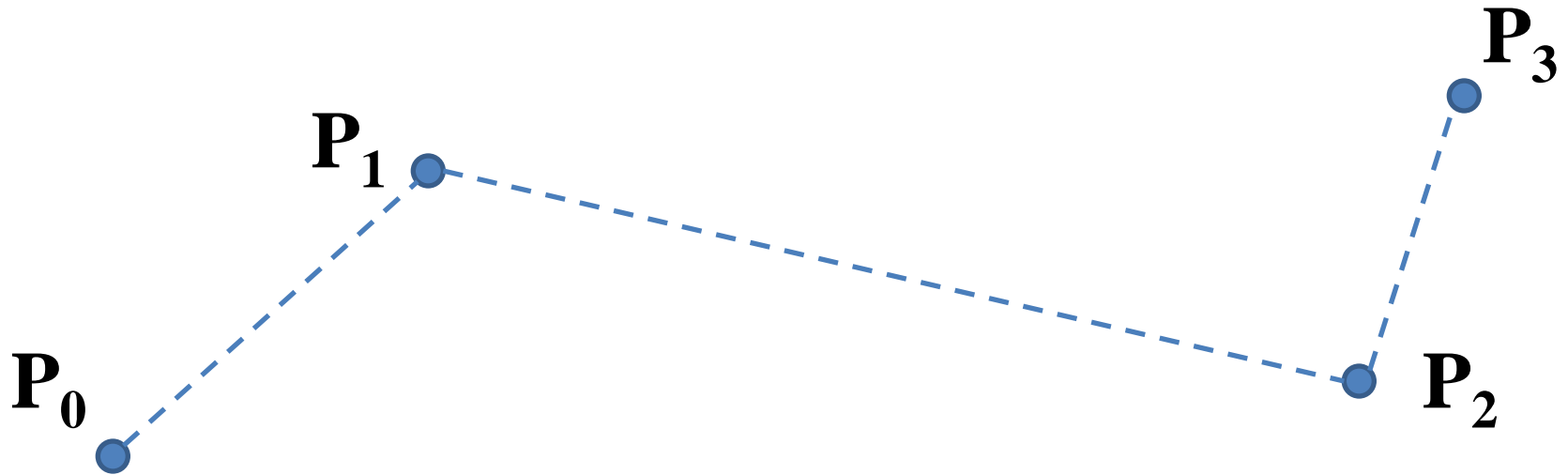# On the Parameterization of Catmull-Rom Curves
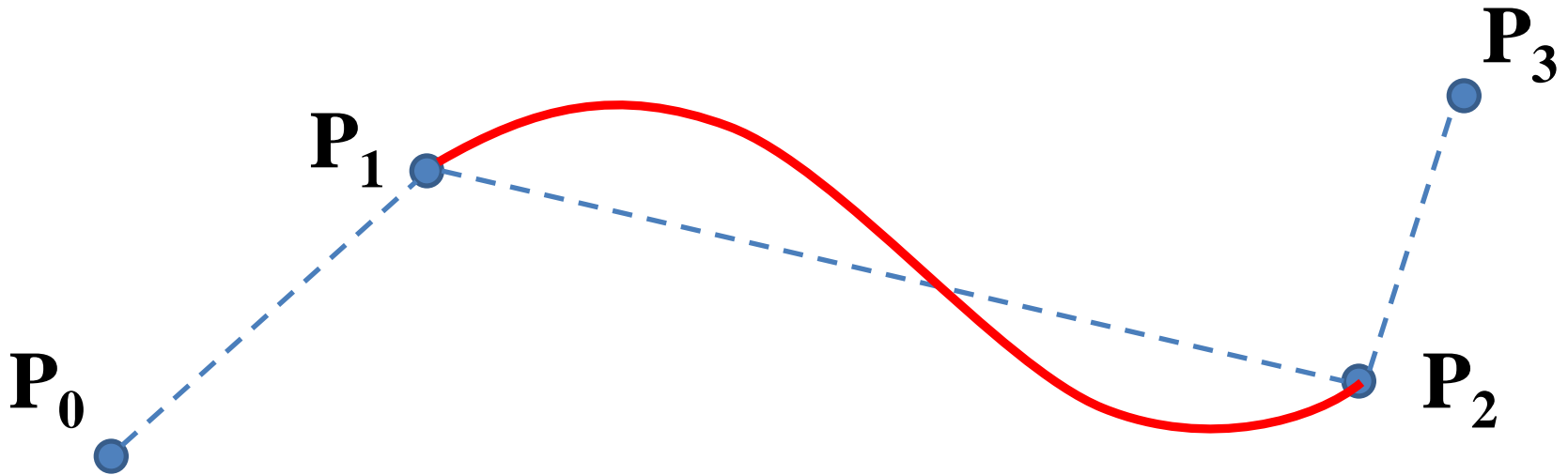
Cem Yuksel   Scott Schaefer   John Keyser

Texas A&M University

Catmull-Rom Curves
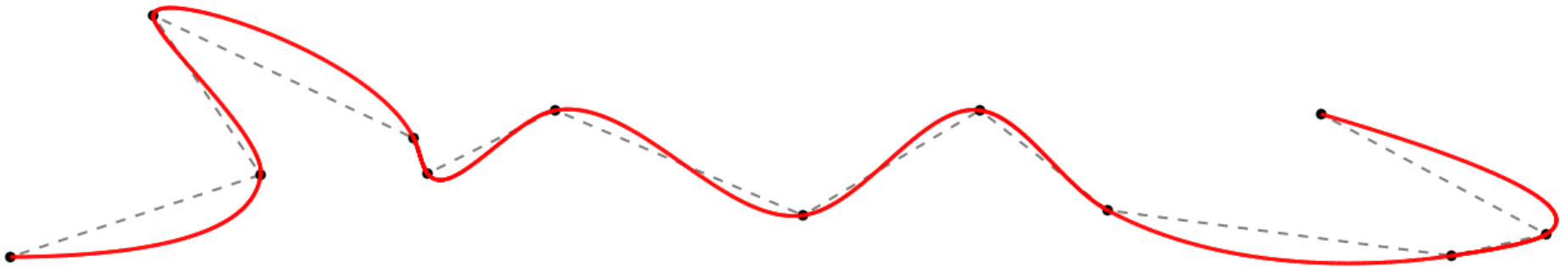
Catmull-Rom Curves
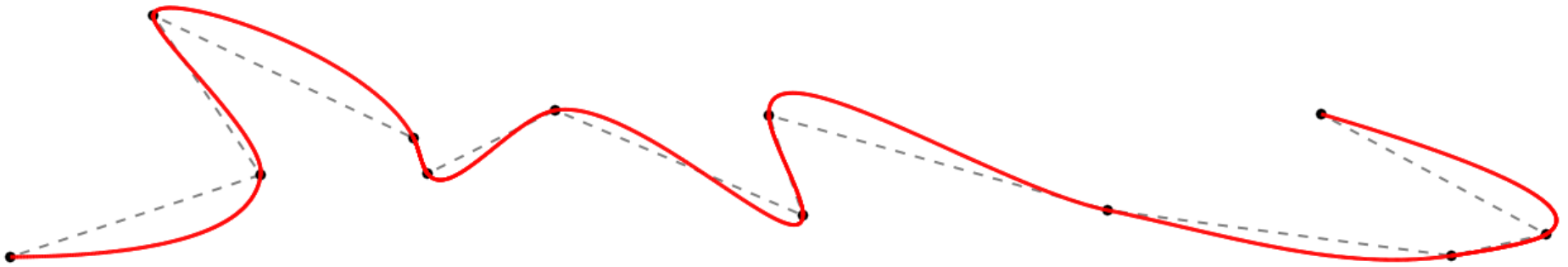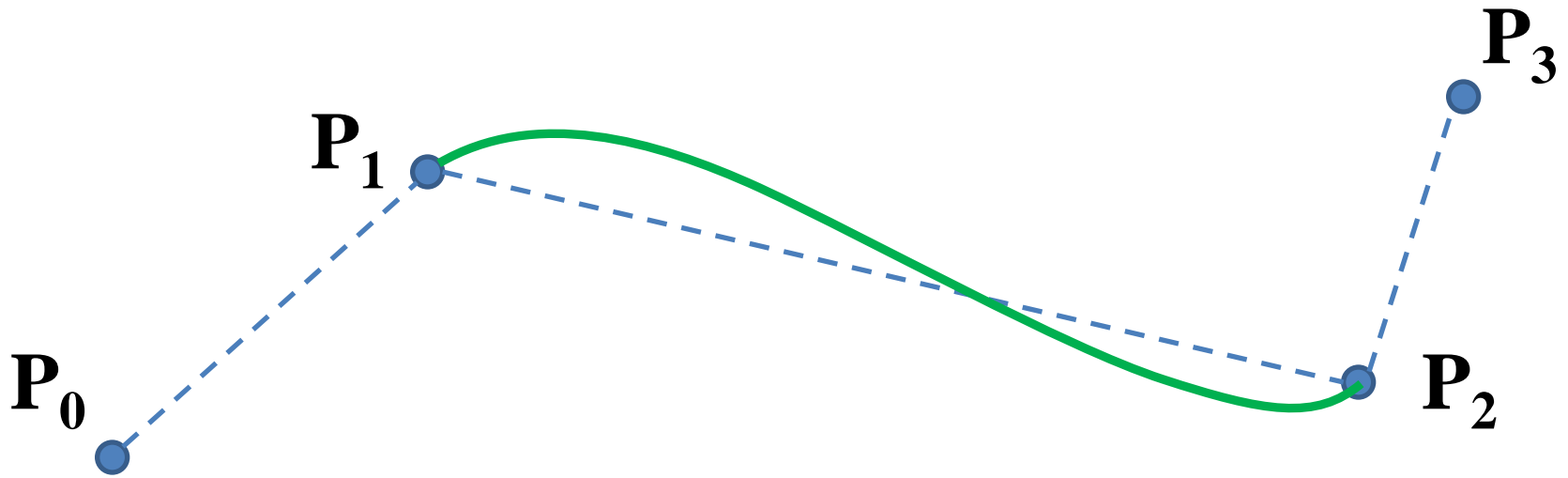
P_3

P_1

P_0

P_2

Catmull-Rom Curves

- Important Properties
  - Interpolate control points
  - Local support
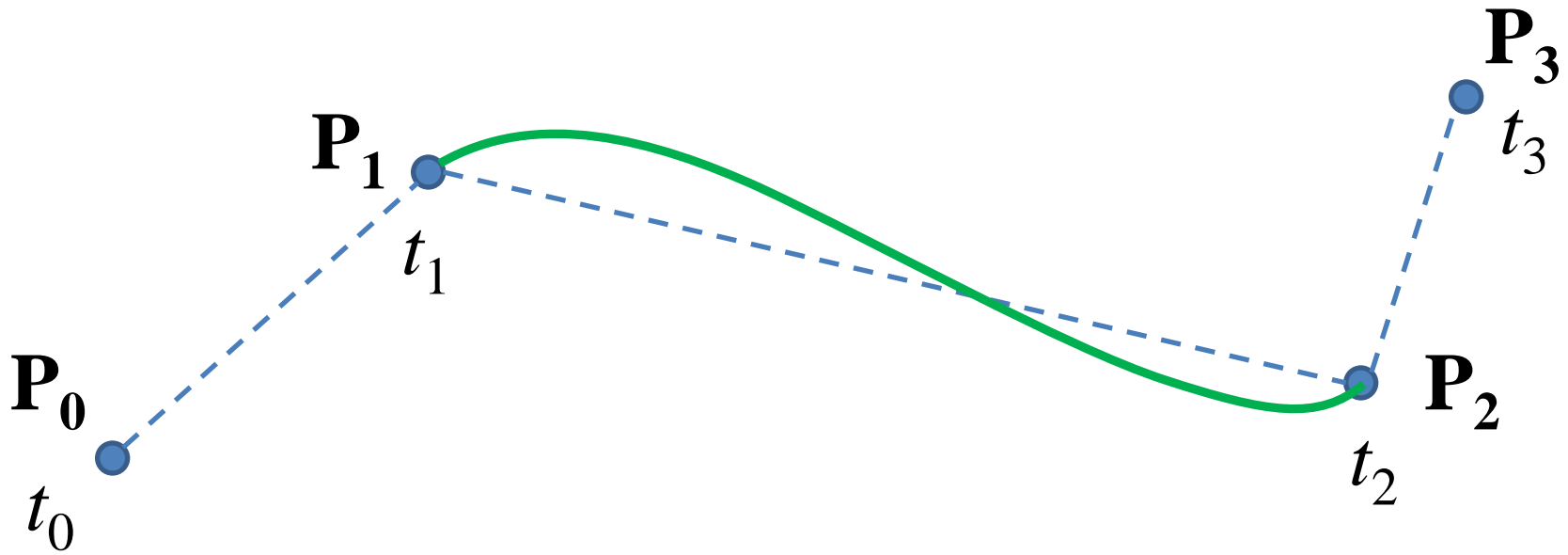 - Piecewise polynomial representation

Catmull-Rom Curves

- Important Properties
  - Interpolate control points
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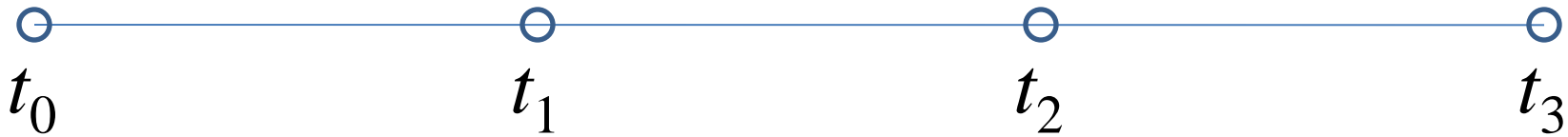
Catmull-Rom Curves
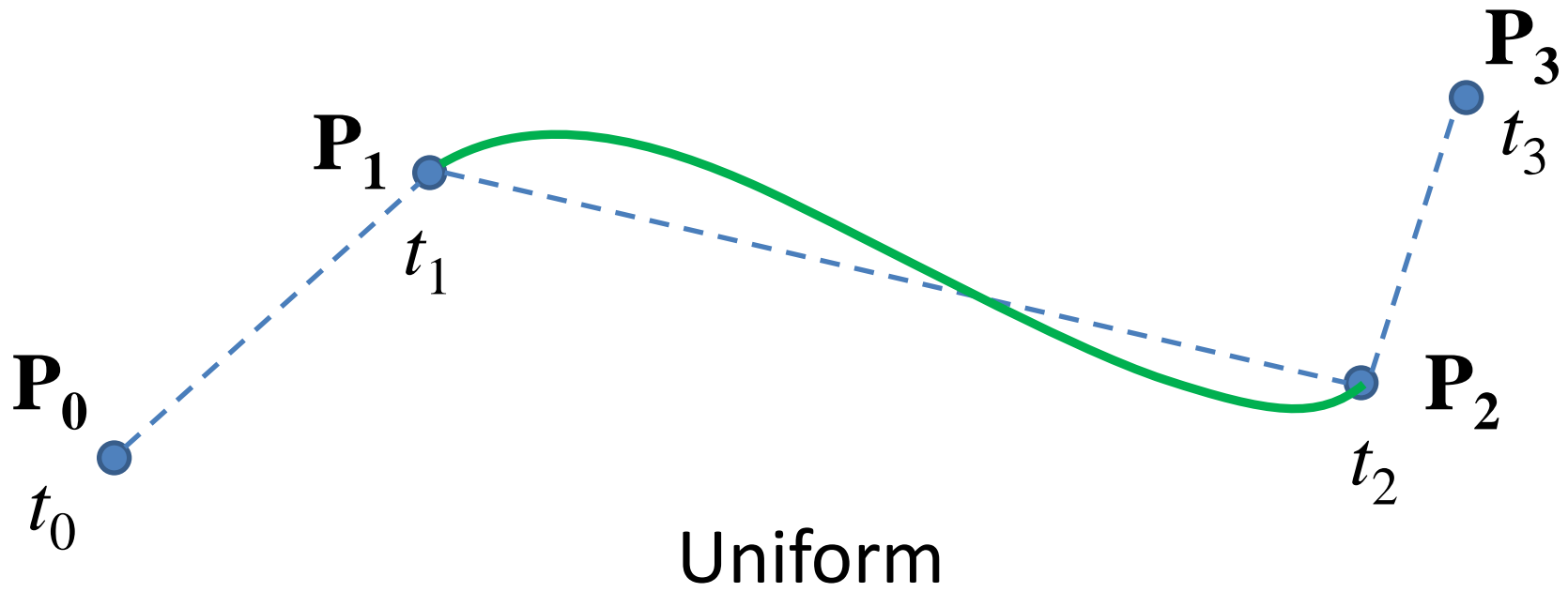
Catmull-Rom Curves

