Barycentric Coordinates for Closed Curves

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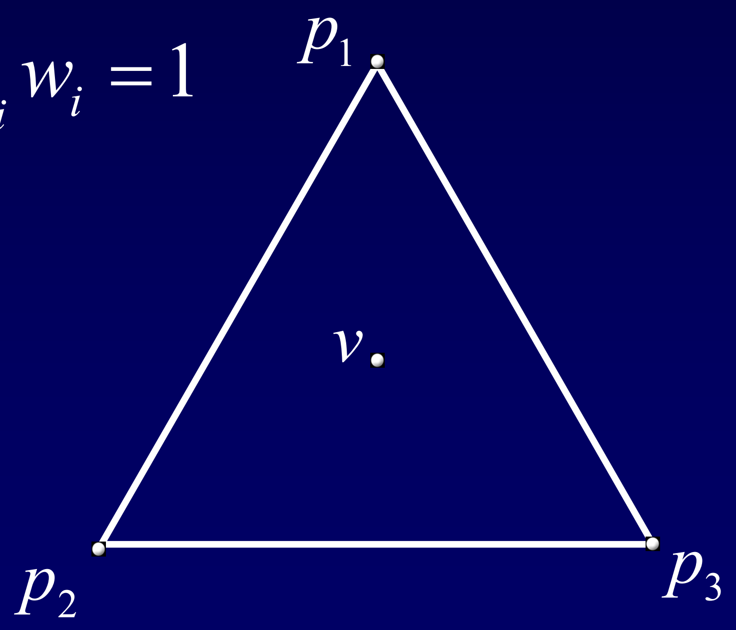
RICE COMPUTER SCIENCE

Barycentric Coordinates

- Given v find weights w_i such that
$$v = \sum_i w_i p_i$$
- Smoothness: $\frac{\partial w_i}{\partial v}$ is continuous
- Translation invariance: $\sum_i w_i = 1$



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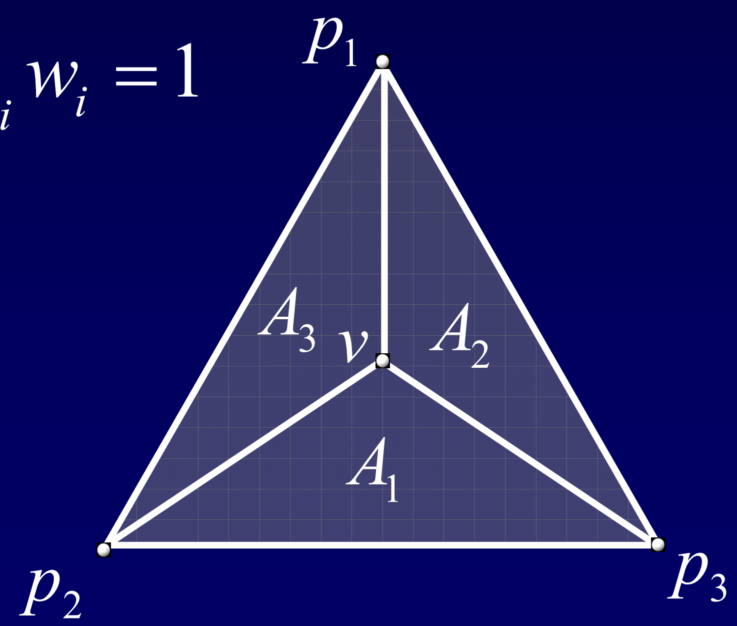

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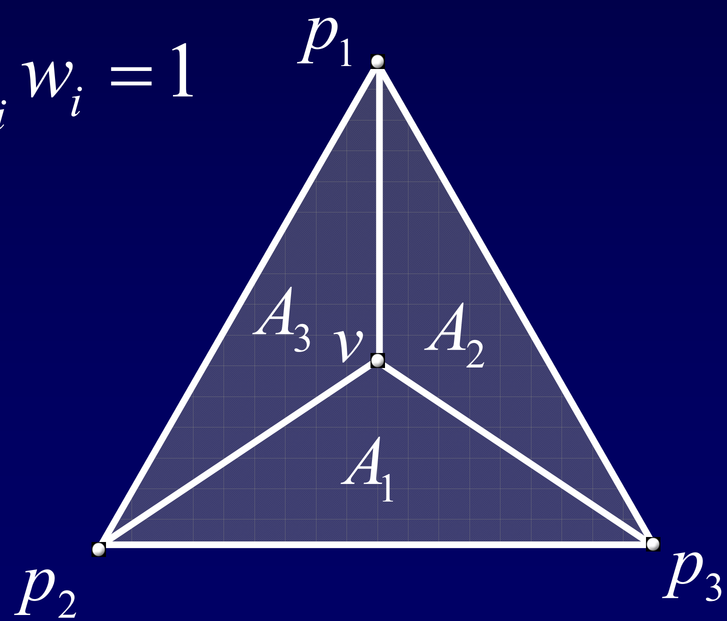

$$w_i = \frac{A_i}{\sum_j A_j}$$

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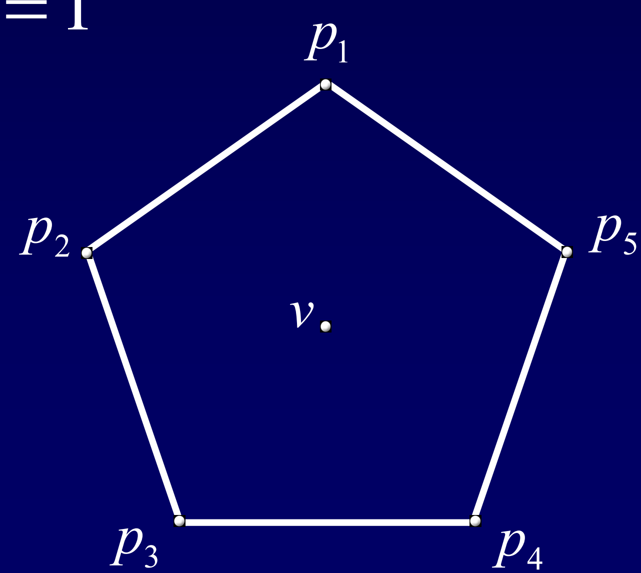
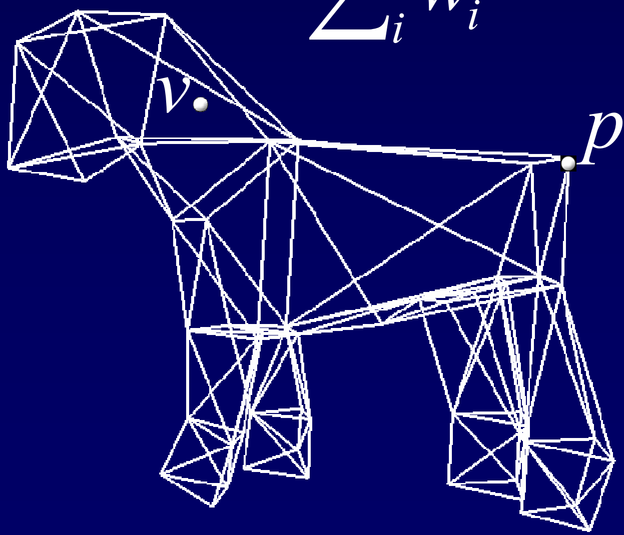


