ECEN 665 (ESS) : *RF Communication Circuits and Systems*

Volterra Series: Introduction & Application

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Part of the material here provided is based on Dr. Chunyu Xin's dissertation

Outline

- Introduction: History, Basics
- Volterra Series in frequency domain
- Applications of Volterra Series:
  - Example1: Common Source Amplifier
  - Example2: Differential Pair
  - Example3: Current Mirror
  - Example4: Gilbert Mixer

Volterra Series: History

- In 1887, Vito Volterra : “Volterra Series” as a model for nonlinear behavior
- In 1942, Norbert Wiener: applied Volterra Series to nonlinear circuit analysis
- In 1957, J. F. Barrett: systematically applied Volterra series to nonlinear system; Later, D.A. George: used the multidimensional Laplace transformation to study Volterra operators
- Nowadays: extensively used to calculate small, but nevertheless troublesome, distortion terms in transistor amplifiers and systems.

Why do we need Volterra Series?

- At high enough frequency, the assumption there's no memory effect due to capacitors and inductors – NOT CORRECT!
- Taylor series analysis: NO memory effect, cannot calculate distortion at high frequency
- Low frequency analysis: high-frequency effect can degrade distortion performance by 100% more than predicted.

eg. Fully differential circuits without mismatch:

Low-frequency analysis predicts the HD_2 to be zero

Volterra series reveals an HD_2 as high as -32dB (see the numerical example later)

When Volterra Series Are Good?

- Can calculate the high-frequency-low-distortion terms for weakly non-linear time-invariant (NLTI) system with memory effect
- "Weakly nonlinear" assumption:
  - Input excitation is "small" → use polynomials to model nonlinearities
  - When small inputs, the 1st term dominates

When Volterra Series Are Bad?

- Results are a sum of infinite numbers of terms, possible to diverge

- Define: “weak enough” system (G1>>G3>>G5; G1, G3, G5 = 1st order, 3rd order, and 5th order Volterra kernels; G5 is small to be negligible), the infinite sums will converge and converge rapidly

- “Strongly nonlinear” system: sum will diverge, Volterra series becomes invalid → Volterra series are impractical in strongly nonlinear problems

Volterra Series: basic

- Linear system without memory :

$$y(t) = h \cdot x(t)$$

Output y at instant t only depends on input x at that instant only.
h: linear gain

- Linear, discrete, causal and time-invariant system with memory (described by summing all the effects of past inputs with proper “weights”):

$$y(n) = \sum_{i=0}^{n} h(\tau_i) \cdot x(n - \tau_i)$$

n: time index h(τ): impulse response

continuous time domain (the convolution sum becomes a convolution integral):

$$y(t) = \int_0^t h(\tau) x(t - \tau) d\tau$$

Volterra Series: basic

System with 2nd order nonlinearity:

Memory-less system: $y_2(t) = h_2 \cdot x^2(t)$

System with memory: (firstly: discrete and assume all the terms sum with equal weights)

$$y(n) = x(n)\cdot x(n) + x(n)\cdot x(n-1) + ... + x(n)\cdot x(0)$$
$$+x(n-1)\cdot x(n-1) + x(n-1)\cdot x(n-2) + ... + x(n-1)\cdot x(0)$$
$$+... + x(0)\cdot x(0) = \sum_{i=0}^{n} x(n)\cdot x(j) + \sum_{i=0}^{n} x(n-1)\cdot x(j) + ... = \sum_{j=0}^{n}\sum_{i=0}^{n} x(i)\cdot x(j)$$

Next: Add proper weights to make it a weighed double sum:

$$y(n) = \sum_{j=0}^{n}\sum_{i=0}^{n} h_2\left(p_i, p_j\right) x(n-p_i)\cdot x(n-p_j)$$

h_2: a function of time index p_i , p_j (2nd order impulse response)

Continuous time domain (the convolution sum becomes a convolution integral) :

$$y_2(t) = \int\int_0^t h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$

Volterra Series: basic

n^{th}-order nonlinear system :

