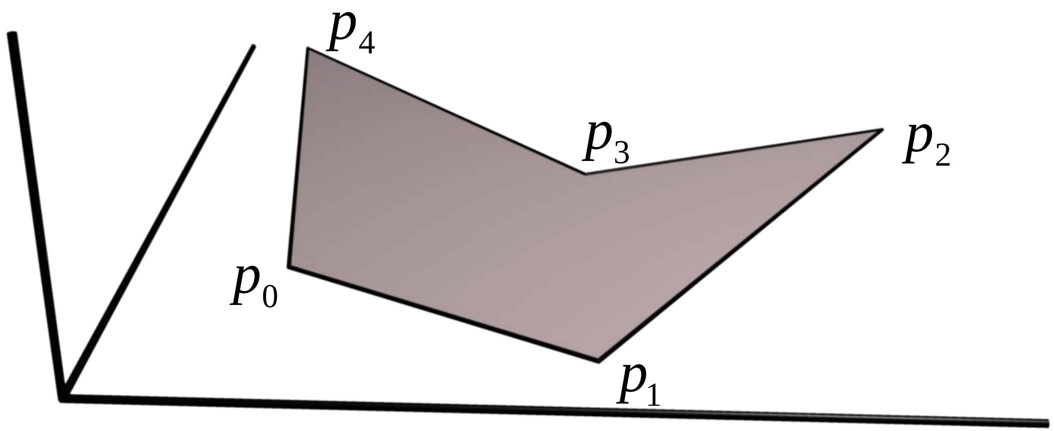
Moving Least Squares Coordinates

Josiah Manson and Scott Schaefer
Texas A&M University

Barycentric Coordinates

- Polygon Domain



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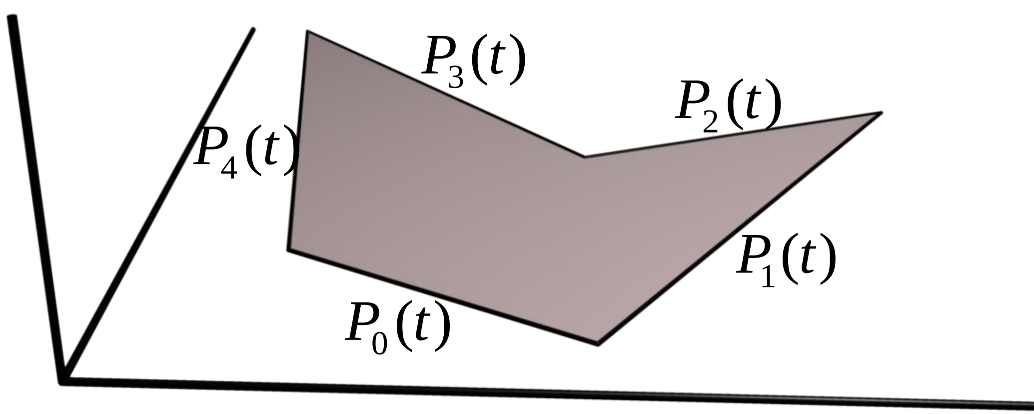

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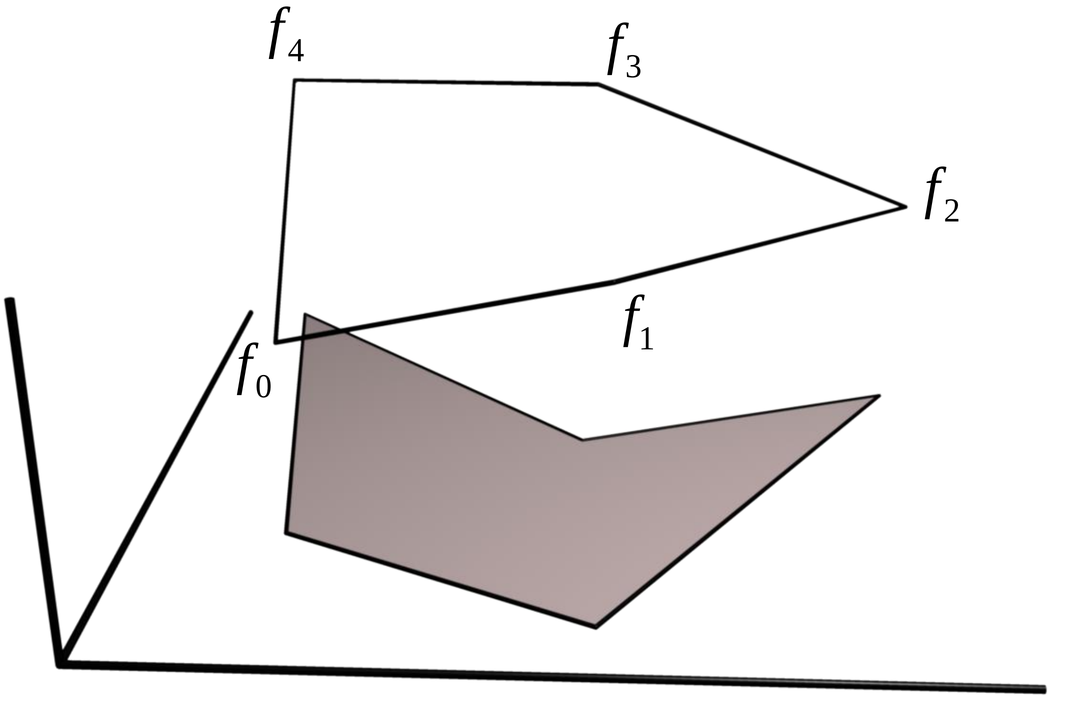

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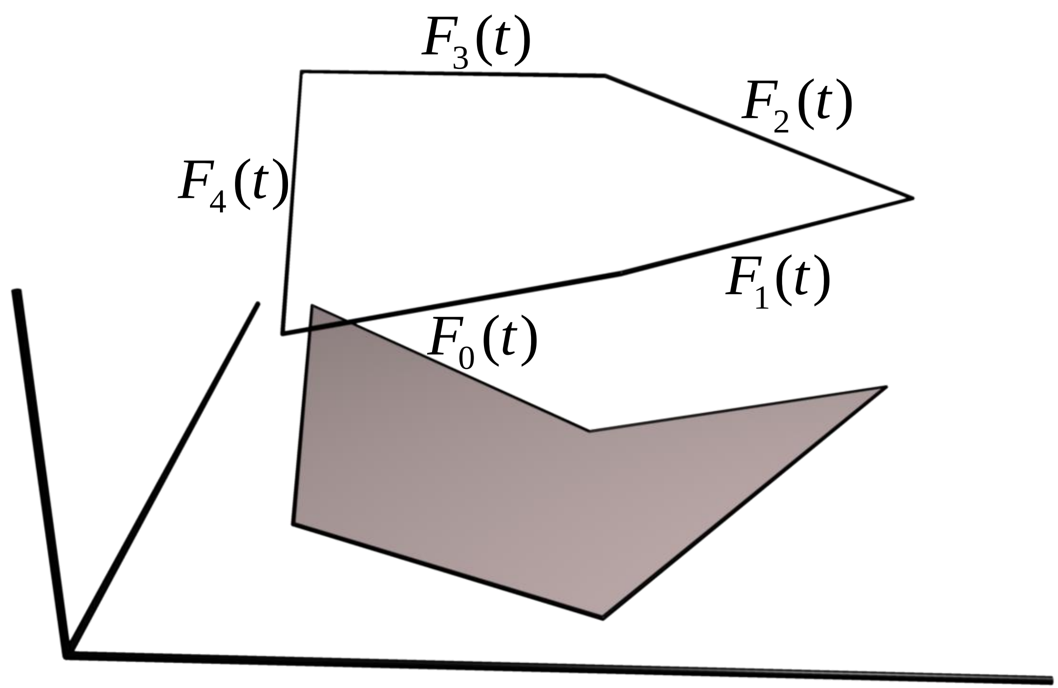
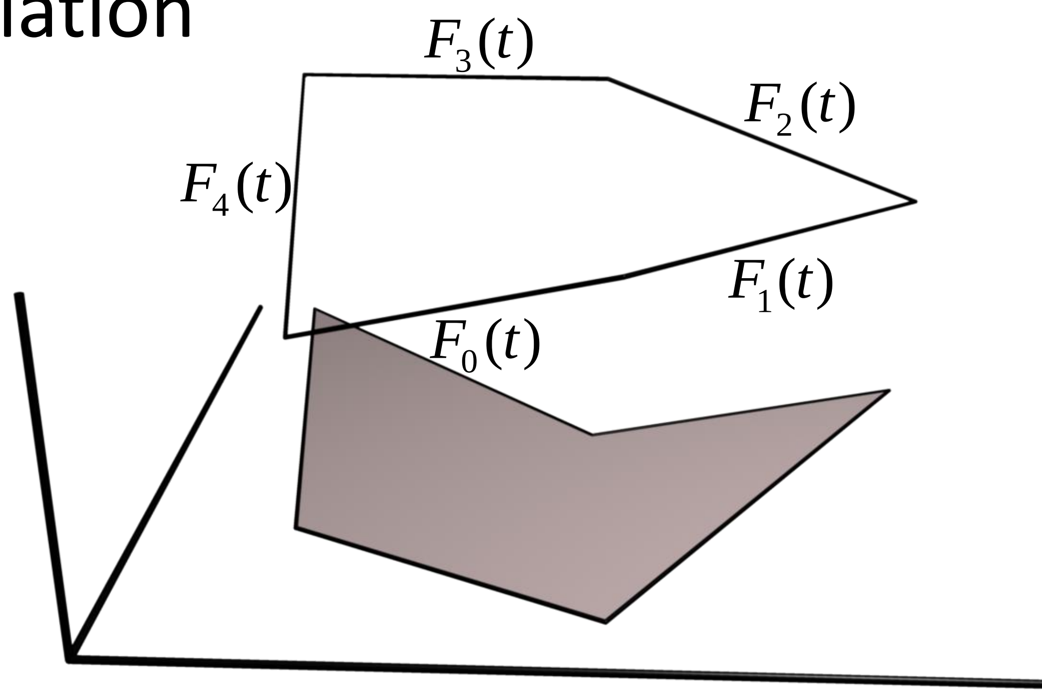