Applications

- Free-Form Deformations

  [Sederberg et al 86], [MacCracken et al 96], [Ju et al 05]

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$



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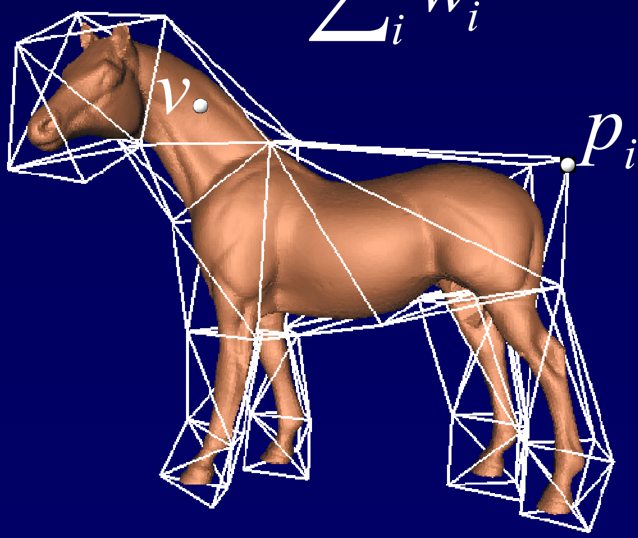

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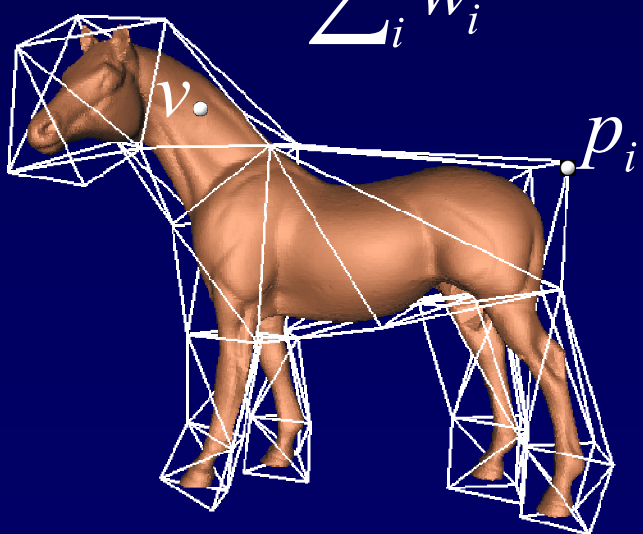
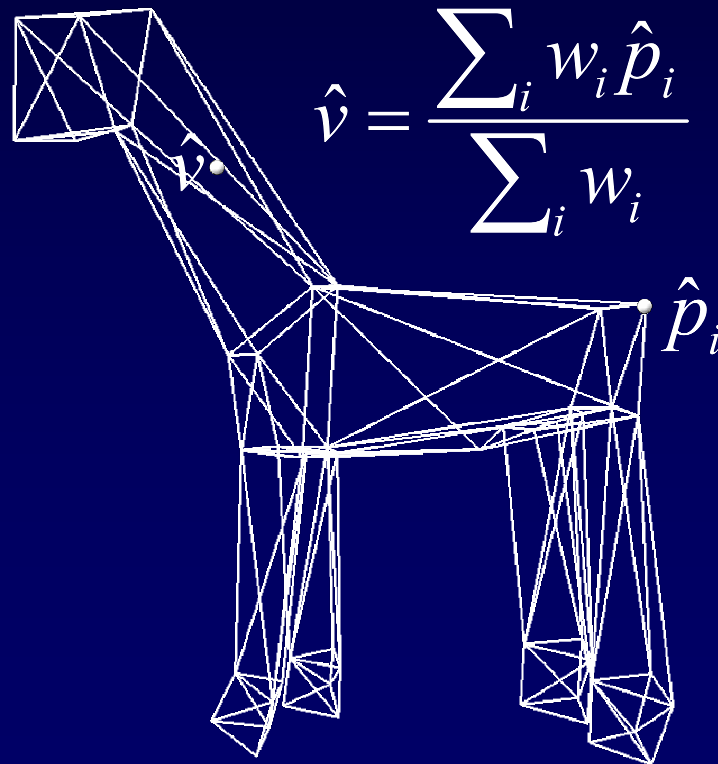


$$\hat{v} = \frac{\sum_i w_i \hat{p}_i}{\sum_i w_i}$$



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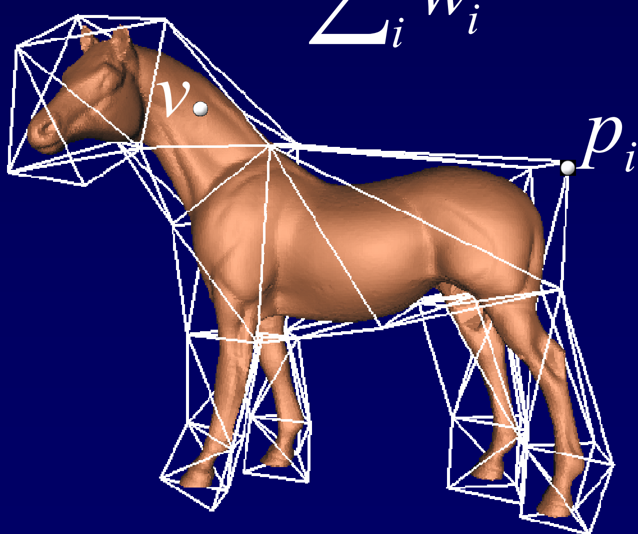

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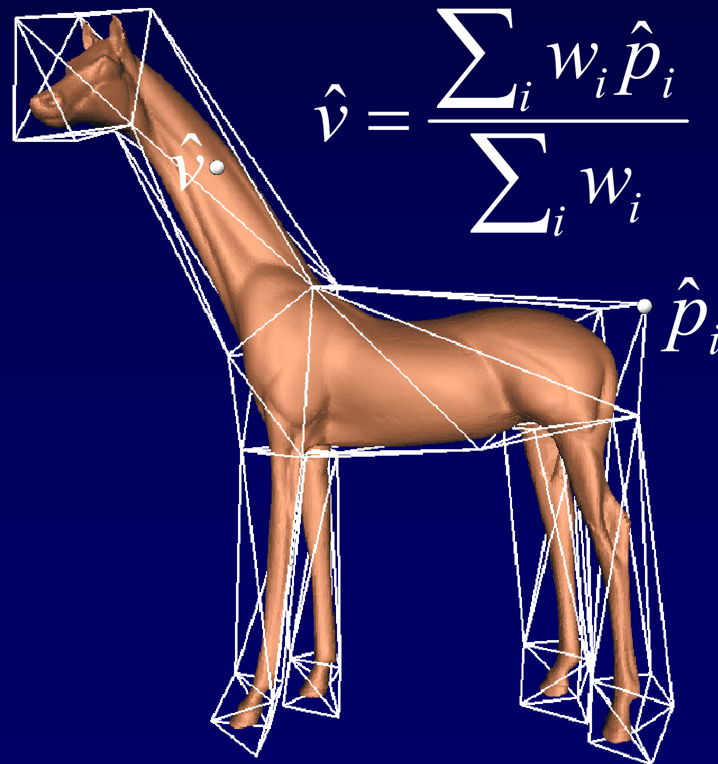

