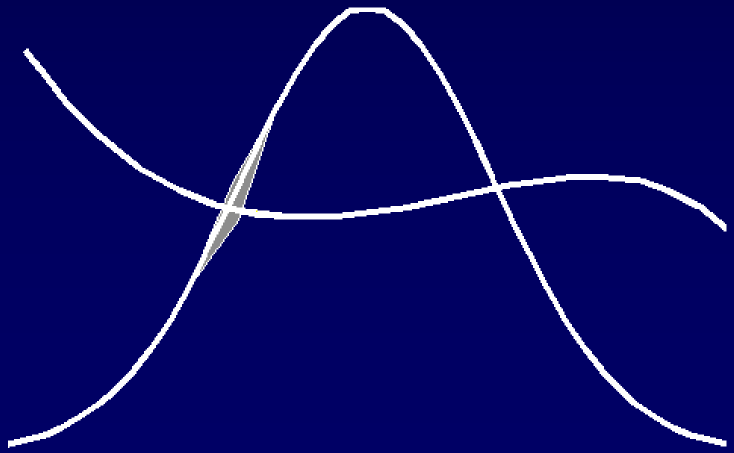

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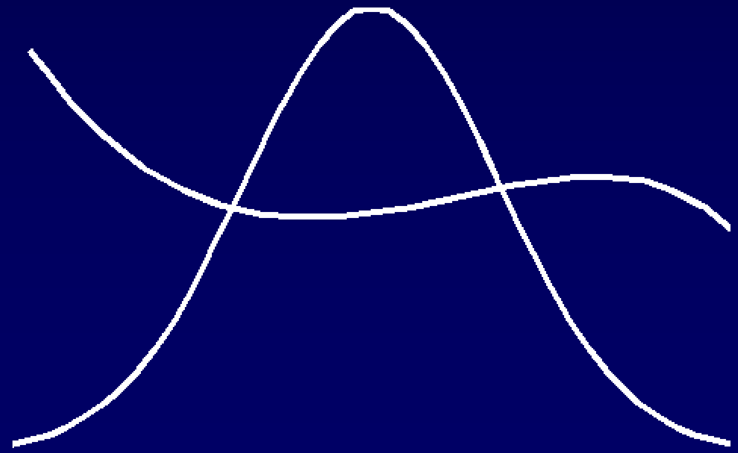

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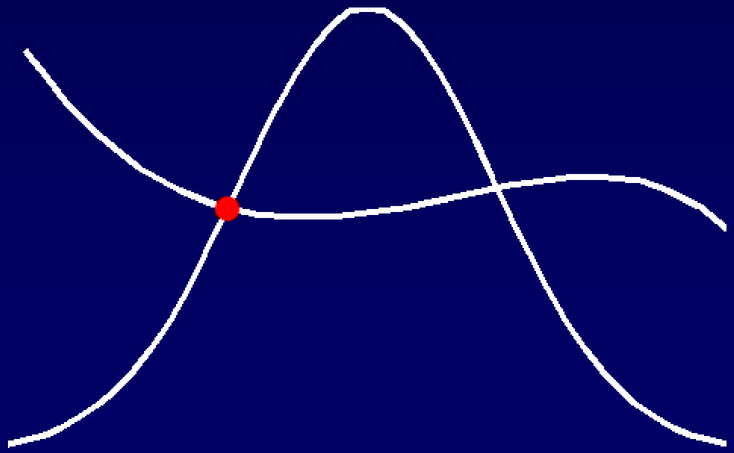

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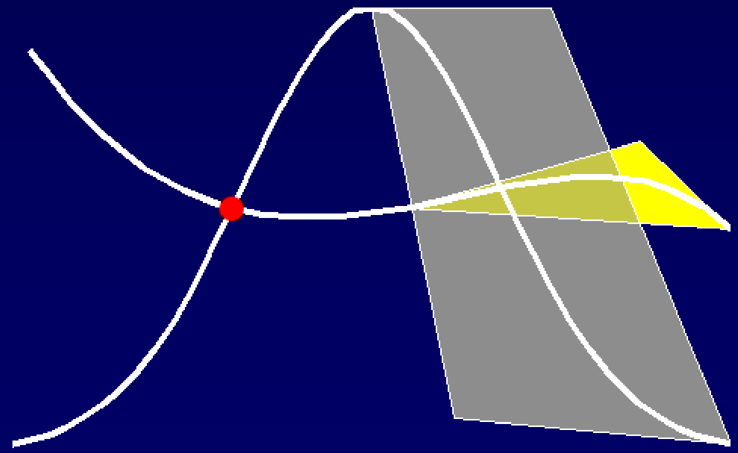

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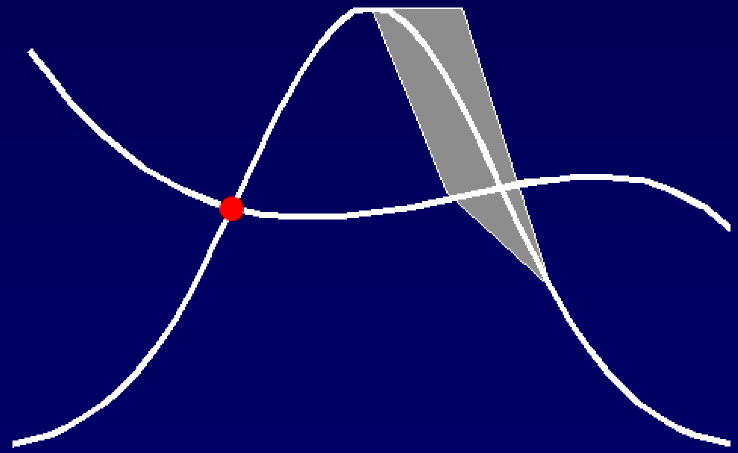