Memory-less system: $y(t) = h_1 \cdot x(t) + h_2 \cdot x^2(t) + \ldots + h_n \cdot x^n(t)$

System with memory:

$$
\begin{aligned}
y(t) &= \int_0^t h_1(\tau_1)x(t-\tau_1)d\tau_1 \\
&+ \int\int_0^t h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \\
&+ \iint\int_0^t h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \\
&+ \cdots + \int \ldots \int_0^t h_n(\tau_1,\tau_2 \ldots \tau_n)x(t-\tau_1)x(t-\tau_2)\ldots x(t-\tau_n)d\tau_1 d\tau_2 \ldots d\tau_n
\end{aligned}
$$

Volterra Series expansion: an infinite sum of multidimensional convolution integrals

$h_n(\tau_1, \cdots, \tau_n)$:n^{th} order Volterra Kernels (n^{th} order impulse response of the system)

Volterra Series vs. Taylor Series

- Nonlinear memory-less system represented using Taylor series:

$$y(t) = K_1 \cdot x(t) + K_2 \cdot x^2(t) + K_3 \cdot x^3(t) + \ldots + K_n \cdot x^n(t) + \ldots$$

- Nonlinear system with memory represented using Volterra series:

$$y(t) = \mathrm{H}_1[x(t)] + \mathrm{H}_2[x(t)] + \mathrm{H}_3[x(t)] + \cdots + \mathrm{H}_n[x(t)] + \cdots$$

in which $H_n[x(t)] = \int \ldots \int h_n(\tau_1, \tau_2 \ldots \tau_n) x(t-\tau_1) x(t-\tau_2) \ldots x(t-\tau_n) d\tau_1 d\tau_2 \ldots d\tau_n$

$\mathrm{H}_n[\cdot]$: n^{th} order Volterra operator.

$h_n(\tau_1, \cdots, \tau_n)$: n^{th} order Volterra Kernels

Volterra kernel

$h_n(\tau_1, \cdots, \tau_n)$: n^{th} order Volterra Kernels

$h_n(\tau_1, \cdots, \tau_n) = 0$: for any $\tau_j < 0$, $j = 1, 2, \ldots, n$

$h_n(\tau_1, \cdots, \tau_n)$ is not necessarily symmetrical to its variables

Always possible to construct a symmetrical kernel from the asymmetrical kernel

$$h_n^{(s)}(\tau_1, \cdots, \tau_n) = \frac{1}{n!} \sum_p h_n^{(a)}(\tau_1, \cdots, \tau_n)$$

Summation runs through all possible permutations

Volterra series in frequency domain

- Why study the nonlinear system in frequency domain?

- Frequency domain Volterra kernels are needed to calculate the distortion. eg. HD_2, HD_3, IM_3,…

- Fourier transform: time domain Volterra → frequency domain Volterra

eg. the n-dimensional Fourier transform for an n^{th} order Volterra kernel $h_n(\tau_1,\cdots,\tau_n)$ is:

$$H_n(\omega_1,\cdots,\omega_n) = F\{h_n(\tau_1,\cdots,\tau_n)\}$$

$$= \int\cdots\int h_n(\tau_1,\cdots,\tau_n) e^{-j\omega_1\tau_1}\cdots e^{-j\omega_n\tau_n} d\tau_1 \ldots d\tau_n$$

H_n is the frequency domain Volterra kernel.

