G^2 Tensor Product Splines over Extraordinary Vertices

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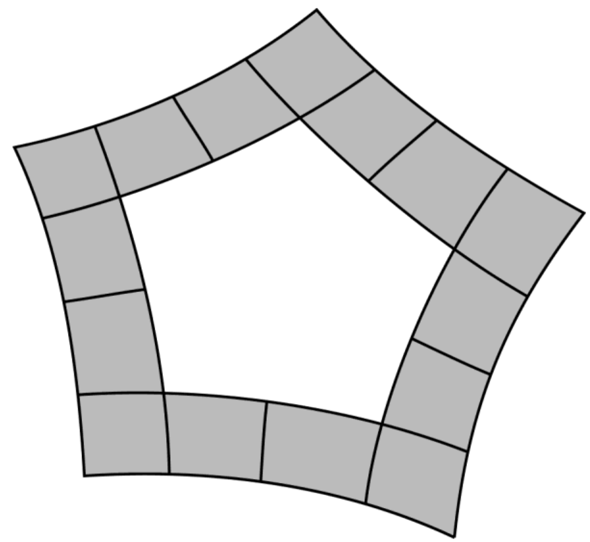
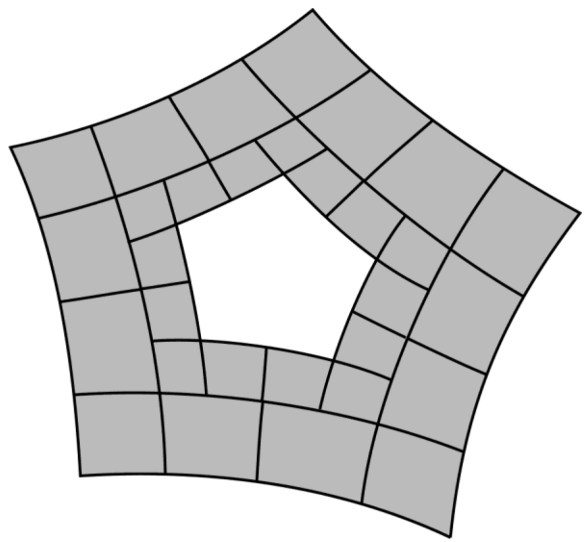
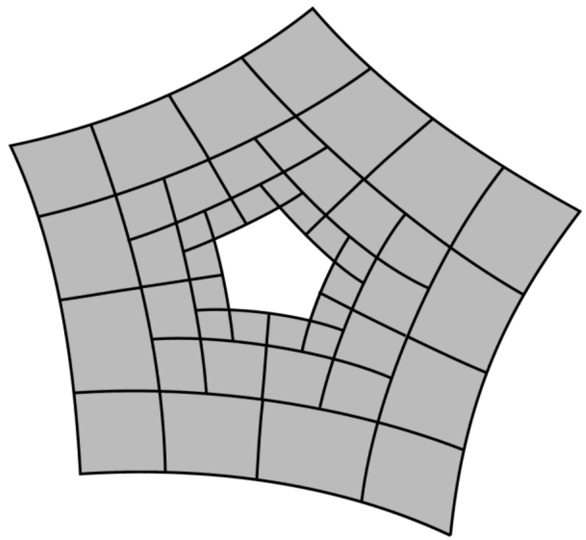
Spline Rings



Spline Rings



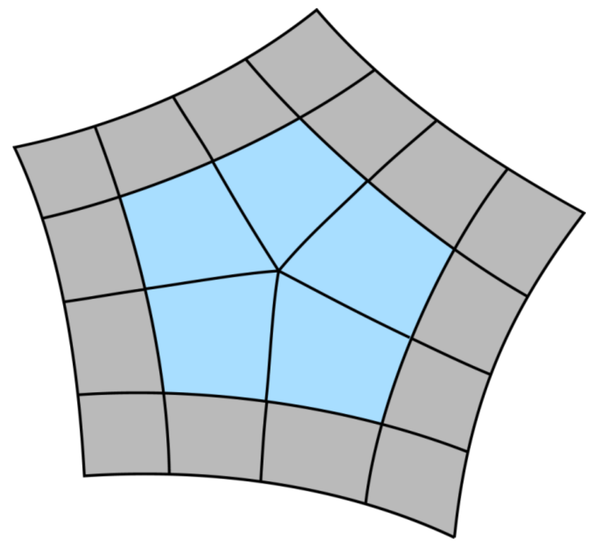
Spline Rings



Problems with Catmull-Clark Subdivision

- Composed of an infinite number of patches
  - Hard to evaluate/analyze/process
  - Not well suited to GPU pipeline
- Not C^2 at extraordinary vertices
  - Not “Class A” surface
  - Limits use to entertainment scenarios

Problem Statement



Fill the hole in an n-valent Catmull-Clark spline ring with n tensor product patches that join each other and the spline ring with second order smoothness.

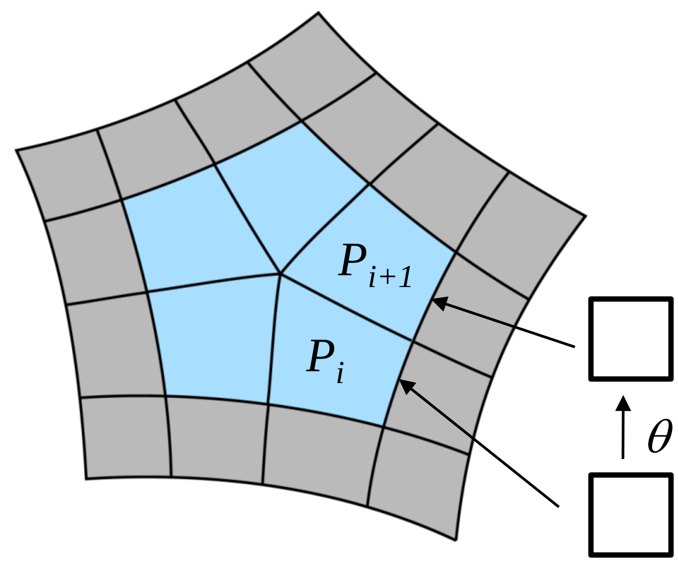
Geometric Continuity

- Definition $P_i \stackrel{G^k}{=} P_{i+1} \Rightarrow P_i \stackrel{C^k}{=} P_{i+1} \circ \theta$ [DeRose '85]

- Correspondence Map $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$
  - Assumed to be identity on edge