$\mathbf{C_{12}}\left(\mathbf{P_{0,1,2,3}},\ t_{0,1,2,3}\right)$

Catmull-Rom Curves

P₃

t_3

P₁

t_1

P₀

t_0

P₂

t_2

Uniform

t_0 t_1 t_2 t_3

$$t_{i+1} = t_i + 1$$

Catmull-Rom Curves
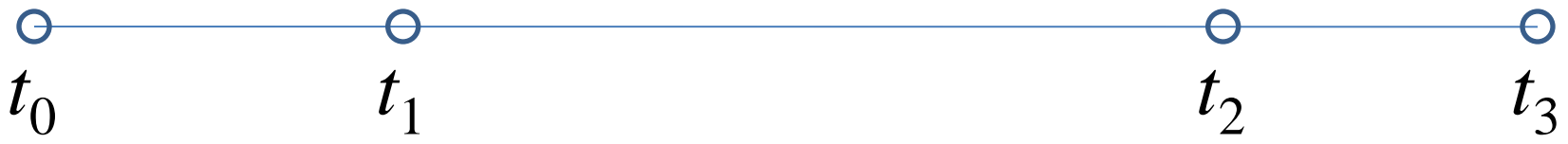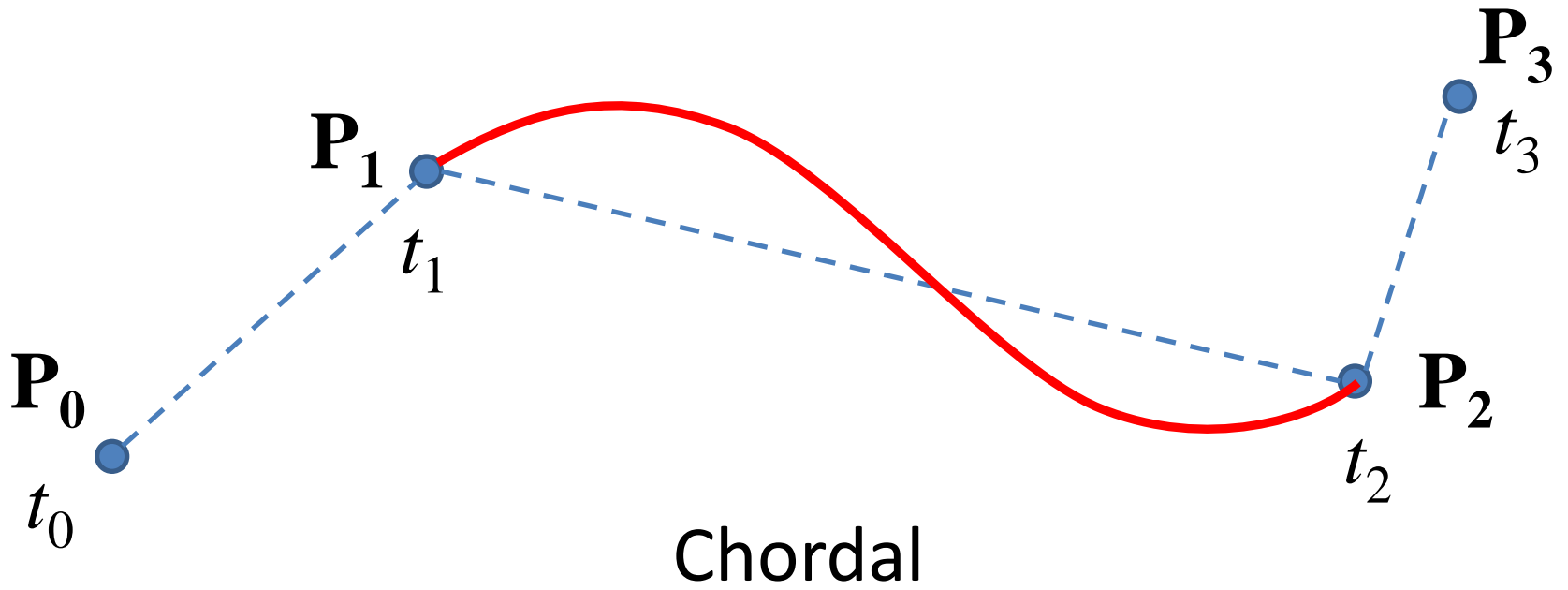
Chordal

$$t_{i+1} = t_i + |\mathbf{P}_{i+1} - \mathbf{P}_i|$$

Catmull-Rom Curves

- Parameterization
  - Uniform:

    $$t_{i+1} = t_i + 1$$

  - Chordal:

    $$t_{i+1} = t_i + |\mathbf{P}_{i+1} - \mathbf{P}_i|$$

Catmull-Rom Curves

- Parameterization
  - Uniform:

    $$t_{i+1} = t_i + |\mathbf{P}_{i+1} - \mathbf{P}_i|^0$$

  - Chordal:

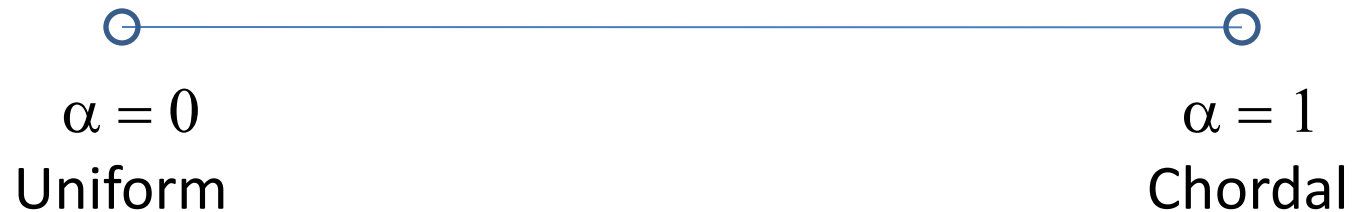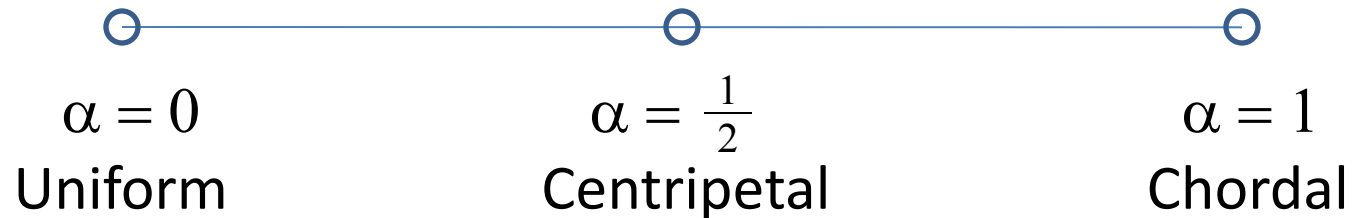 $$t_{i+1} = t_i + |\mathbf{P}_{i+1} - \mathbf{P}_i|^1$$

Catmull-Rom Curves

- Parameterization

$$t_{i+1} \;=\; t_i \;+\; | \mathbf{P}_{i+1} - \mathbf{P}_i |^{\alpha}$$

Catmull-Rom Curves

- Parameterization

$$t_{i+1} \;=\; t_i \;+\; |\, \mathbf{P}_{i+1} - \mathbf{P}_i \,|^{\alpha}$$

$\alpha = 0$

Uniform

$\alpha = 1$

Chordal

Catmull-Rom Curves

- Parameterization

$$t_{i+1} = t_i + |\mathbf{P}_{i+1} - \mathbf{P}_i|^{\alpha}$$

$\alpha = 0$

Uniform

$\alpha = \frac{1}{2}$

Centripetal

$\alpha = 1$
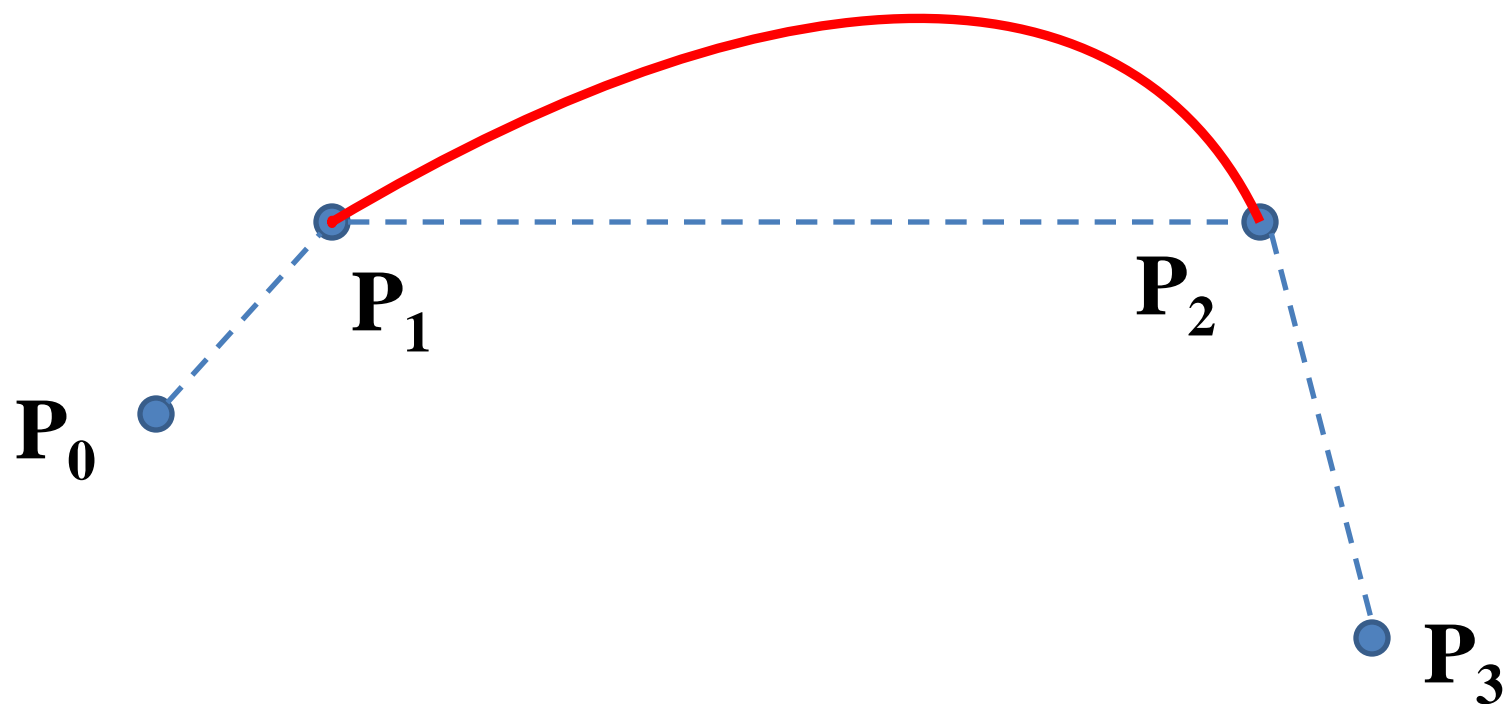
Chordal

On the Parameterization of Catmull-Rom Curves
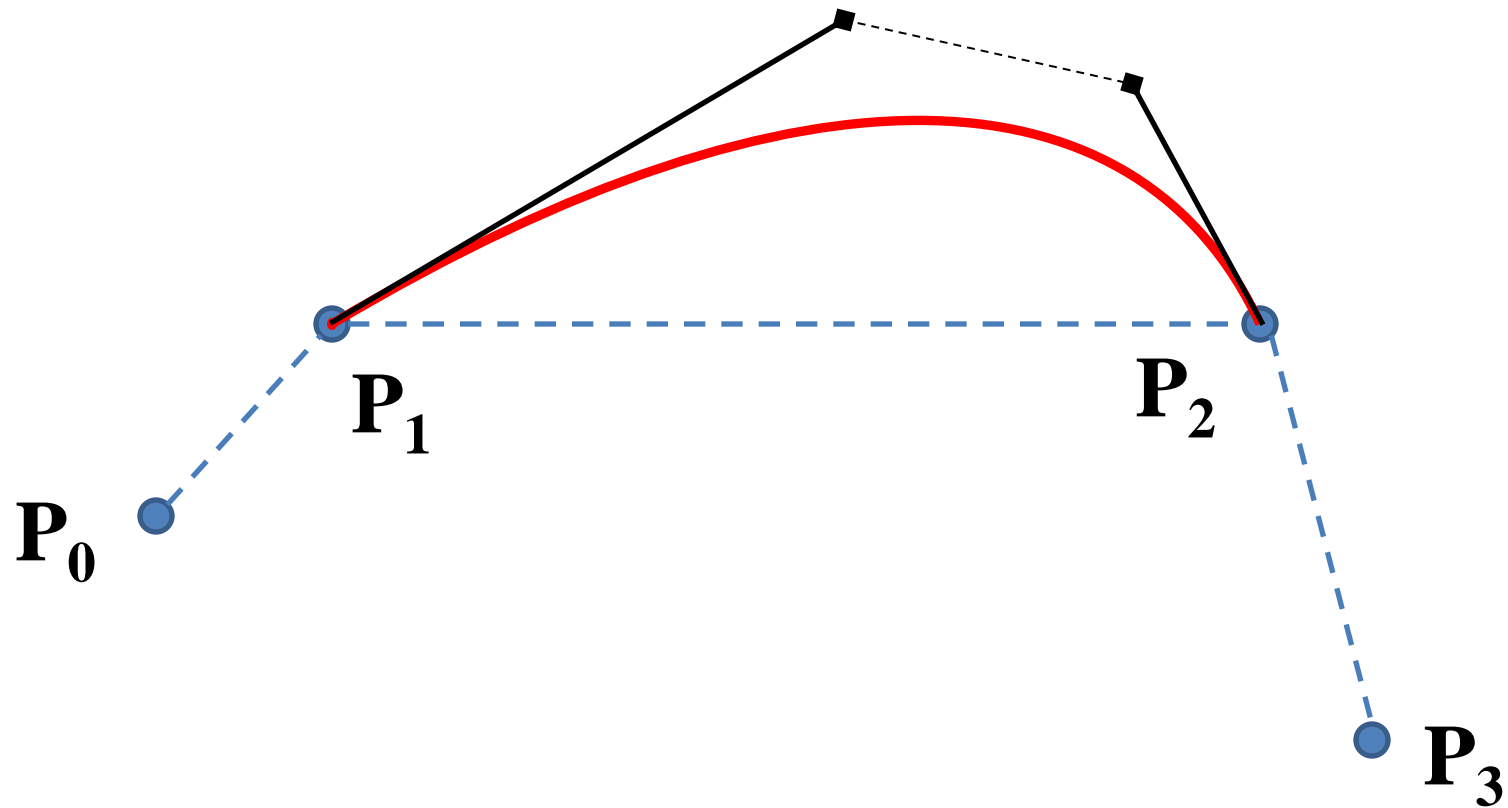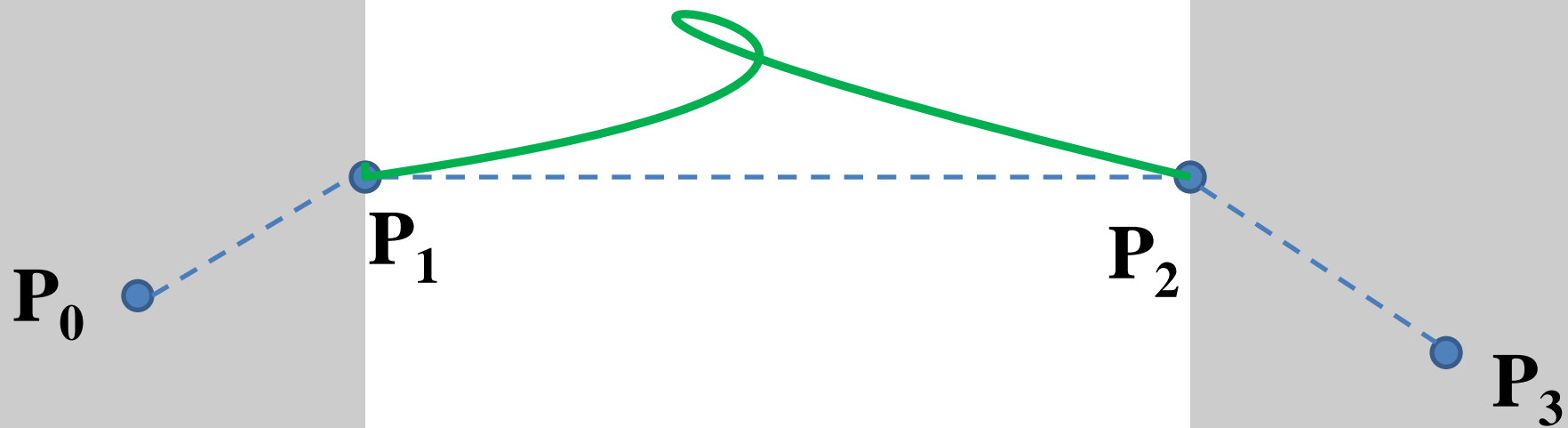
DEMO