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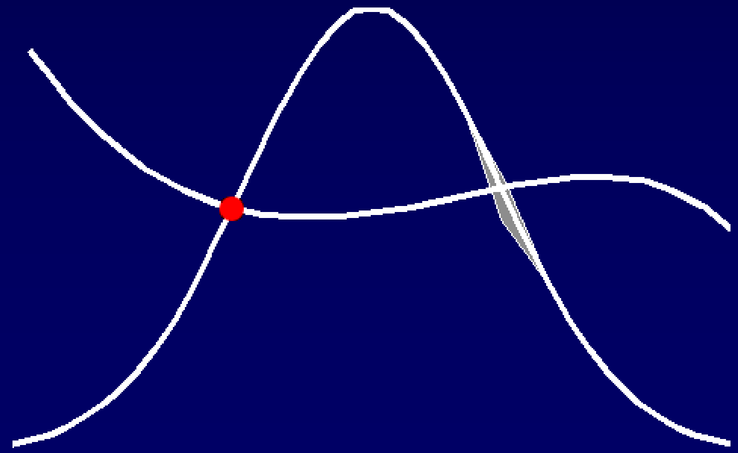

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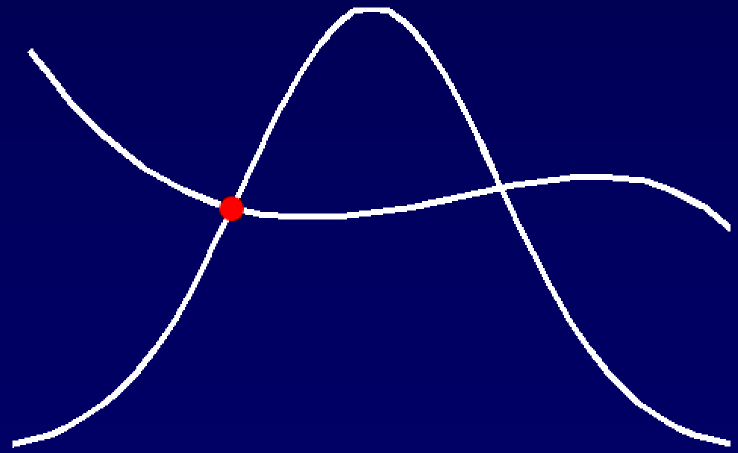

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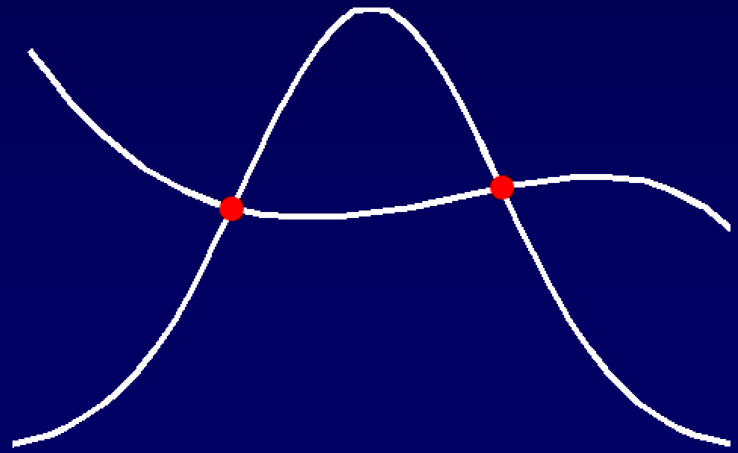

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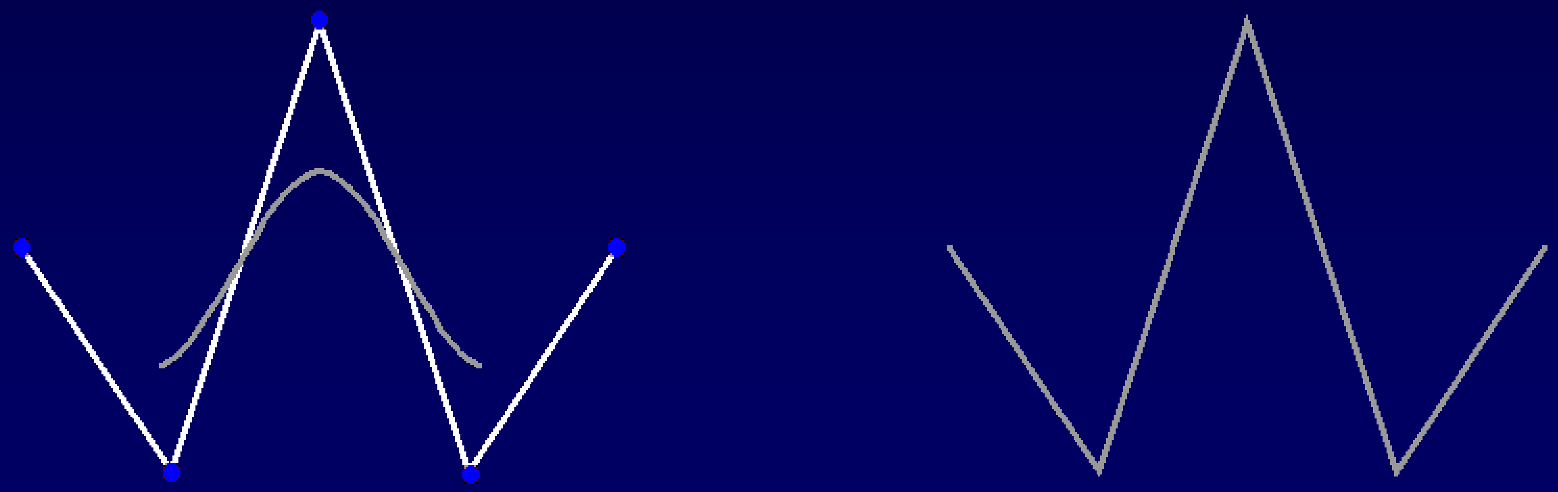