- Matrix Equation $D_i = D_{i+1} \cdot \Theta$

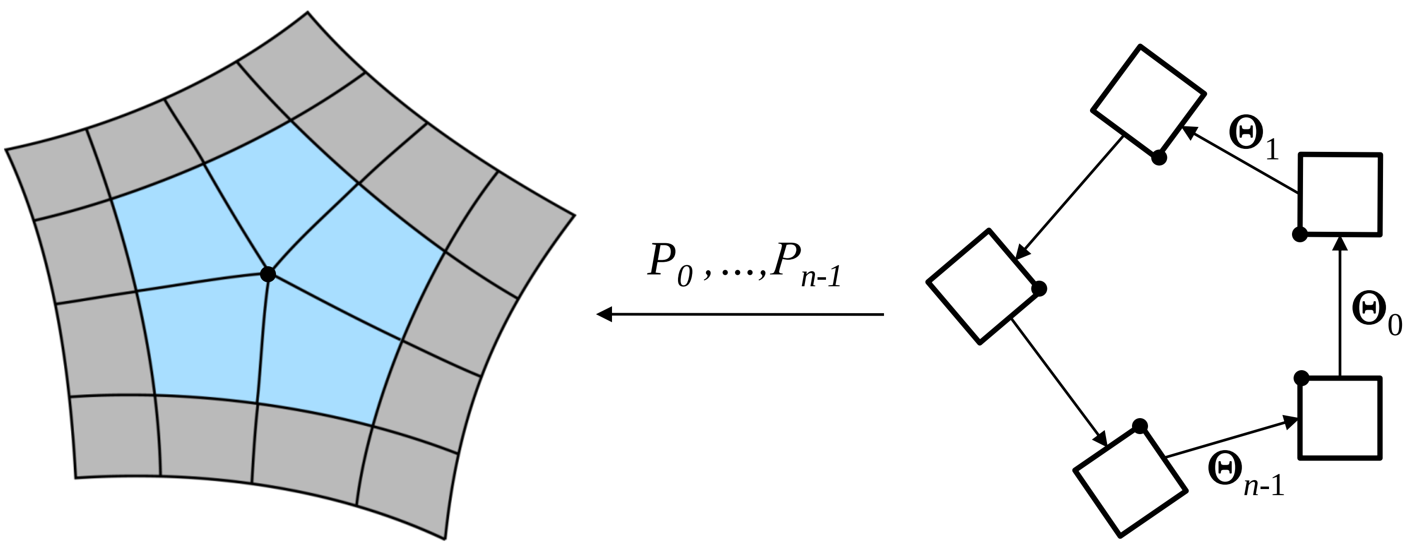
- Chain Rule Matrix Θ



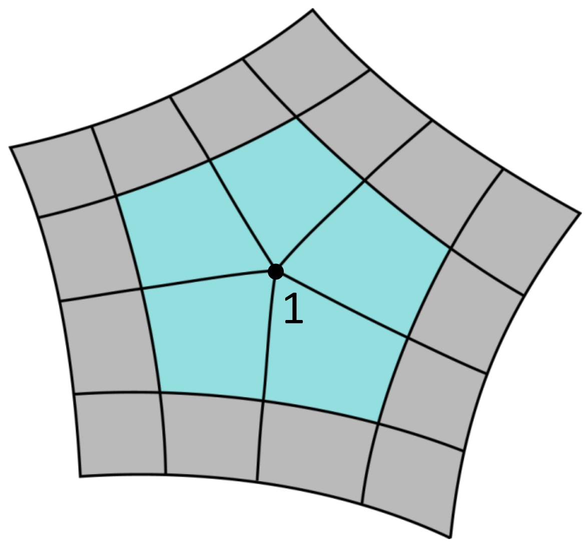
Cocycle Condition

- For cyclic collection of patches incident on a common vertex

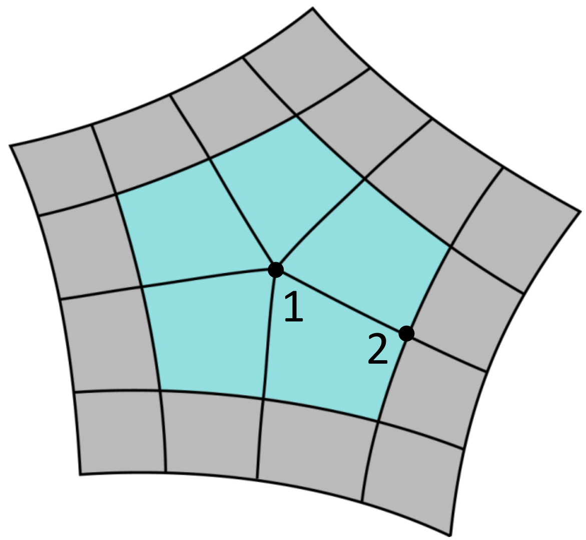
$$D_0 = D_0 \cdot \overbrace{\Theta_{n-1} \cdot \Theta_{n-2} \cdots \Theta_1 \cdot \Theta_0}^{\mathrm{I}}$$ [Hahn ’89]



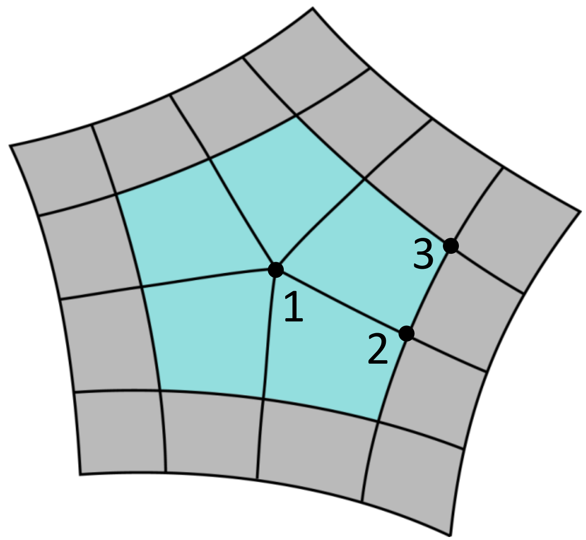
Correspondence Maps



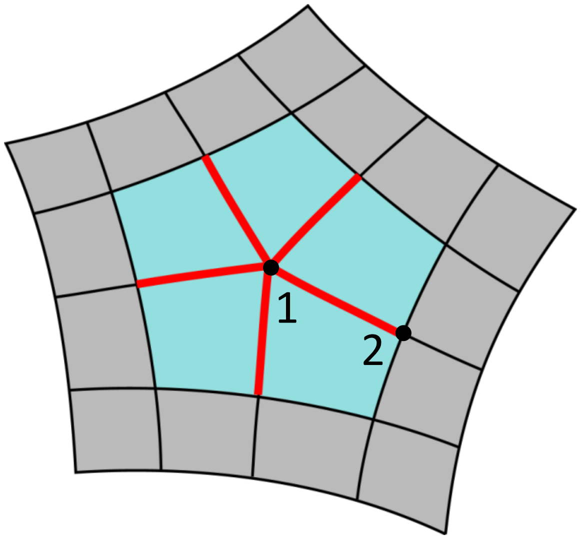
Correspondence Maps



Correspondence Maps

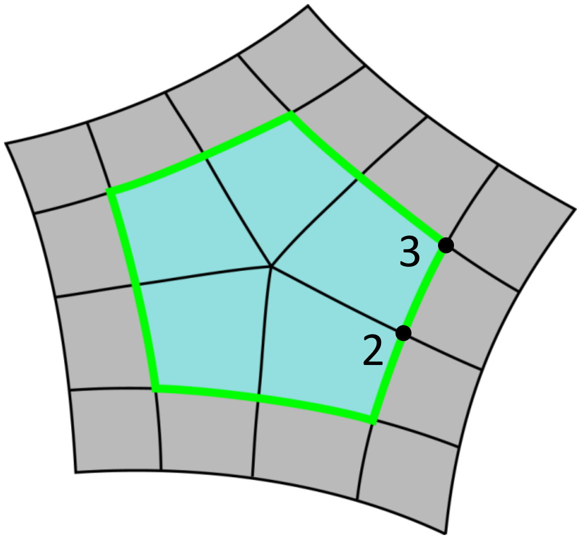


Correspondence Maps



interior

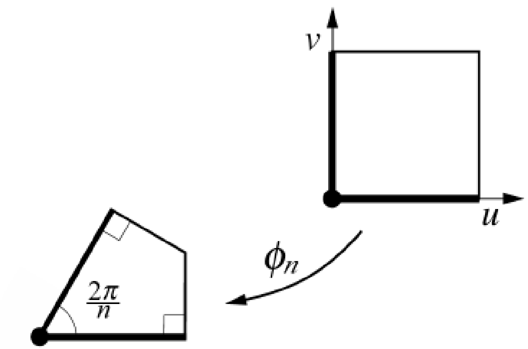
Correspondence Maps



exterior

Interior Correspondence Maps

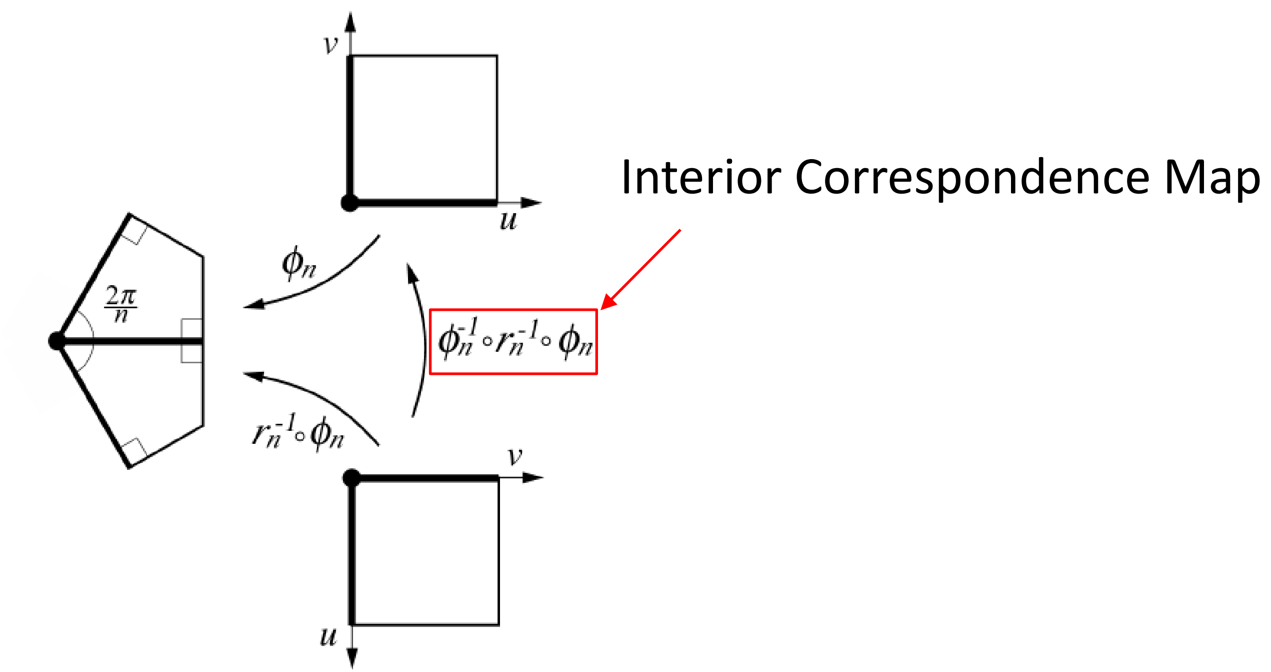
$$\phi_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