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- Boundary Value Problems

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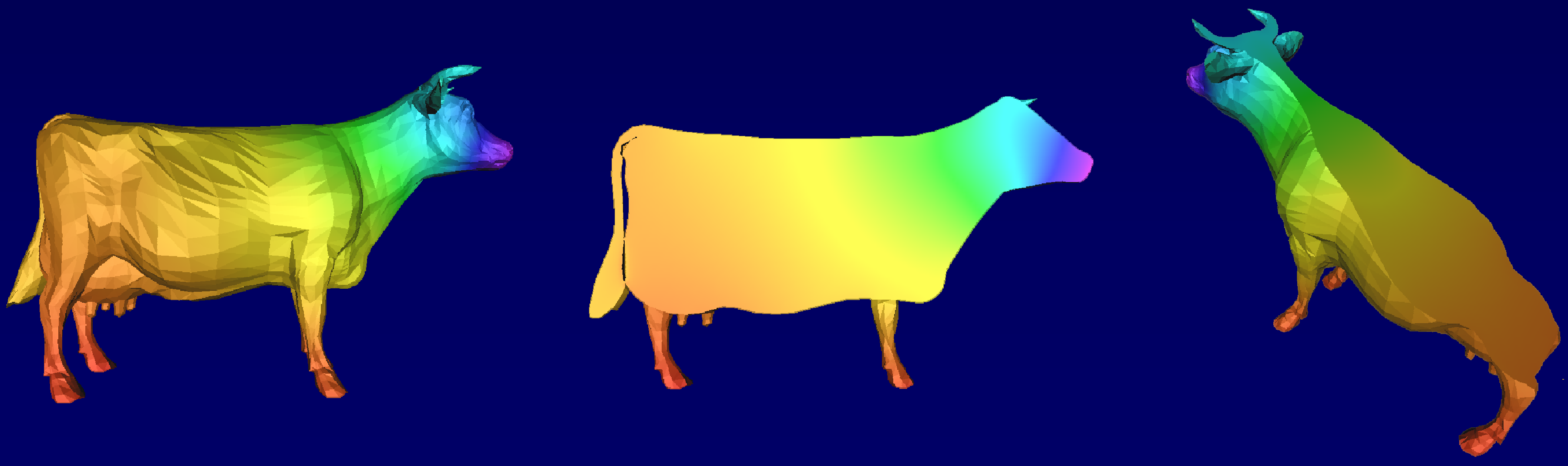


Applications

- Free-Form Deformations

  [Sederberg et al 86], [MacCracken et al 96], [Ju et al 05]

- Boundary Value Problems

  [Ju et al 05]

- Surface Parameterization

  [Hormann et al 00], [Desbrun et al 02]

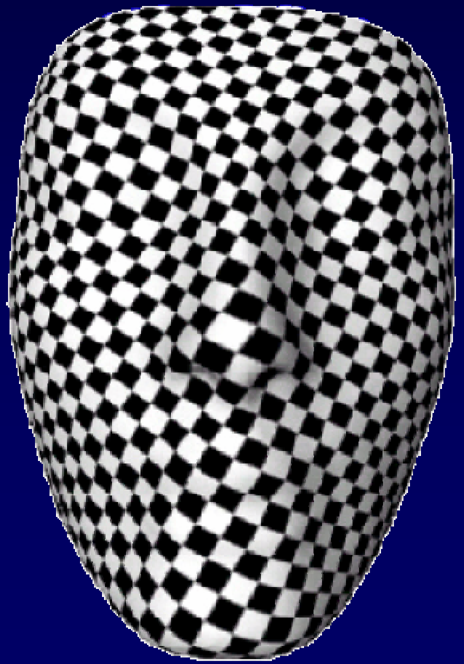


Image taken from [Desbrun et al 02]

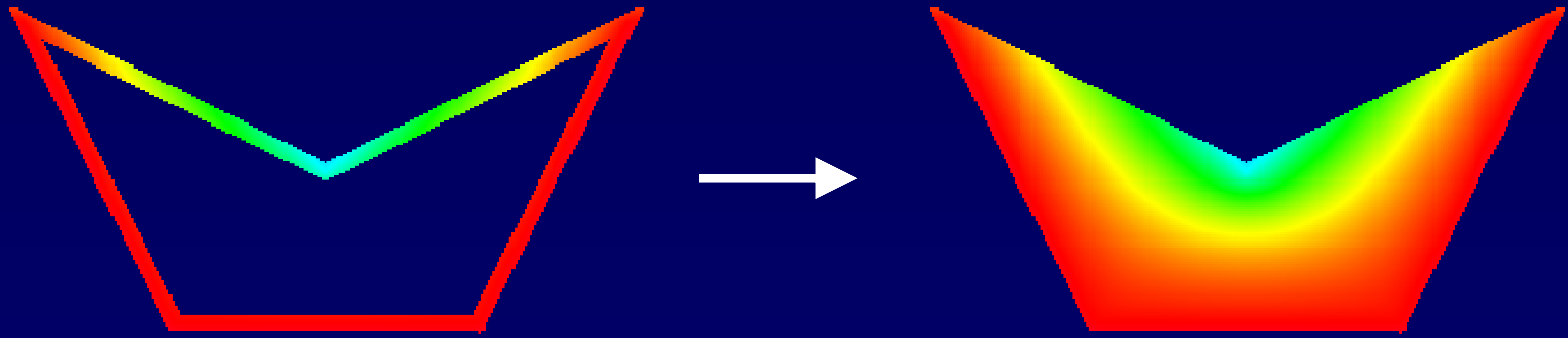
Previous Work

- Closed Polygons

  [Wachspress 75], [Floater 03], [Hormann 06]
- Closed Polyhedra

  [Warren 96], [Floater et al 05], [Ju et al 05], [Ju et al 07]
- Smooth Convex Curves/Surfaces/…

  [Schaefer et al 03]