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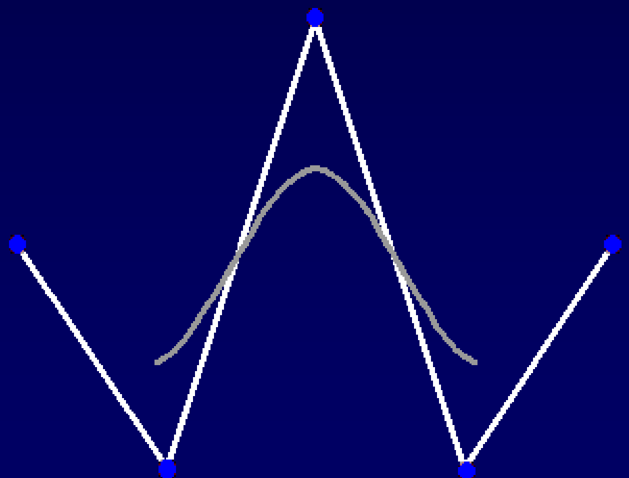

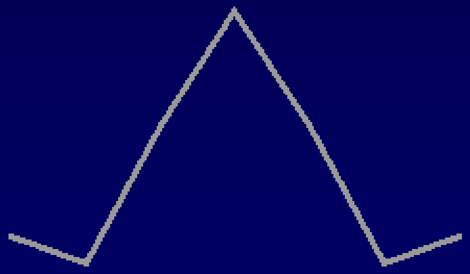

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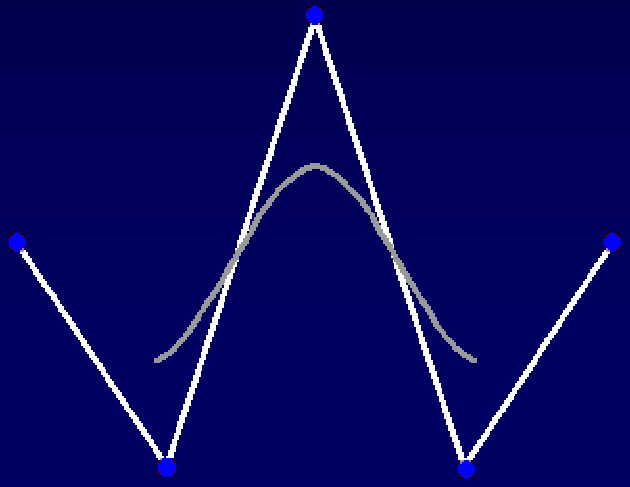

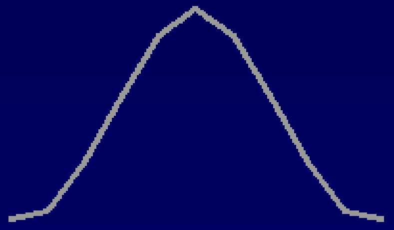

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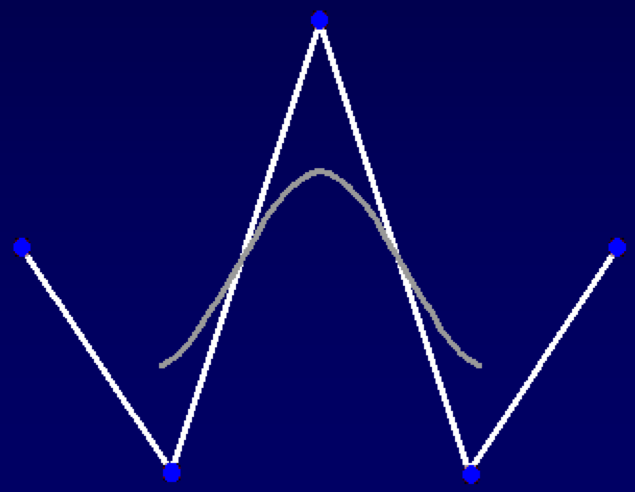