On the Parameterization of Catmull-Rom Curves

CUSPS & SELF-INTERSECTIONS

Cusps & Self-Intersections

Cusps & Self-Intersections

Cusps & Self-Intersections

Cusps & Self-Intersections

Cusps & Self-Intersections

P_0

P_1

P_2
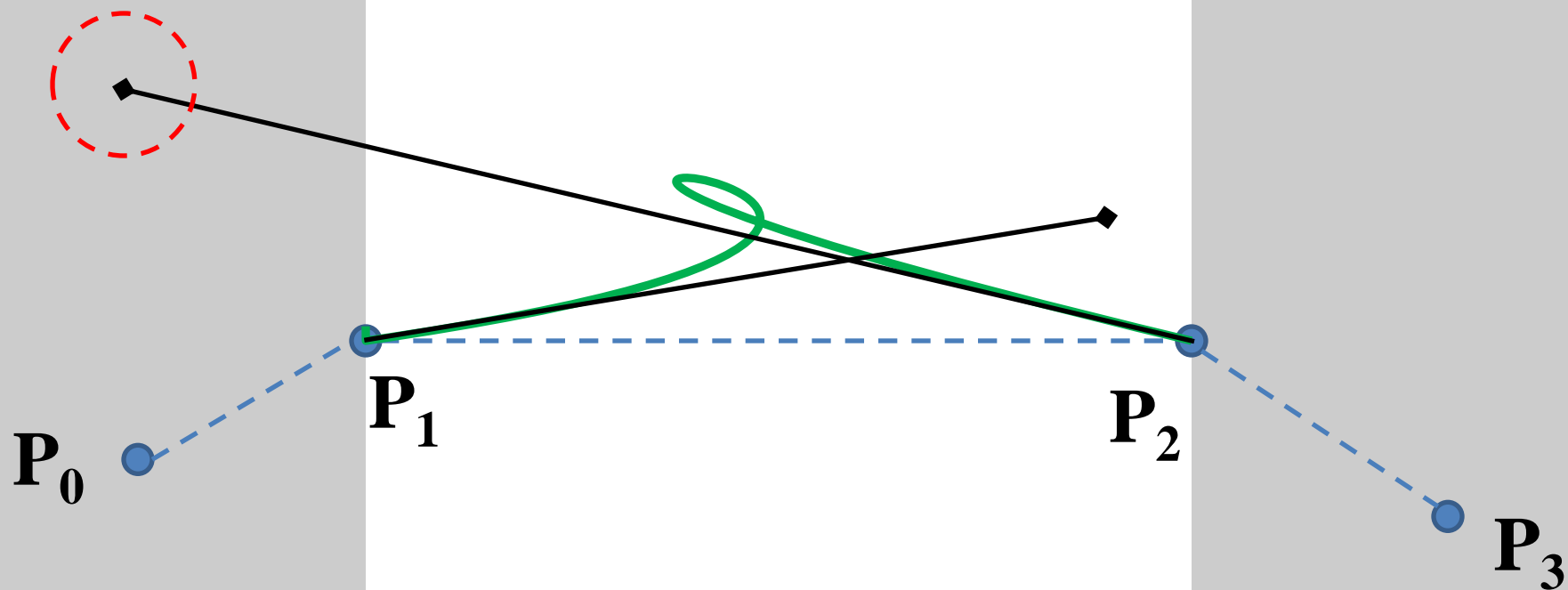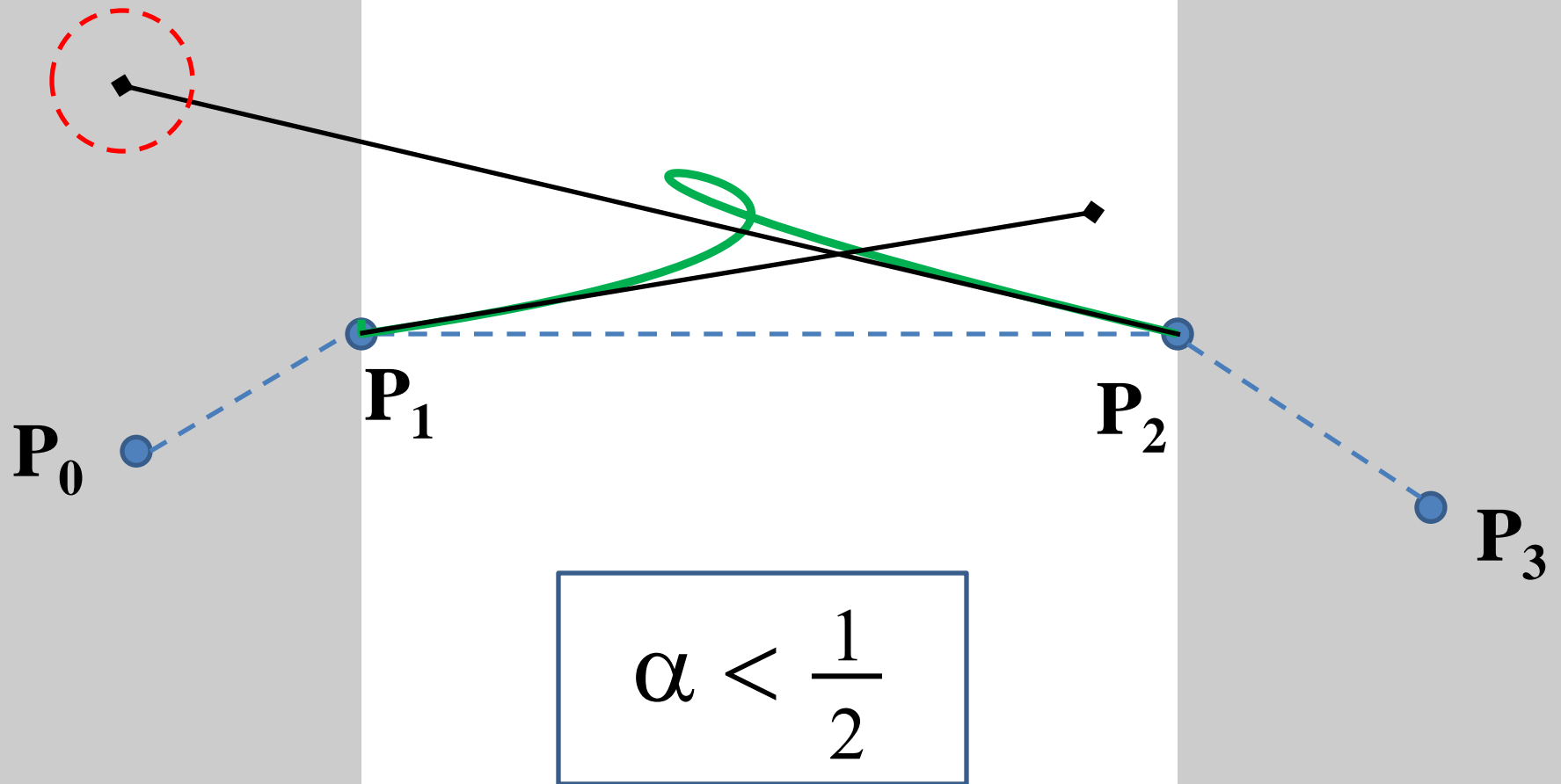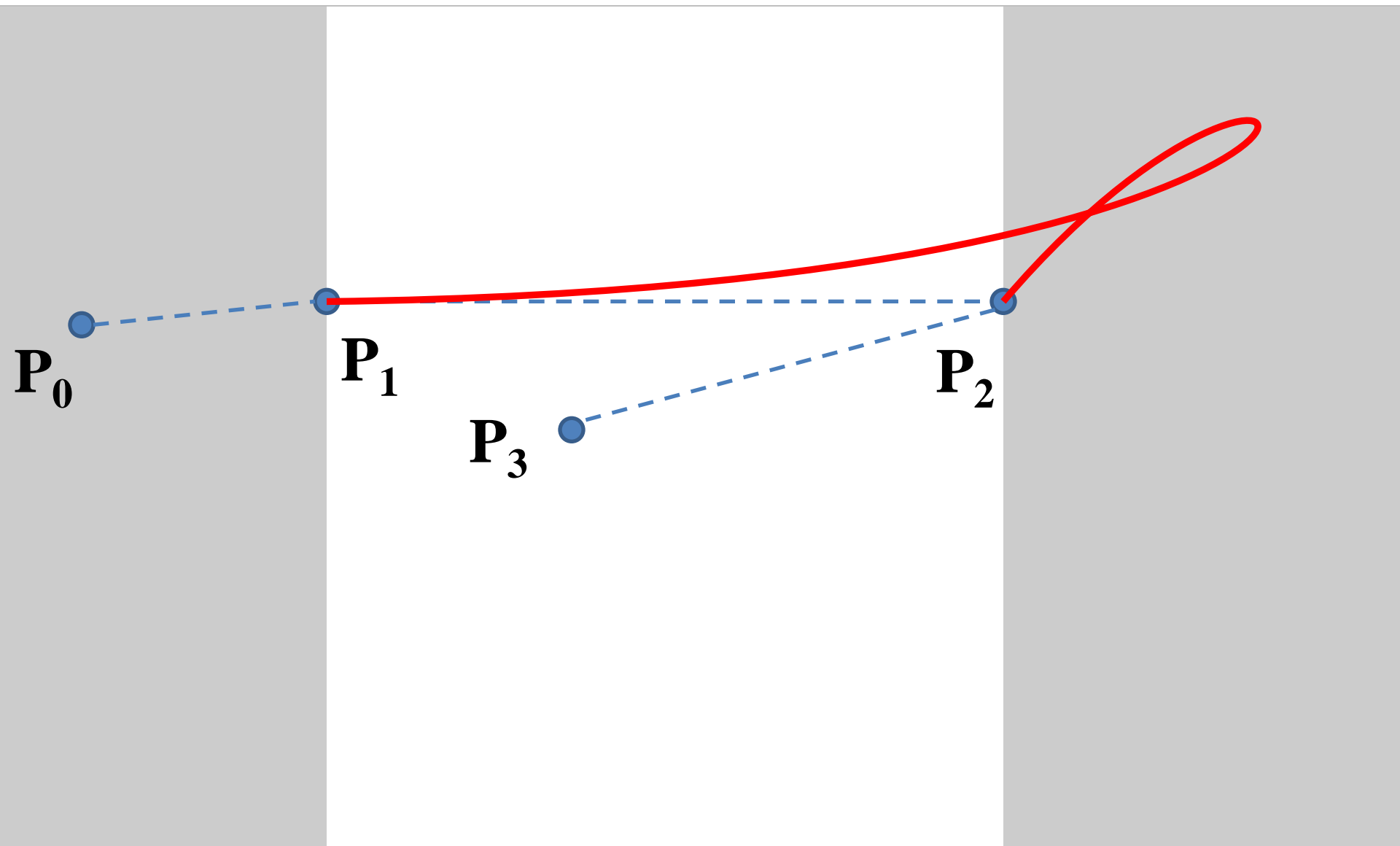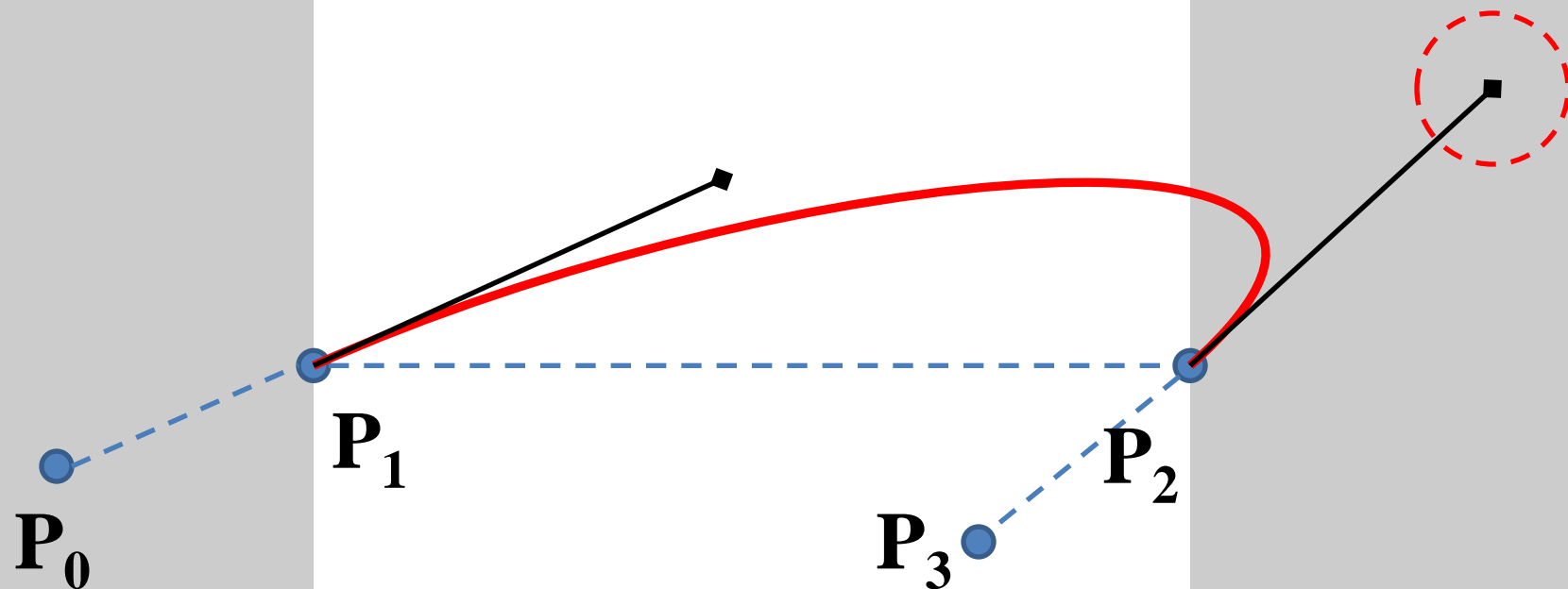
P_3