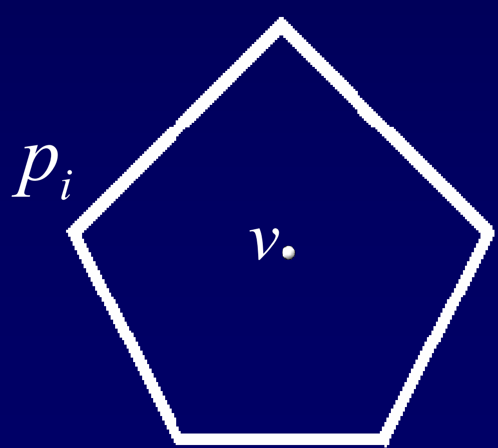
Continuous Barycentric Coordinates

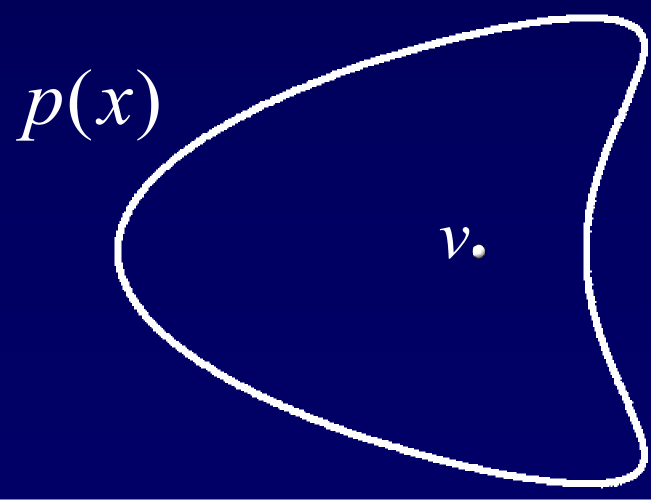
Discrete

$$v = \sum_i w_i p_i$$

Continuous

$$v = \int_x w(x,v)\, p(x)\, dx$$





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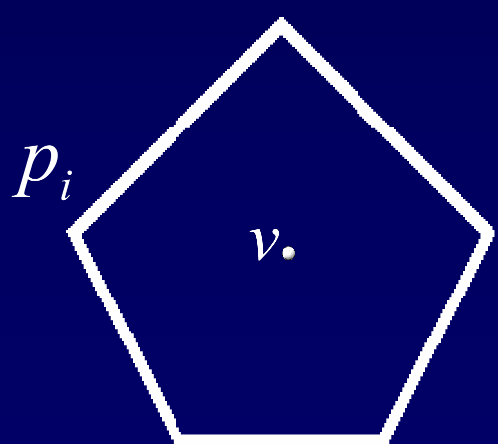

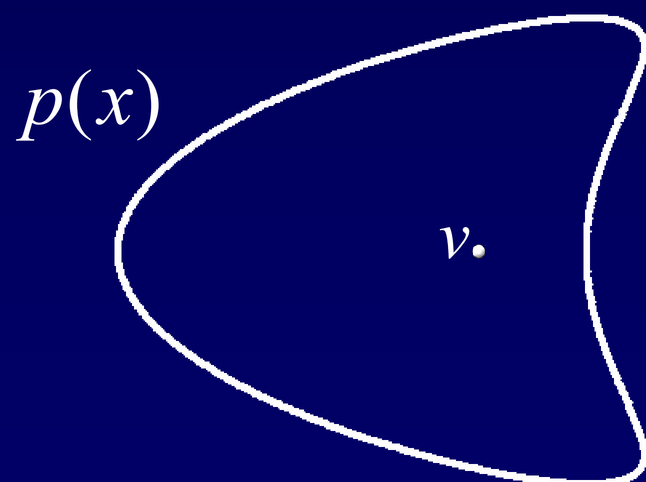

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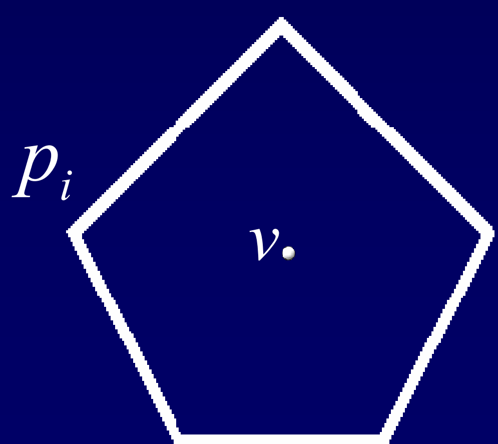

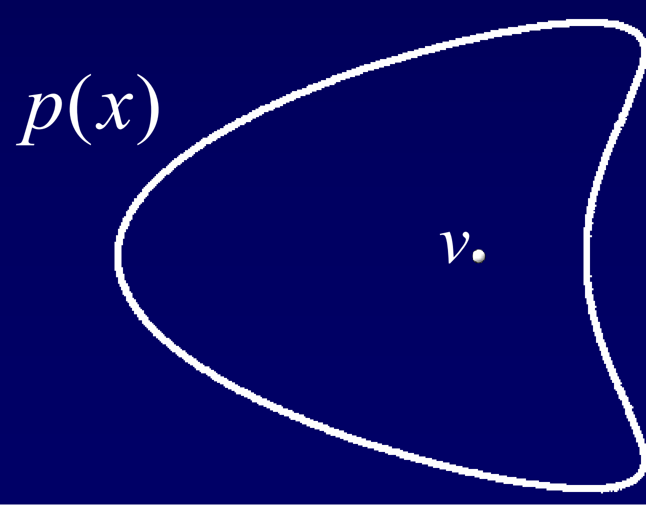

Continuous Barycentric Coordinates

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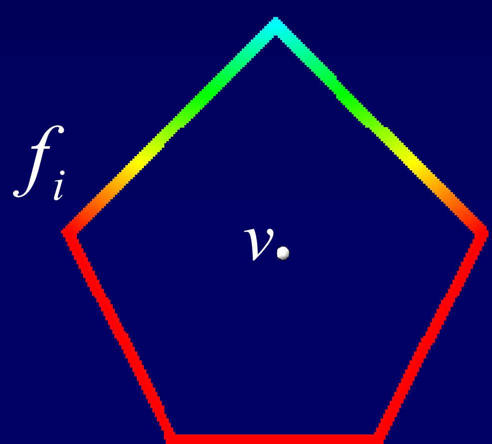
$$\hat{f}(v) = \sum_i w_i f_i$$



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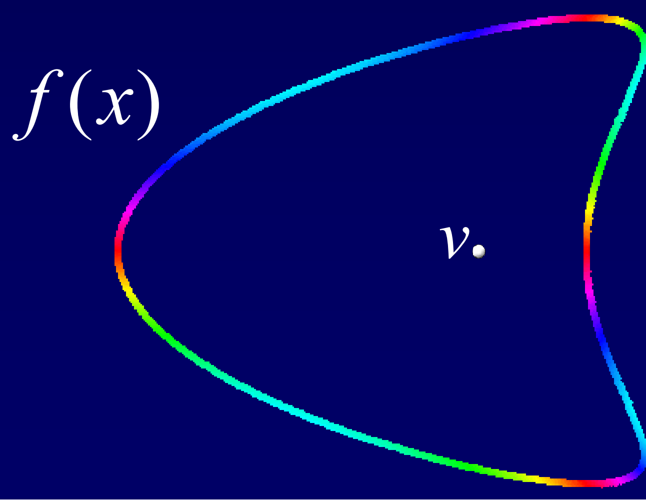
$$\hat{f}(v) = \int_x w(x, v) f(x) dx$$