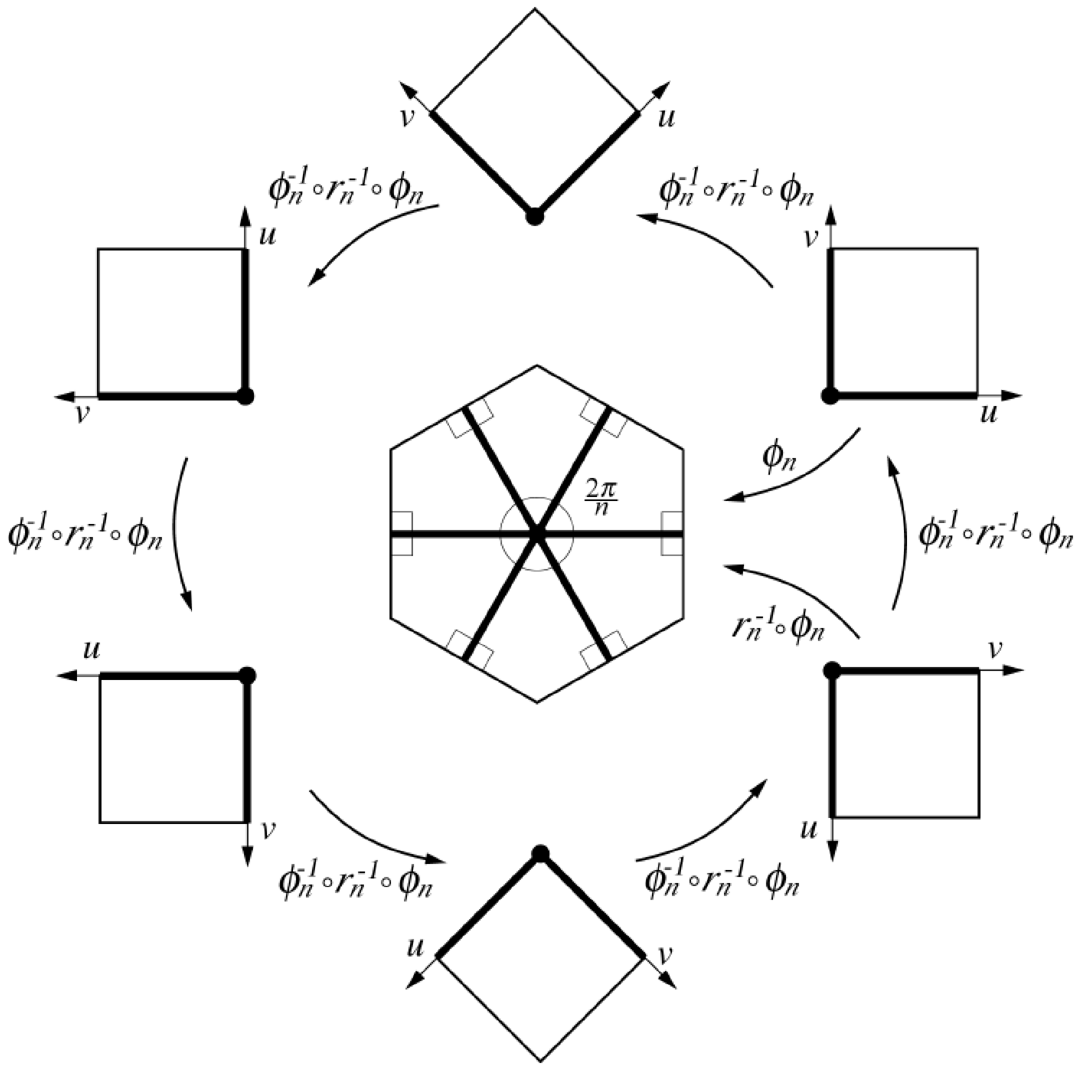
$$\phi_{n,x}\left(u,v\right) = \mathrm{b}^1\left(u\right)\begin{bmatrix} 0 & \cos\left(\frac{2\pi}{n}\right) \\ 1 & 1 \end{bmatrix}\mathrm{b}^1\left(v\right)$$

$$\phi_{n,y}\left(u,v\right) = \mathrm{b}^1\left(u\right)\begin{bmatrix} 0 & \sin\left(\frac{2\pi}{n}\right) \\ 0 & \tan\left(\frac{\pi}{n}\right) \end{bmatrix}\mathrm{b}^1\left(v\right)$$

Interior Correspondence Maps



Interior Correspondence Maps

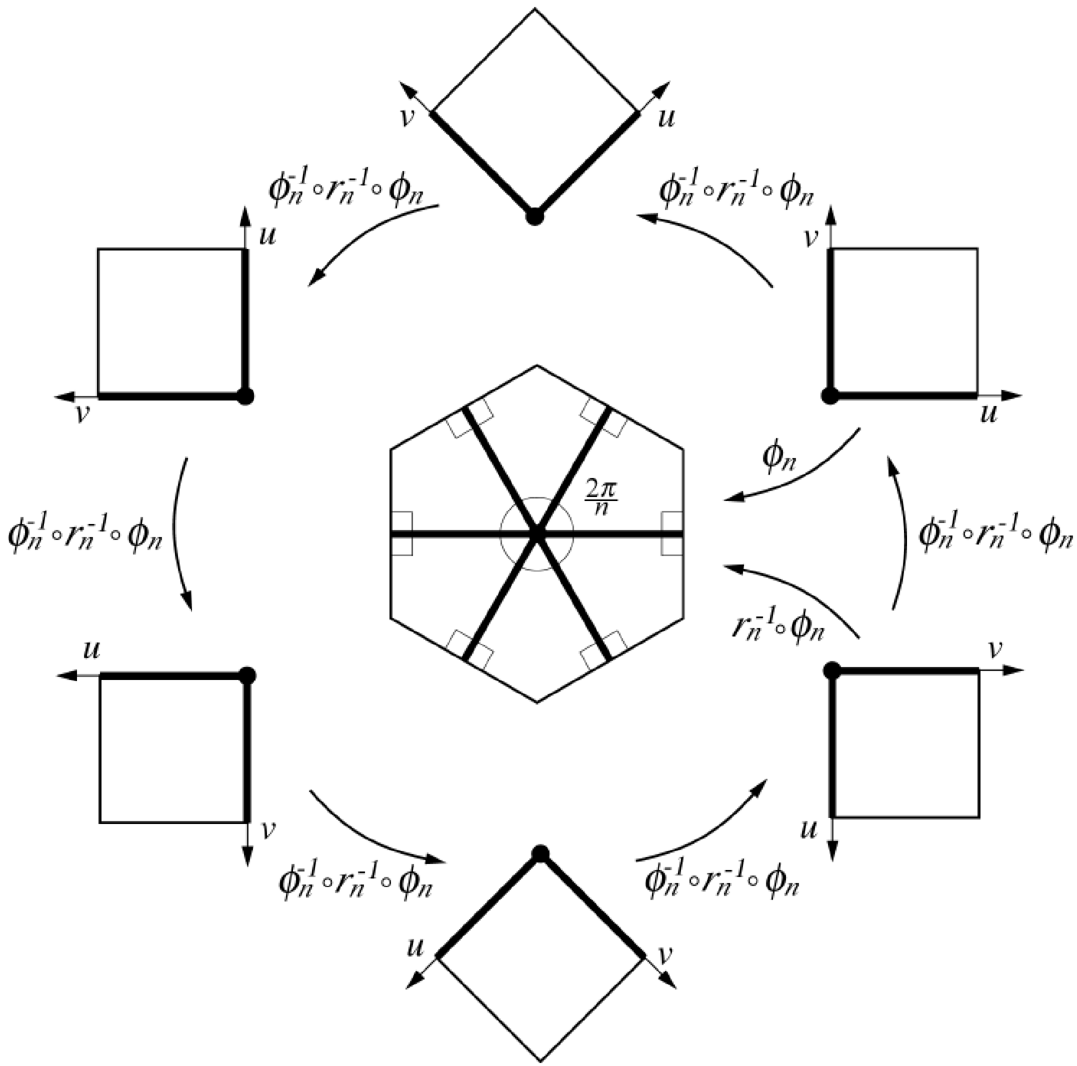


Cocycle Condition:

$$\left(\Phi_n^{-1} \cdot R_n^{-1} \cdot \Phi_n\right)^n = \mathrm{I}$$

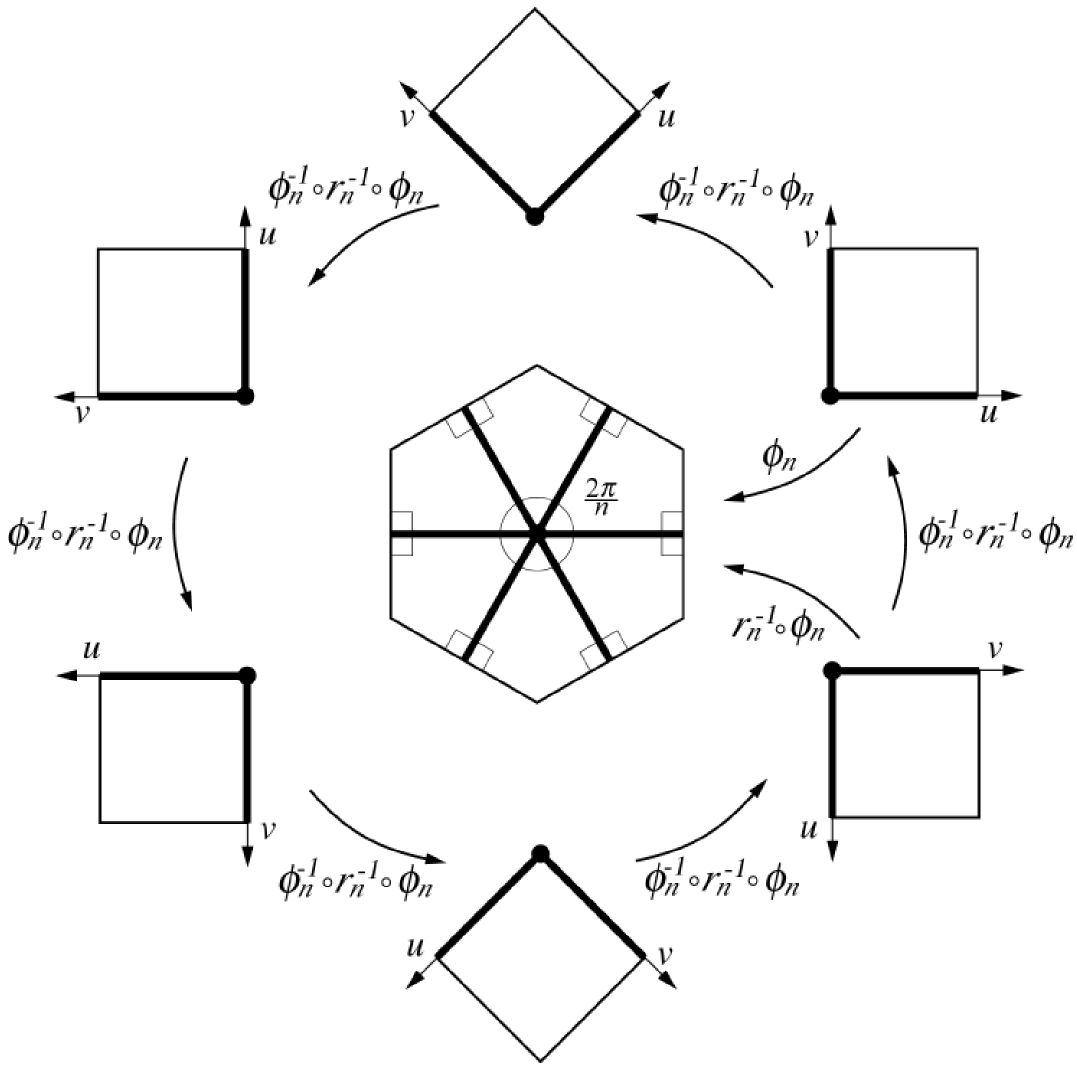
at type 1 vertex

Interior Correspondence Maps



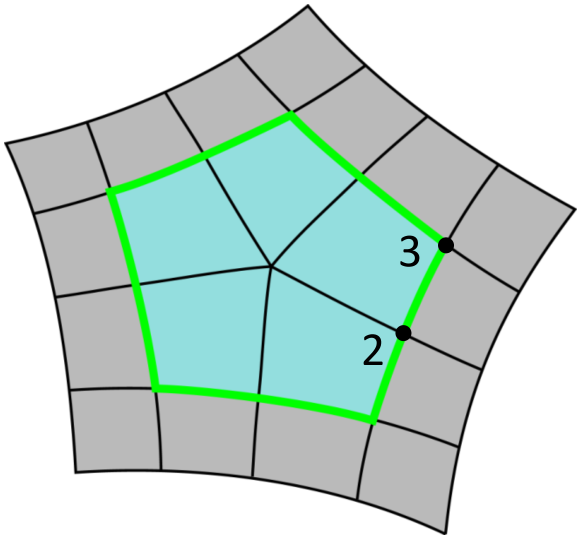
$$\Phi_n^{-1} \cdot \left(R_n^{-1}\right)^n \cdot \Phi_n = \mathrm{I}$$

Interior Correspondence Maps

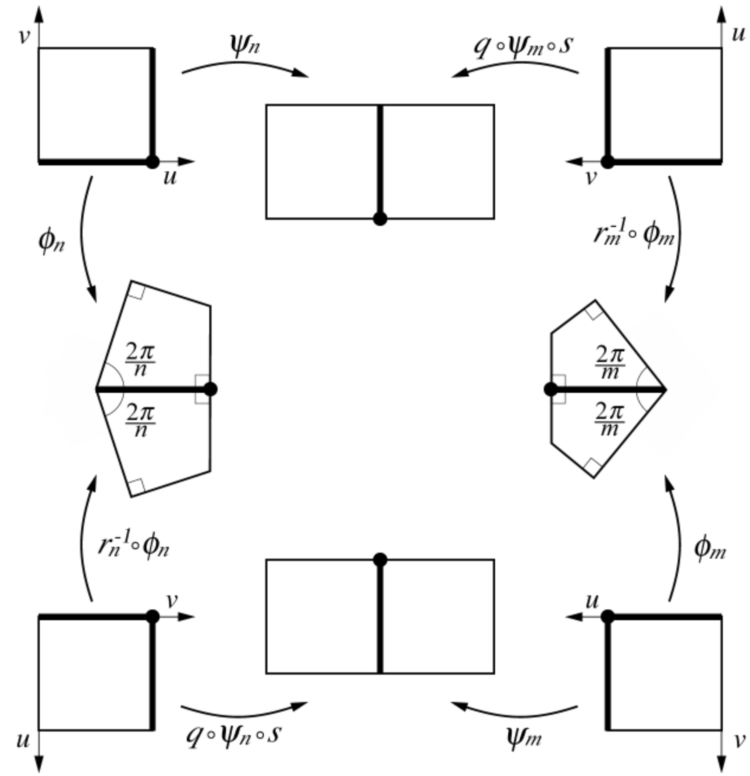


$$\Phi_n^{-1} \cdot \Phi_n = \mathrm{I}$$

Exterior Correspondence Maps



Exterior Correspondence Maps

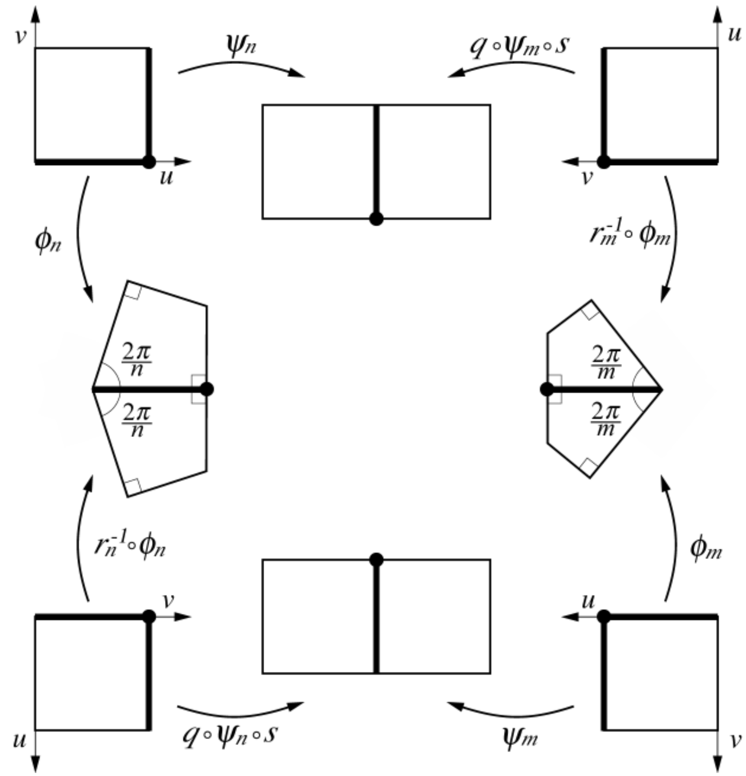


$$q(u,v) = (u,-v)$$
$$s(u,v) = (v,u)$$

Cocycle Condition at type 2 vertex:

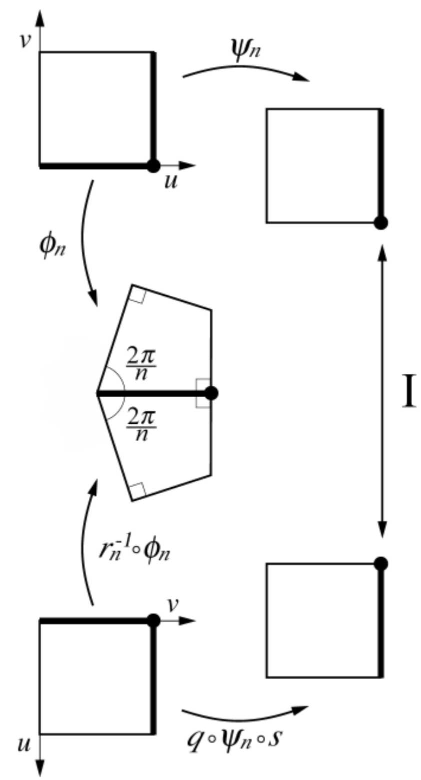
$$\mathrm{I} = Q \cdot \Psi_n \cdot S \cdot \left(R_n^{-1} \cdot \Phi_n\right)^{-1} \cdot \Phi_n \cdot \Psi_n^{-1} \cdot Q \cdot \Psi_m \cdot S \cdot \left(R_m^{-1} \cdot \Phi_m\right)^{-1} \cdot \Phi_m \cdot \Psi_m^{-1}$$

Exterior Correspondence Maps



$$\mathrm{I} = \overbrace{Q \cdot \Psi_n \cdot S \cdot \left(R_n^{-1} \cdot \Phi_n\right)^{-1} \cdot \Phi_n \cdot \Psi_n^{-1}}^{\mathrm{I}} \cdot \overbrace{Q \cdot \Psi_m \cdot S \cdot \left(R_m^{-1} \cdot \Phi_m\right)^{-1} \cdot \Phi_m \cdot \Psi_m^{-1}}^{\mathrm{I}}$$