Barycentric Coordinates

- Polygon Domain
- Boundary Interpolation

$$\hat{F}(P_i(t)) = F_i(t)$$



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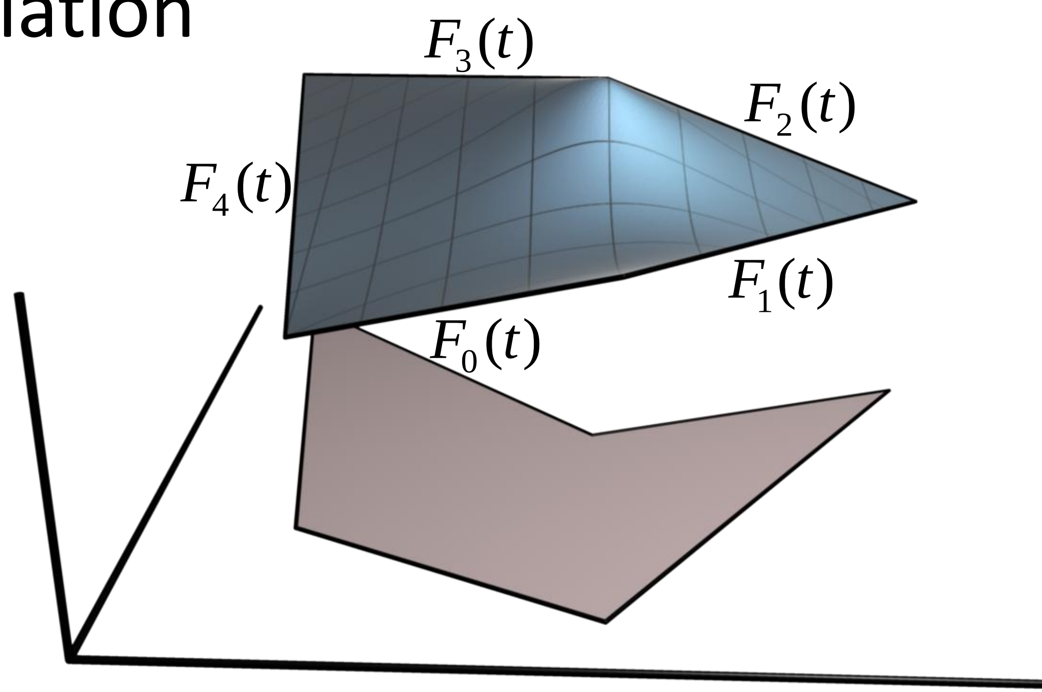
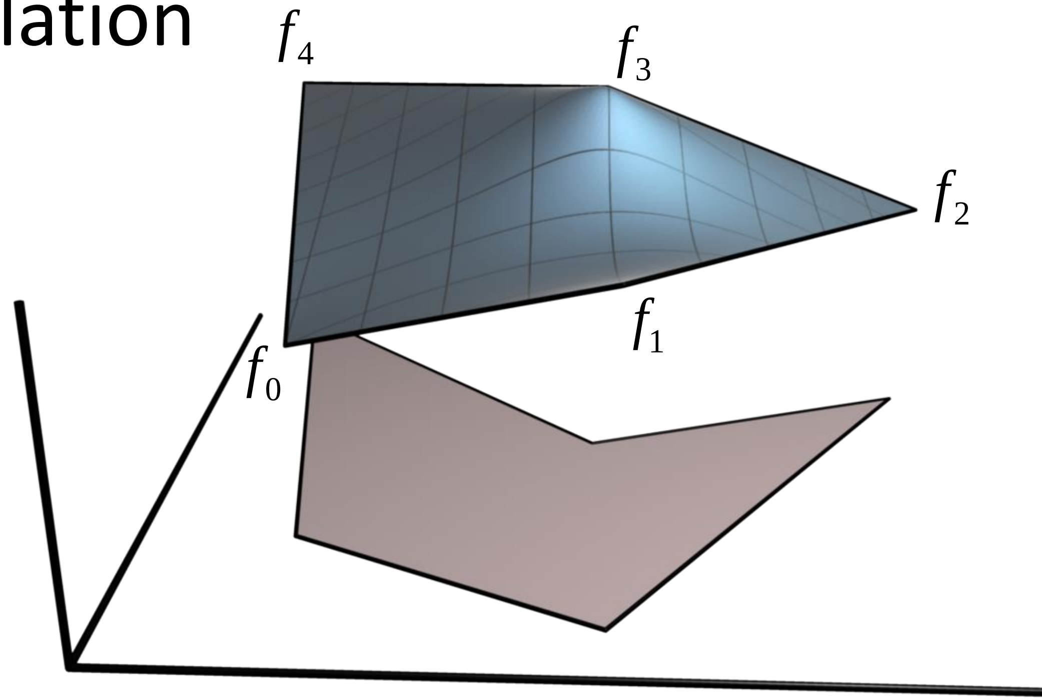


Barycentric Coordinates

- Polygon Domain
- Boundary Interpolation
- Basis Functions



$$\hat{F}(x) = \sum_{i}^{n} b_i(x) f_i$$

Barycentric Coordinates

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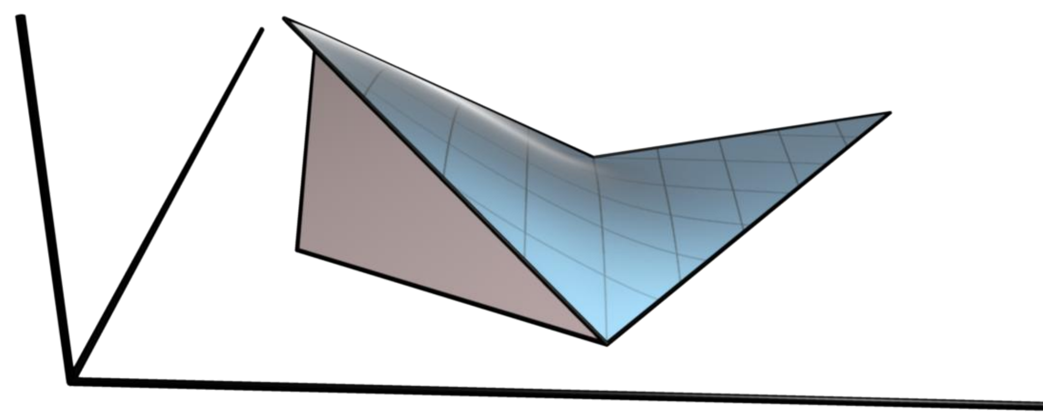

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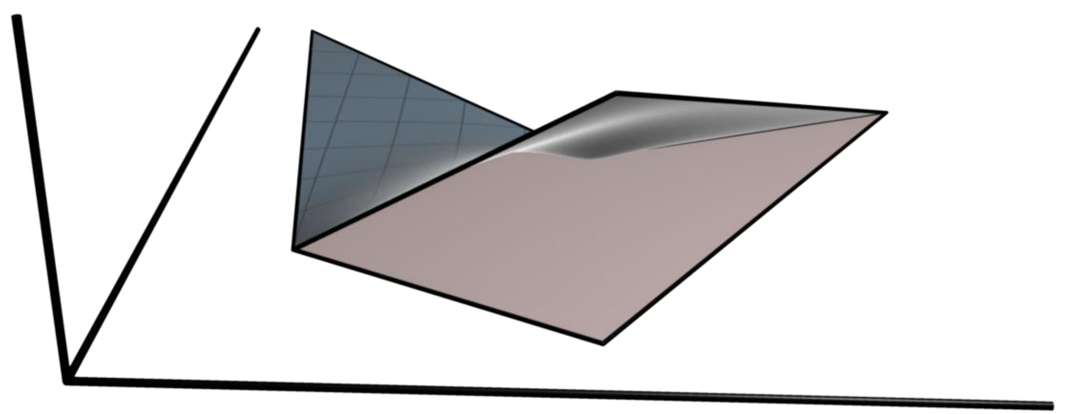

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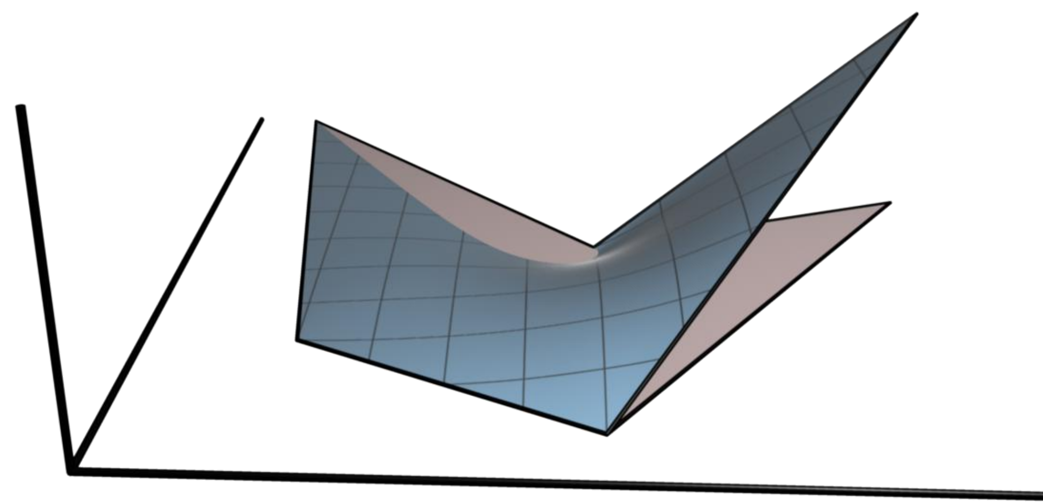

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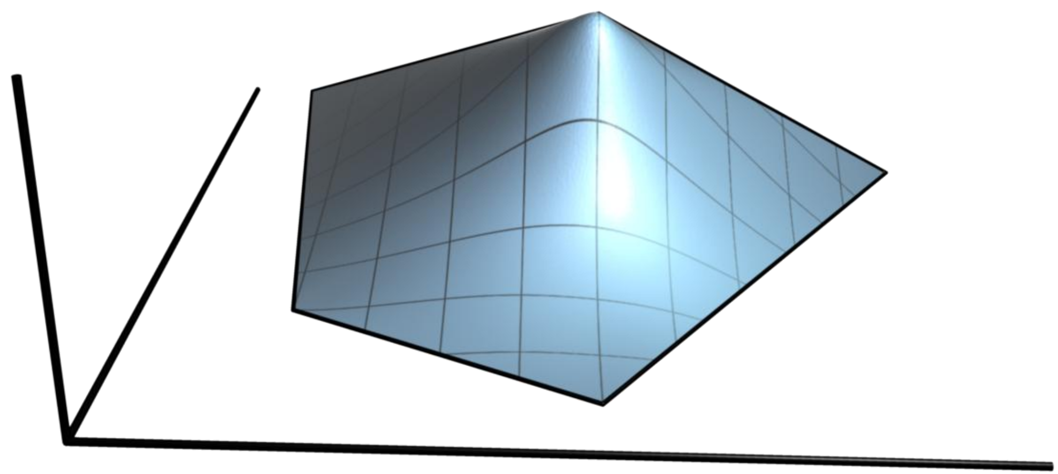

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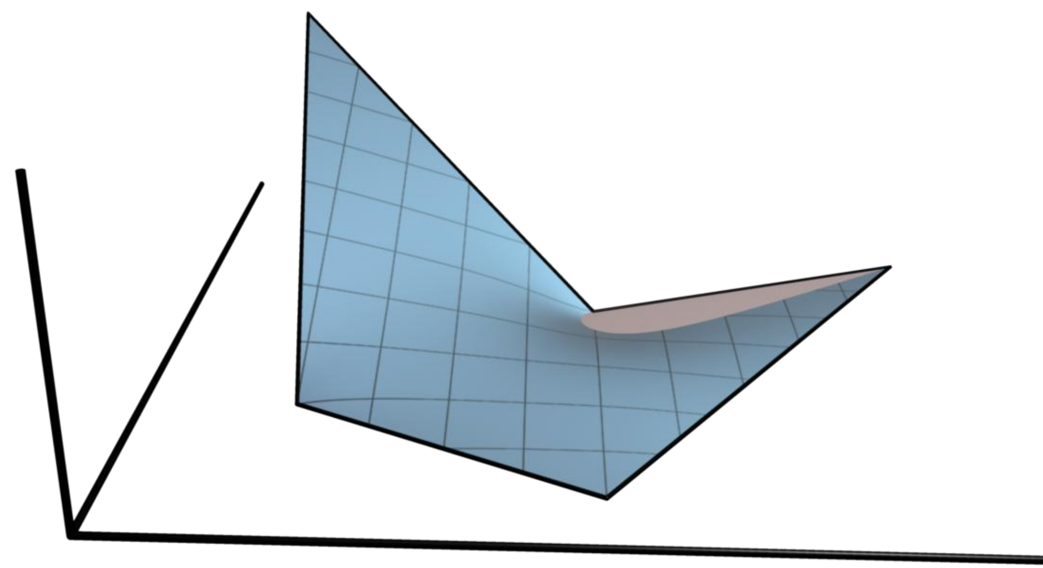
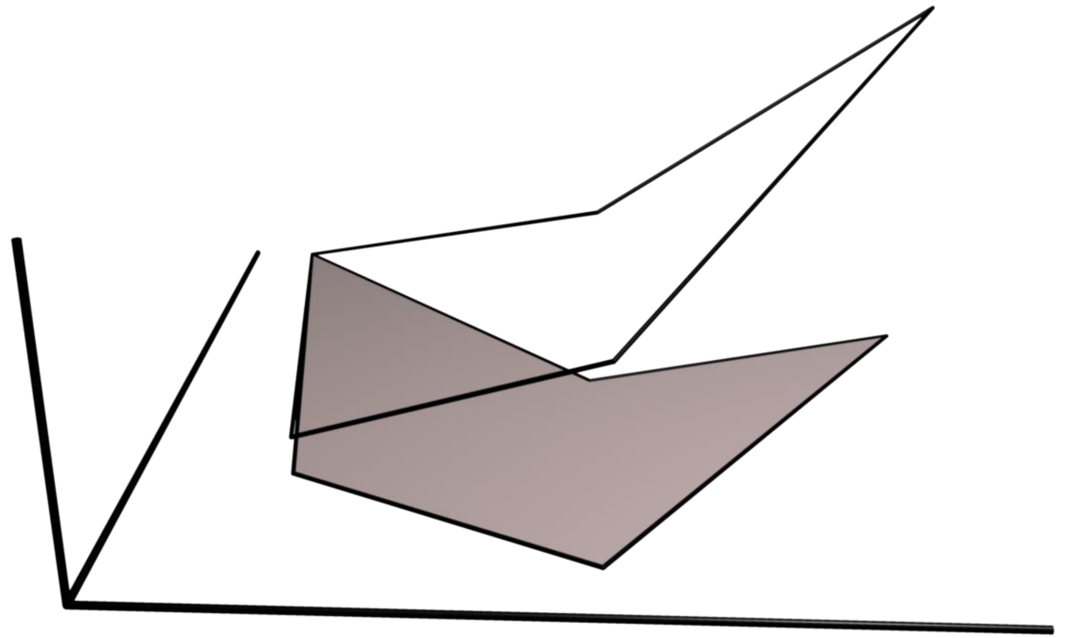