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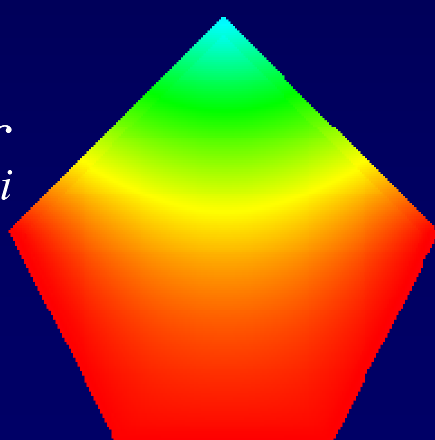
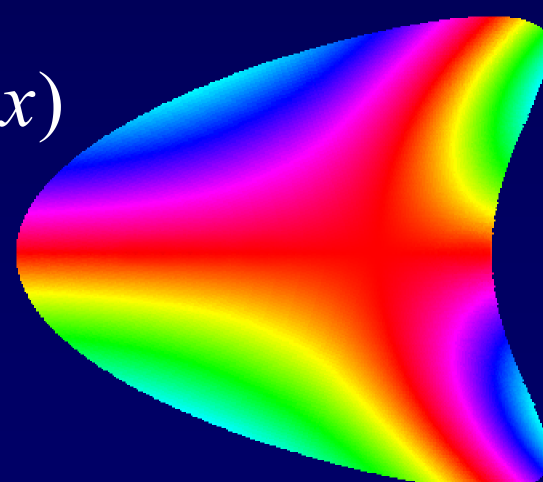
f_i



$f(x)$

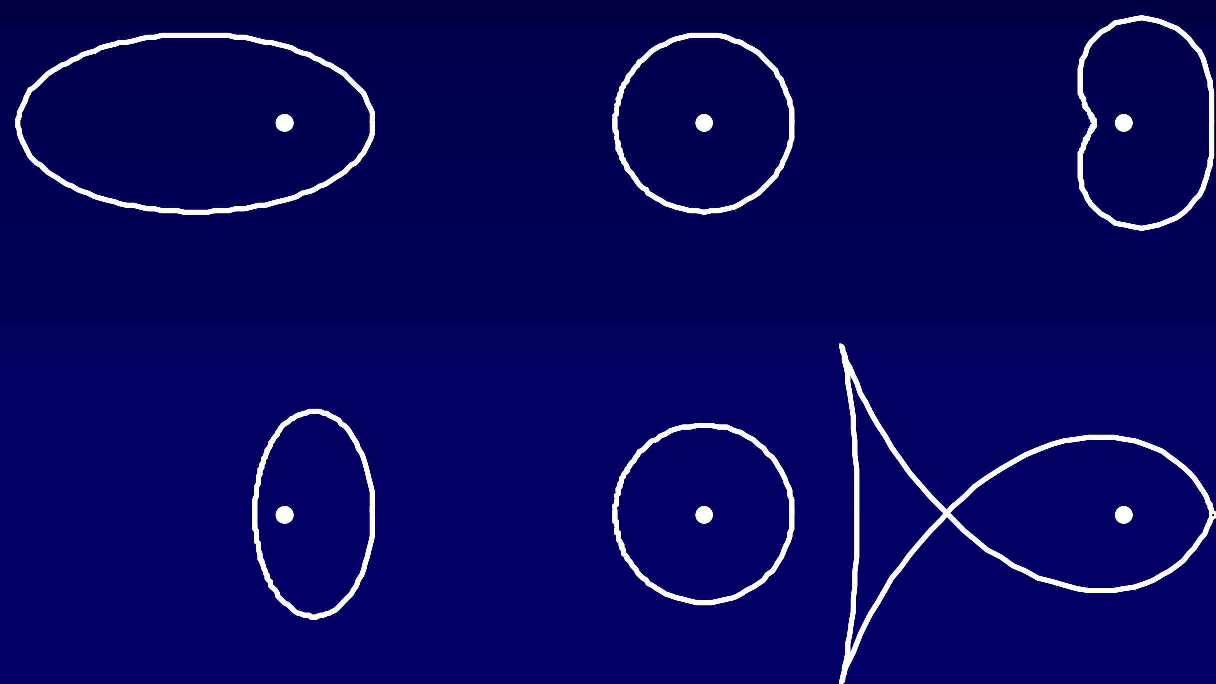


The Polar Dual

$$d(p(x)) = \frac{p^{\perp}(x)}{p^{\perp}(x) \cdot p(x)}$$

$p(x):$

$d(p(x)):$



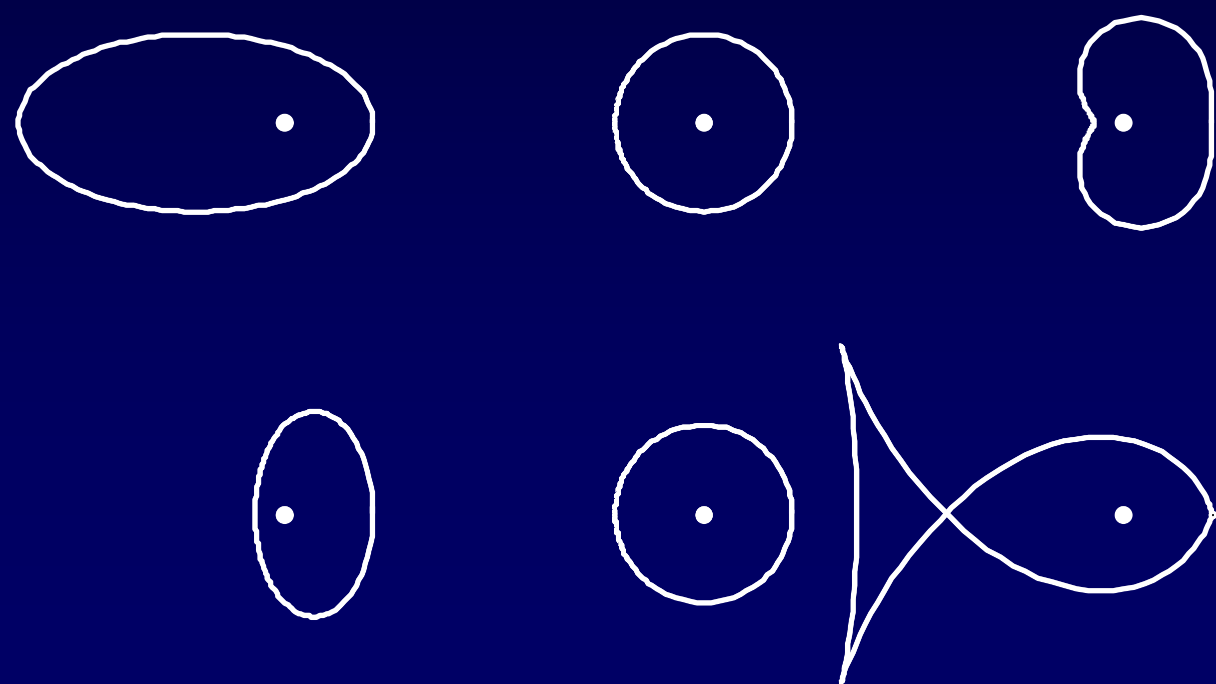
The Polar Dual

$$d(d(p(x))) = p(x)$$

as long as $p(x) \cdot p^{\perp}(x) \neq 0$

$p(x):$

$d(p(x)):$



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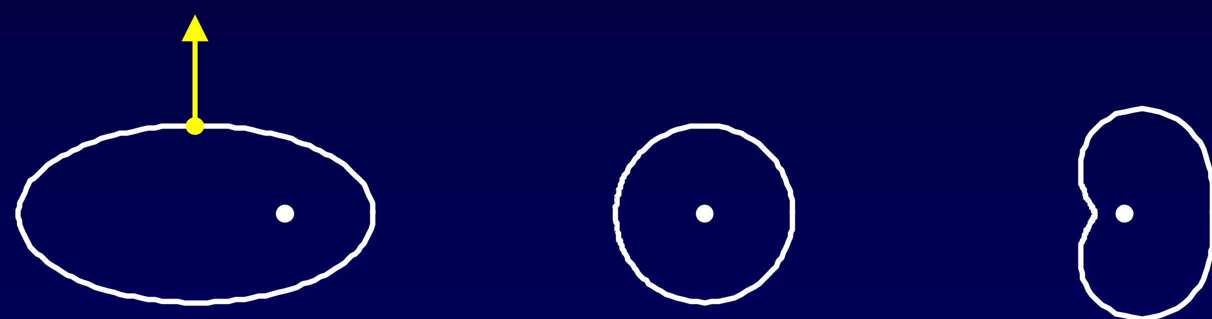