Volterra series in frequency domain

- An input with m frequency components:

$$X = A(\cos\omega_1 t + \cos\omega_2 t + \ldots + \cos\omega_m t)$$

- The output of the n^{th} order nonlinear system can be denoted as:

$$Y = H_1(j\omega_{p1}) \circ X + H_2(j\omega_{p1}, j\omega_{p2}) \circ X^2 + \ldots + H_n(j\omega_{p1}, j\omega_{p2}, \ldots j\omega_{pn}) \circ X^n$$

where $\omega_{p1}, \omega_{p2}, \ldots \omega_{pn}$ can be chosen from $\pm\omega_1, \pm\omega_2, \ldots, \pm\omega_m$

They can be equal or different, and have both the + and – combination

For each term, the frequency components in H_n are the same as in X^n

The operator $\circ$ means:

1) Multiply each frequency component in X^n by: $|H_n(j\omega_{p1}, j\omega_{p2}, \cdots j\omega_{pn})|$

2) Shift phase by: $\angle H_n(j\omega_{p1}, j\omega_{p2}, \cdots j\omega_{pn})$

(analogy to filtering operation)

Volterra series in frequency domain

eg, input with two frequency components: $X = A(\cos w_1 t + \cos w_2 t)$

$w_{p1}, w_{p2}, \ldots w_{pn}$ can be chosen from: $\pm w_1, \pm w_2$

$H_2(jw_{p1}, jw_{p2}) \circ X^2$ represents the following terms (eliminates all the overlap terms):

$$|H_2(jw_1, jw_1)| X^2 \angle H_2(jw_1, jw_1) = \frac{1}{2}|H_2(jw_1, jw_1)| A^2 \cos(2w_1 t + \angle H_2(j2w_1))$$

$$|H_2(jw_1, -jw_1)| X^2 \angle H_2(jw_1, -jw_1) = \frac{1}{2}|H_2(jw_1, -jw_1)| A^2$$

$$|H_2(jw_1, jw_2)| X^2 \angle H_2(jw_1, jw_2) = |H_2(jw_1, jw_2)| A^2 \cos((w_1 + w_2)t + \angle H_2(j(w_1 + w_2)))$$

$$|H_2(jw_1, -jw_2)| X^2 \angle H_2(jw_1, -jw_2) = |H_2(jw_1, -jw_2)| A^2 \cos((w_1 - w_2)t + \angle H_2(j(w_1 - w_2)))$$

$$|H_2(jw_2, jw_2)| X^2 \angle H_2(jw_2, jw_2) = \frac{1}{2}|H_2(jw_2, jw_2)| A^2 \cos(2w_2 t + \angle H_2(j2w_2))$$

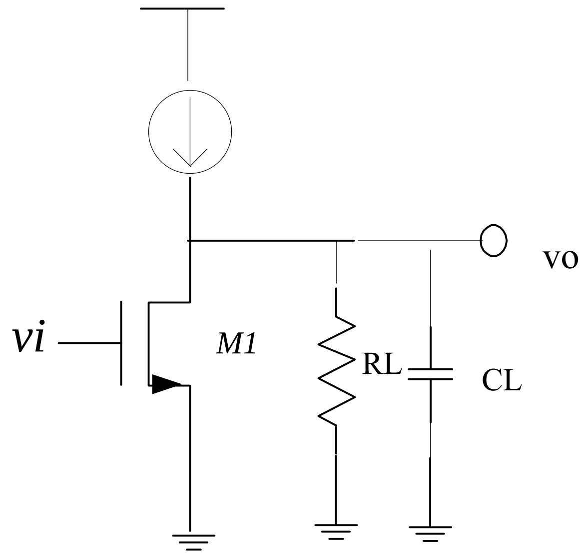
Definition of HD_2, HD_3 and IM_3

	Volterra Series	Taylor Series
HD_2	$\frac{1}{2}\frac{\lvert H_2(jw_1, jw_1)\rvert}{\lvert H_1(jw_1)\rvert}A$	$\frac{1}{2}\frac{a_2}{a_1}A$
HD_3	$\frac{1}{4}\frac{\lvert H_3(j\omega_1, j\omega_1, j\omega_1)\rvert}{\lvert H_1(j\omega_1)\rvert}A^2$	$\frac{1}{4}\frac{a_3}{a_1}A^2$
IM_3	$\frac{3}{4}\frac{\lvert H_3(j\omega_1, j\omega_1, -j\omega_2)\rvert}{\lvert H_1(j\omega_1)\rvert}A^2$	$\frac{3}{4}\frac{a_3}{a_1}A^2$

Observation:

1. Volterra series incorporates the frequency dependent effects
2. $H_3(jw_1, jw_1, jw_1) \neq H_3(jw_1, jw_1, -jw_2) \longrightarrow IM_3 \neq 3HD_3$

Example 1: Common Source Amplifier



■ MOS transistor: nonlinearity mainly introduced by transconductance; output impedance also contributes to distortion.