Barycentric Coordinates

- Polygon Domain
- Boundary Interpolation
- Basis Functions
- Linear Precision

$$L(x) = \sum_{i}^{n} b_i(x) L(p_i)$$



Barycentric Coordinates

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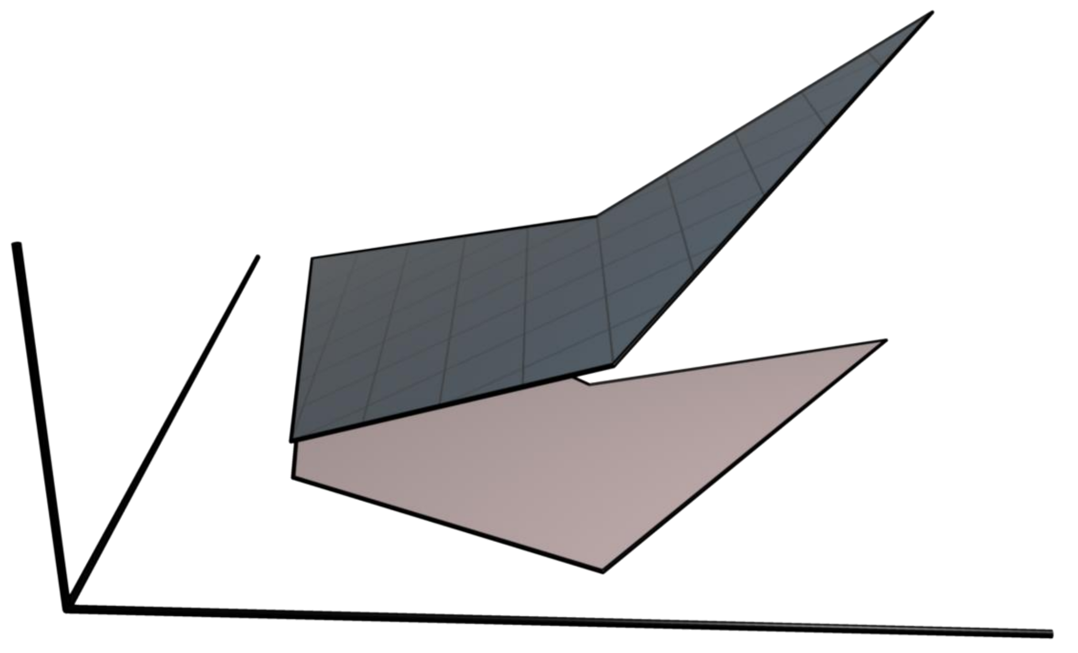

Barycentric Coordinates

- Polygon Domain
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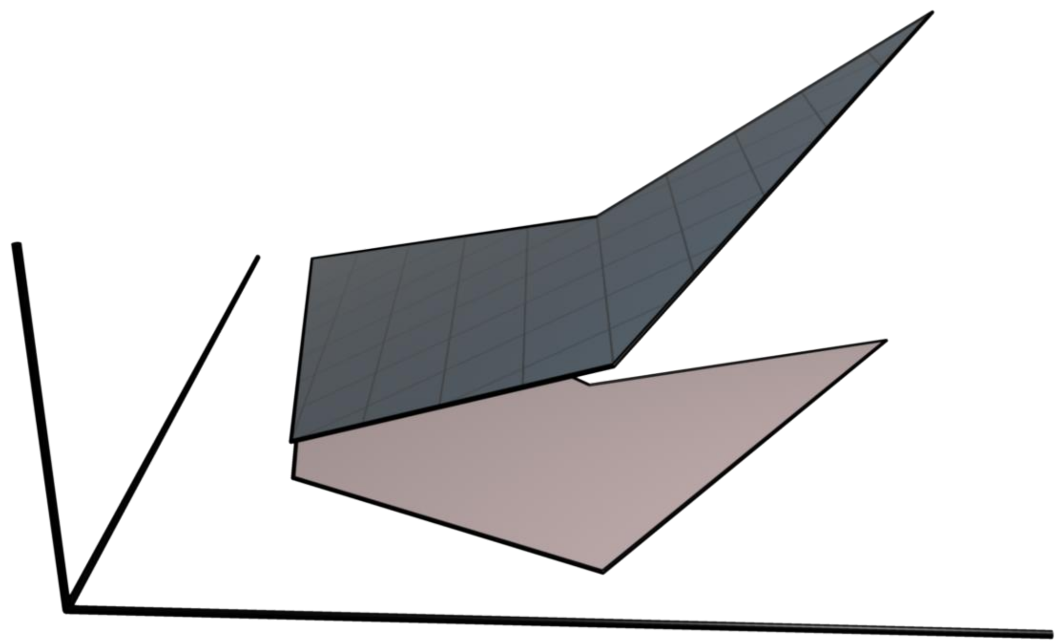
- Linear Precision

$$1 = \sum_{i}^{n} b_i(x)$$

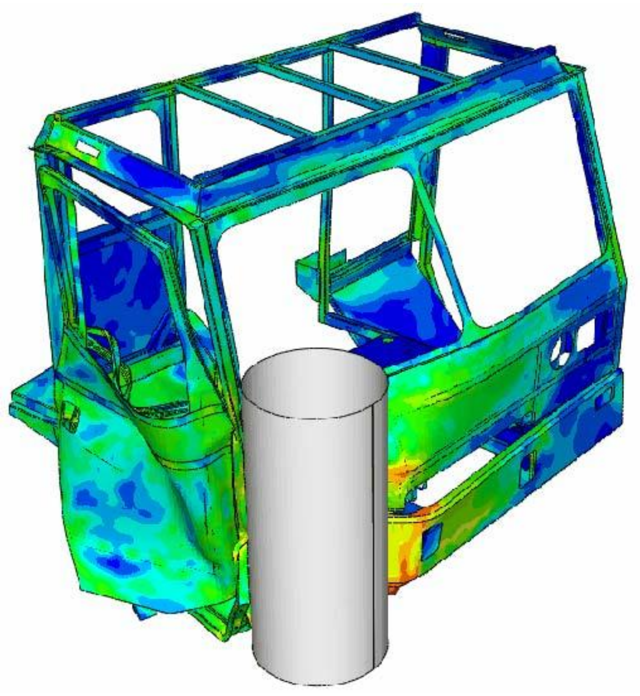


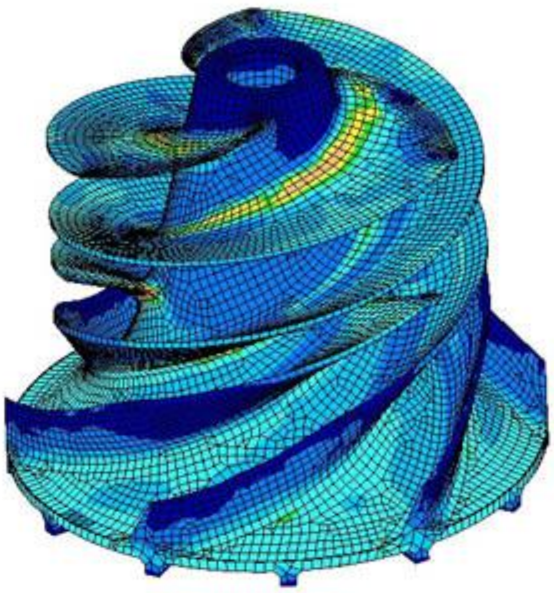
Other Properties

- Desirable Features
  - Smoothness
  - Closed-form solution
  - Positivity
- Extended Coordinates
  - Polynomial Boundary Values
  - Polynomial Precision
  - Interpolation of Derivatives
  - Curved Boundaries

Applications

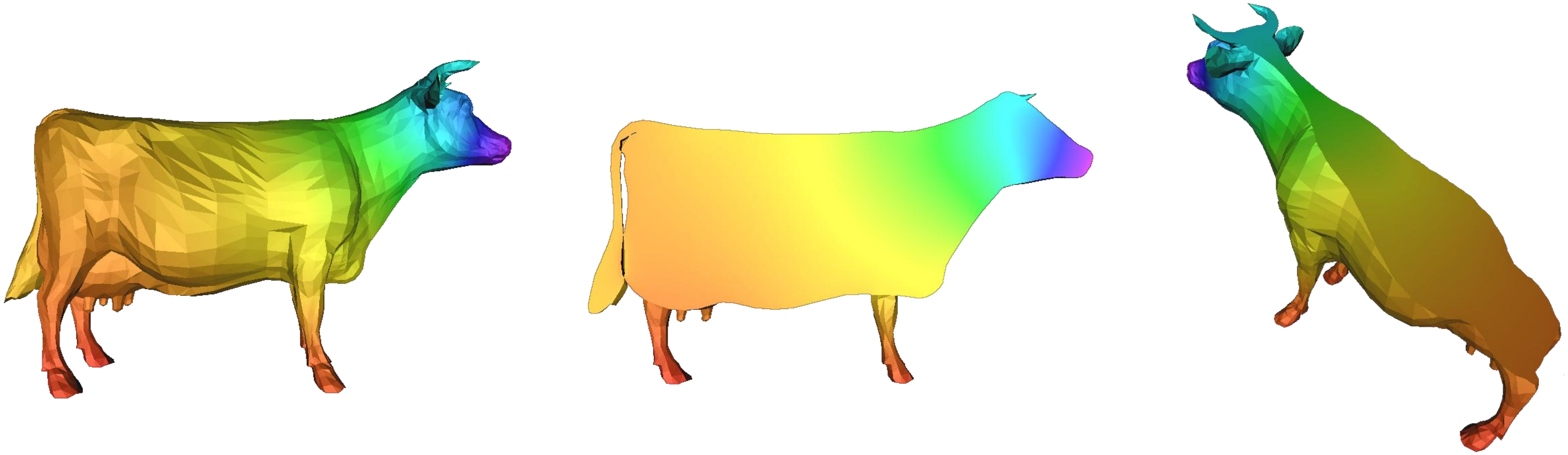
- Finite Element Methods [Wachspress 1975]