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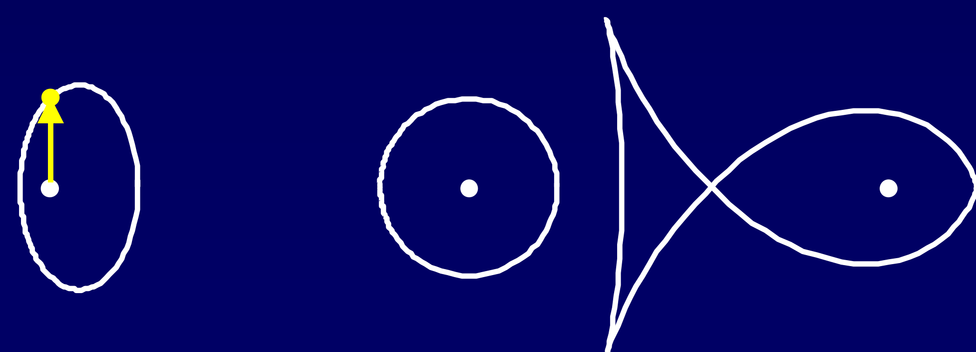

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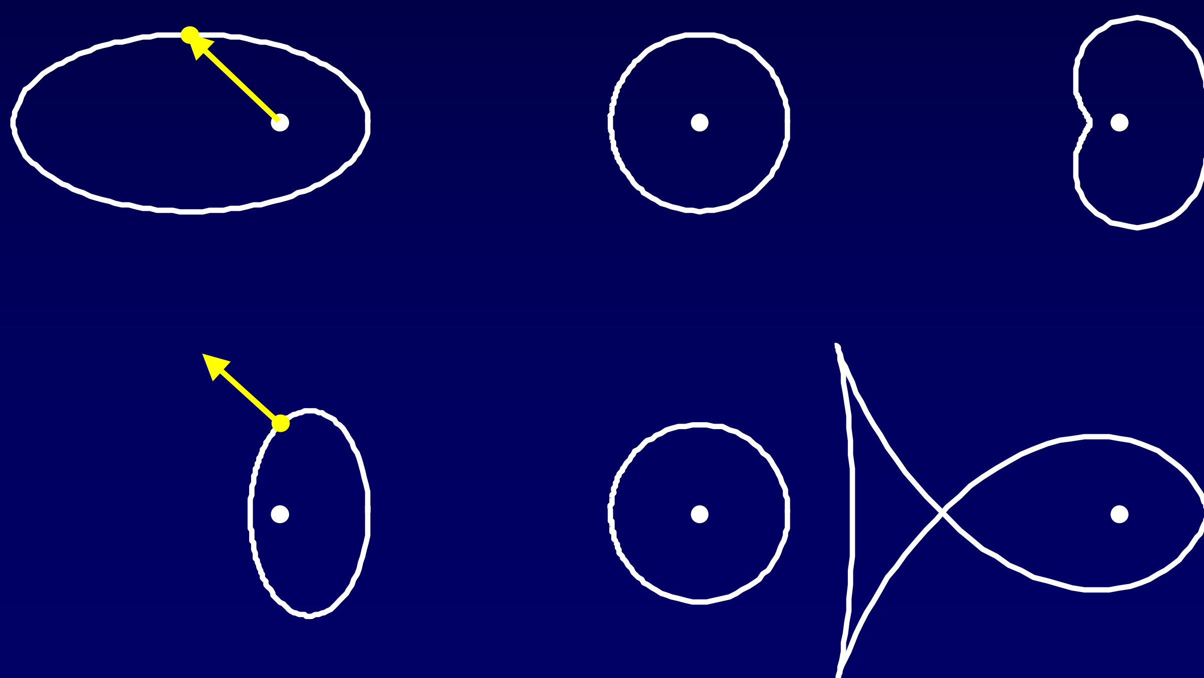

Continuous Barycentric Interpolation

$$\hat{f}(v) = \frac{\int_x \frac{f(x)}{|p(x)-v|}\, d\overline{P}_v}{\int_x \frac{1}{|p(x)-v|}\, d\overline{P}_v}$$

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<u>Properties</u>

$$1 = \int_x w(x,v)\,dx$$

$\frac{\partial w(x,v)}{\partial v}$ continuous

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Linear Precision

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normal of $\overline{p}_v(x)$

Continuous Family of Coordinates

$$a_v(x) = \left| p(x) - v \right|^k$$

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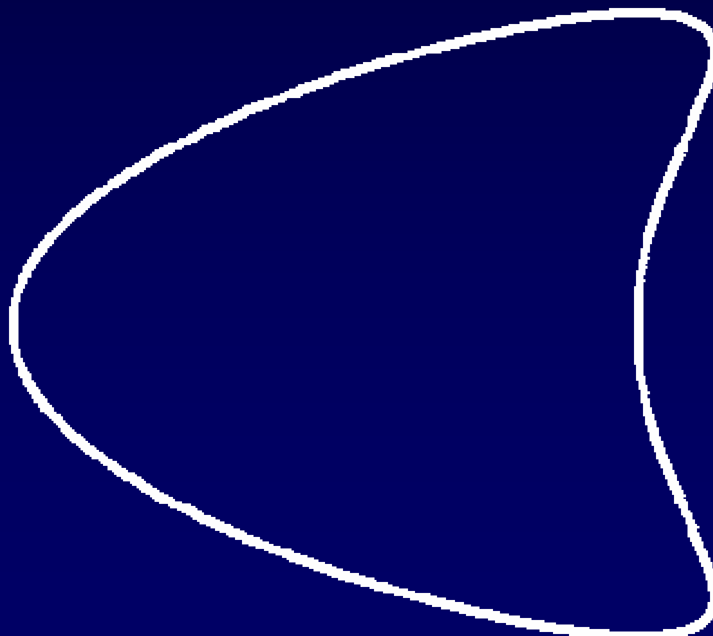
- k=1: mean value coordinates [Ju et al 05]
  - $\overline{P}_v$ = sphere centered at v