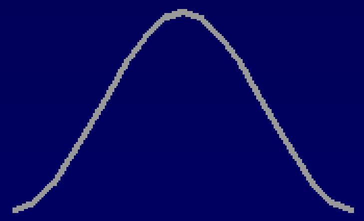

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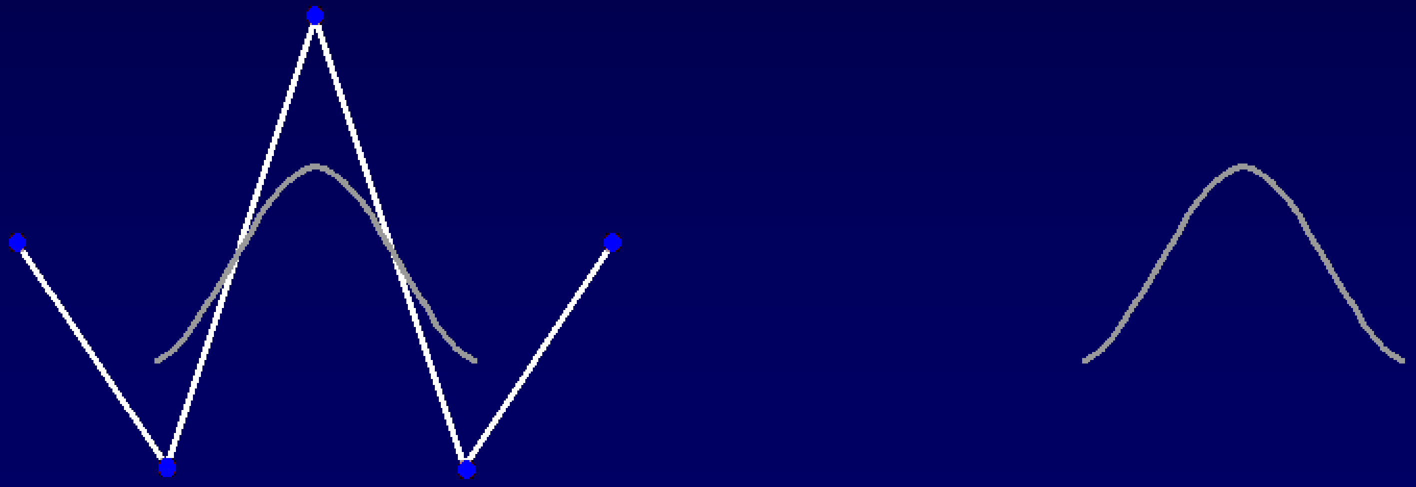

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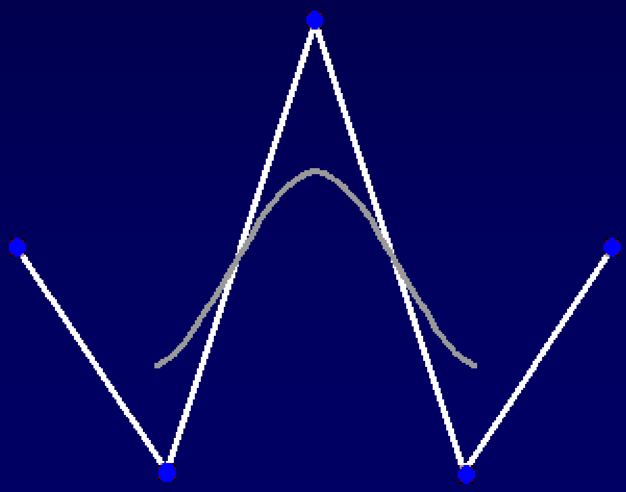

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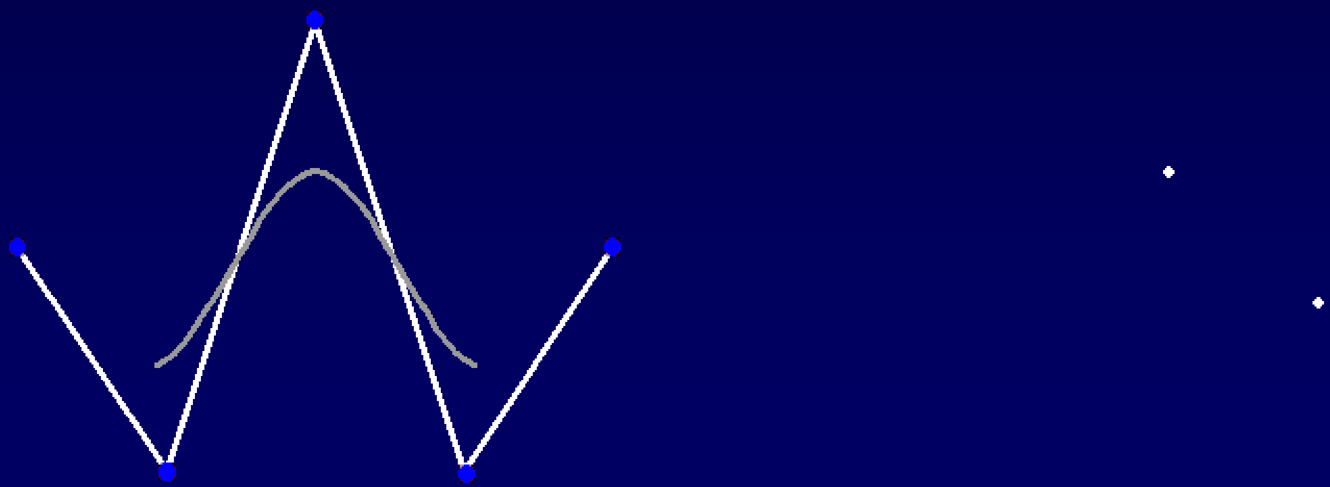