Cusps & Self-Intersections

$$\alpha < \frac{1}{2}$$
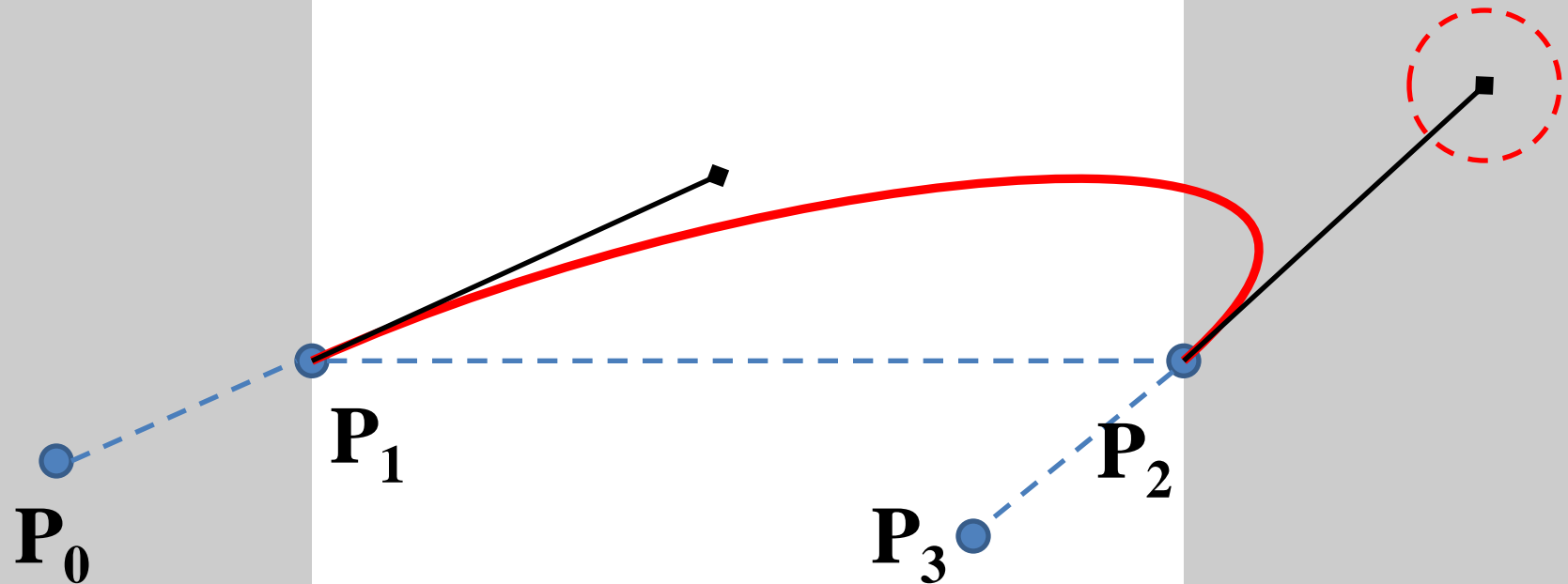
Cusps & Self-Intersections
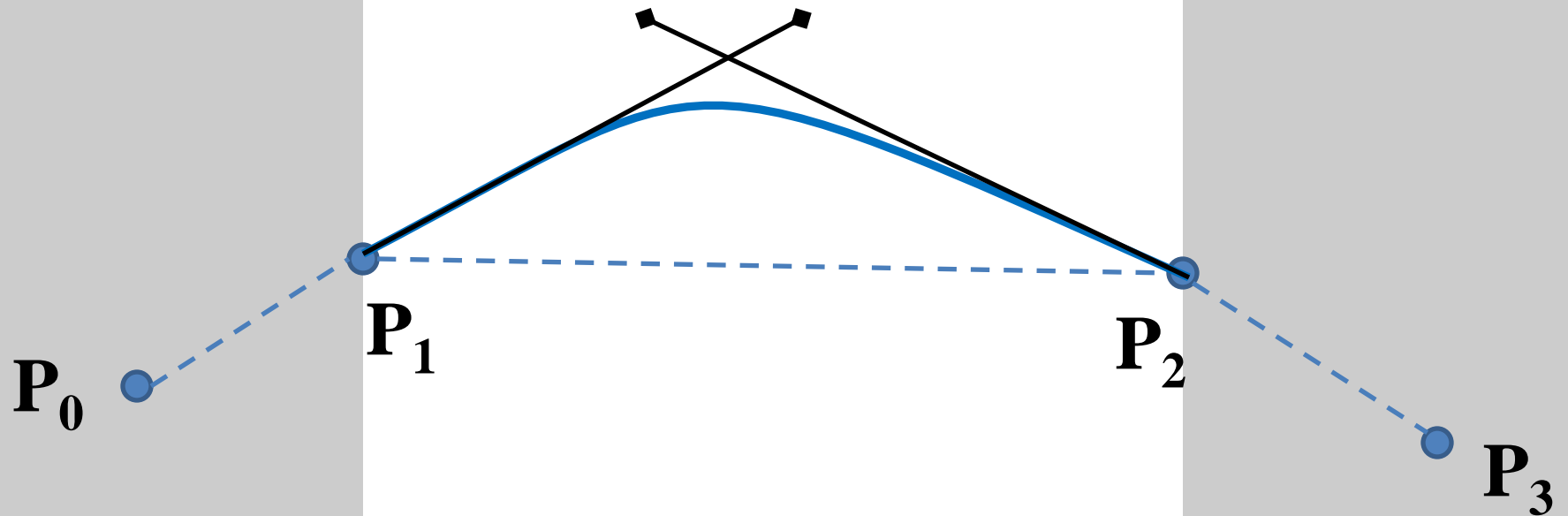
Cusps & Self-Intersections

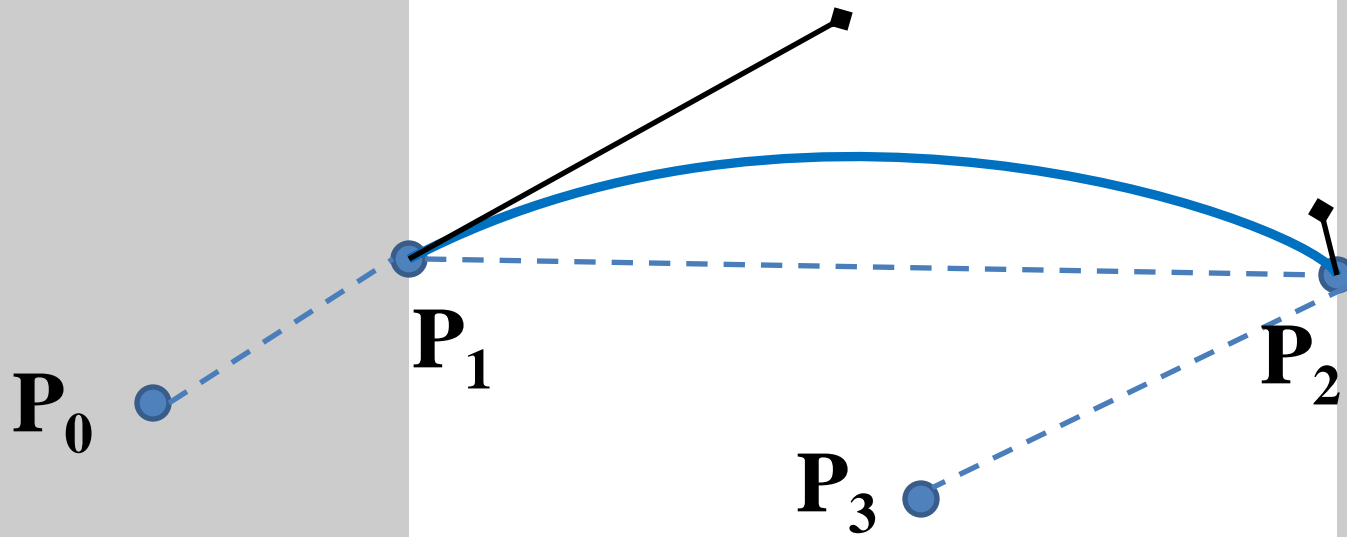# Cusps & Self-Intersections

$$\alpha \neq \frac{1}{2}$$

Cusps & Self-Intersections

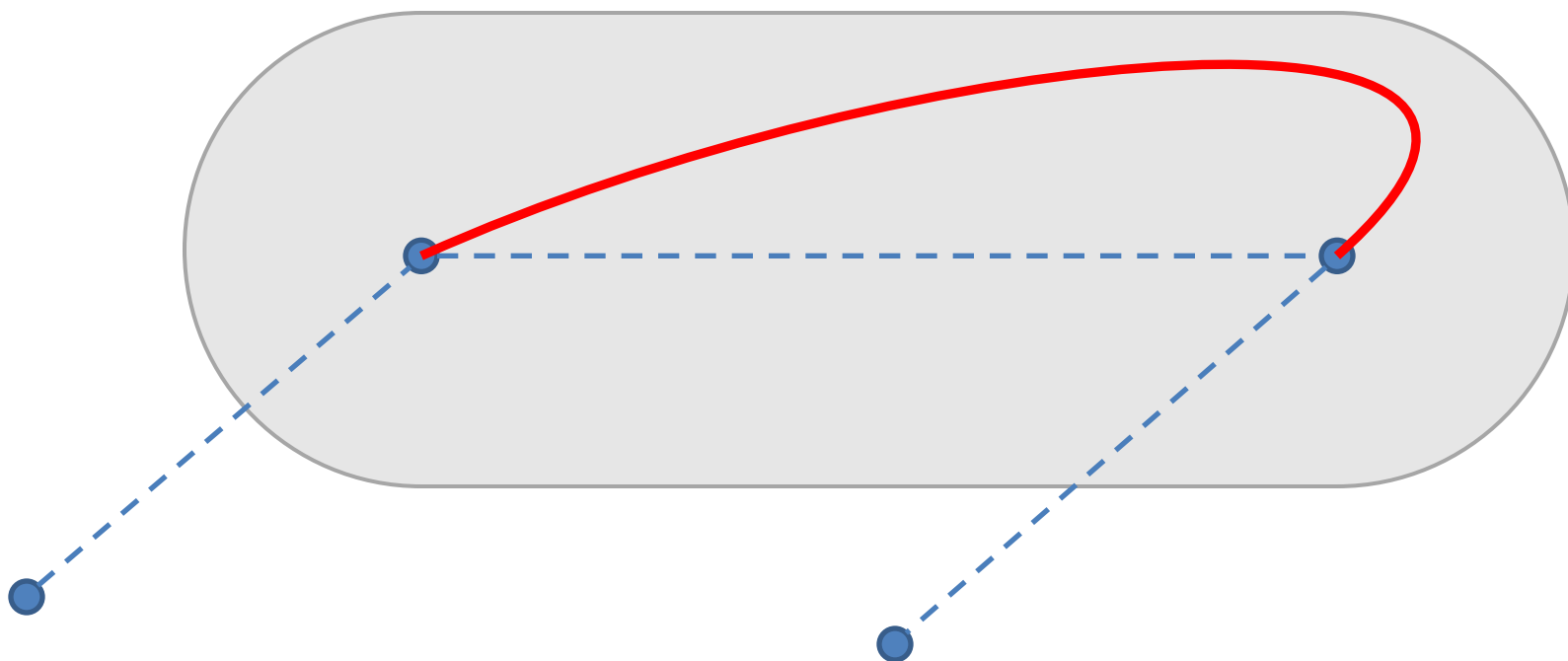

$$\alpha = \frac{1}{2}$$

Cusps & Self-Intersections

$$\alpha = \frac{1}{2}$$

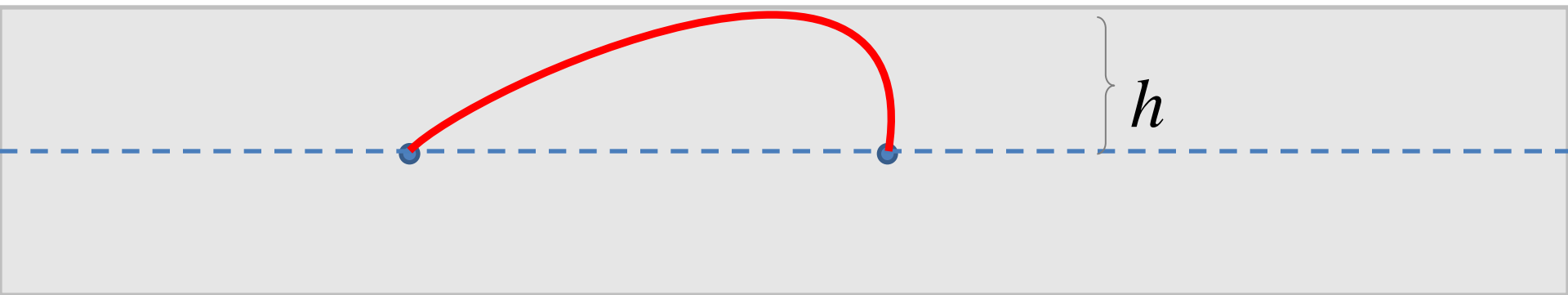On the Parameterization of Catmull-Rom Curves