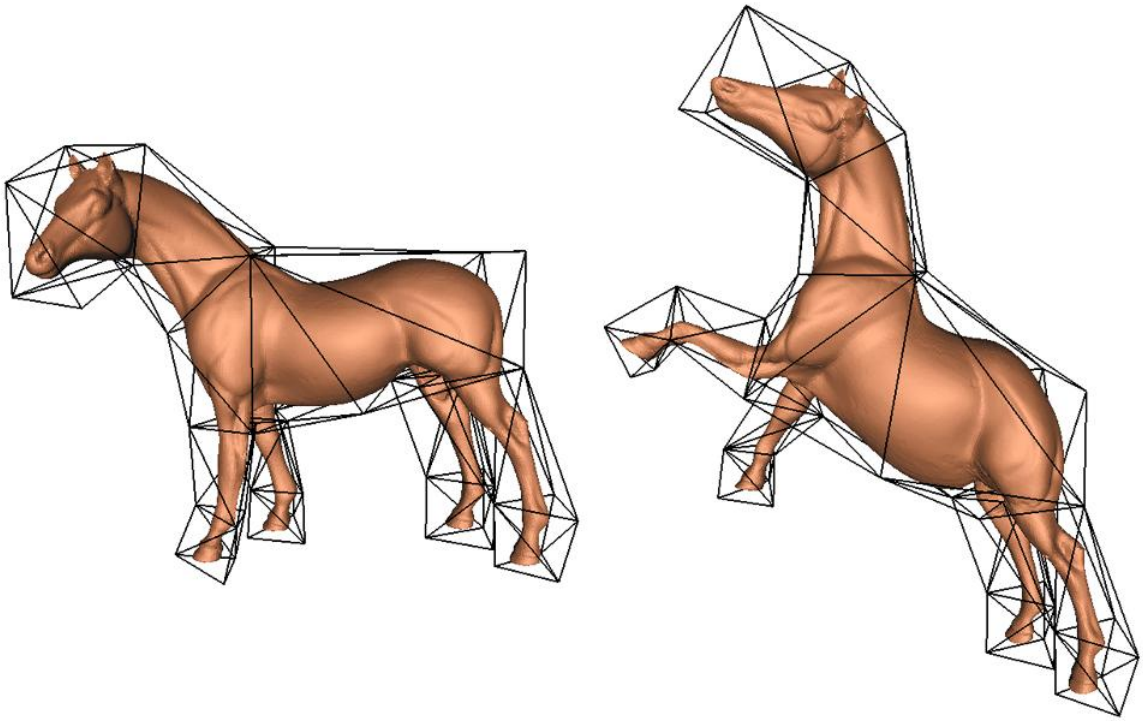
Applications

- Boundary Value Problems
  [Ju et al. 2005]



Applications

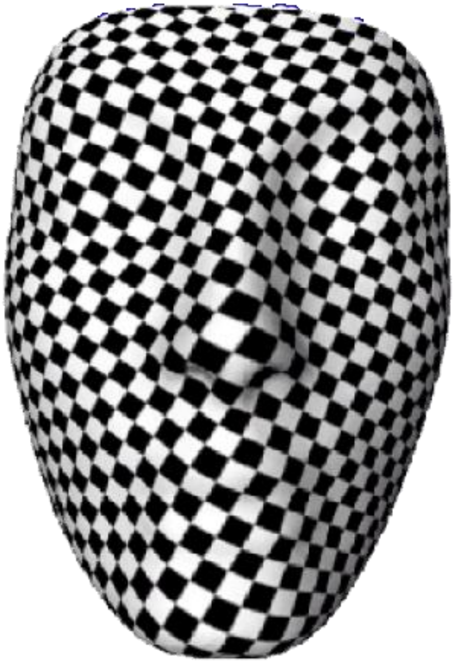
- Free-Form Deformations
  [Sederberg et al. 1986], [MacCracken et al. 1996], [Ju et al. 2005], [Joshi et al. 2007]



Applications

- Surface Parameterization

[Hormann et al. 2000], [Desbrun et al. 2002]



Comparison of Methods

	Concave Shapes	Closed-Form	Smooth	Positive	Poly. Boundary	Derivatives	Poly. Precision	Open Boundaries
Wachspress	✗	✓	✓	✗	✗	✗	✗	✗
Mean Val.	✓	✓	✓	✗	✗	✗	✗	✗
Pos. Mean Val.	✓	✗	✗	✓	✗	✗	✗	✗
Max Entropy	✓	✗	✓	✓	✗	✗	✗	✗
Harmonic	✓	✗	✓	✓	✗	✗	✗	✗
Hermite MVC	✓	✗	✓	✗	✓	✓	*	✗
Moving Least Sqr.	✓	✓	✓	✗	✓	✓	✓	✓

Moving Least Squares Coordinates

- A new family of barycentric coordinates
- Solves a least squares problem
- Solution depends on point of evaluation

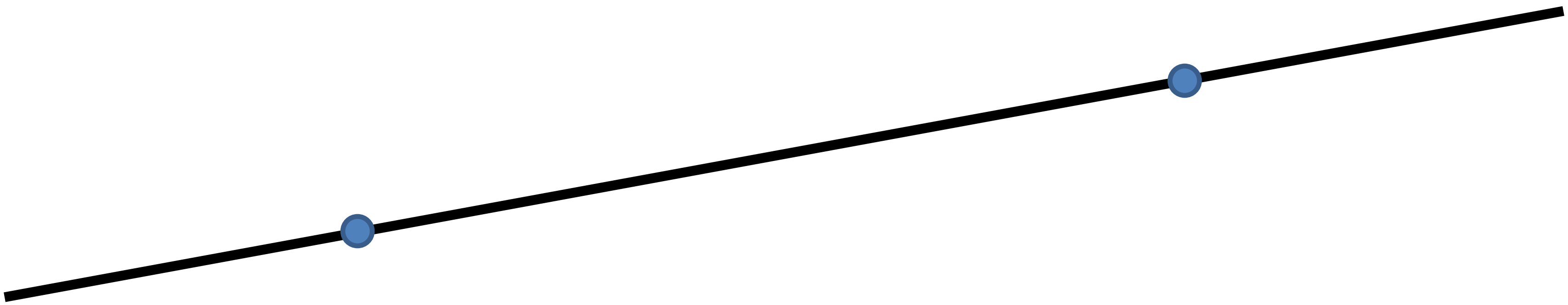
Fit a Polynomial to Points

$$V_1(x) = (1 \quad x_1 \quad x_2)$$

$$\underset{C}{\operatorname{argmin}} \sum_{i}^{n} \left[V_1(p_i)C - F(p_i) \right]^2$$

$$\hat{F}(x) = V_1(x)C$$

Fit a Polynomial to Points



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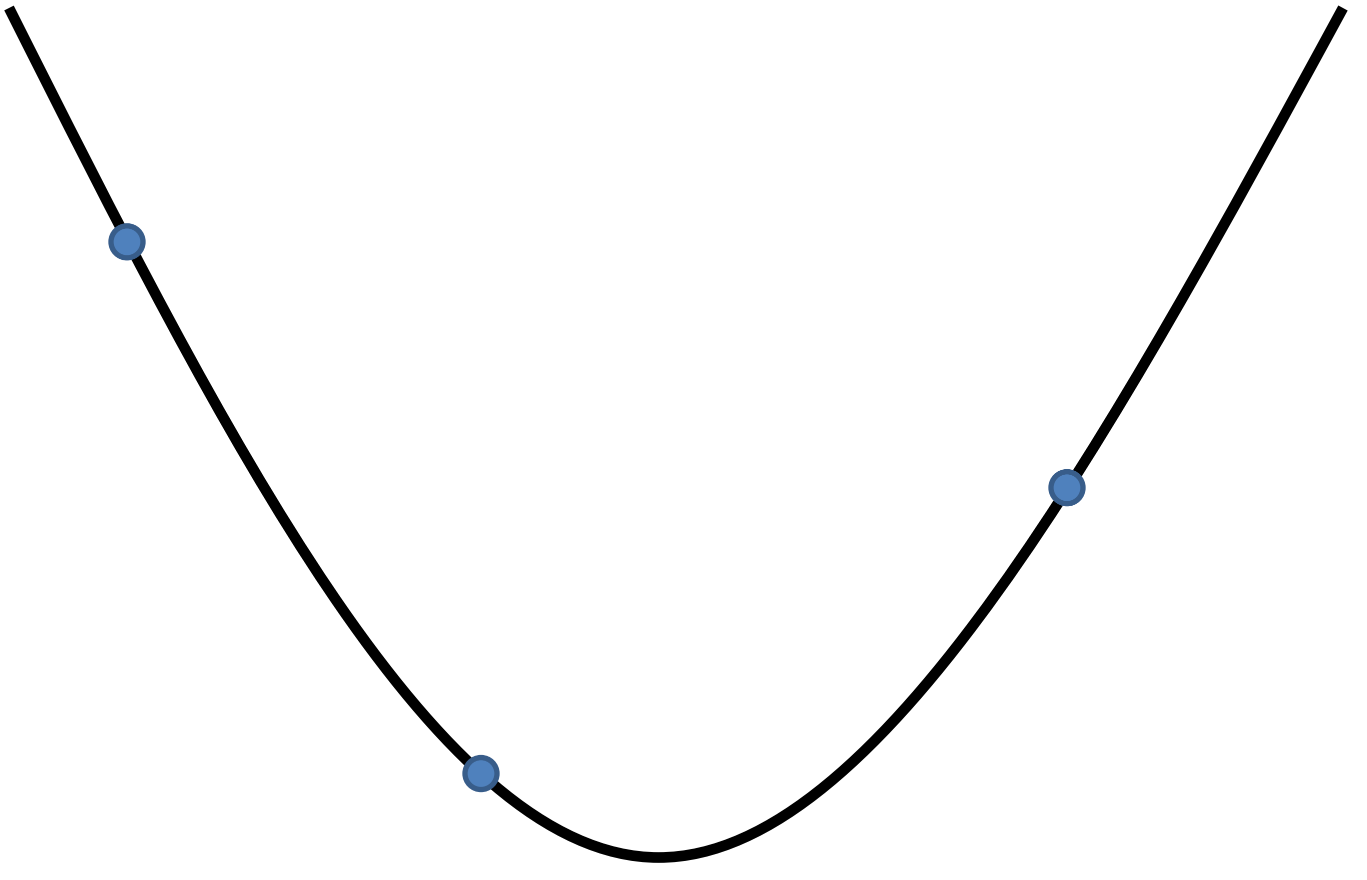
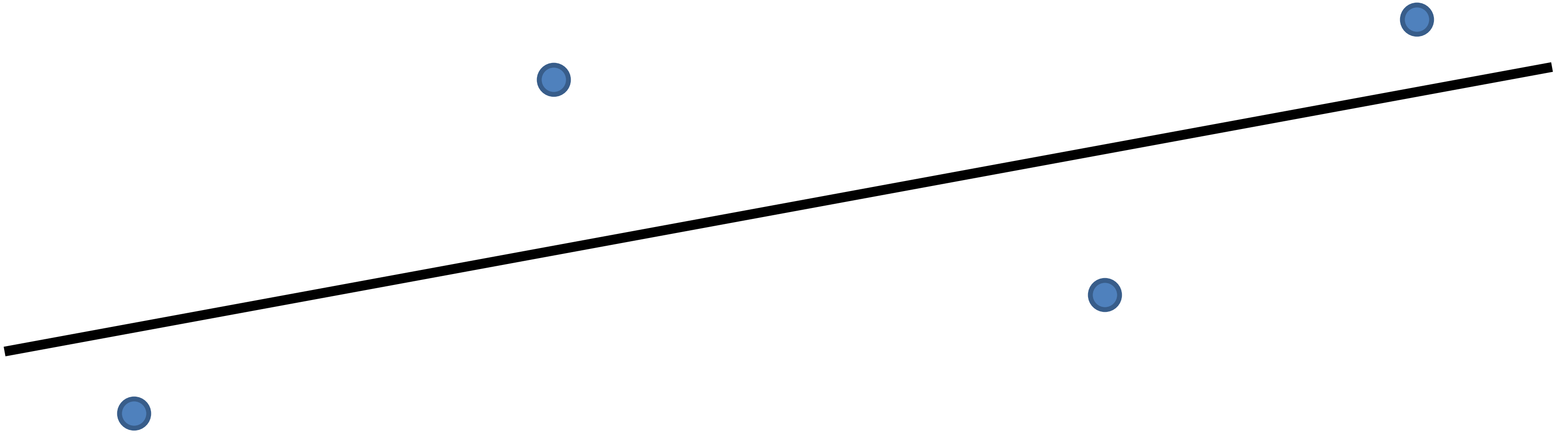