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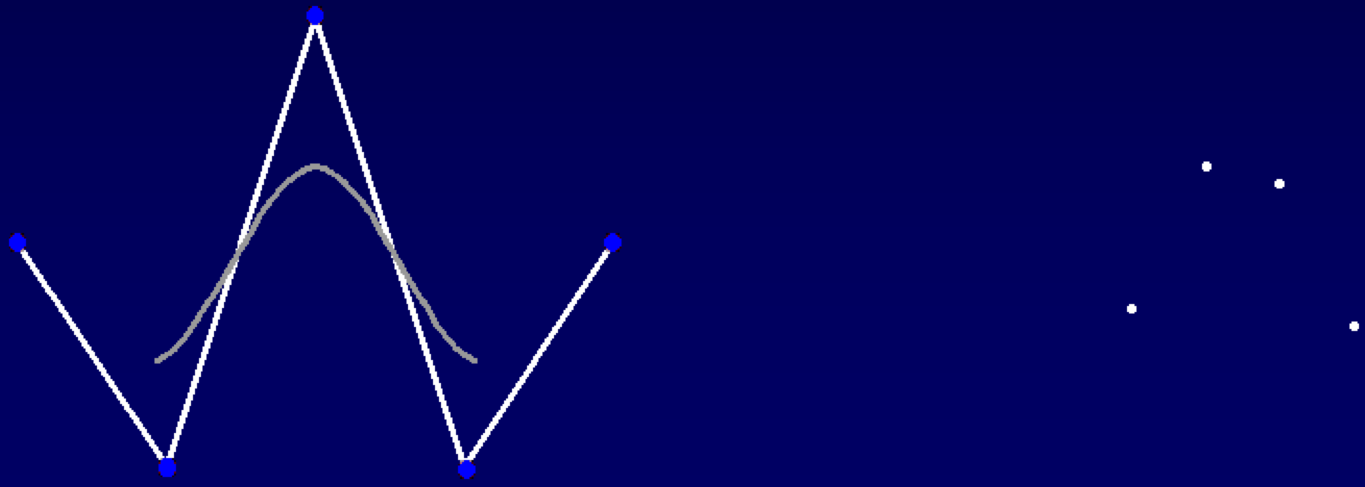

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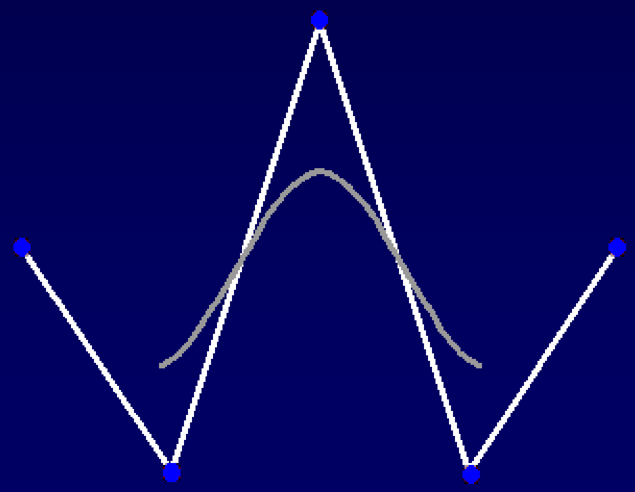

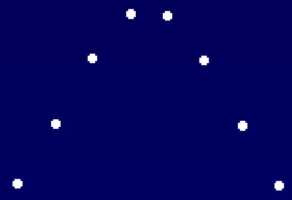

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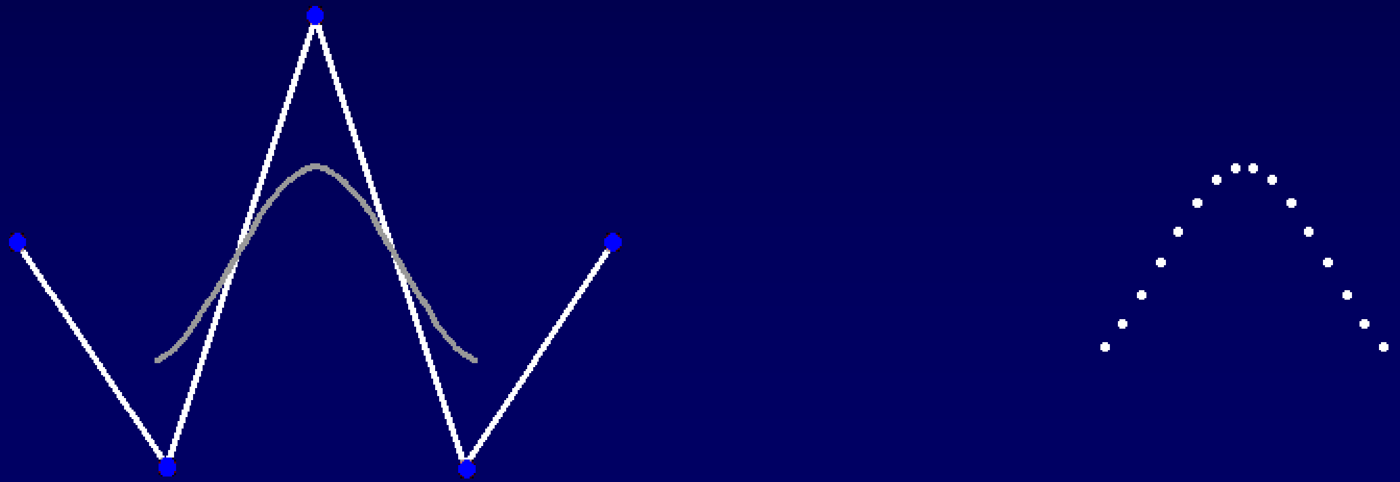

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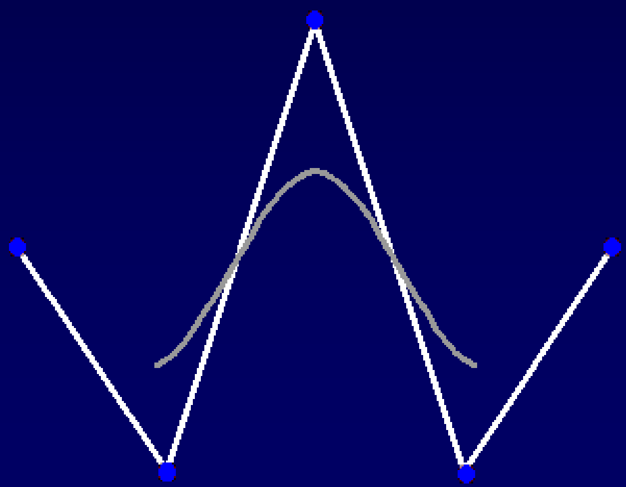