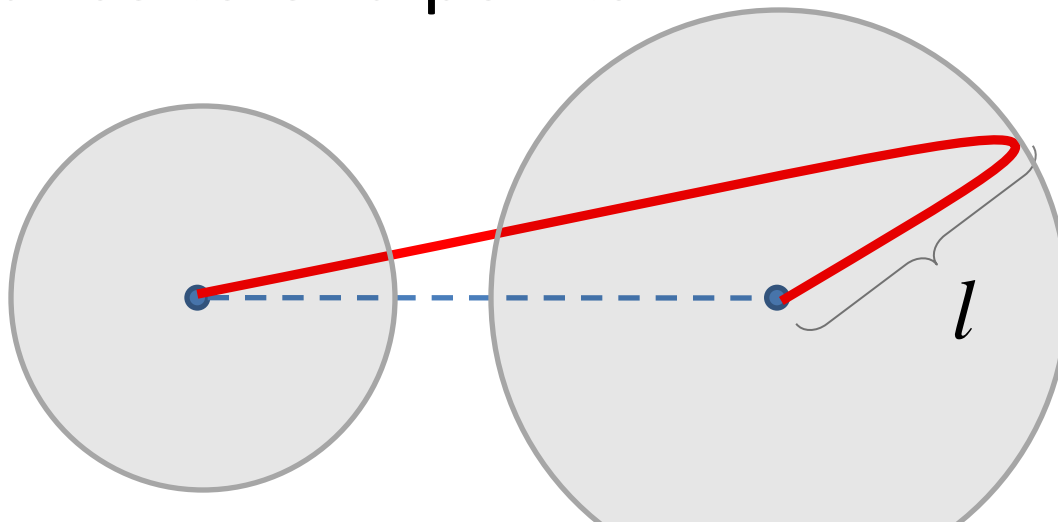# DISTANCE BOUND

Distance Bound

Distance Bound

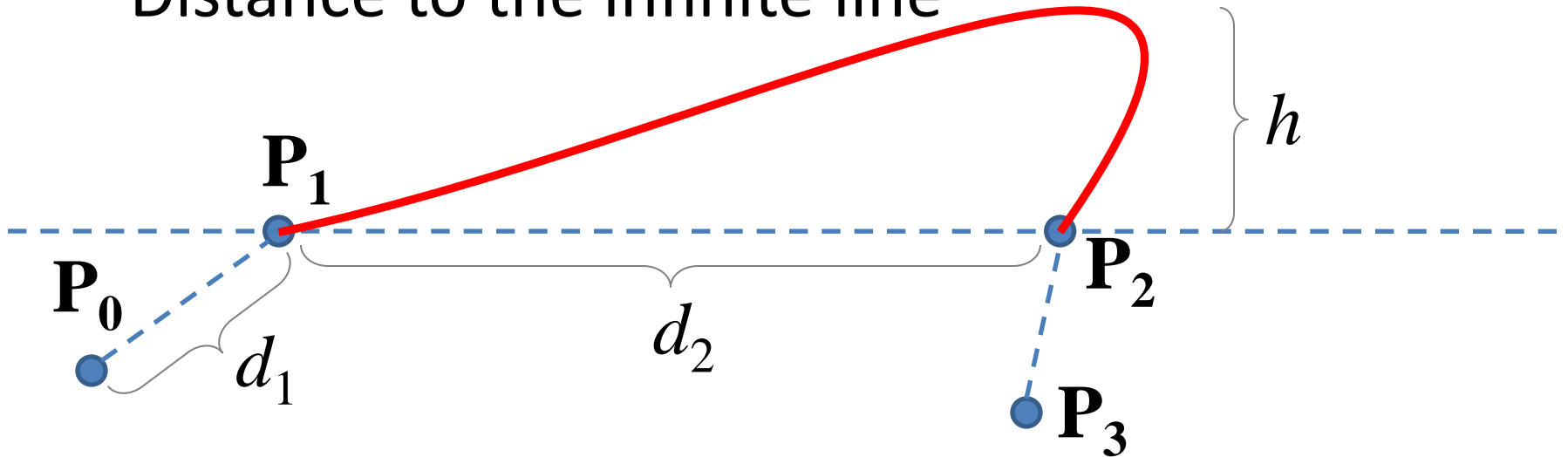- Distance to the infinite line

h
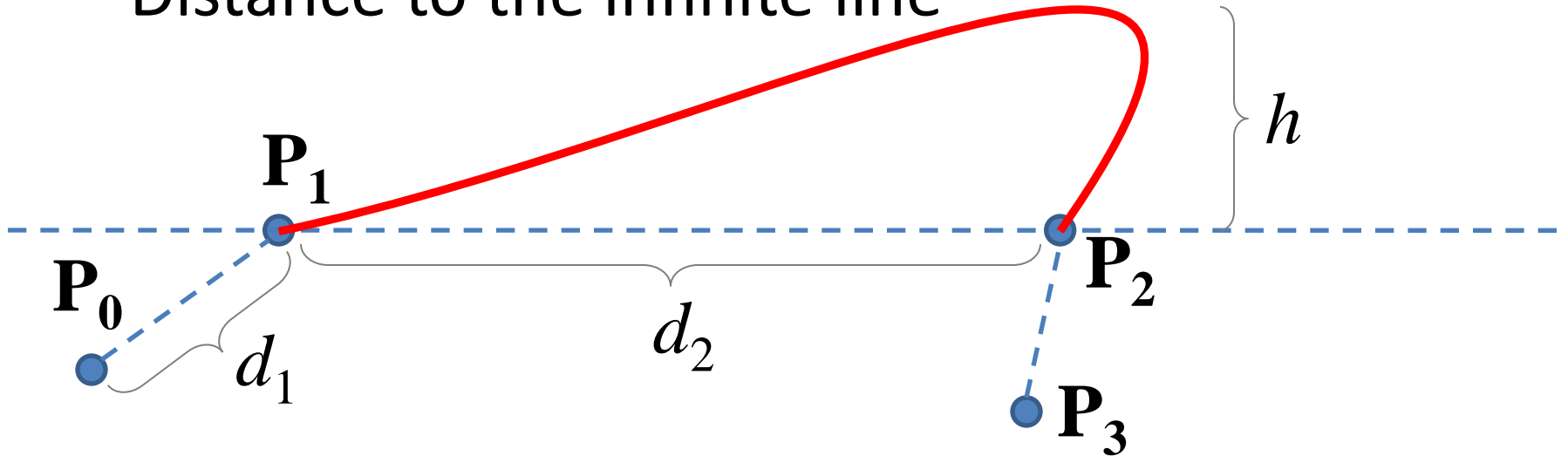
- Distance to end points

l

Distance Bound

- Distance to the infinite line

Distance Bound

- Distance to the infinite line

$$h \ \leq\ d_2 \ \times\ A$$

Distance Bound

- Distance to the infinite line

$$r = \frac{d_1}{d_2}$$

$$h \leq d_2 \, \frac{r^{1-\alpha}}{4\,(1 + r^{\alpha})}$$

Distance Bound

- Distance to the infinite line

$$r = \frac{d_1}{d_2}$$

$$h \le d_2 \frac{r^{1-\alpha}}{4(1 + r^\alpha)}$$

$\alpha < 1/2 \quad h \le \infty$

$\alpha = 1/2 \quad h \le d_2/4$

$\alpha = 2/3 \quad h \le d_2/8$

$\alpha = 1 \quad \ \ h \le d_2/4$
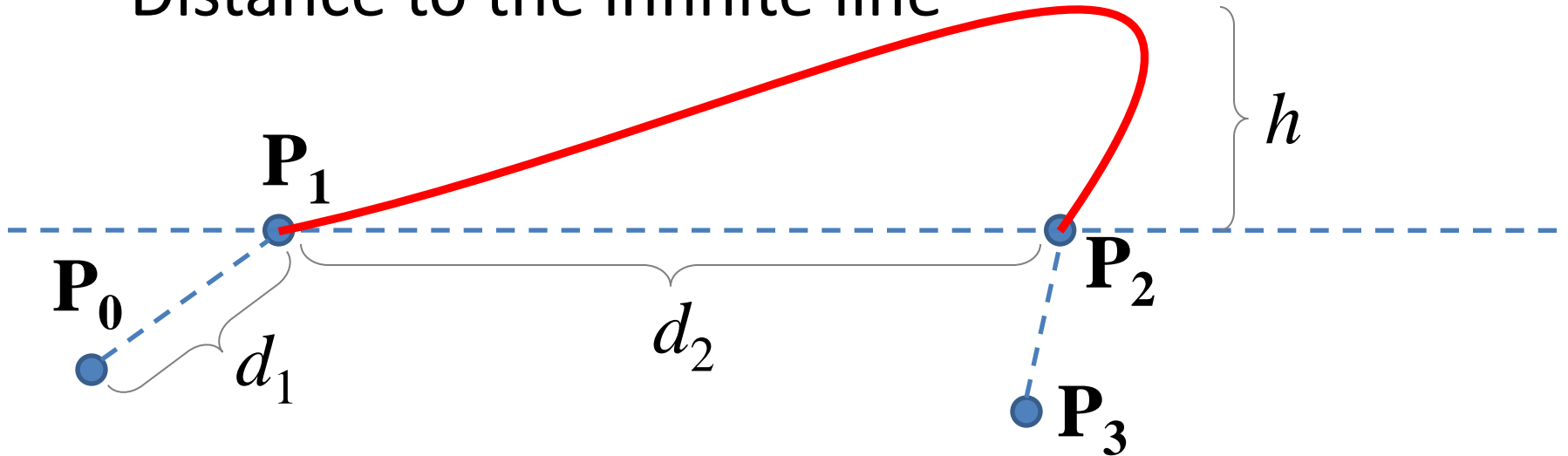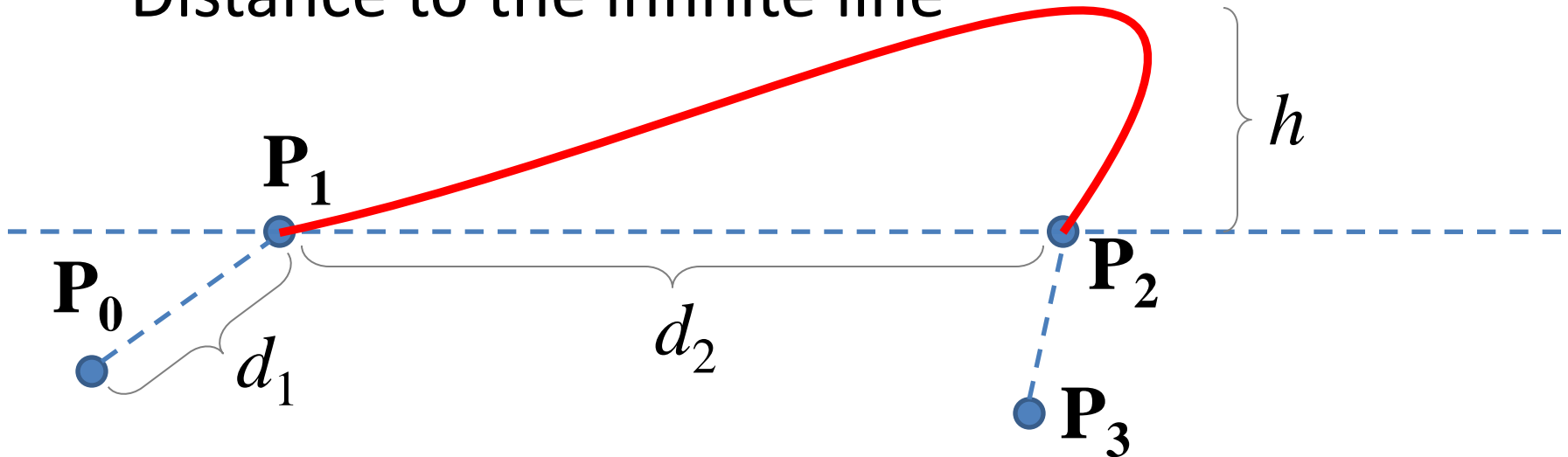
Distance Bound

- Distance to the infinite line

$$r = \frac{d_1}{d_2}$$

$$h \le d_2 \frac{r^{1-\alpha}}{4\,(1 + r^\alpha)}$$

$$\alpha < 1/2 \quad h \le \infty$$
$$\alpha = 1/2 \quad h \le d_2\,/4$$
$$\alpha = 2/3 \quad h \le d_2\,/8$$
$$\alpha = 1 \qquad h \le d_2\,/4$$

Distance Bound

- Distance to the infinite line

$$r = \frac{d_1}{d_2}$$

$$h \leq d_2 \frac{r^{1-\alpha}}{4(1+r^\alpha)}$$
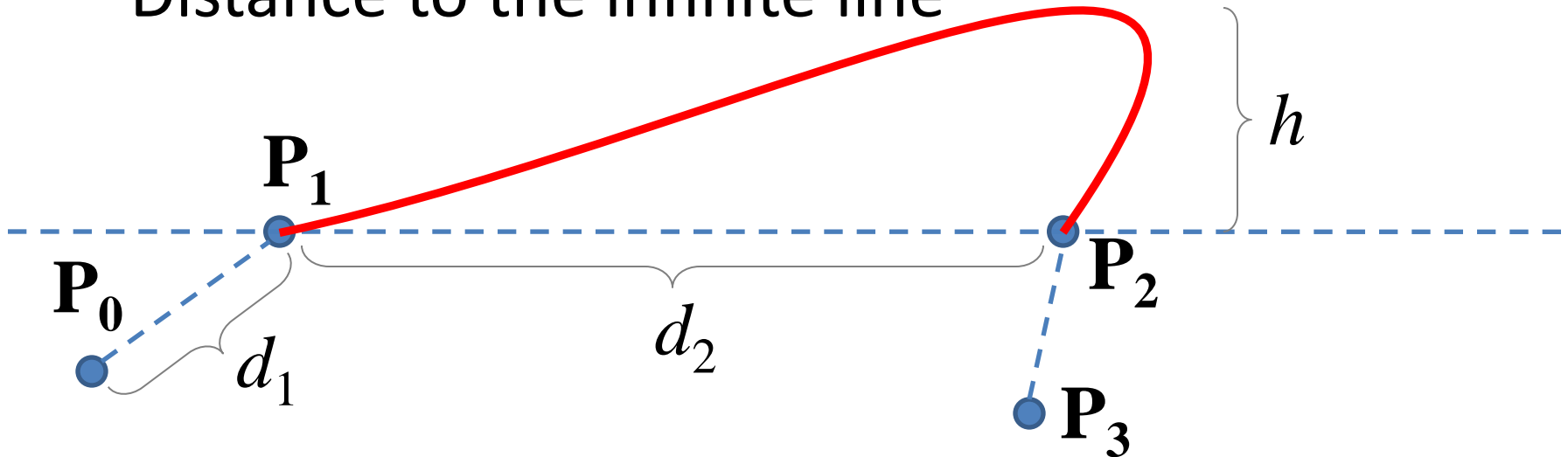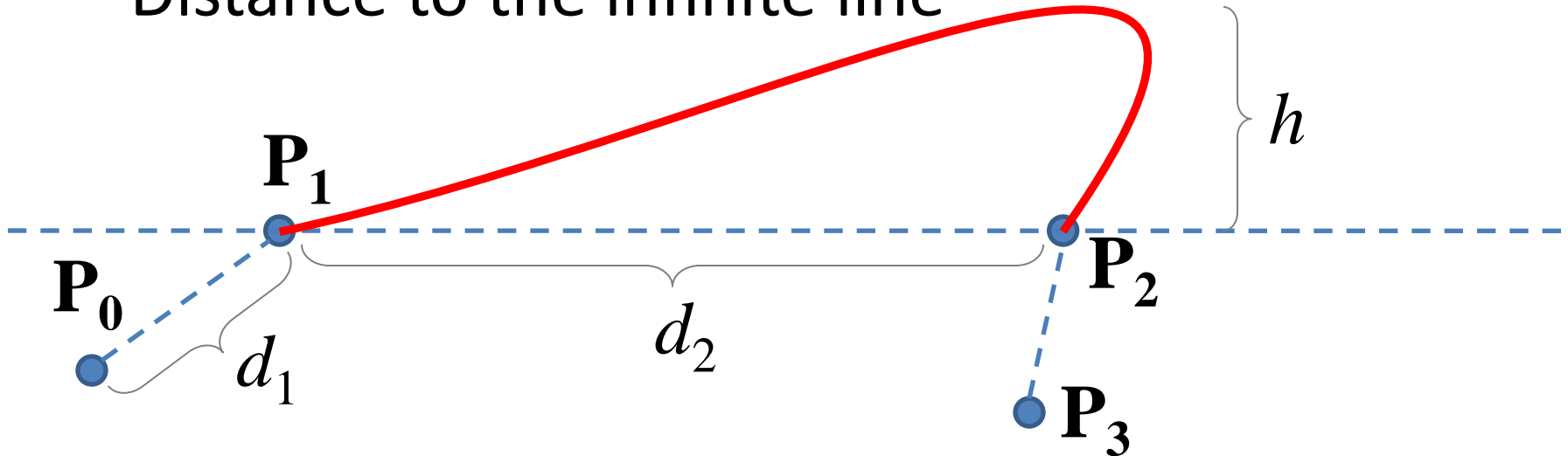
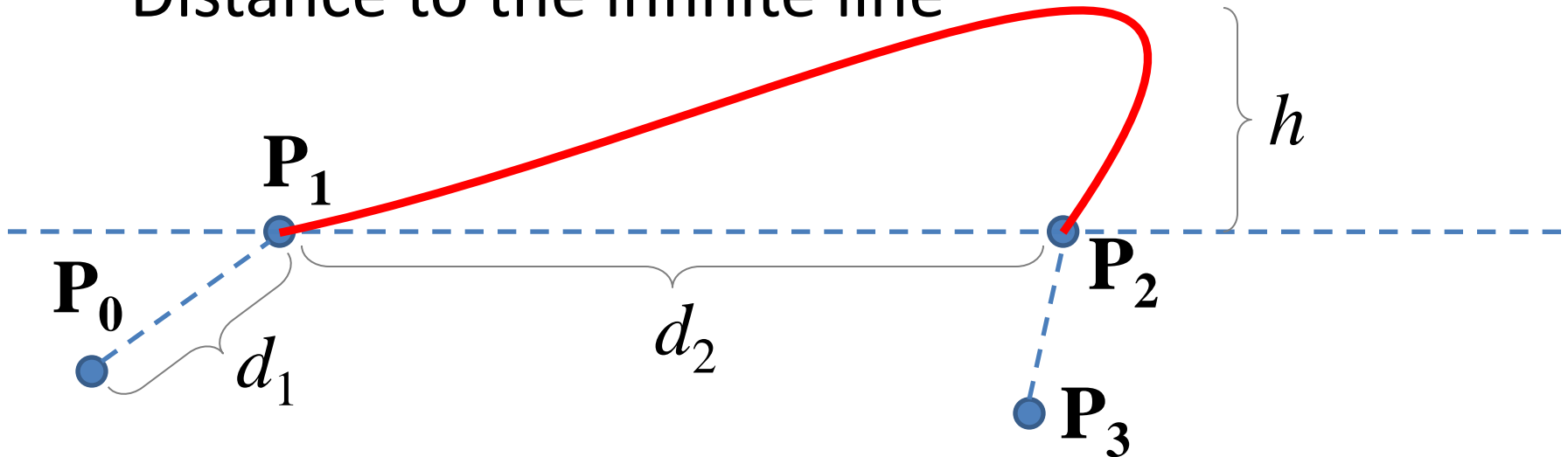| $\alpha < 1/2$	$h \leq \infty$
$\alpha = 1/2$	$h \leq d_2/4$
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$\alpha = 1$	$h \leq d_2/4$