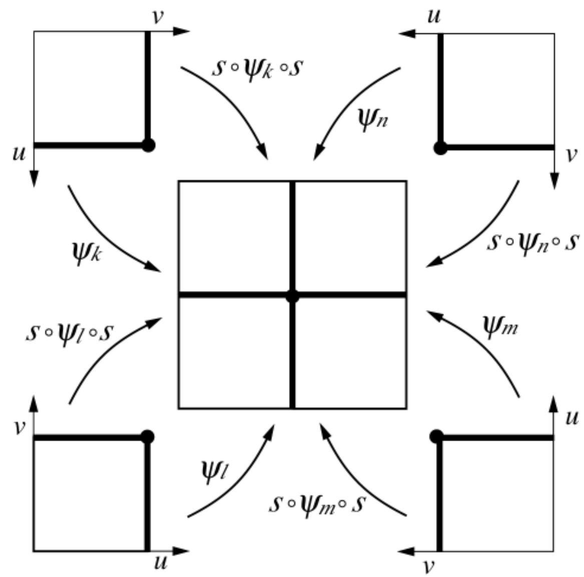
Exterior Correspondence Maps



$$\mathrm{I} = Q \cdot \Psi_n \cdot S \cdot \left(R_n^{-1} \cdot \Phi_n\right)^{-1} \cdot \Phi_n \cdot \Psi_n^{-1}$$

Exterior Correspondence Maps

$$\psi_n : \mathbb{R}^2 \to \mathbb{R}^2$$



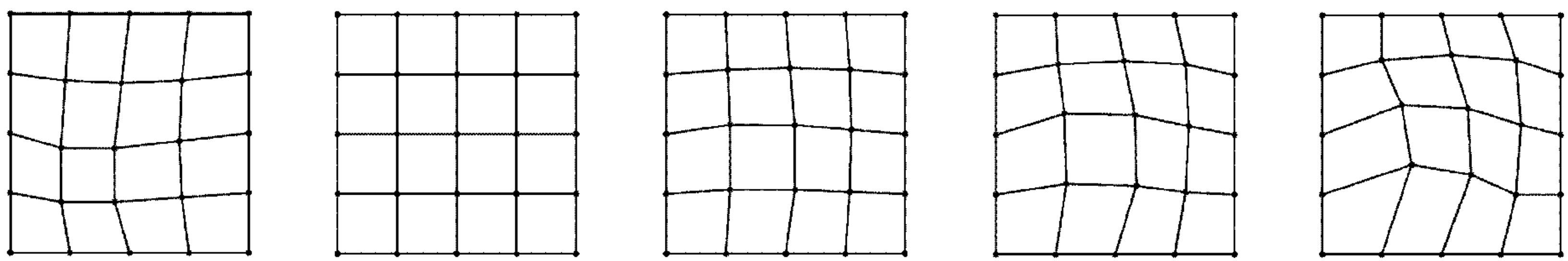
$$s(u,v) = (v,u)$$

Cocycle Condition at type 3 vertex:

$$\mathrm{I} = \overbrace{\Psi_k \cdot \left(S \cdot \Psi_k \cdot S\right)^{-1}}^{\mathrm{I}} \cdot \overbrace{\Psi_l \cdot \left(S \cdot \Psi_l \cdot S\right)^{-1}}^{\mathrm{I}} \cdot \overbrace{\Psi_m \cdot \left(S \cdot \Psi_m \cdot S\right)^{-1}}^{\mathrm{I}} \cdot \overbrace{\Psi_n \cdot \left(S \cdot \Psi_n \cdot S\right)^{-1}}^{\mathrm{I}}$$

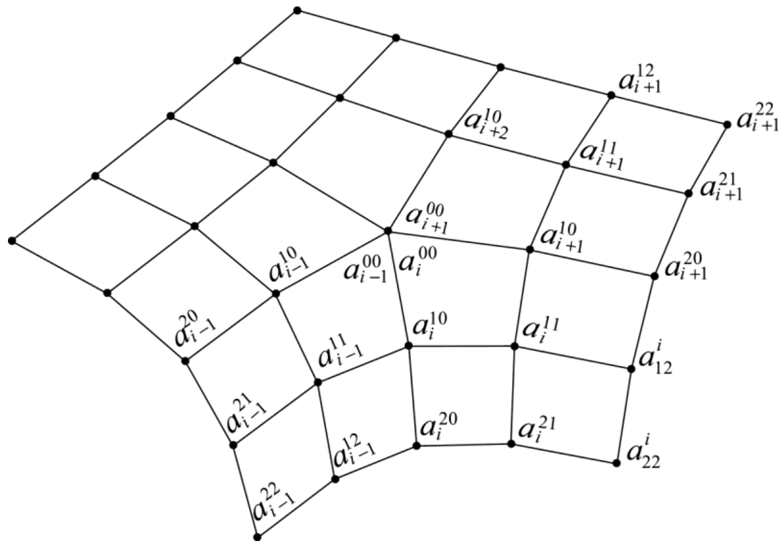
Exterior Correspondence Maps

$$\psi_n(u,v)$$



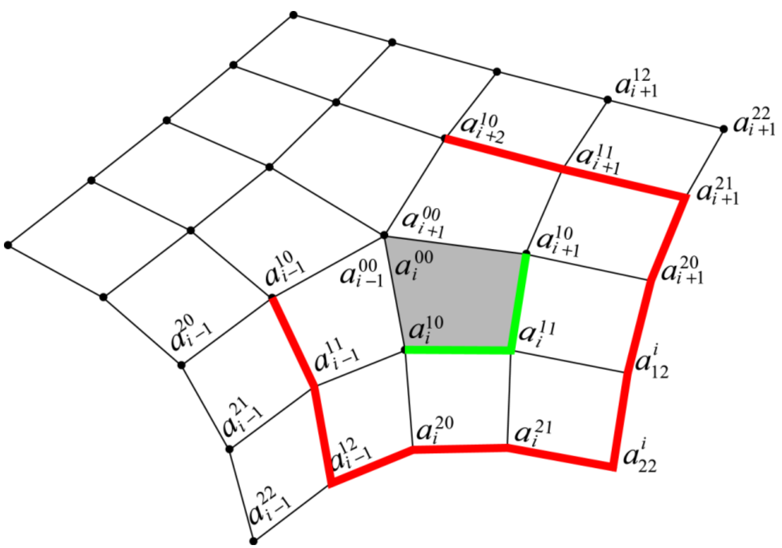
$n =$ 3 4 5 8 ∞

Boundary Data



$$H_i(u,v) = \mathrm{B}^3(u)^T \begin{bmatrix} \cdot & a_{i+2}^{10} & a_{i+1}^{11} & a_{i+1}^{21} \\ a_{i-1}^{10} & a_i^{00} & a_{i+1}^{10} & a_{i+1}^{20} \\ a_{i-1}^{11} & a_i^{10} & a_i^{11} & a_i^{12} \\ a_{i-1}^{12} & a_i^{20} & a_i^{21} & a_i^{22} \end{bmatrix} \mathrm{B}^3(v)$$

Boundary Data



$$H_i(u,v) = \mathrm{B}^3(u)^T \begin{bmatrix} \cdot & a_{i+2}^{10} & a_{i+1}^{11} & a_{i+1}^{21} \\ a_{i-1}^{10} & a_i^{00} & a_{i+1}^{10} & a_{i+1}^{20} \\ a_{i-1}^{11} & a_i^{10} & a_i^{11} & a_i^{12} \\ a_{i-1}^{12} & a_i^{20} & a_i^{21} & a_i^{22} \end{bmatrix} \mathrm{B}^3(v)$$

Patch Smoothness Constraints

- External Constraints

$$\frac{\partial^j}{\partial u^j} P_i(1,t) = \frac{\partial^j}{\partial u^j}\left(H_i \circ \psi_n\right)(1,t), \quad j = 0,1,2$$

- Internal Constraints

$$\frac{\partial^{j+k}}{\partial u^j v^k} P_i(0,t) = \frac{\partial^{j+k}}{\partial u^j v^k}\left(P_{i+1} \circ \phi_n^{-1} \circ r_n^{-1} \circ \phi_n\right)(t,0), \quad j+k = 0,1,2$$

- Constraint System

$$\mathbf{Cp} = \mathbf{Wa} \qquad \mathbf{C} \in \mathbb{R}^{55n \times 64n}, \mathbf{W} \in \mathbb{R}^{7n \times 64n}$$

Bicubic Energy

$$energy = \int_0^1 \int_0^1 \left(\frac{\partial^4}{\partial u^4} P_i\left(u,v\right) \right)^2 + \left(\frac{\partial^4}{\partial v^4} P_i\left(u,v\right) \right)^2 du\, dv$$

Bicubic Energy

$$energy = \int_0^1 \int_0^1 \left(\frac{\partial^4}{\partial u^4} P_i\left(u, v\right) \right)^2 + \left(\frac{\partial^4}{\partial v^4} P_i\left(u, v\right) \right)^2 du\, dv$$

$$p_i^T \cdot E \cdot p_i$$

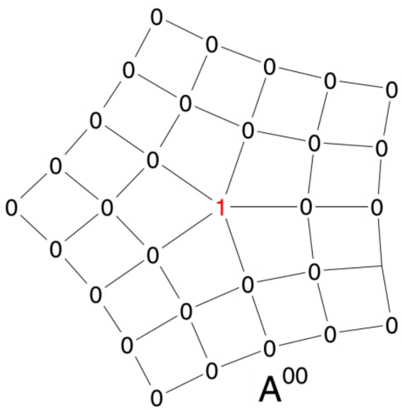
Constrained Minimization

$$\left[\begin{array}{c|c} \mathbf{E} & \mathbf{C}^T \\ \hline \mathbf{C} & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{p} \\ \hline \Lambda \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline \mathbf{W} \end{array}\right]\mathbf{a}$$