■ At high frequency, load impedance dominated by C_L → memory effect cannot be neglected, need to use Volterra series to analyze nonlinearity.

Procedure:

1. Express output voltage Vout using Volterra series:

$$v_o = H_0 + H_1 \circ v_i + H_2 \circ v_i^2 + H_3 \circ v_i^3 + \cdots \qquad (1)$$

2. Large signal transfer function using long channel device model:

$$i_o = K\left(v_i + V_{od}\right)^2\left(1 + \lambda v_o\right) \qquad (2)$$

$$v_o = -i_o Z_L = -K\left(v_i + V_{od}\right)^2\left(1 + \lambda v_o\right) Z_L \qquad (3)$$

where $V_{od} = V_{gs0} - V_{th}$, Z_L : impedance of R and C in parallel

3. Substitute (1) into (3):

$$\begin{aligned} H_0 + H_1 \circ v_i + H_2 \circ v_i^2 + H_3 \circ v_i^3 &= D_0 + D_1 v_i + D_2 v_i^2 \\ &+ \lambda D_0\left(H_o + H_1 \circ v_i + H_2 \circ v_i^2 + H_3 \circ v_i^3\right) \\ &+ \lambda D_1\left(H_o + H_1 \circ v_i + H_2 \circ v_i^2 + H_3 \circ v_i^3\right) v_i \\ &+ \lambda D_2\left(H_o + H_1 \circ v_i + H_2 \circ v_i^2 + H_3 \circ v_i^3\right) v_i^2 \end{aligned} \qquad (4)$$

where $D_0 = -K V_{od}^2 Z_L$ $\quad D_1 = -2K V_{od} Z_L$ $\quad D_2 = -K Z_L$

Procedure:

4. To find the Volterra kernel H_k, equate same order terms of v_i in both sides of (4) and use the following relationships:

$$KV_{od}^2 R_L \ll \frac{1}{\lambda} \quad (5) \qquad g_m = 2KV_{od} \quad (7)$$

$$\omega C_L \gg \frac{1}{R_L} \quad (6) \qquad g_o = K\lambda V_{od}^2 \quad (8)$$

It can be found that: $H_0 = \frac{D_0}{1-\lambda D_0} \approx -KV_{od}^2 R_L (1 - g_o R_L)$ (9)

$$H_1(w) = \frac{1+\lambda H_0}{1-\lambda D_0} D_1 \approx -\frac{g_m}{j\omega C_L}(1 - g_o R_L) \quad (10)$$

$$H_2(w_1, w_2) = \frac{D_2 + \lambda (D_1 H_1 + D_2 H_0)}{1-\lambda D_0} \approx -\frac{K(1 - g_o R_L)}{j(\omega_1 + \omega_2) C_L} \quad (11)$$

$$H_3(w_1, w_2, w_3) = \frac{\lambda (D_1 H_2 + D_2 H_1)}{1-\lambda D_0} \approx -\frac{K\lambda g_m Z_3(\omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 + \omega_3) C_L^2} \quad (12)$$