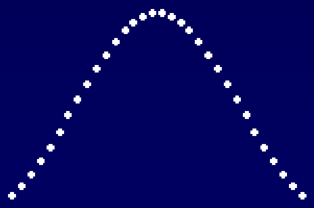

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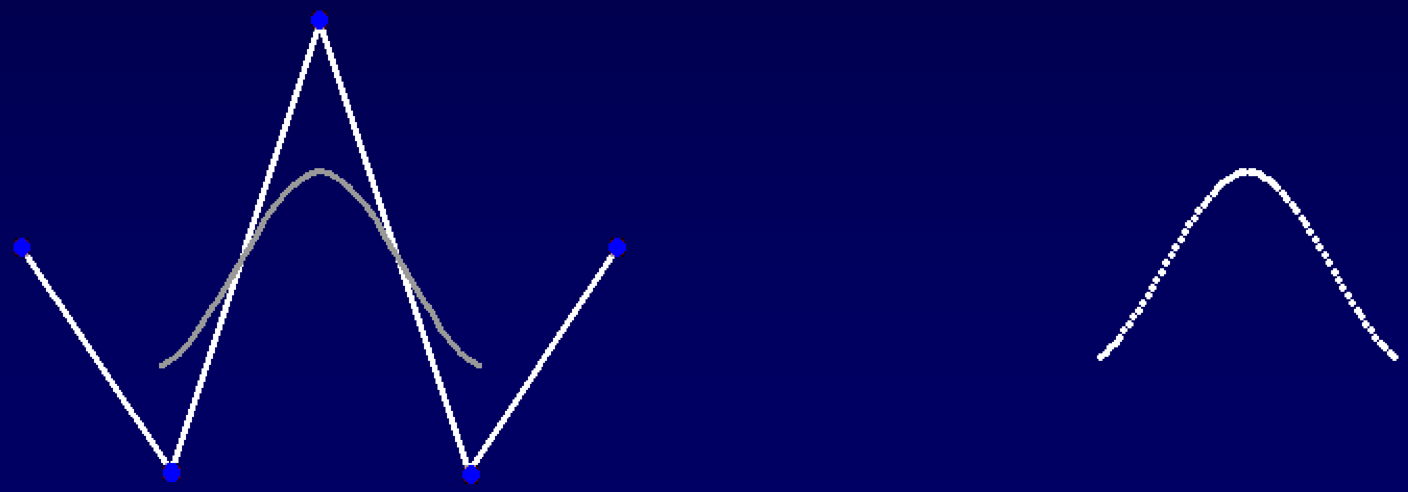

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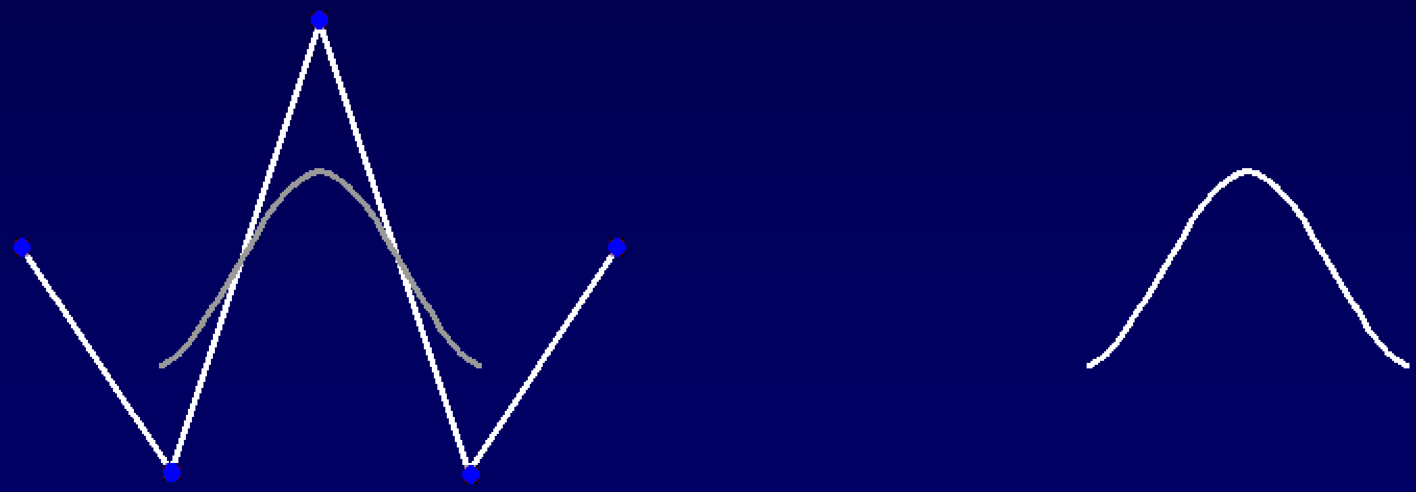

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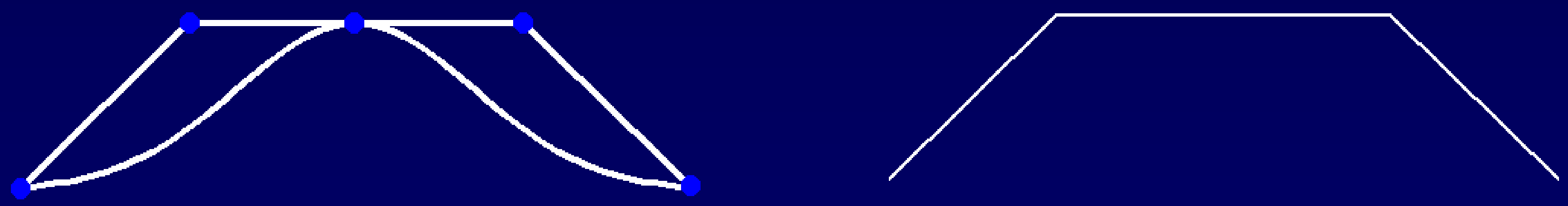