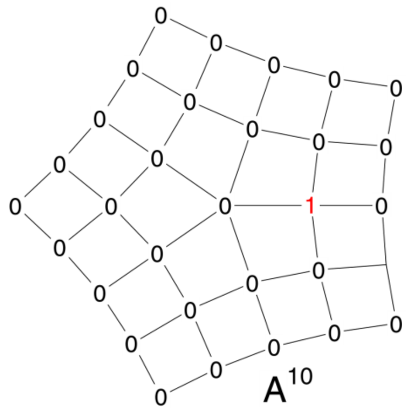
$$\mathbf{E} = \begin{bmatrix} E & 0 & \cdots & 0 \\ 0 & E & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} E & \hat{\mathbf{c}}_j^H \\ \hat{\mathbf{c}}_j & 0 \end{bmatrix}\begin{bmatrix} \hat{p}_j \\ \hat{\lambda}_j \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{w}_j \end{bmatrix}, \quad j = 0, \ldots n-1$$

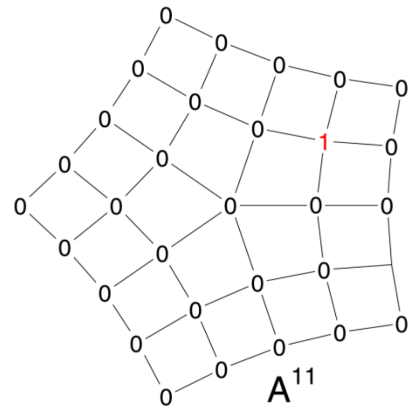
Basis Functions



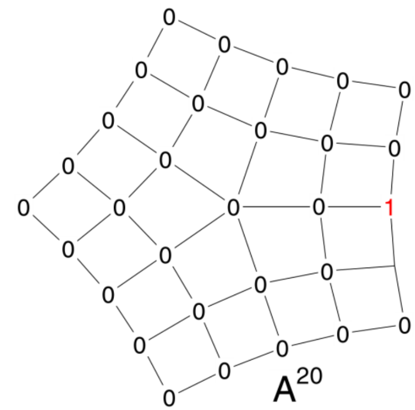
A^{00}



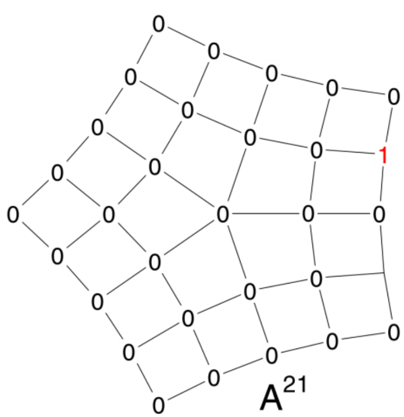