Distance Bound

- Distance to the infinite line

$$r = \frac{d_1}{d_2}$$

$$h \le d_2 \frac{r^{1-\alpha}}{4\,(1 + r^\alpha)}$$
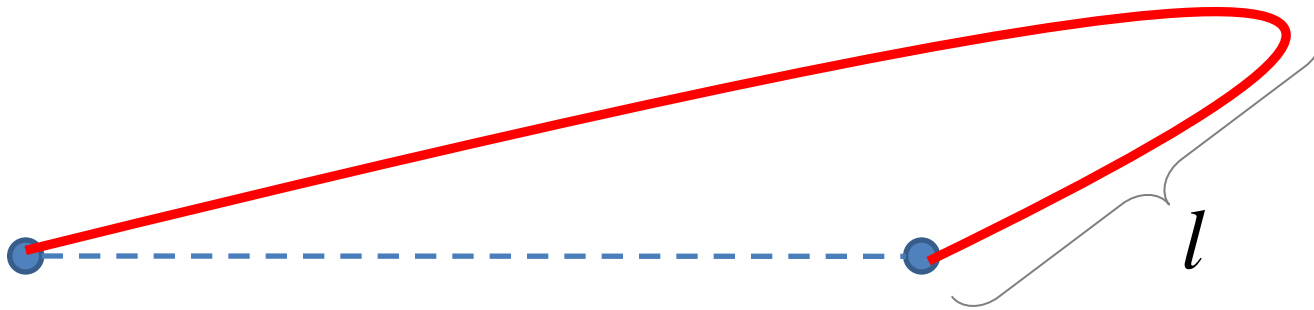
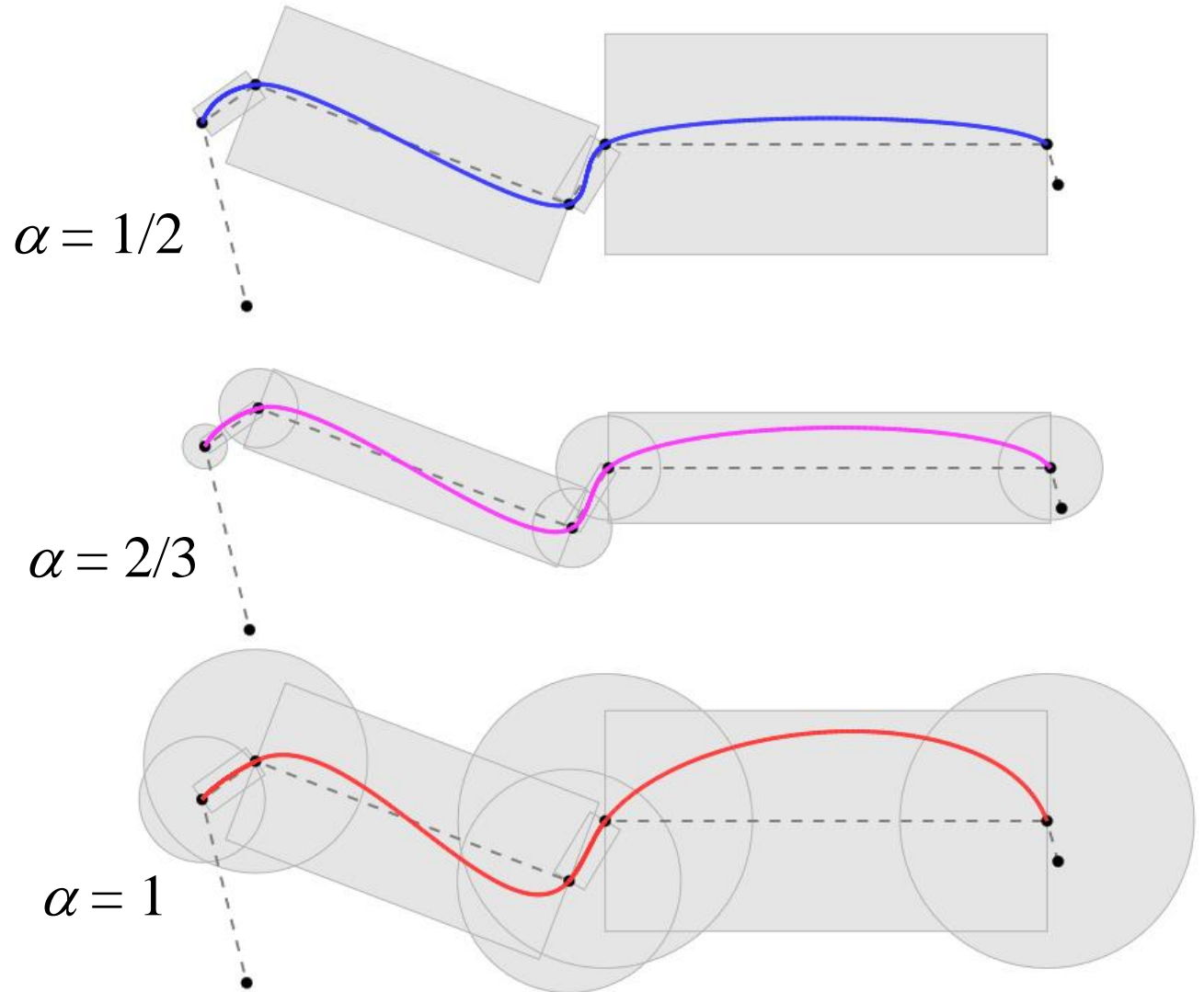|	
$\alpha < 1/2$	$h \le \infty$
$\alpha = 1/2$	$h \le d_2/4$
$\alpha = 2/3$	$h \le d_2/8$
$\alpha = 1$	$h \le d_2/4$

Distance Bound

- Distance to the end points

$$l \leq \frac{d_2 \sqrt{r^2 - r^{4\alpha}}}{3\, r^\alpha (1 + r^\alpha)}$$

Distance Bound

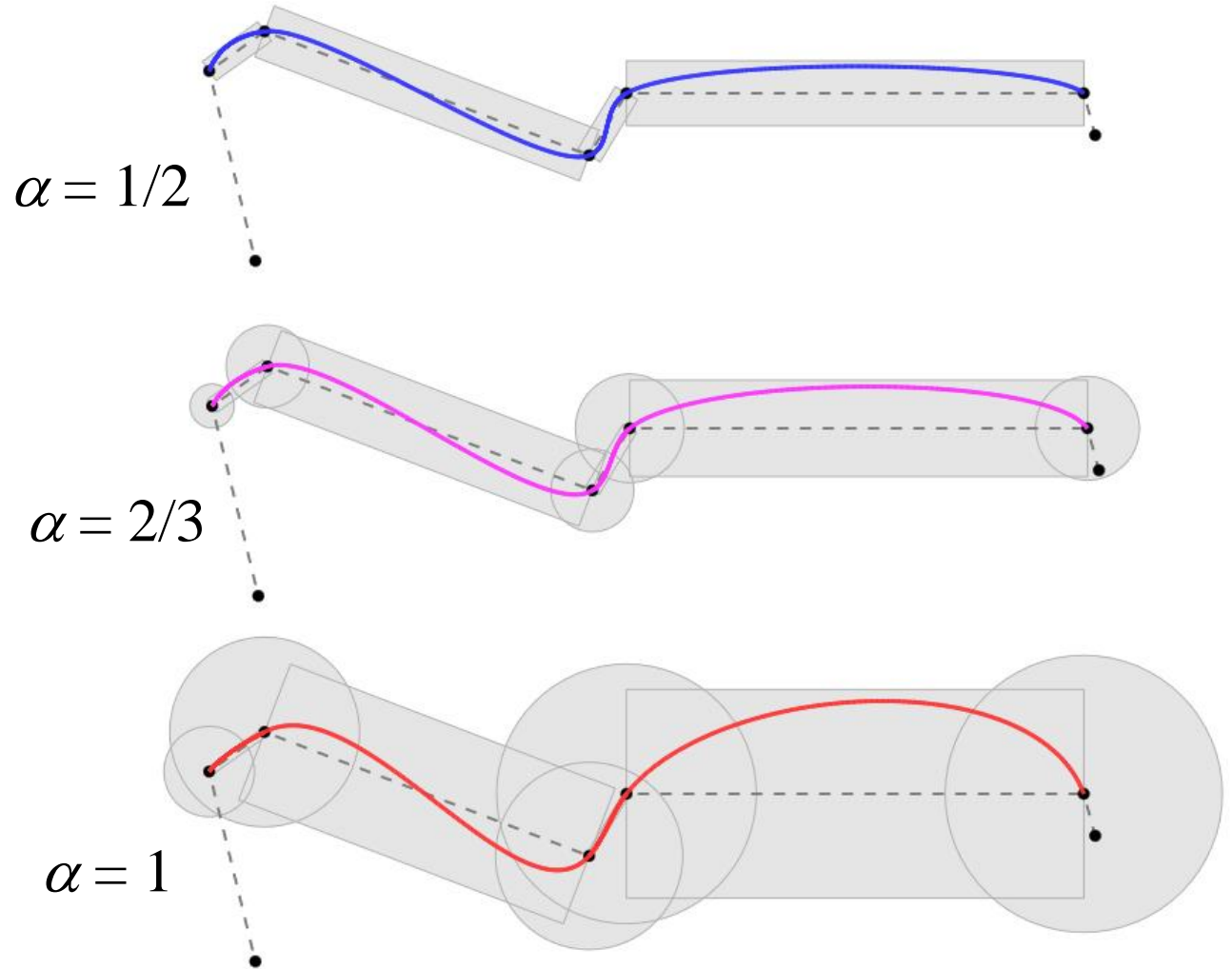
α only

$\alpha = 1/2$

$\alpha = 2/3$

$\alpha = 1$

Distance Bound

α and r

$\alpha = 1/2$

$\alpha = 2/3$

$\alpha = 1$
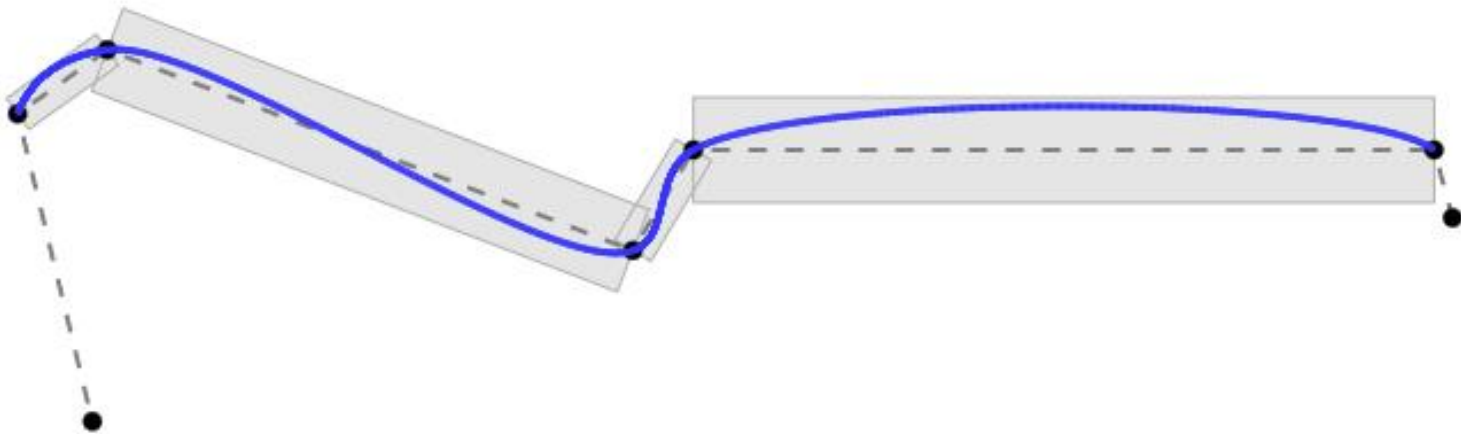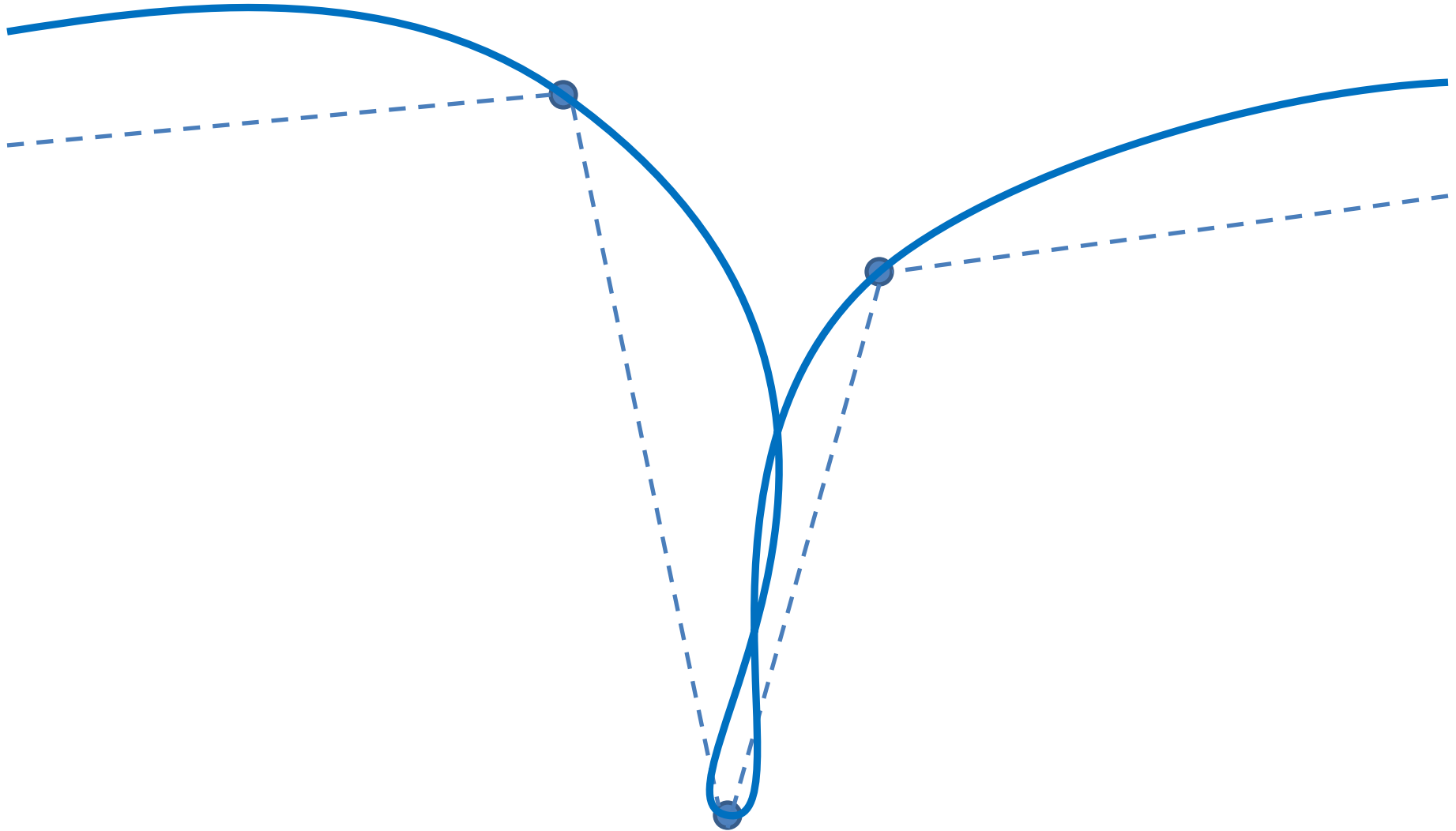
On the Parameterization of Catmull-Rom Curves
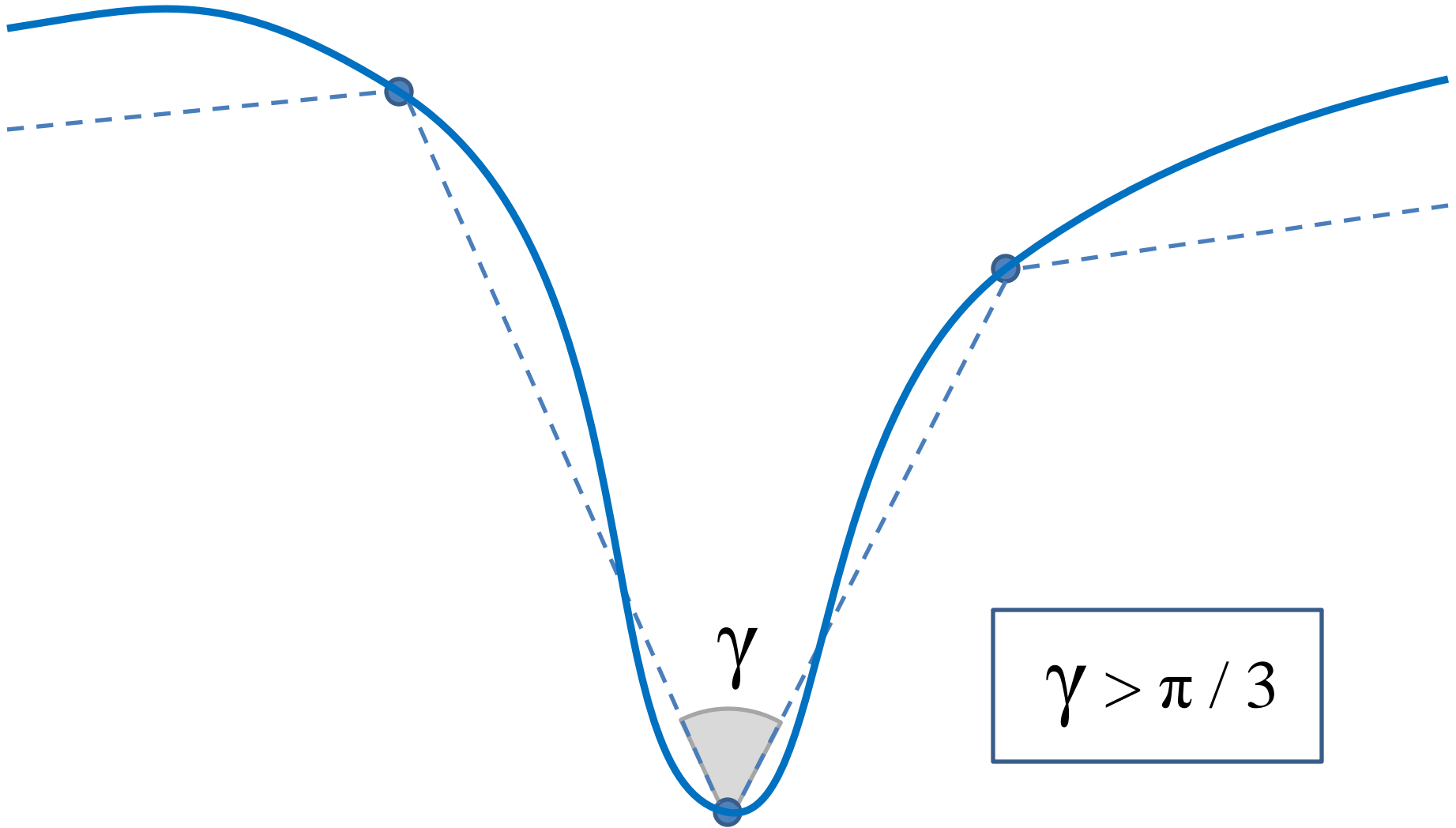
INTERSECTION-FREE CURVES