Interpolating Points



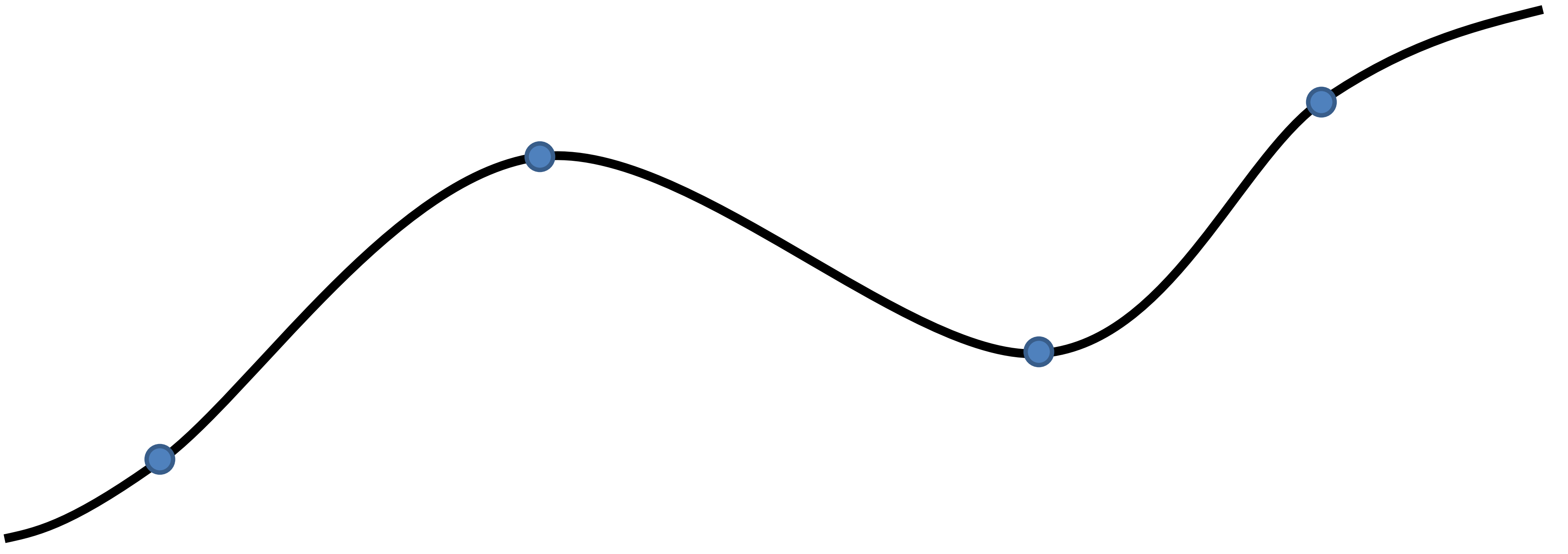
Interpolating Points

$$W(x,p) = \frac{1}{\|x-p\|^{2\alpha}}$$

$$\underset{C}{\operatorname{argmin}} \sum_{i}^{n} W(x,p_i)\left[V_1(p_i)C - F(p_i)\right]^2$$

$$\hat{F}(x) = V_1(x)C$$

Interpolating Points



Interpolating Line Segments

$$P_i(t) = \begin{pmatrix} 1-t & t \end{pmatrix} \begin{pmatrix} P_{i,1} \\ P_{i,2} \end{pmatrix}$$

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$$P_i(t) = \begin{pmatrix} 1-t & t \end{pmatrix} \begin{pmatrix} P_{i,1} \\ P_{i,2} \end{pmatrix} \quad F_i(t) = \begin{pmatrix} 1-t & t \end{pmatrix} \begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}$$

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$$W_i(x,t) = \frac{\left\| P_i'(t) \right\|}{\left\| x - P_i(t) \right\|^{2\alpha}}$$

Interpolating Line Segments

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$$W_i(x,t) = \frac{\left\| P_i'(t) \right\|}{\left\| x - P_i(t) \right\|^{2\alpha}}$$

$$\underset{C}{\operatorname{argmin}} \sum_{i}^{n} \int_{0}^{1} W_i(x,t) \left[V_1(P_i(t))C - F_i(t) \right]^2 dt$$

Line Basis Functions

$$A = \sum_{i}^{n} \int_{0}^{1} W_i(x,t) V_i^T(P_i(t)) V_1(P_i(t)) dt$$

$$C = \sum_{i}^{n} A^{-1} \int_{0}^{1} W_i(x,t) V_i^T(P_i(t)) F_i(t) dt$$

Line Basis Functions

$$\hat{F}(x) = V_1(x)C$$

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$$\hat{F}(x) = V_1(x)C$$

$$= V_1(x)\sum_{i}^{n} A^{-1}\int_{0}^{1} W_i(x,t)V_i^T(P_i(t))F_i(t)dt$$

Line Basis Functions

$$
\begin{aligned}
\hat{F}(x) &= V_1(x)C \\
&= V_1(x)\sum_{i}^{n} A^{-1}\int_{0}^{1} W_i(x,t)V_i^T(P_i(t))F_i(t)dt \\
&= V_1(x)\sum_{i}^{n} A^{-1}\int_{0}^{1} W_i(x,t)V_i^T(P_i(t))(1-t \quad t)\begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}dt
\end{aligned}
$$

Line Basis Functions

$$\hat{F}(x) = V_1(x)C$$

$$= V_1(x)\sum_i^n A^{-1}\int_0^1 W_i(x,t)V_i^T(P_i(t))F_i(t)dt$$

$$= V_1(x)\sum_i^n A^{-1}\int_0^1 W_i(x,t)V_i^T(P_i(t))(1-t \quad t)\begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}dt$$

$$= \sum_i^n V_1(x)A^{-1}\int_0^1 W_i(x,t)V_i^T(P_i(t))(1-t \quad t)dt\begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}$$

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$$\hat{F}(x) = V_1(x)C$$

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$$= V_1(x)\sum_i^n A^{-1}\int_0^1 W_i(x,t)V_i^T(P_i(t))\begin{pmatrix}1-t & t\end{pmatrix}\begin{pmatrix}F_{i,1}\\ F_{i,2}\end{pmatrix}dt$$

$$= \sum_i^n V_1(x)A^{-1}\int_0^1 W_i(x,t)V_i^T(P_i(t))\begin{pmatrix}1-t & t\end{pmatrix}dt\begin{pmatrix}F_{i,1}\\ F_{i,2}\end{pmatrix}$$

$$= \sum_i^n \begin{pmatrix}B_{i,1}(x) & B_{i,2}(x)\end{pmatrix}\begin{pmatrix}F_{i,1}\\ F_{i,2}\end{pmatrix}$$

Polygon Basis Functions

$$\hat{F}(x) = \sum_{i}^{n} \begin{pmatrix} B_{i,1}(x) & B_{i,2}(x) \end{pmatrix} \begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}$$

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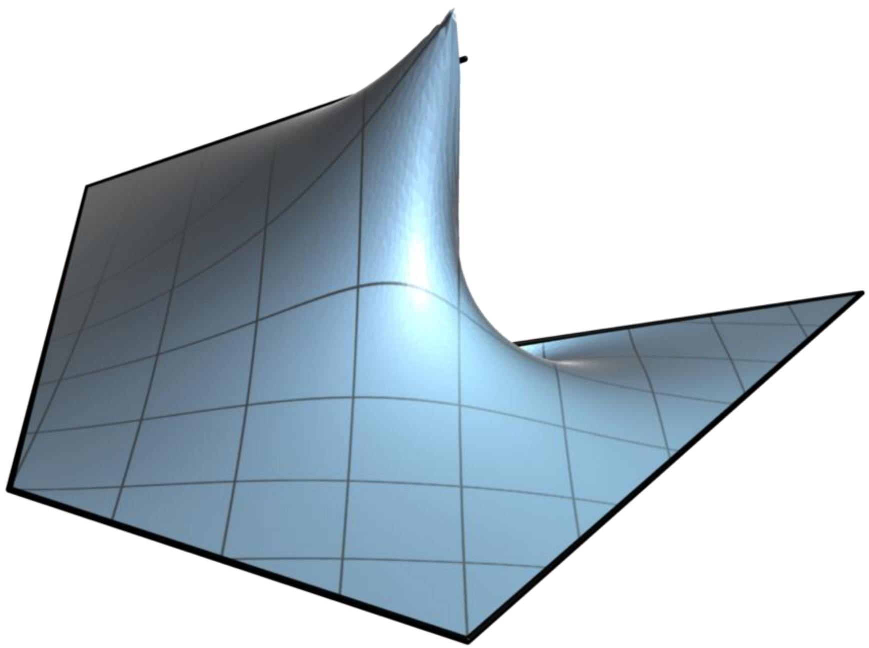
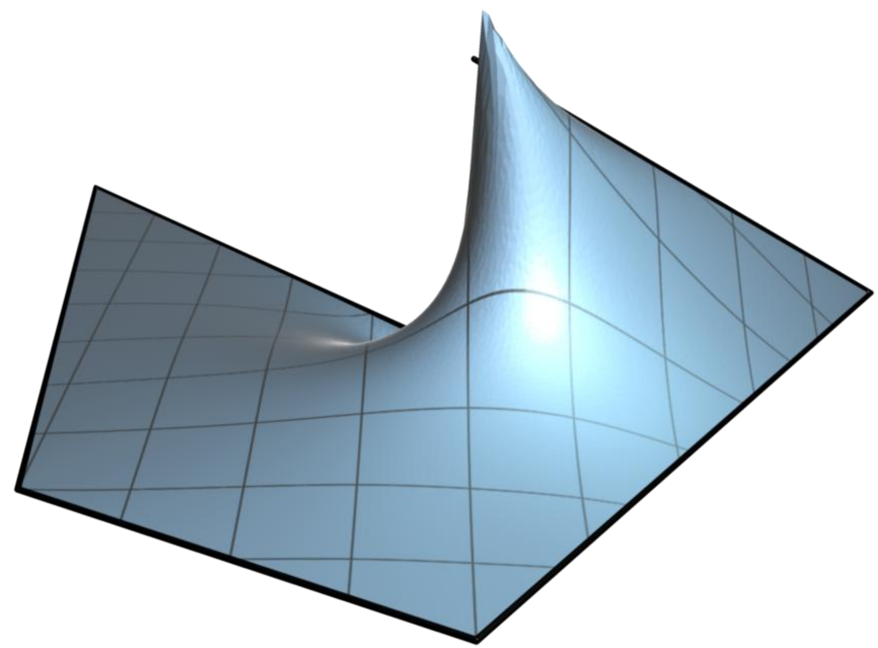
$$f_i = F_{i,1} = F_{i-1,2}$$