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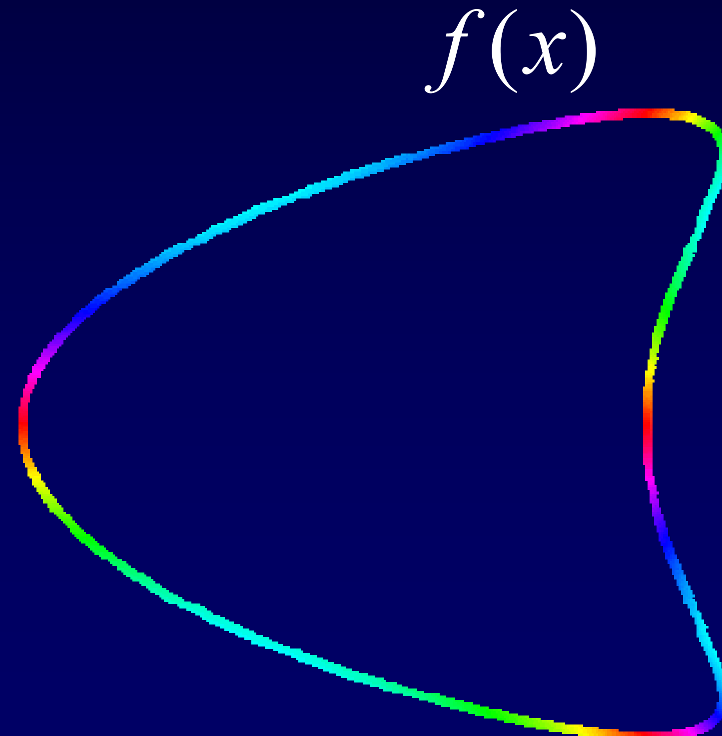

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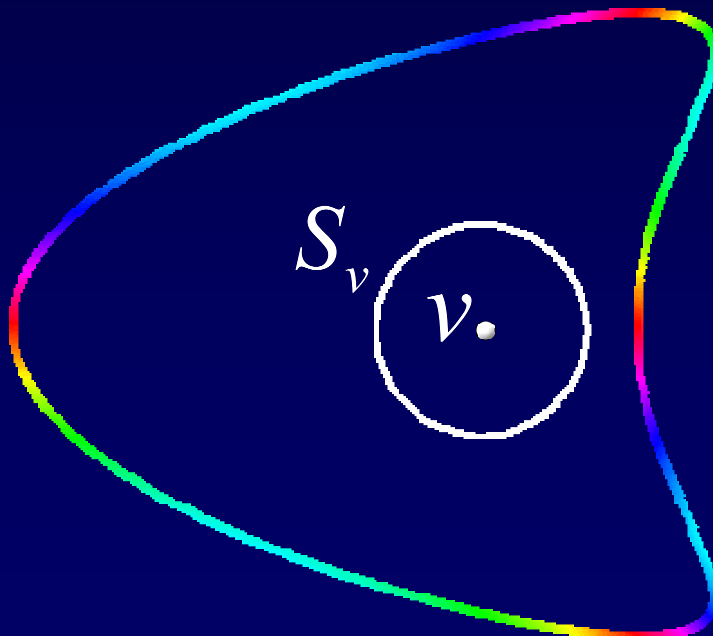

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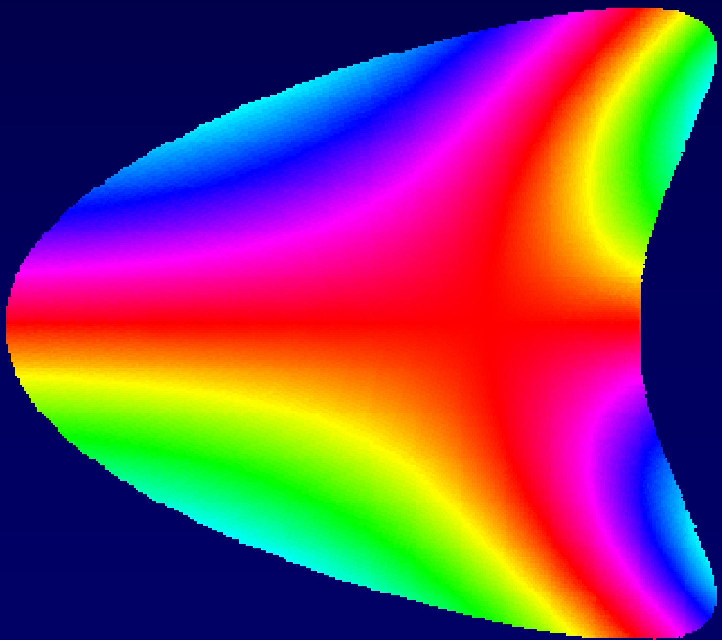


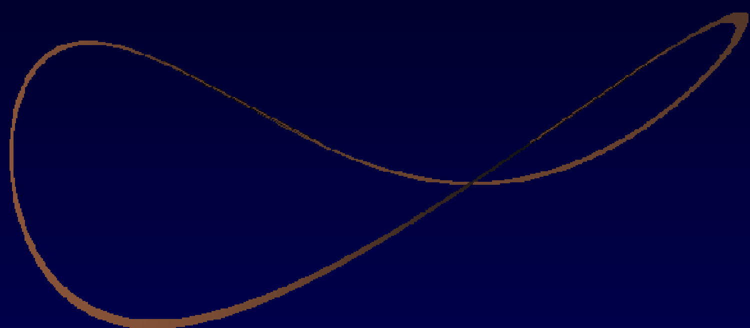
Continuous Family of Coordinates

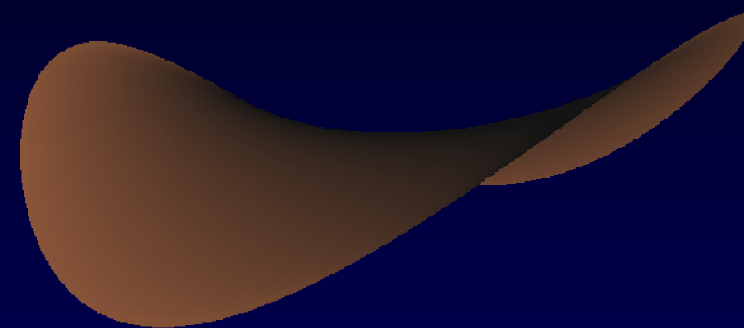
$$a_v(x) = \left| p(x) - v \right|^k$$

- k=1: mean value coordinates [Ju et al 05]
- k=0: continuous Wachspress [Schaefer et al 03]

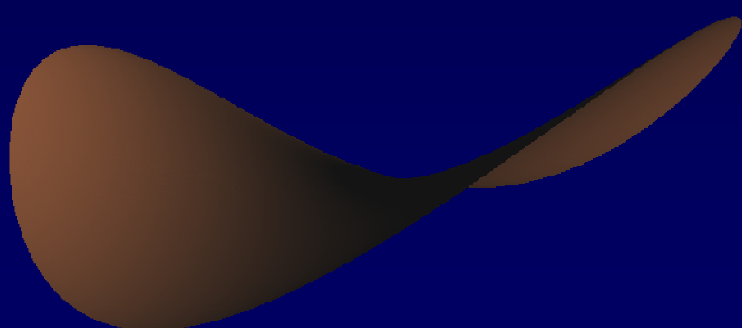
$$\hat{f}(v) = \frac{\int_x \frac{f(x)\kappa(x)}{(n(x)\cdot(p(x)-v))^m}\, dP}{\int_x \frac{\kappa(x)}{(n(x)\cdot(p(x)-v))^m}\, dP}$$