Intersection-Free Curves

Intersection-Free Curves

Intersection-Free Curves

γ

$\gamma > \pi / 3$

Intersection-Free Curves

γ

$\gamma > \pi\,/\,3$
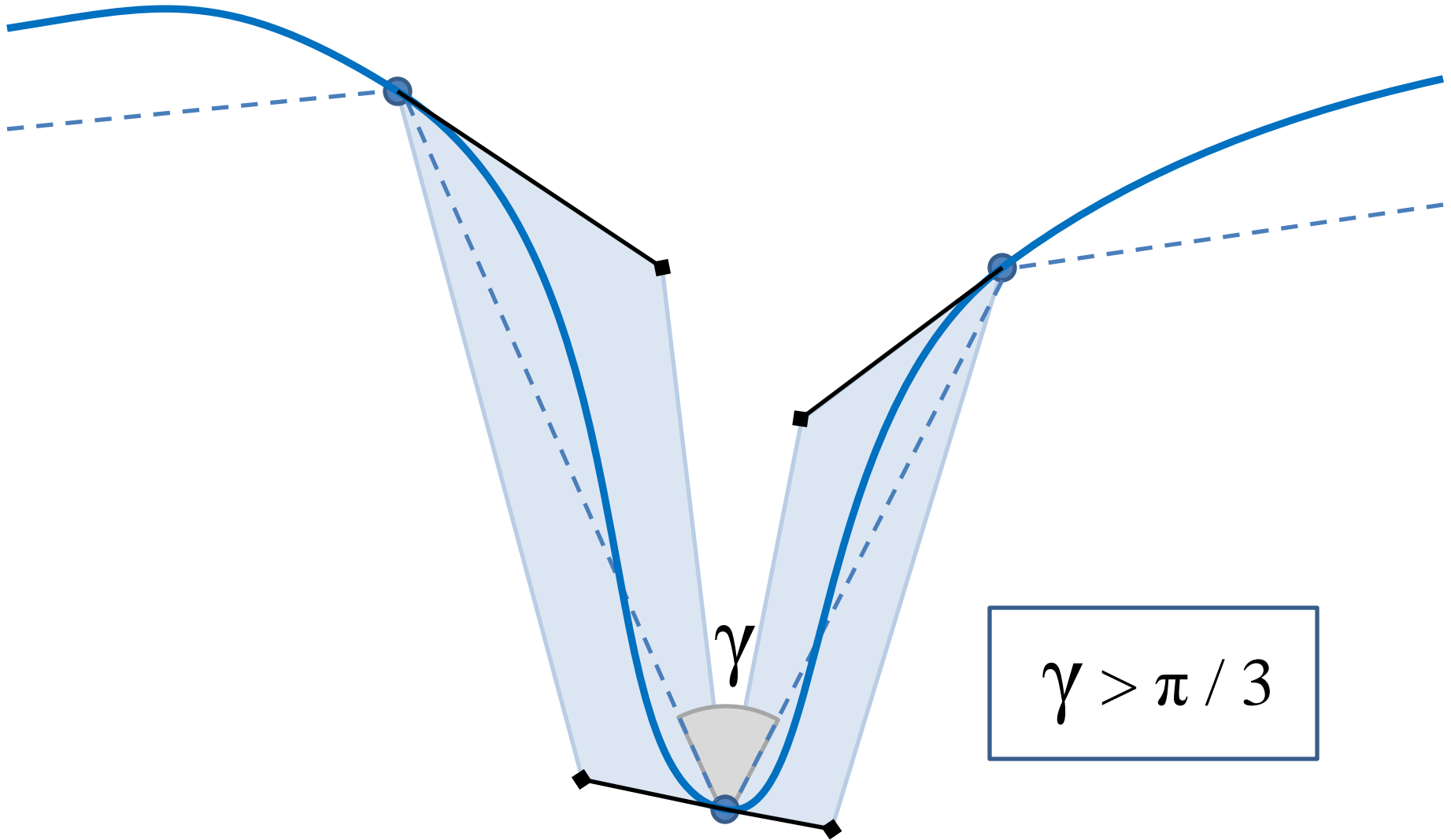
Intersection-Free Curves

- Avoid self-intersections
  - Centripetal parameterization

- Avoid adjacent segment intersections
  - Control polygon angle $> \pi / 3$

- Avoid non-adjacent segment intersections
  - Bounding box
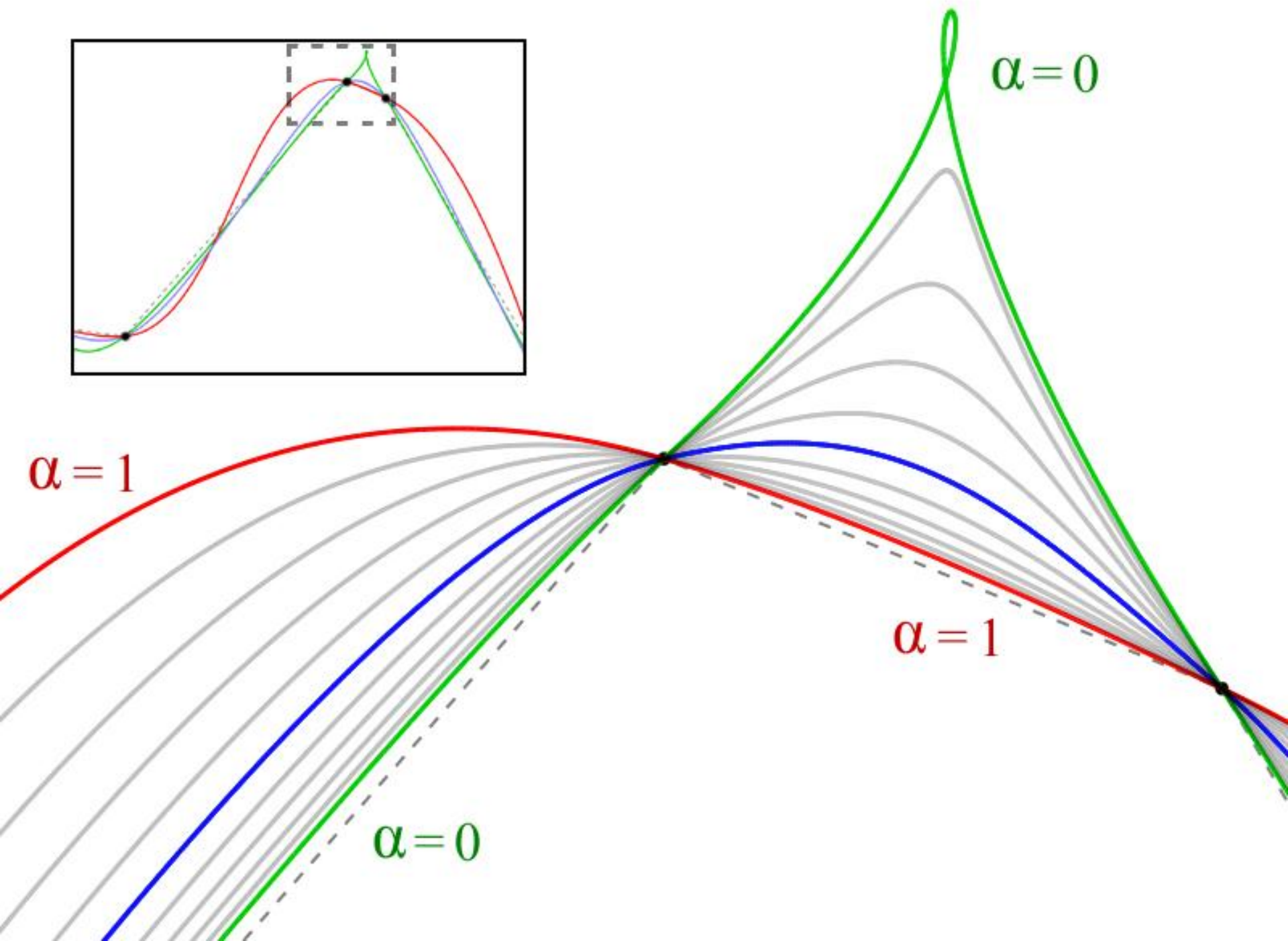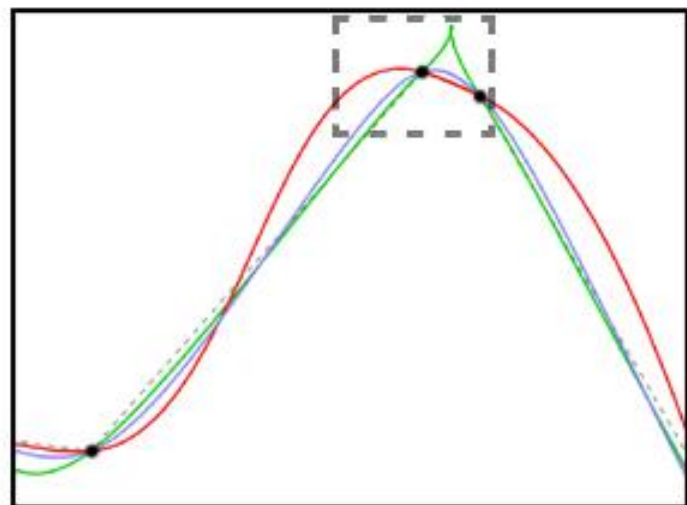
On the Parameterization of Catmull-Rom Curves

DISCUSSION

Discussion

- Distance to Control Polygon
  - Uniform is closer for longer segments
  - Chordal is closer for shorter segments

$\alpha = 0$

$\alpha = 1$

$\alpha = 1$

$\alpha = 0$

Discussion

- Edge Direction
  - Chordal has extreme sensitivity to short edge directions

Discussion

- Edge Direction
  - Chordal has extreme sensitivity to short edge directions
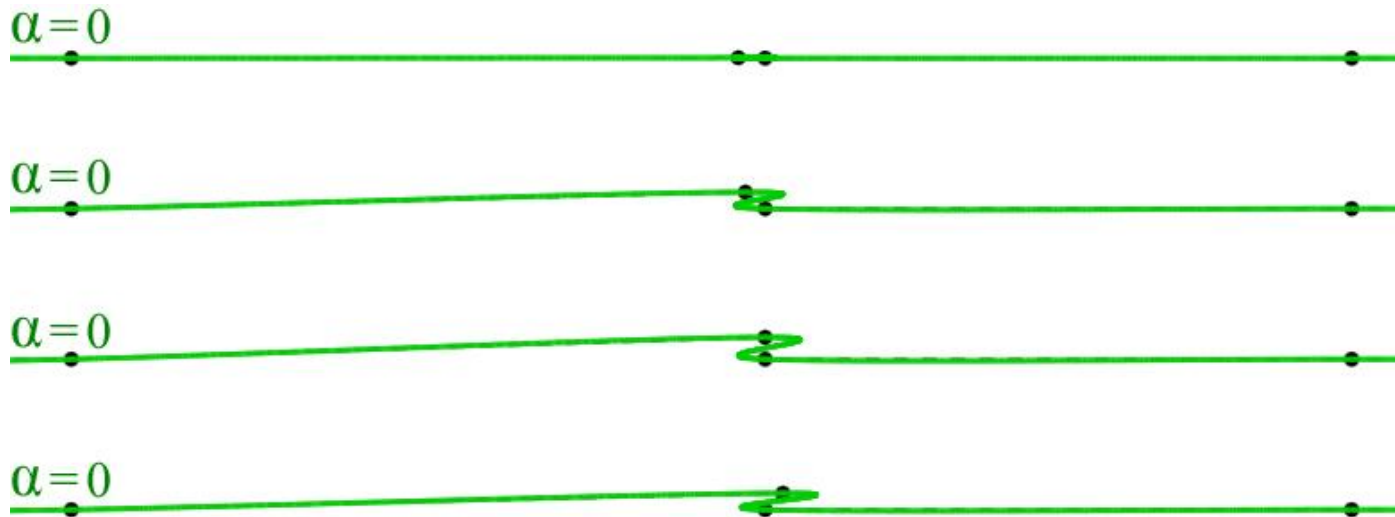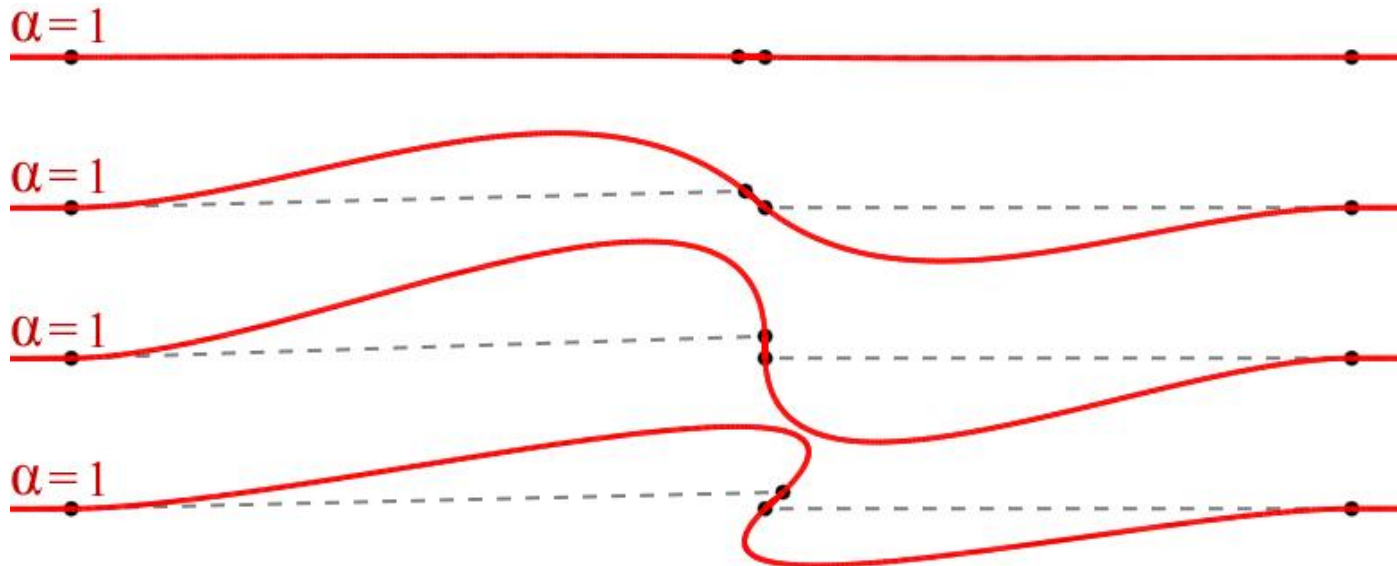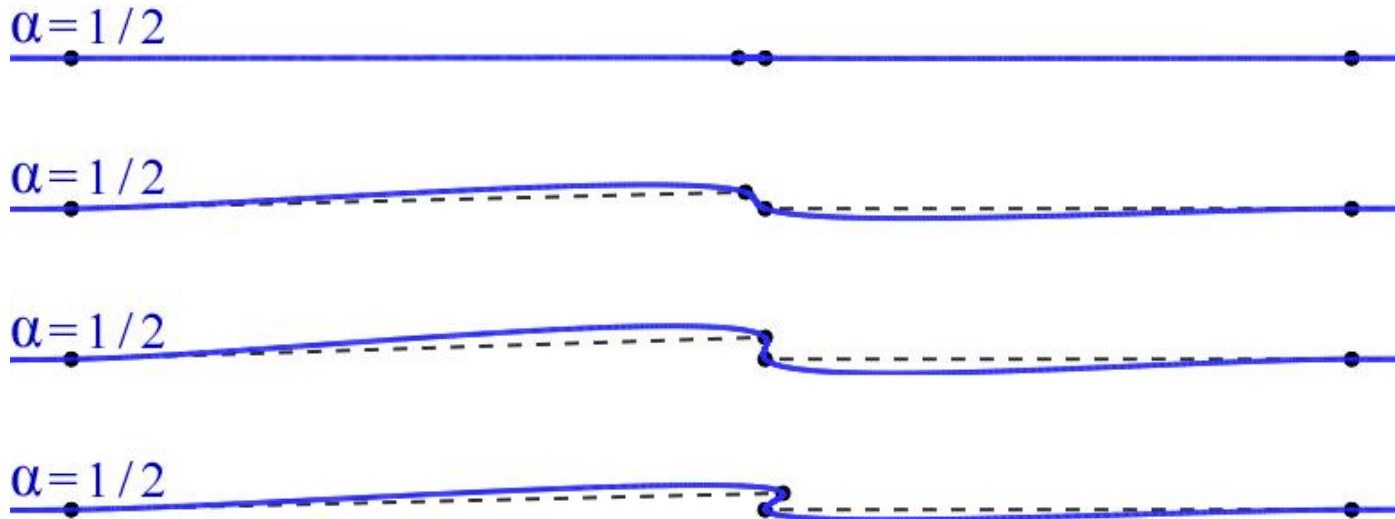
Discussion