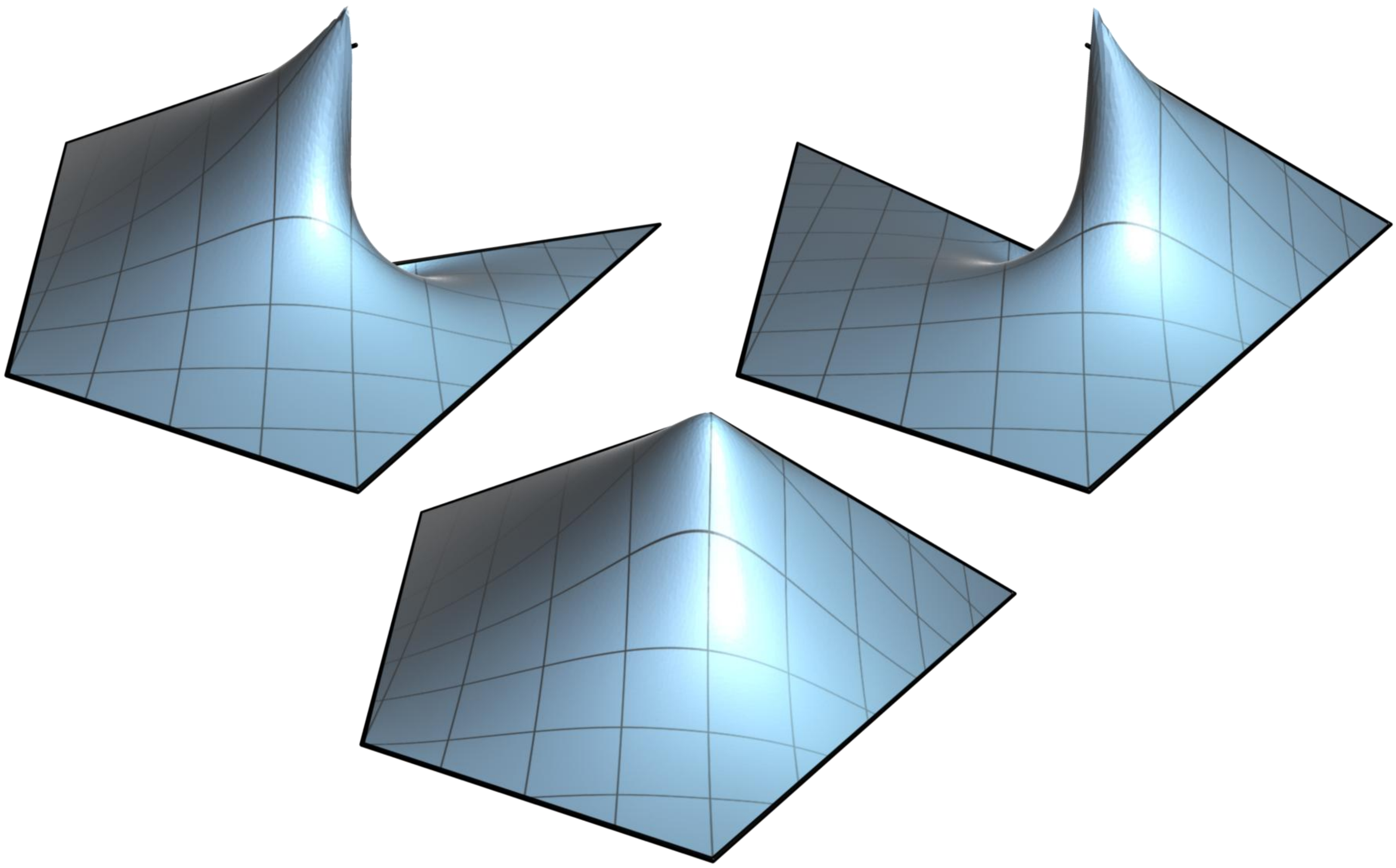
$$b_i(x) = B_{i,1}(x) + B_{i-1,2}(x)$$

Polygon Basis Functions





Polygon Basis Functions



Polygon Basis Functions

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$$f_i = F_{i,1} = F_{i-1,2}$$

$$b_i(x) = B_{i,1}(x) + B_{i-1,2}(x)$$

$$\hat{F}(x) = \sum_{i}^{n} b_i(x) f_i$$

Polynomial Boundary Values

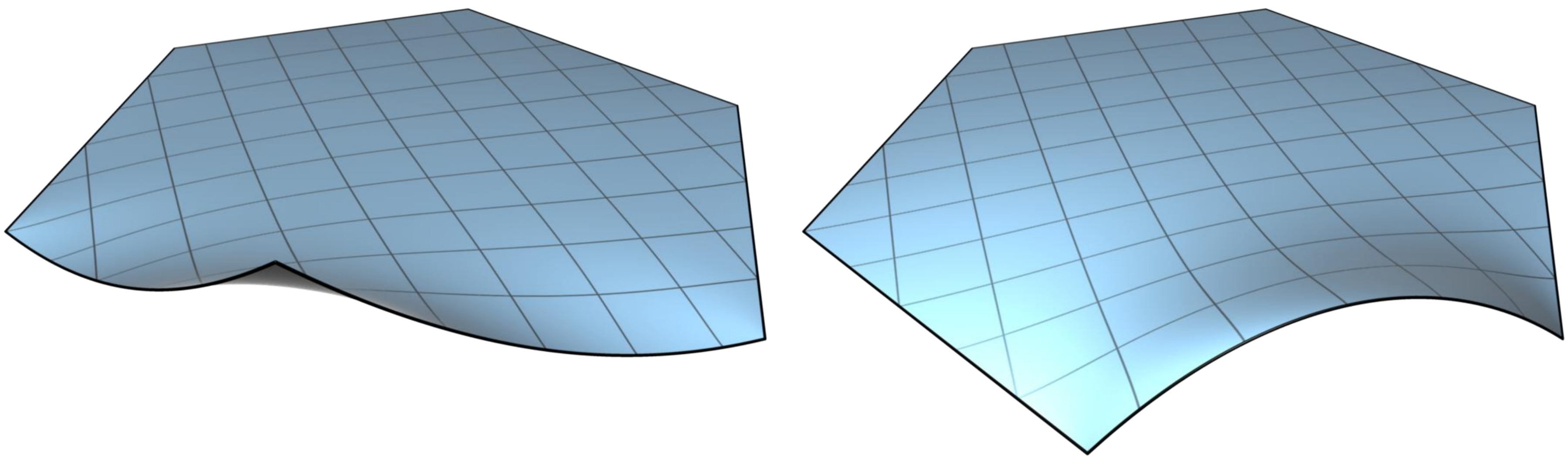
$$F_i(t) = \begin{pmatrix} 1-t & t \end{pmatrix} \begin{pmatrix} F_{i,1} \\ F_{i,2} \end{pmatrix}$$

$$F_i(t) = \begin{pmatrix} (1-t)^2 & 2(1-t)t & t^2 \end{pmatrix} \begin{pmatrix} F_{i,1} \\ F_{i,2} \\ F_{i,3} \end{pmatrix}$$

$$F_i(t) = \begin{pmatrix} (1-t)^3 & 3(1-t)^2 t & 3(1-t)t^2 & t^3 \end{pmatrix} \begin{pmatrix} F_{i,1} \\ F_{i,2} \\ F_{i,3} \\ F_{i,4} \end{pmatrix}$$

$$\vdots$$

Polynomial Boundary Values



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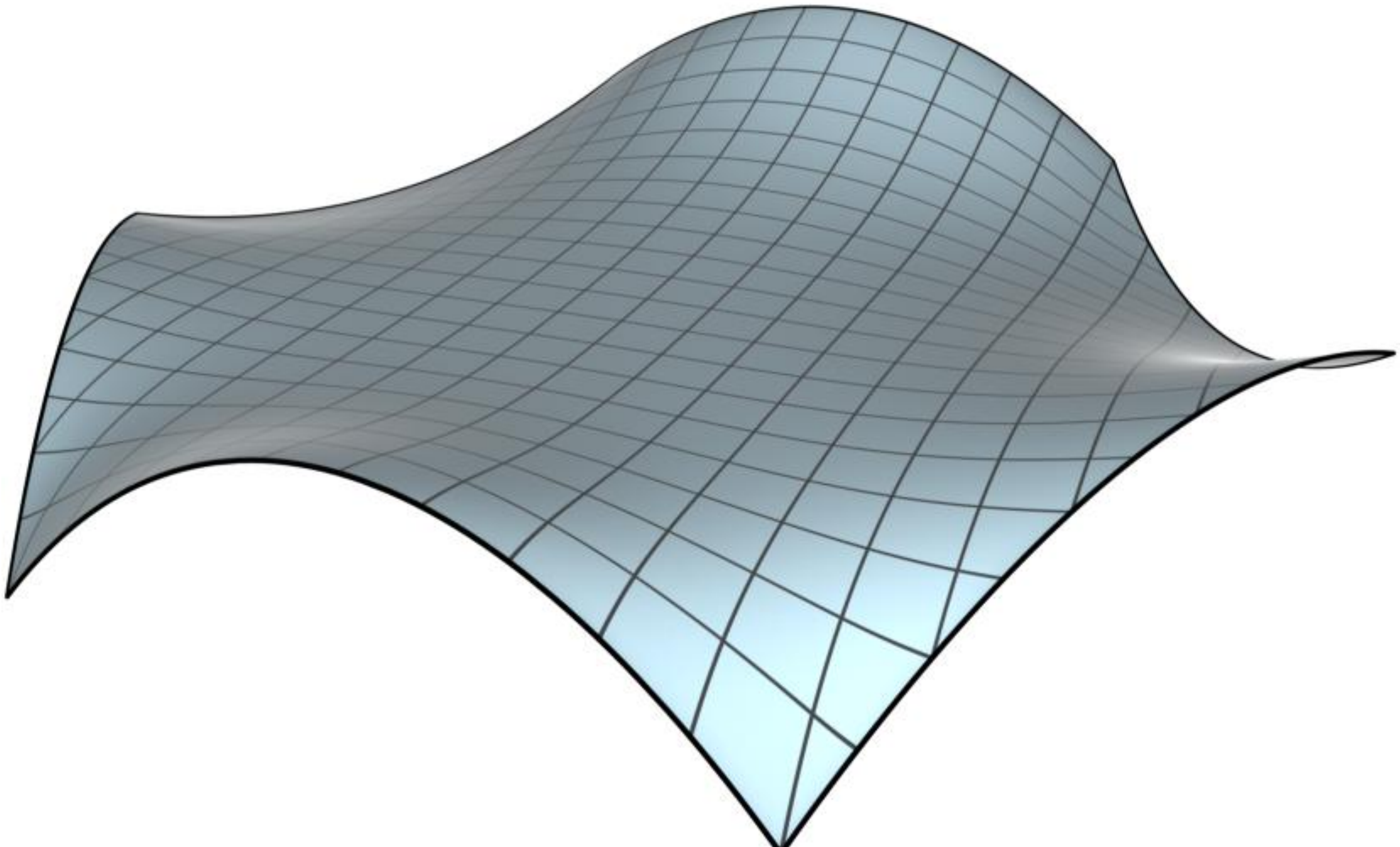