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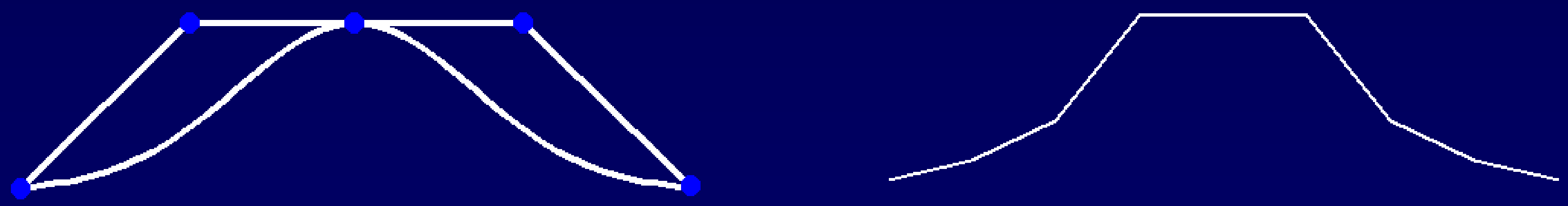

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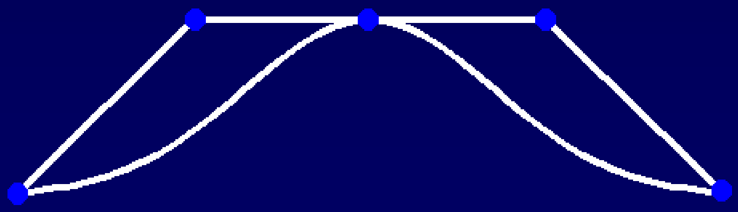

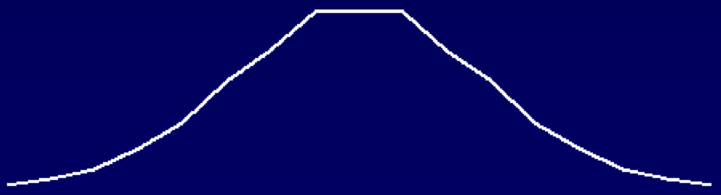

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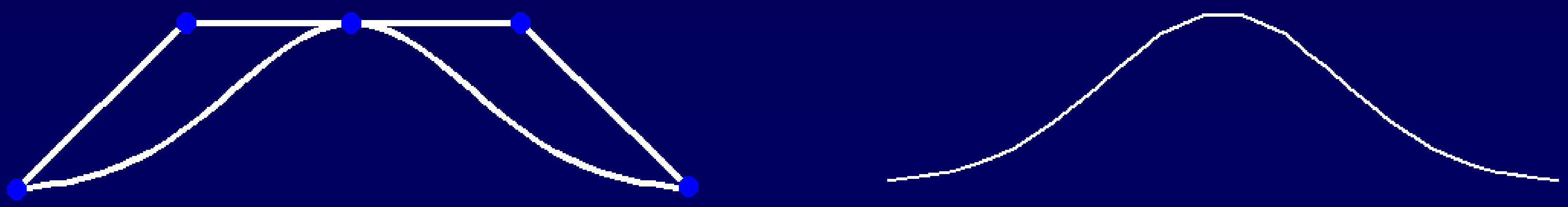

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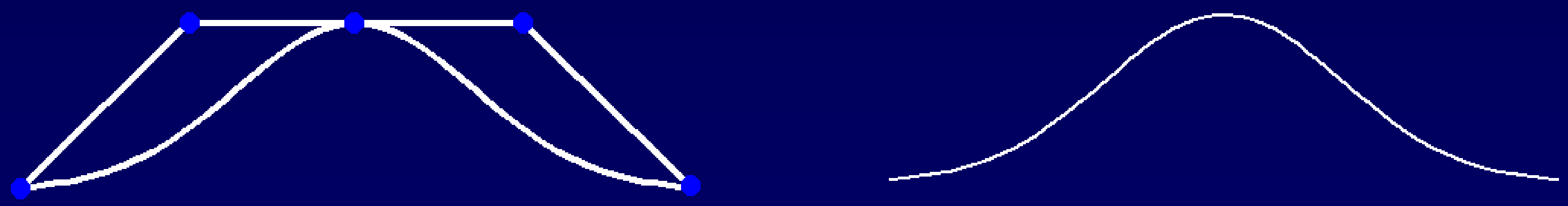

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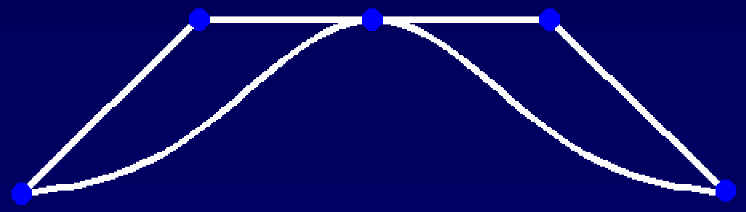