- Edge Direction
  - Chordal has extreme sensitivity to short edge directions

Discussion

- Edge Direction
  - Chordal has extreme sensitivity to short edge directions

Discussion

- Curvature
  - Centripetal *tends* to have higher curvature at control points.

Discussion

- Curvature
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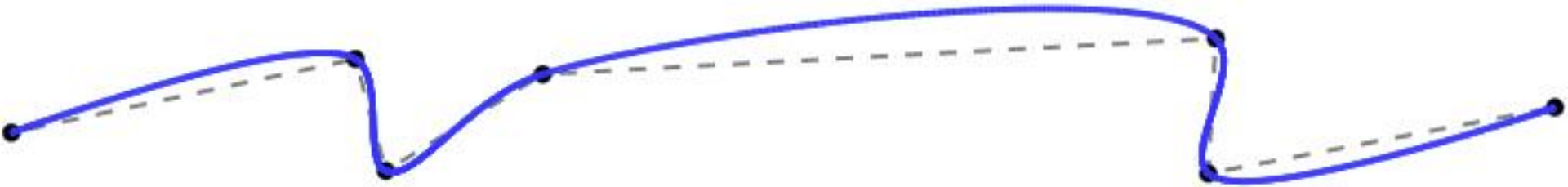
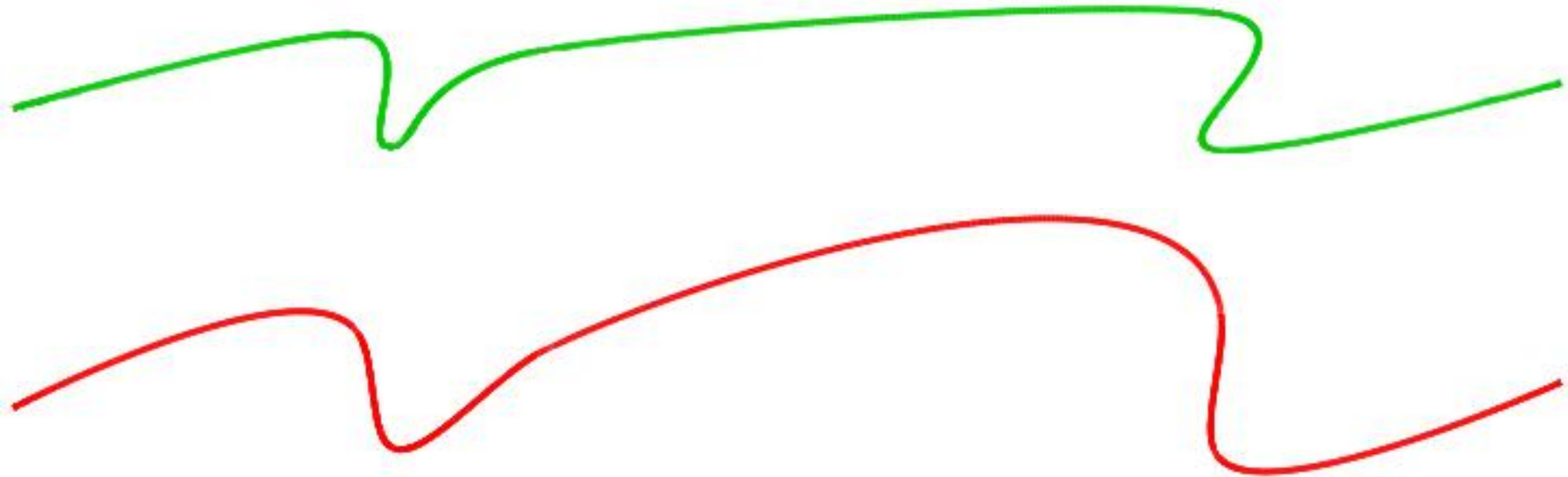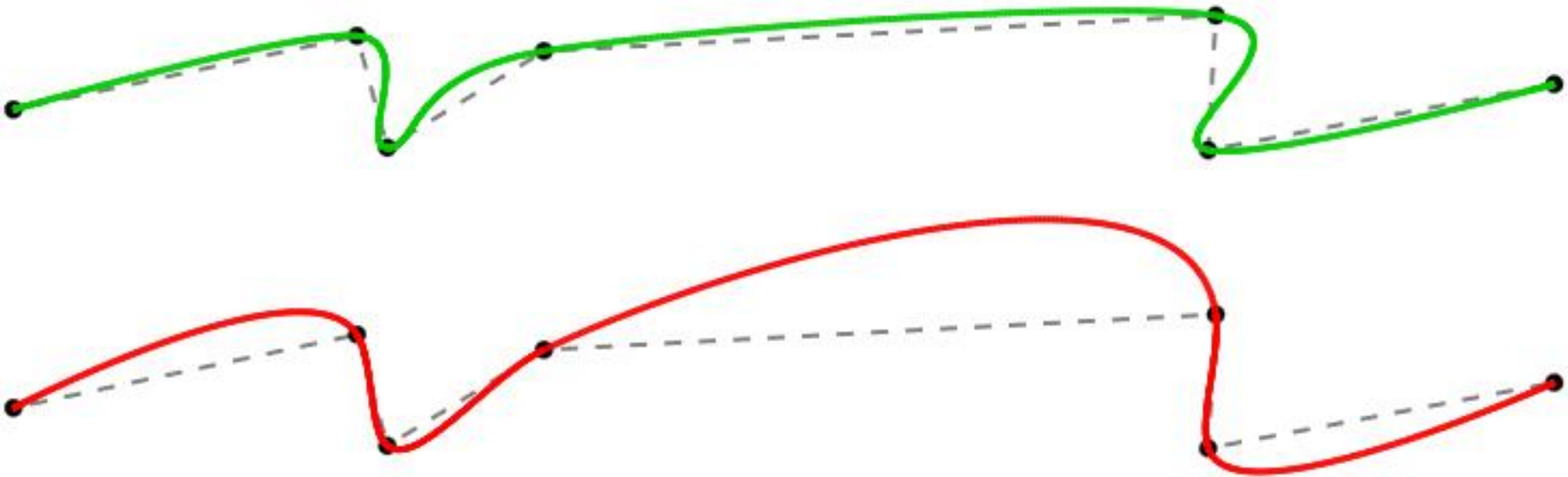# Discussion

- Curvature
  - Centripetal *tends* to have higher curvature at control points.

Discussion

- Curvature
  - Centripetal *tends* to have higher curvature at control points.

Catmull-Rom Curves
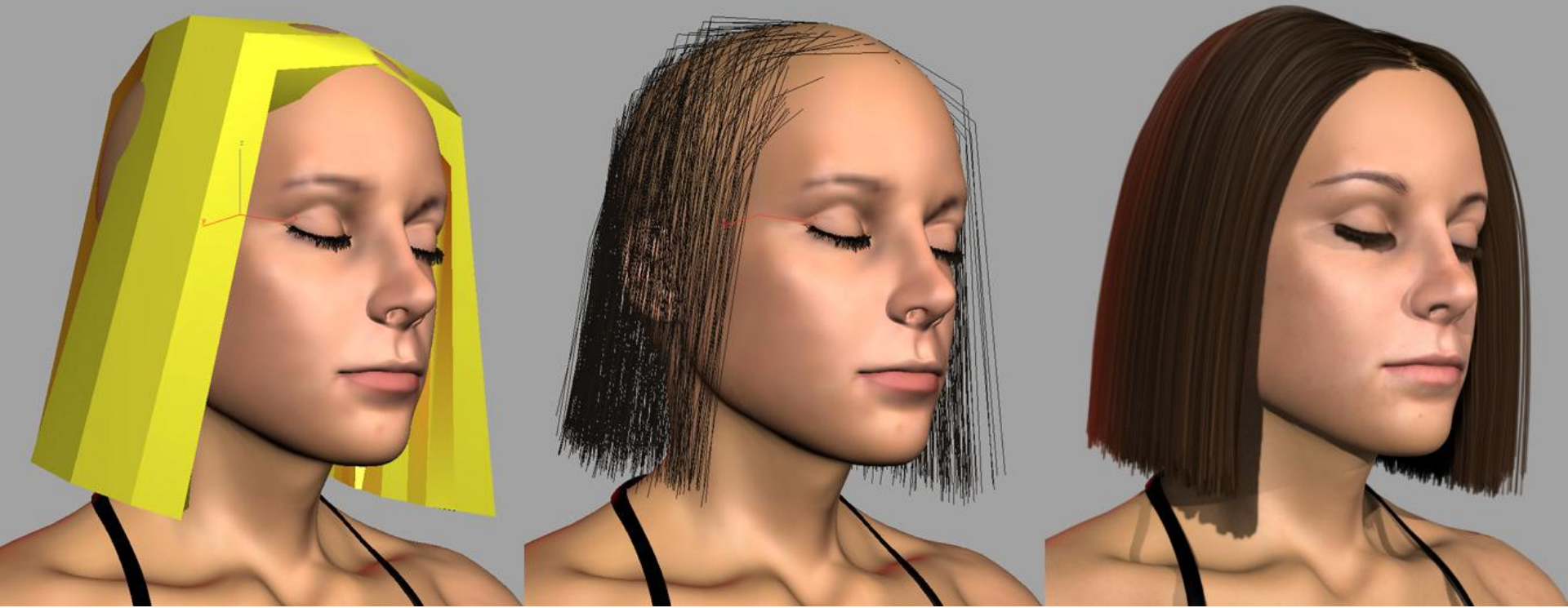

Lee Perry-Smith
Lee Perry-Smith
Alexander Tomchuk

Cem Yuksel, Scott Schaefer, John Keyser, "Hair Meshes," Siggraph Asia 2009
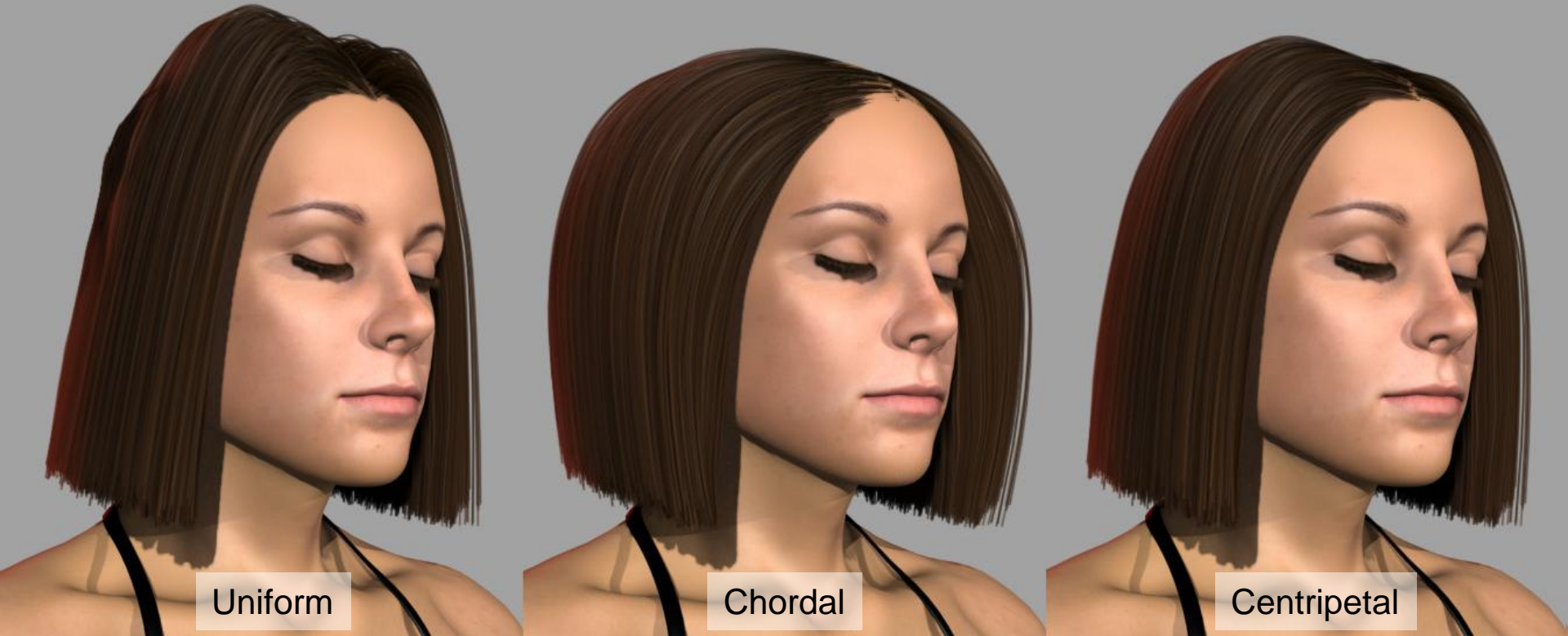
HAIRFARM™
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Catmull-Rom Curves

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Catmull-Rom Curves

Uniform   Chordal   Centripetal

Cem Yuksel, Scott Schaefer, John Keyser, "Hair Meshes," Siggraph Asia 2009

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Summary

- Parameterization of Catmull-Rom curves

$$0 \leq \ \alpha \ \leq \ 1$$

- Cusps and self-intersections

- Distance bound

- Intersection-free curves

- C^1 Catmull-Rom curves only!

On the Parameterization of Catmull-Rom Curves

QUESTIONS?