Polynomial Precision

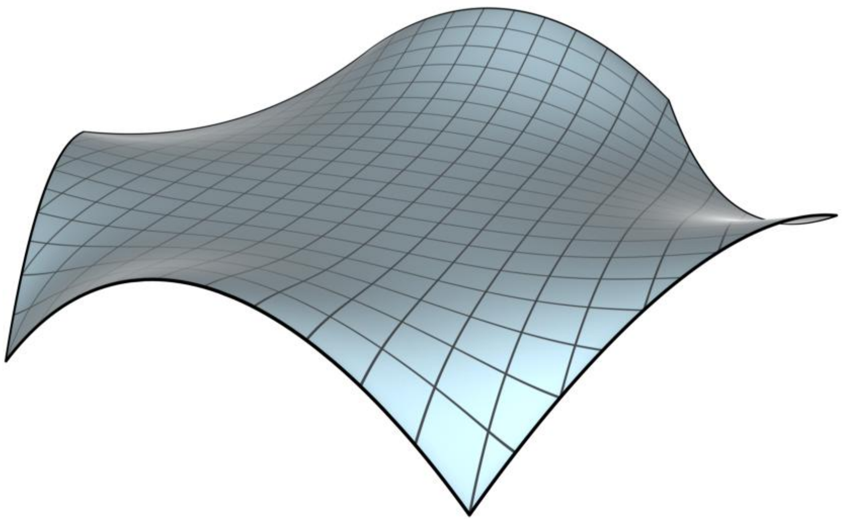
$$V_1(x) = (1 \quad x_1 \quad x_2)$$

$$V_2(x) = (1 \quad x_1 \quad x_2 \quad x_1^2 \quad x_1 x_2 \quad x_2^2)$$

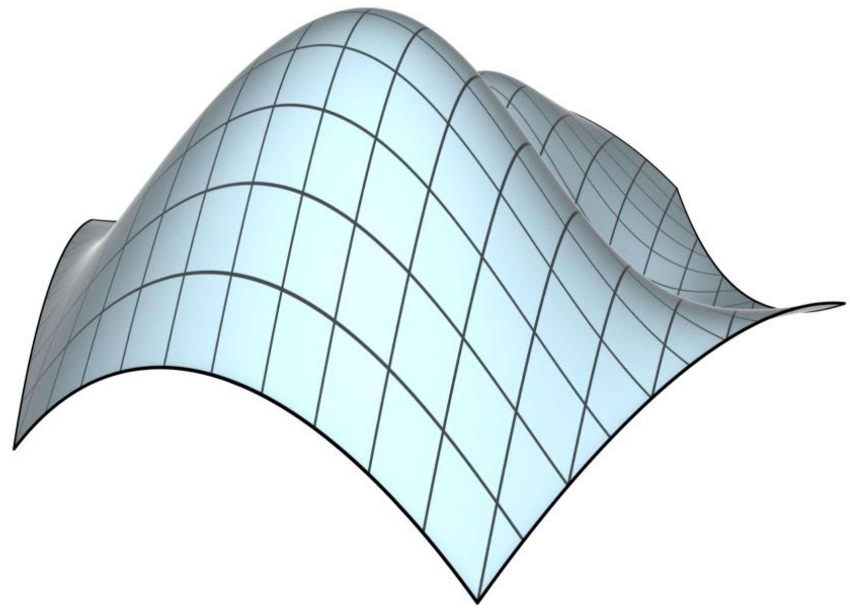
$$V_3(x) = (1 \quad x_1 \quad x_2 \quad x_1^2 \quad x_1 x_2 \quad x_2^2 \quad x_1^3 \quad x_1^2 x_2^1 \quad x_1^1 x_2^2 \quad x_2^3)$$

$$\vdots$$

Polynomial Precision



Linear



Quadratic

Interpolation of Derivatives

$$\underset{C}{\operatorname{argmin}} \sum_{i}^{n} \int_{0}^{1} W_i(x,t)\left[V_1(P_i(t))C - F_i(t)\right]^2 dt$$

Interpolation of Derivatives

$$\underset{C}{\operatorname{argmin}} \sum_{i}^{n} \int_0^1 W_i(x,t)\left[V_1(P_i(t))C - F_i(t)\right]^2 dt$$

$$+ \int_0^1 W_i(x,t)\left[G_{1,i}(t)C - F_i^{\perp}(t)\right]^2 dt$$

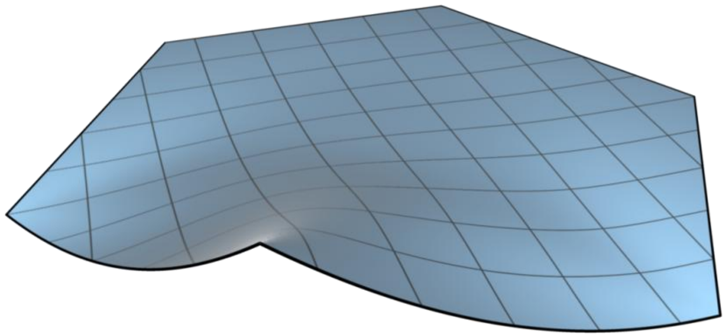
Interpolation of Derivatives

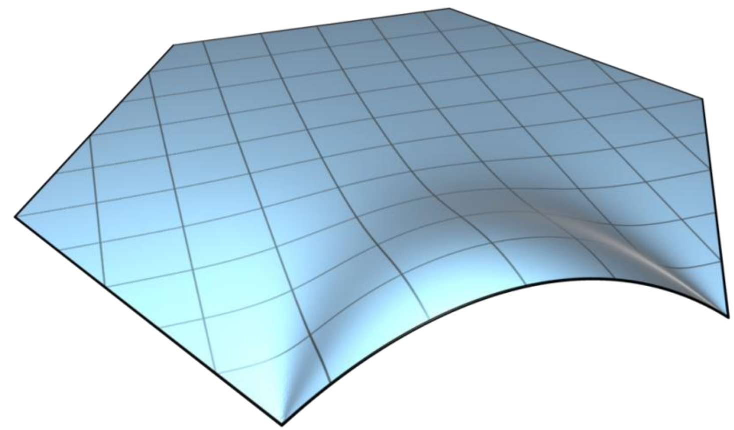
$$G_{1,i}(t) = P_i^{\perp}(t)\begin{pmatrix} \frac{\partial}{\partial x_1} V_1(P_i(t)) \\ \frac{\partial}{\partial x_2} V_1(P_i(t)) \end{pmatrix}$$

$$\underset{C}{\operatorname{argmin}} \sum_i^n \int_0^1 W_i(x,t)\left[V_1(P_i(t))C - F_i(t)\right]^2 dt$$

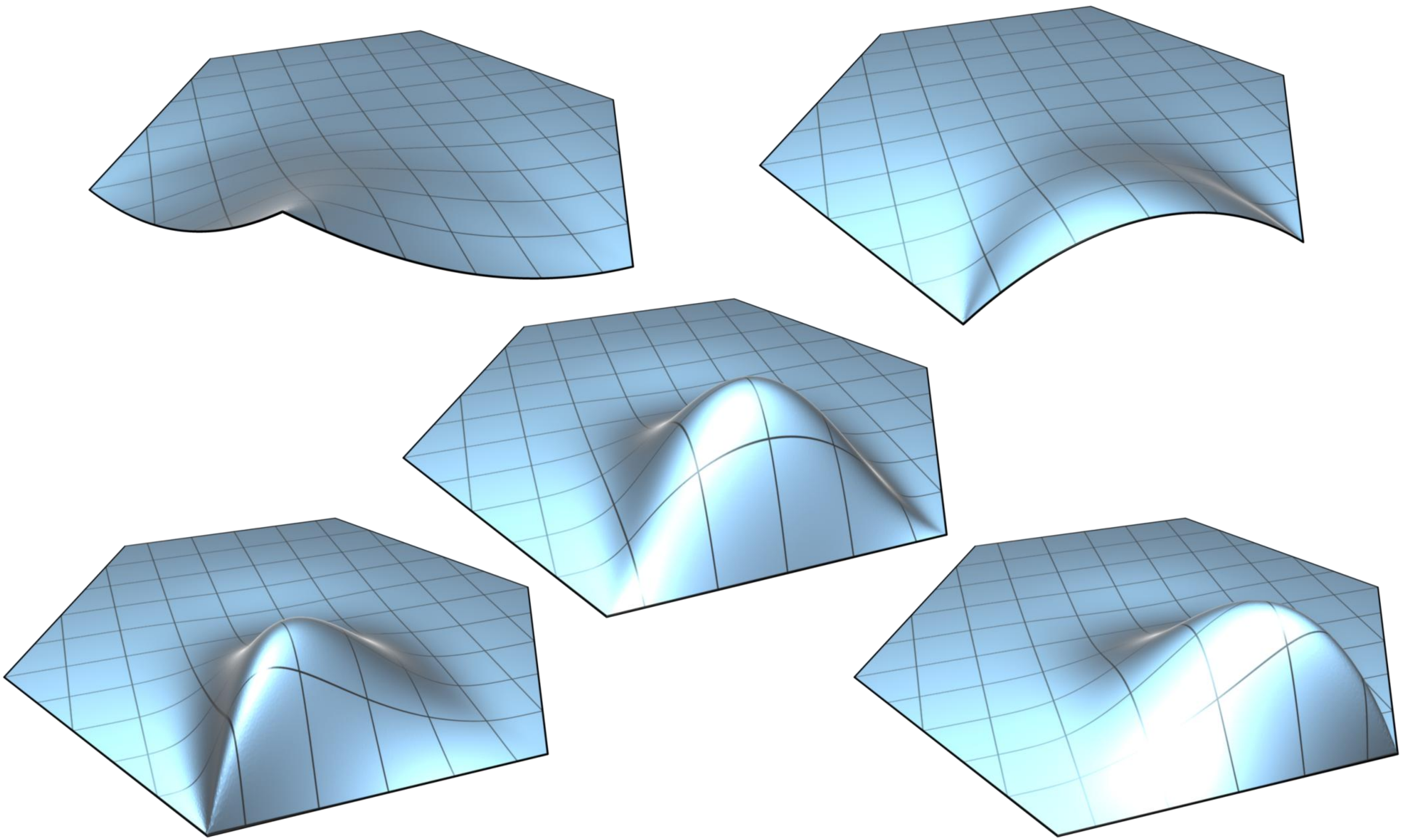
$$+ \int_0^1 W_i(x,t)\left[G_{1,i}(t)C - F_i^{\perp}(t)\right]^2 dt$$

Interpolation of Derivatives





Interpolation of Derivatives



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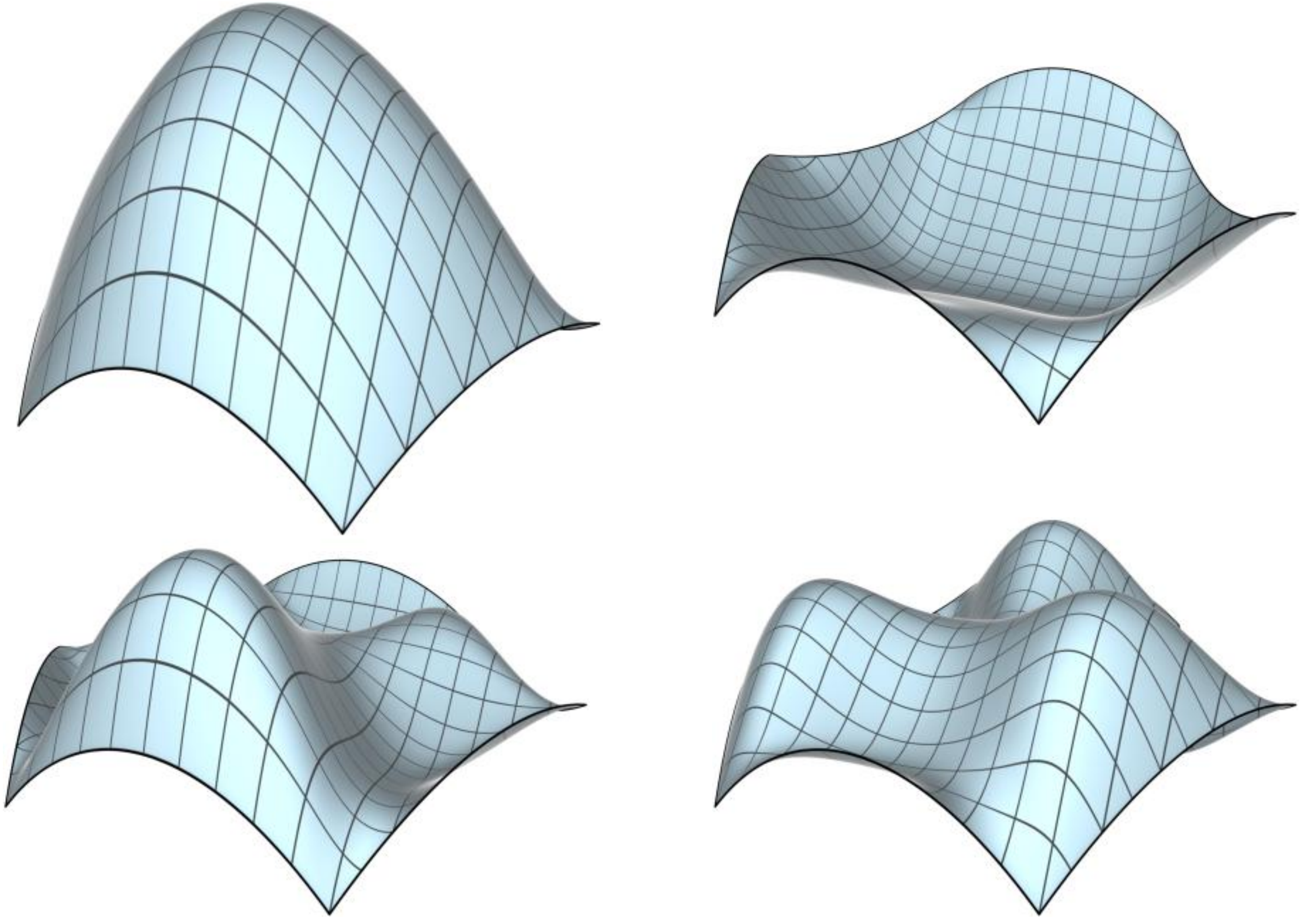