Continuous Family of Coordinates

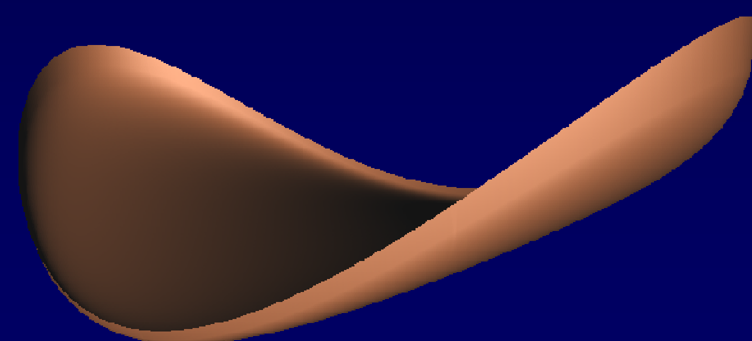




k=0



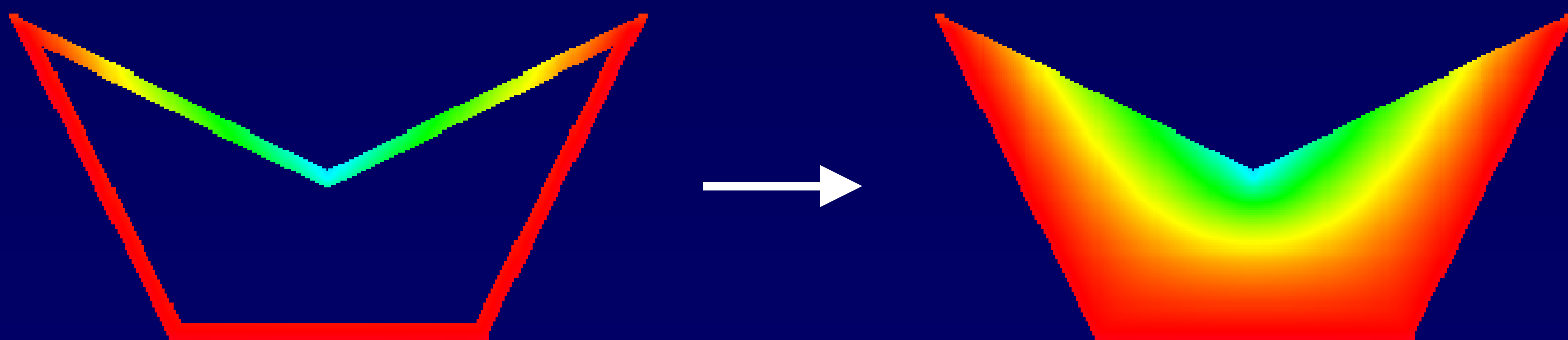
k=1



k=2

Discrete Barycentric Coordinates

- Treat function $p(x)$ and boundary data $f(x)$ as piecewise linear functions
- k=0: discrete Wachspress coordinates
- k=1: discrete mean value coordinates
- k=2: discrete harmonic coordinates



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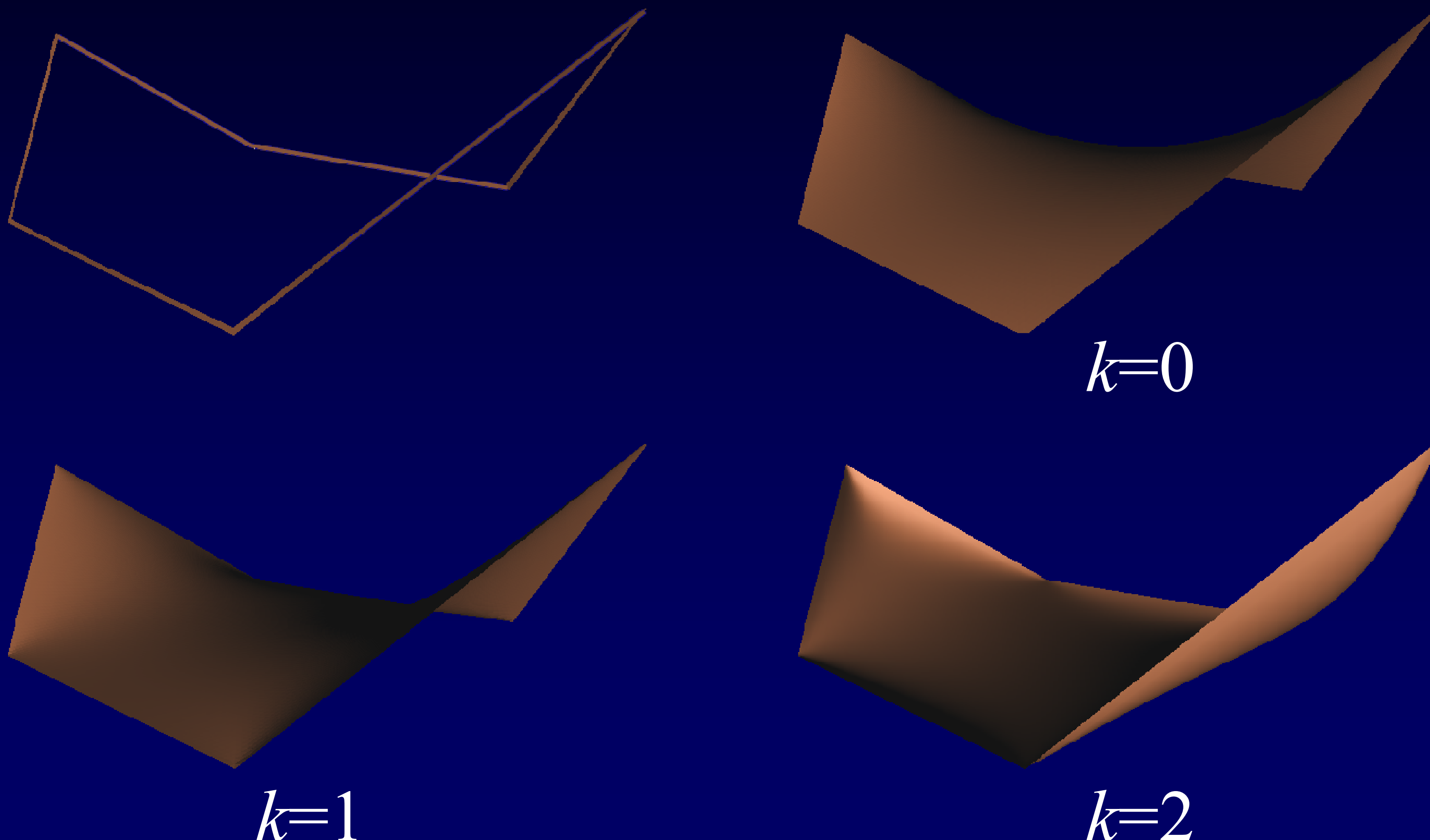

Conclusions

- New barycentric coordinates for curves/surfaces/…
- Reproduces all discrete barycentric coordinates
- Limited to shapes where $\frac{p(x)-v}{a_v(x)} \cdot \left(\frac{p(x)-v}{a_v(x)}\right)^{\perp} \neq 0$