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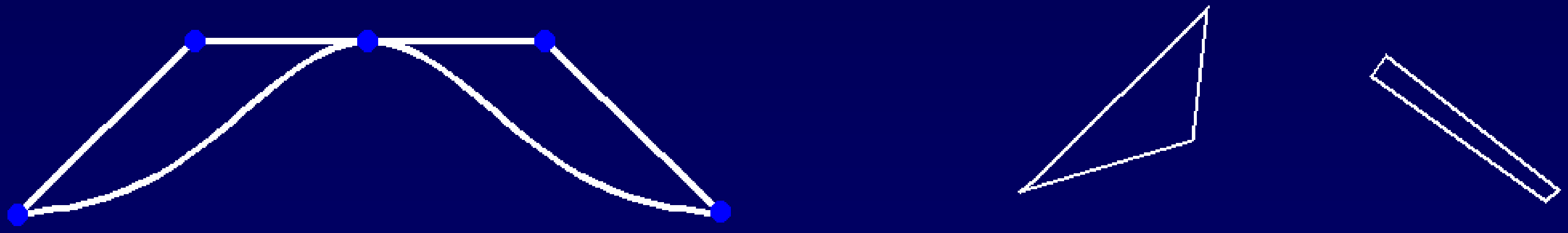

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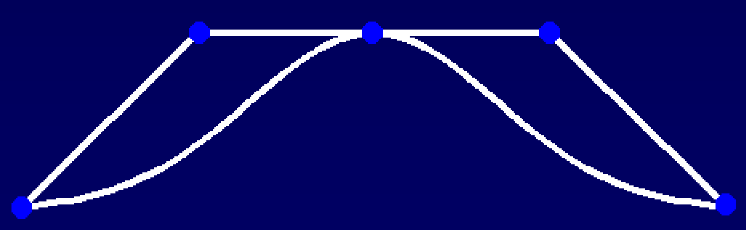
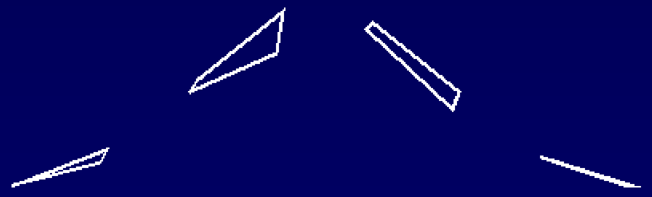

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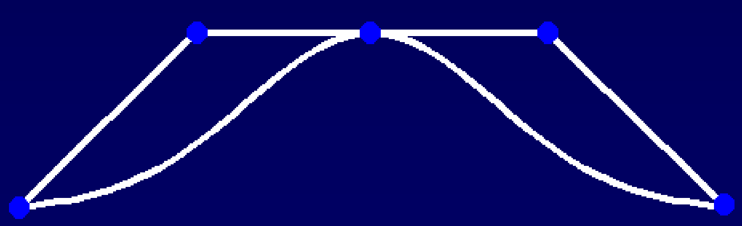

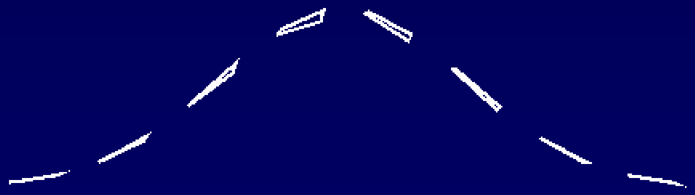

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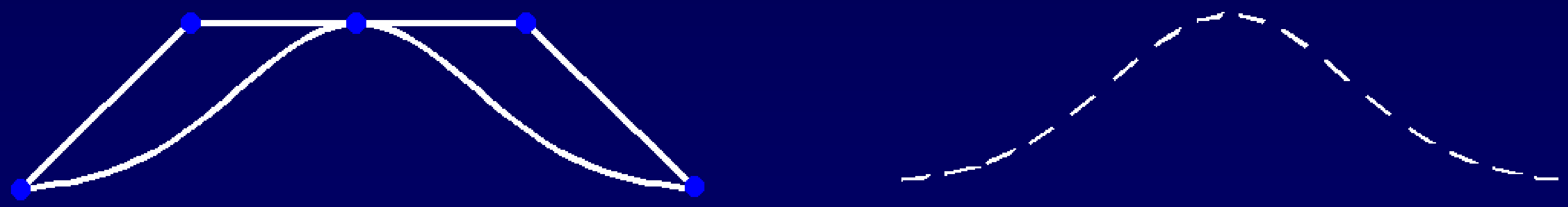

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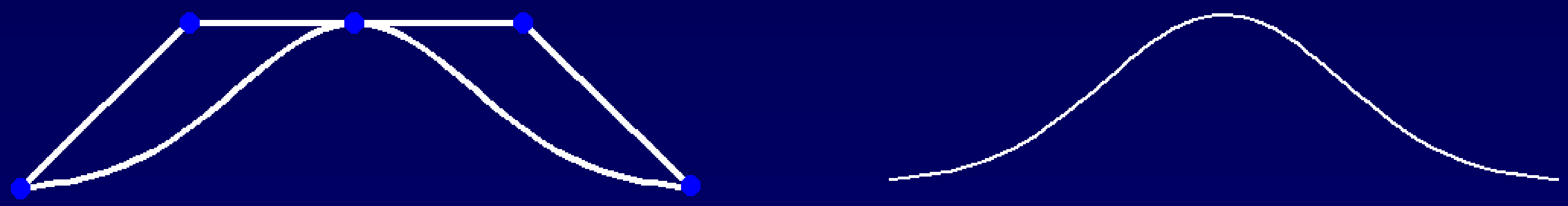

Future Work

- Other types of averaging rules (non-analytic)
  - Lofting curve networks
- Extensions to surfaces
  - Extraordinary points
- Slowing varying non-linear maps