Solutions are Closed-Form

- For polygons $P_i(t)$ is linear
  - $P_i'(t)$ and $P_i^{\perp}(t)$ are constant
  - Polynomial numerator
  - Denominator quadratic to power 2α
  - Integrals have closed-form solutions

Curved Boundaries

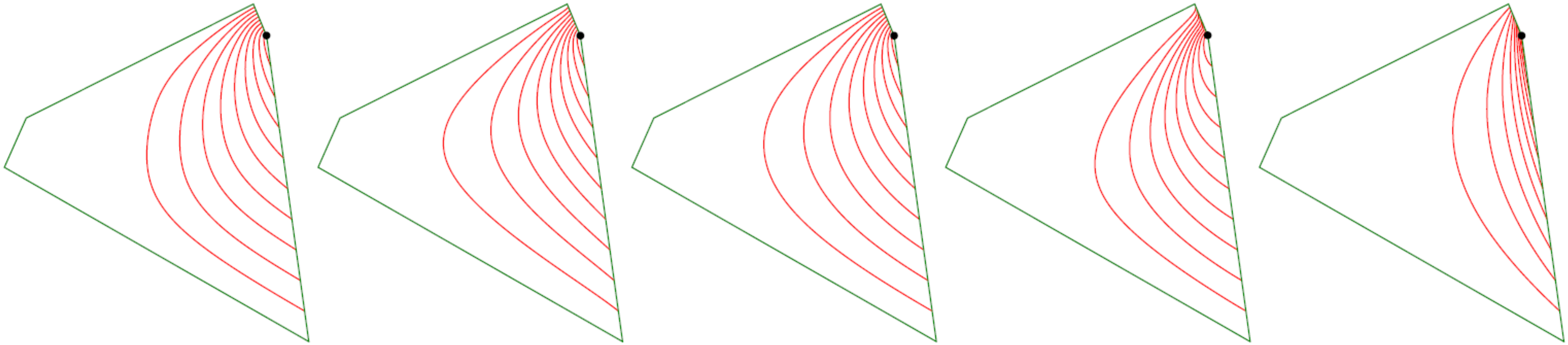
$$P_i(t) = \begin{pmatrix} 1-t & t \end{pmatrix} \begin{pmatrix} P_{i,1} \\ P_{i,2} \end{pmatrix}$$

$$P_i(t) = \begin{pmatrix} (1-t)^2 & 2(1-t)t & t^2 \end{pmatrix} \begin{pmatrix} P_{i,1} \\ P_{i,2} \\ P_{i,3} \end{pmatrix}$$

$$P_i(t) = \begin{pmatrix} (1-t)^3 & 3(1-t)^2 t & 3(1-t)t^2 & t^3 \end{pmatrix} \begin{pmatrix} P_{i,1} \\ P_{i,2} \\ P_{i,3} \\ P_{i,4} \end{pmatrix}$$

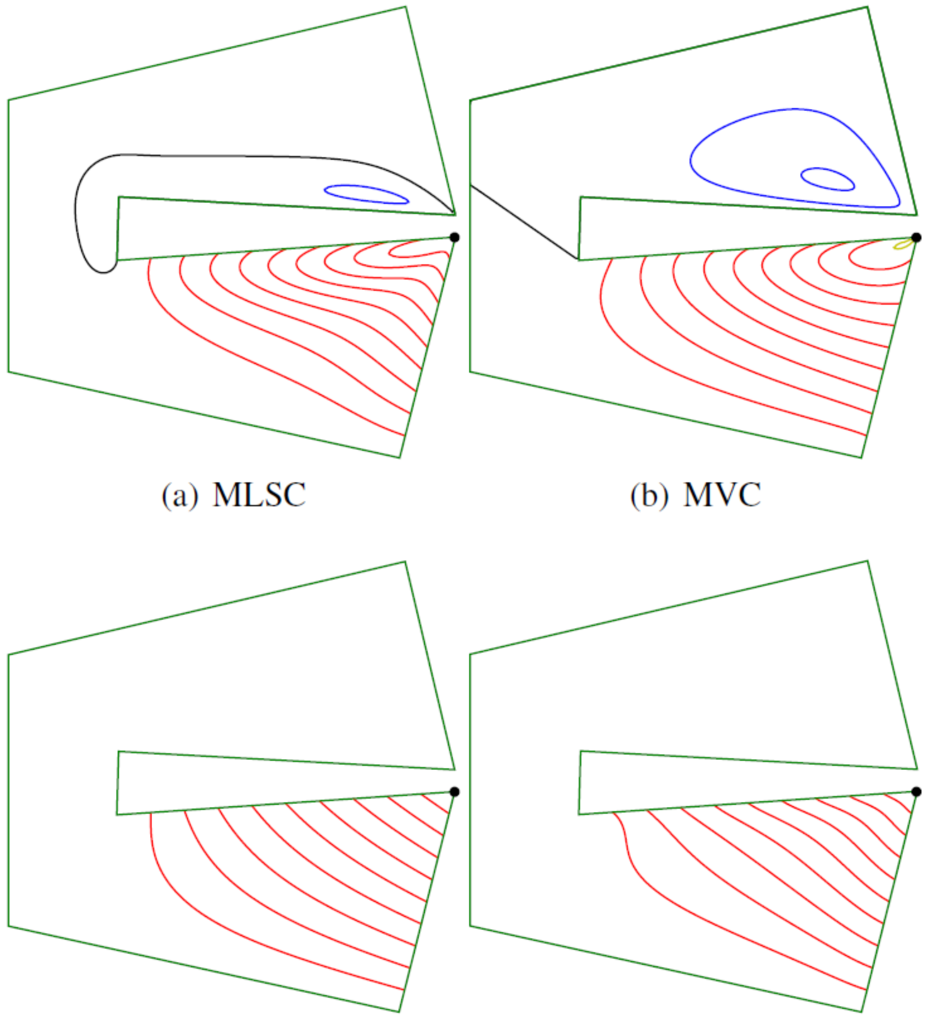
$$\vdots$$

Comparison to Other Methods



(a) MLSC (b) MVC (c) Harmonic (d) Max Entropy (e) Wachspress

Comparison to Other Methods

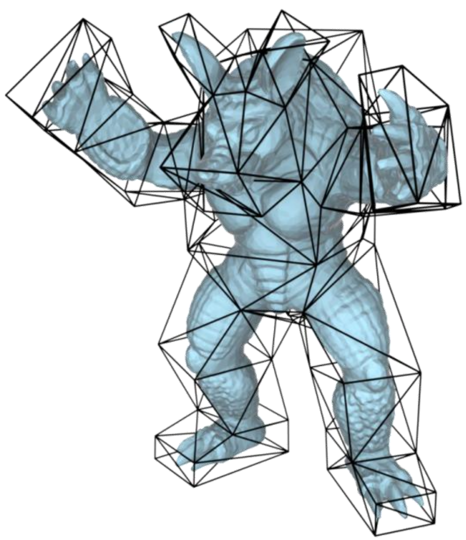


(a) MLSC (b) MVC

(c) Harmonic (d) Max Entropy

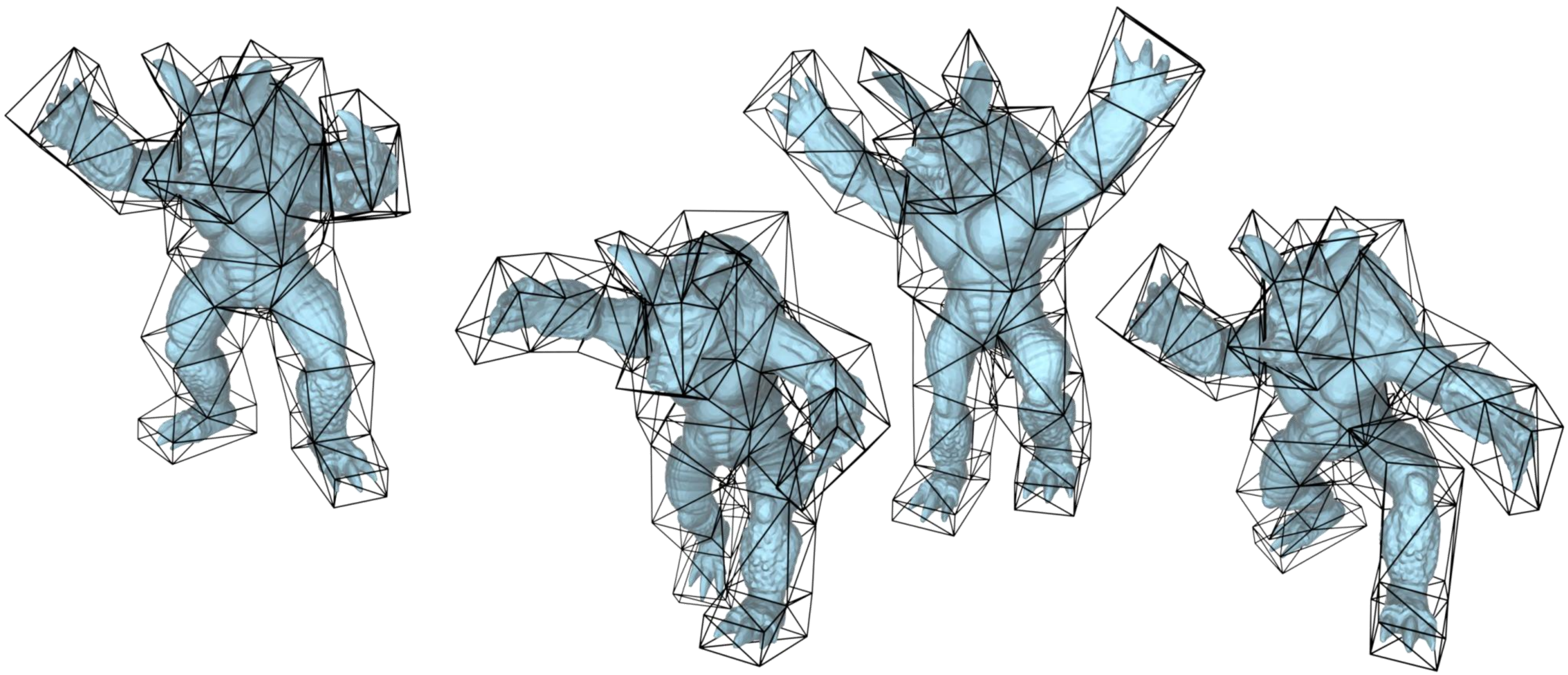
3D Deformation

$$x = \sum_{i}^{n} b_i(x)\, p_i$$



3D Deformation

$$x = \sum_{i}^{n} b_i(x)\, p_i \qquad \hat{x} = \sum_{i}^{n} b_i(x)\, \hat{p}_i$$



Conclusion

- New family of barycentric coordinates
  - Controlled by parameter **α**
  - Polynomial boundaries
  - Polynomial precision
  - Derivative interpolation
  - Open polygons
  - Closed-form