A^{10}



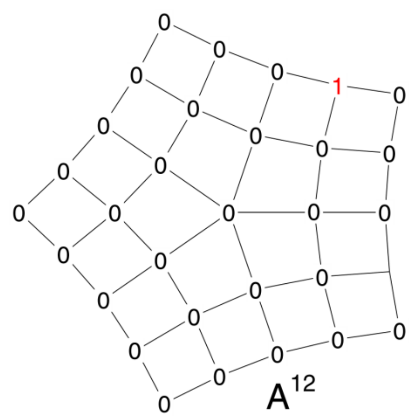
A^{11}



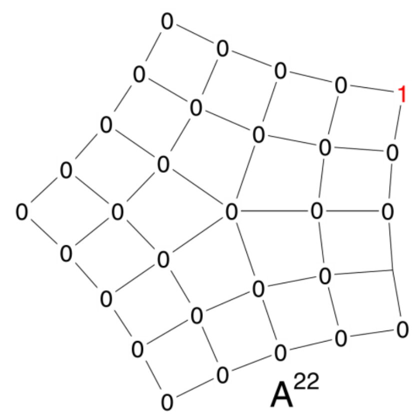
A^{20}



A^{21}

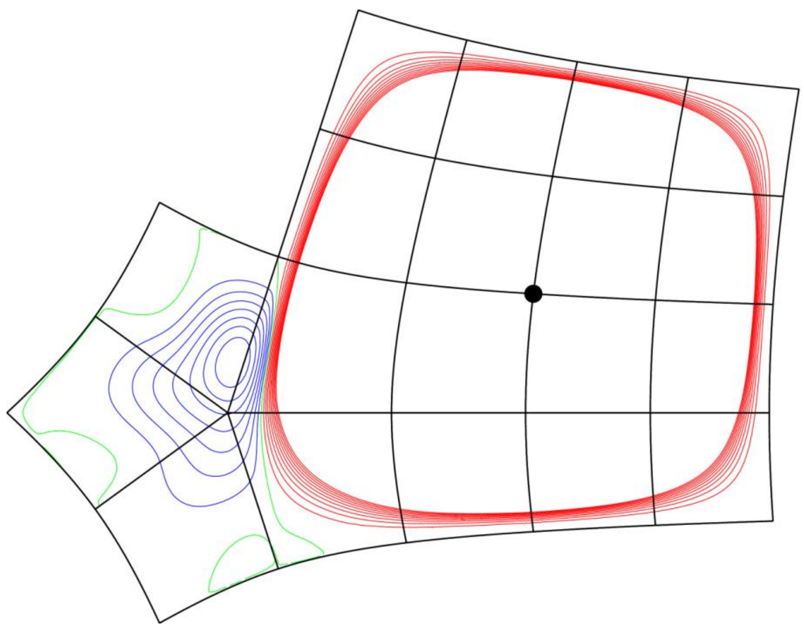


A^{12}

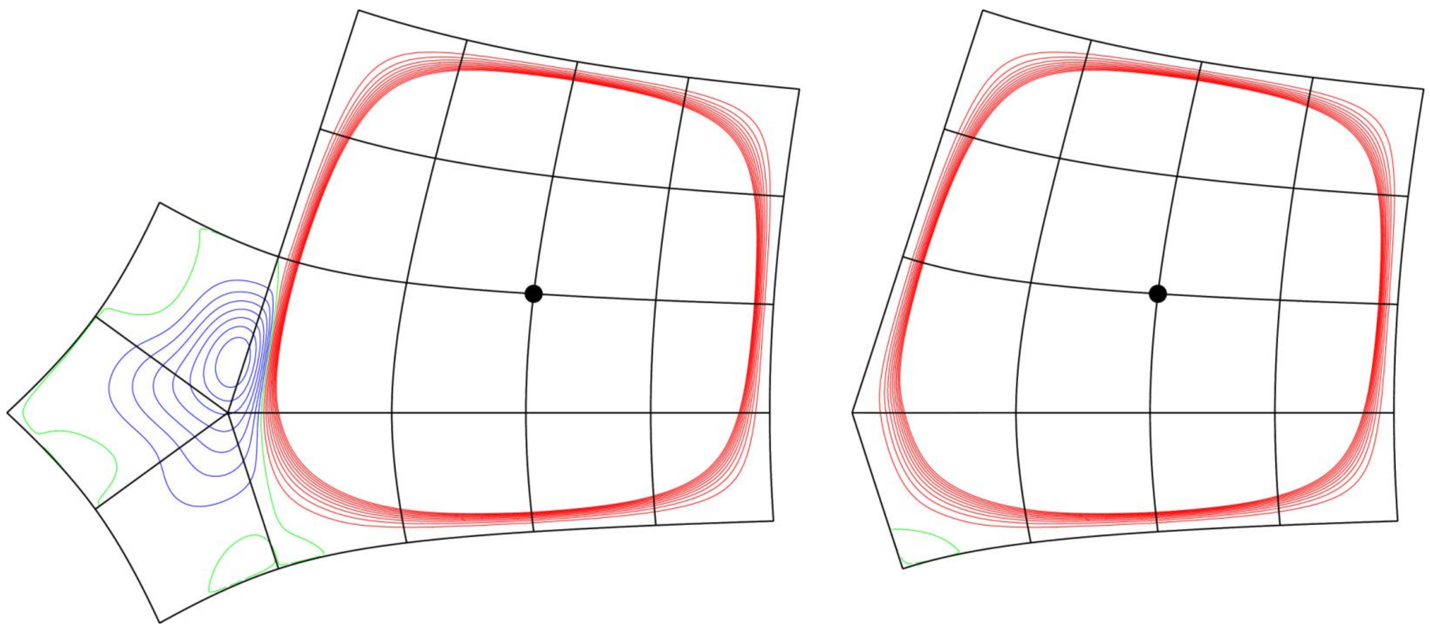


A^{22}

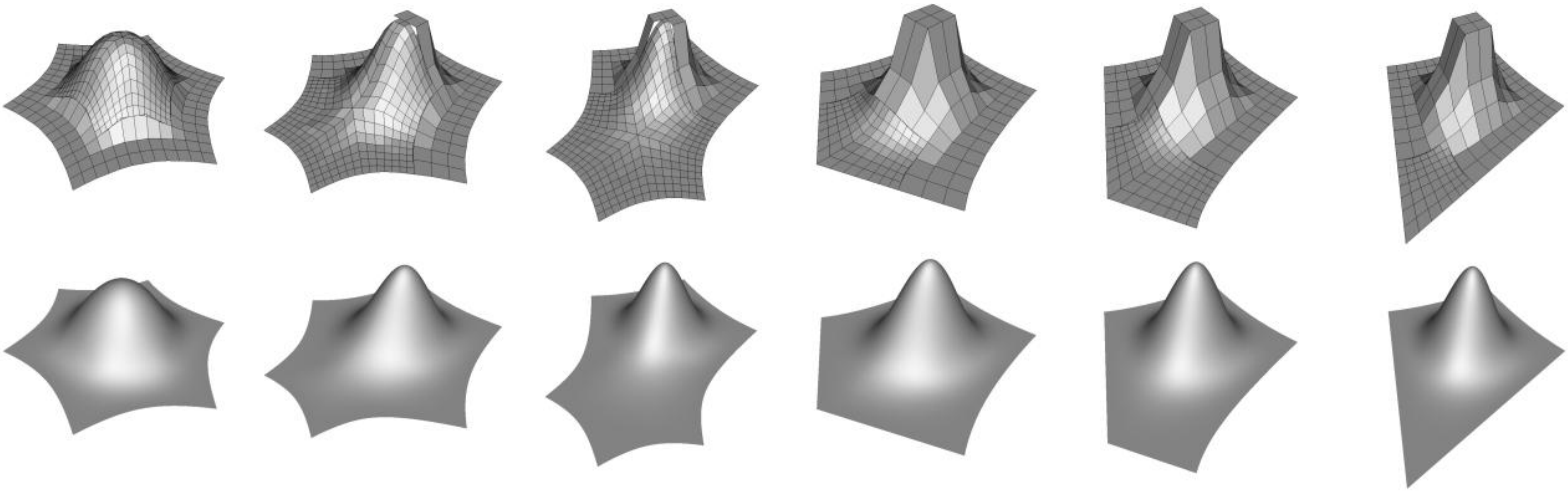
Support Constraints



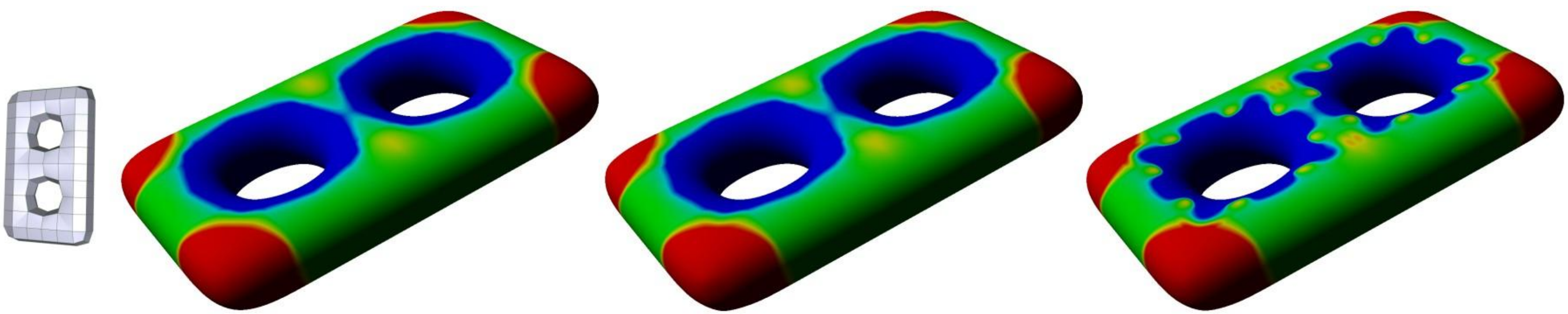
Support Constraints



Basis Function Plots

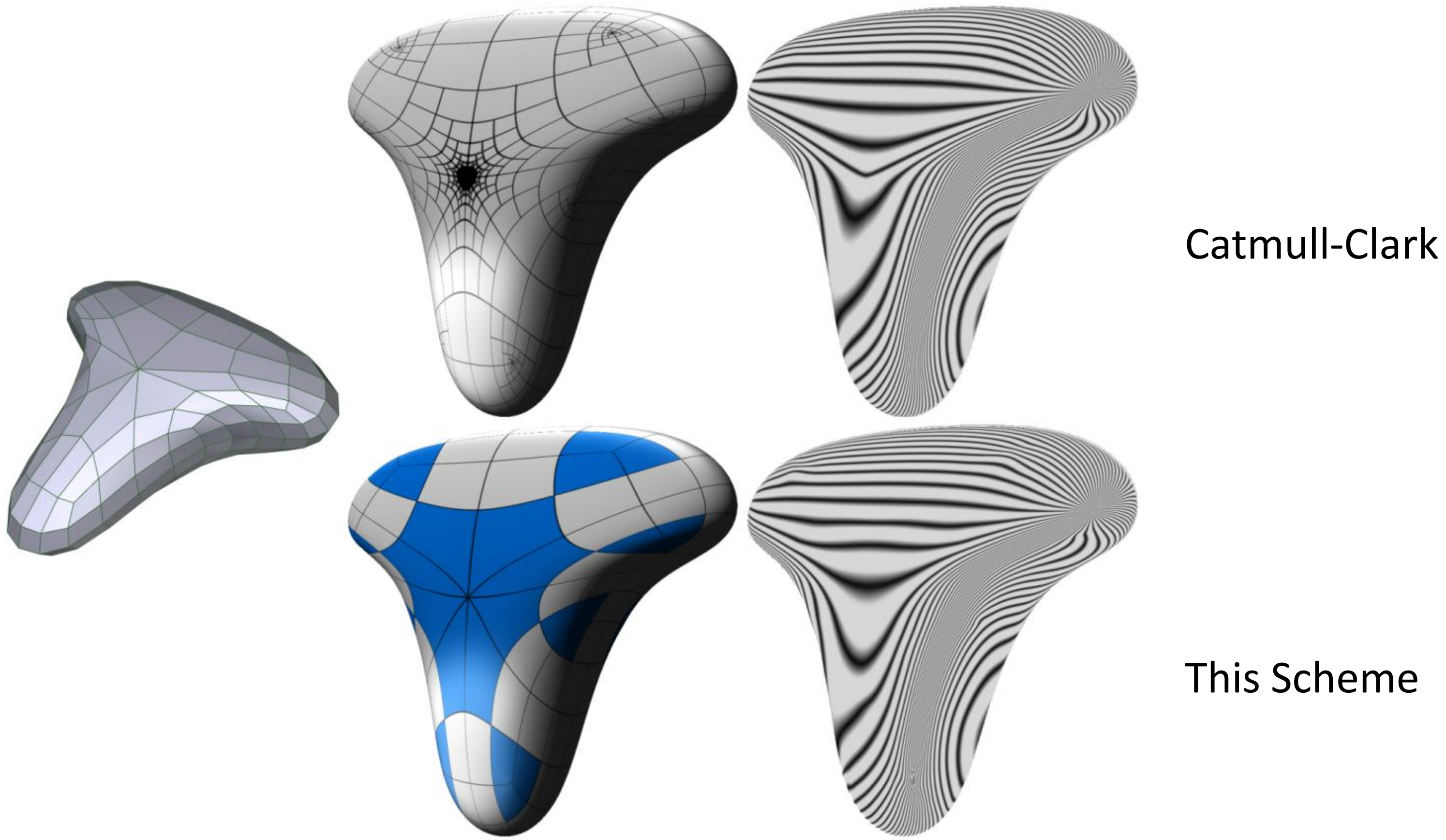


Results

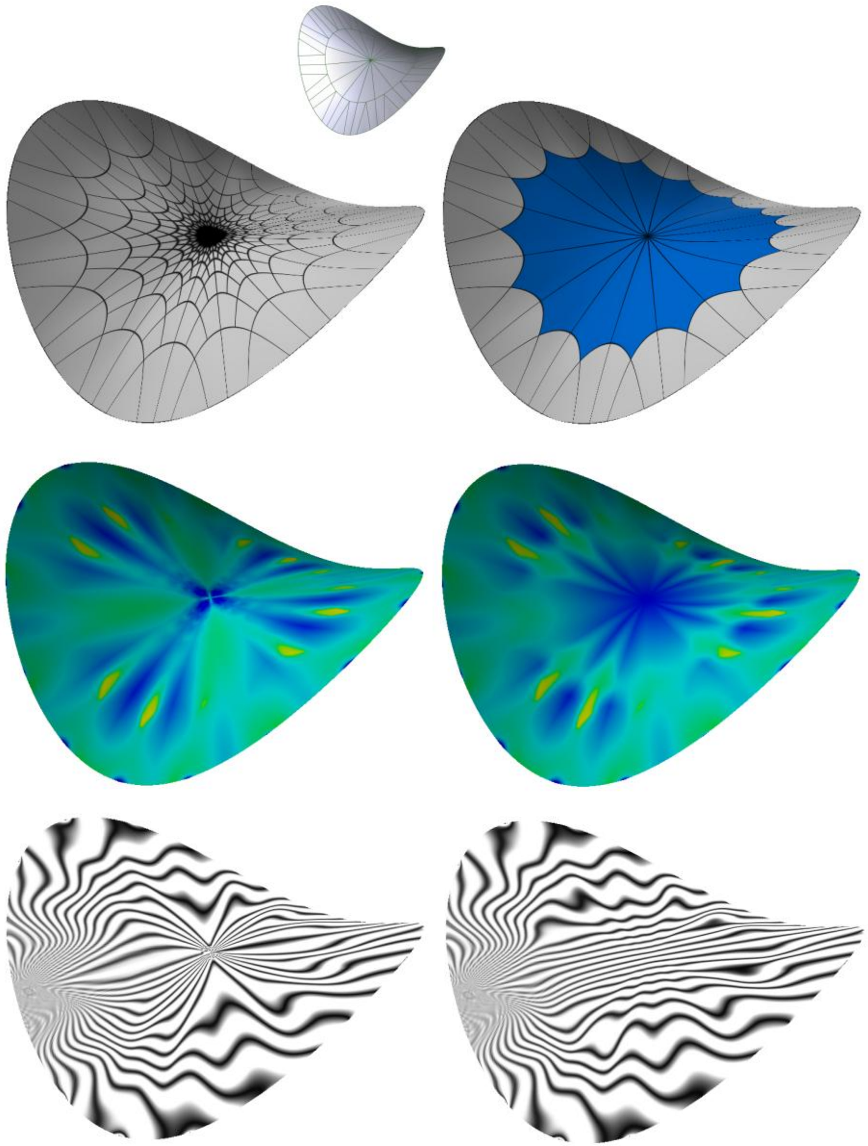


Catmull-Clark This Scheme Loop ‘04

Results



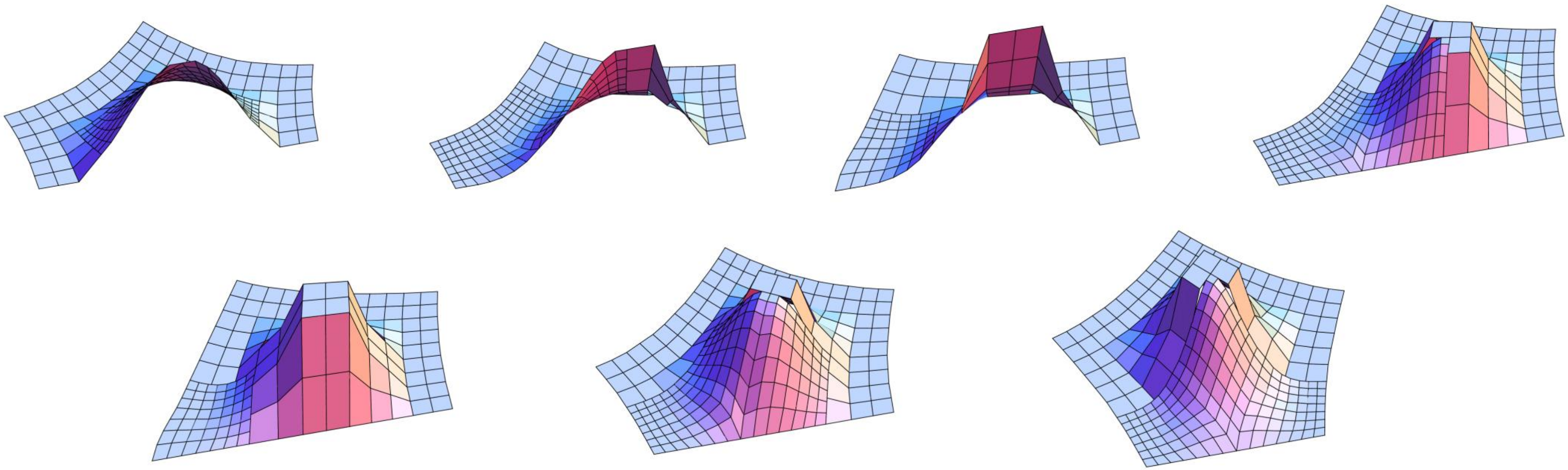
Results



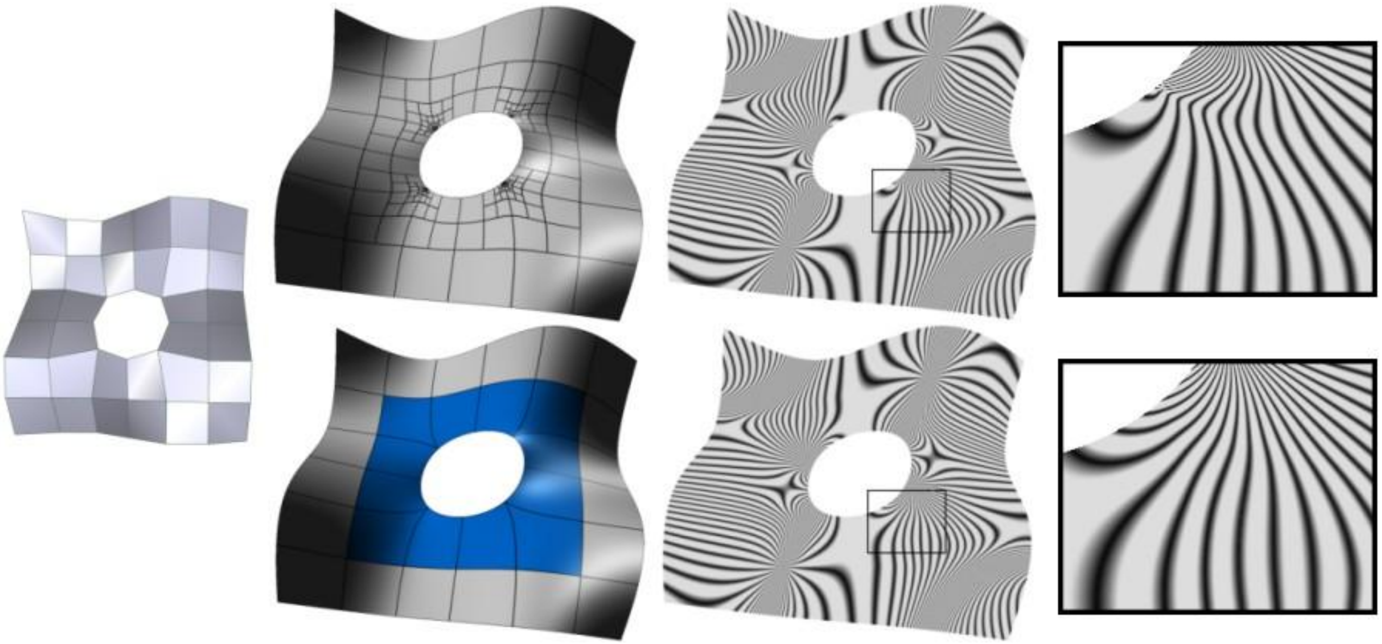
Catmull-Clark

This Scheme

Boundary Basis Functions



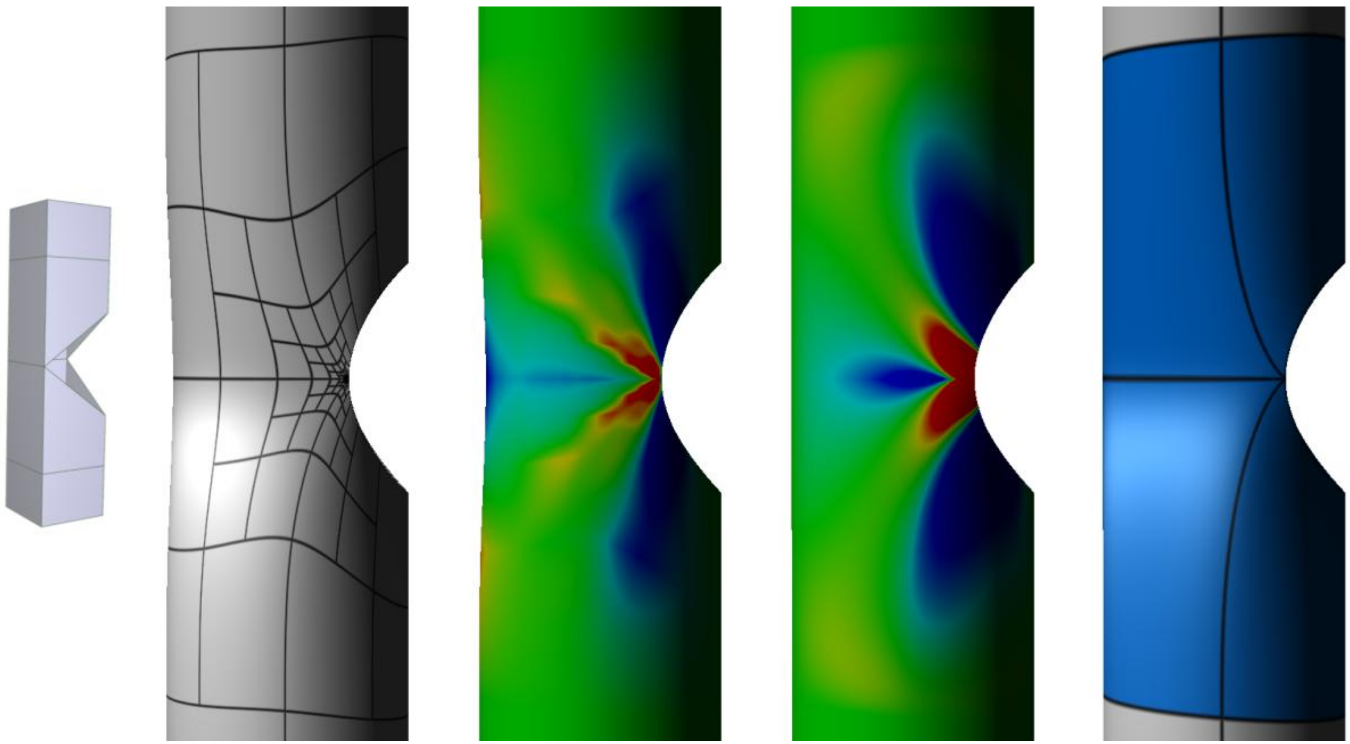
Results



Catmull-Clark

This Scheme

Results



Catmull-Clark

This Scheme

Conclusions

- G^2 piecewise polynomial surface
  - Bidegree 7
  - Finitely many pieces
  - Handle meshes with boundary
- Negative weights
  - Convex minimum?
- Valence 3 has high curvature hot spots
  - Solve as special case?

Thank you for your attention