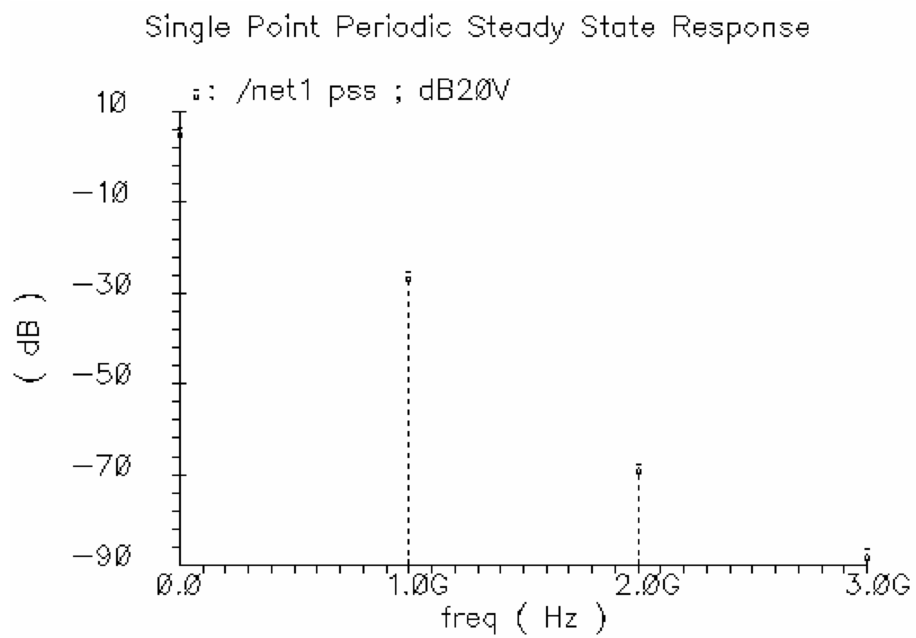
where: $Z_3(\omega_1, \omega_2, \omega_3) = \frac{1}{3}\left(\frac{1}{\omega_1+\omega_2} + \frac{1}{\omega_1+\omega_3} + \frac{1}{\omega_2+\omega_3} + \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3}\right)$

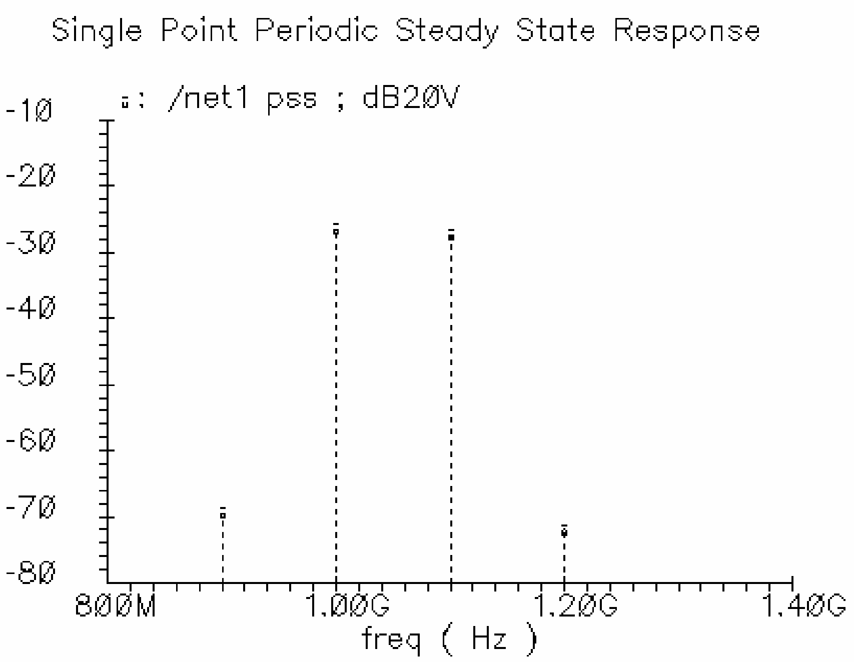
Verification of Volterra Series

- M1=5um/0.6um, K = 130uA/V^2, g_o =100uA/V , gm = 500uA/V, Vod = 1.92V. Input at 0dBm(0.316V), f1 = 1GHz, f2 = 1.1GHz

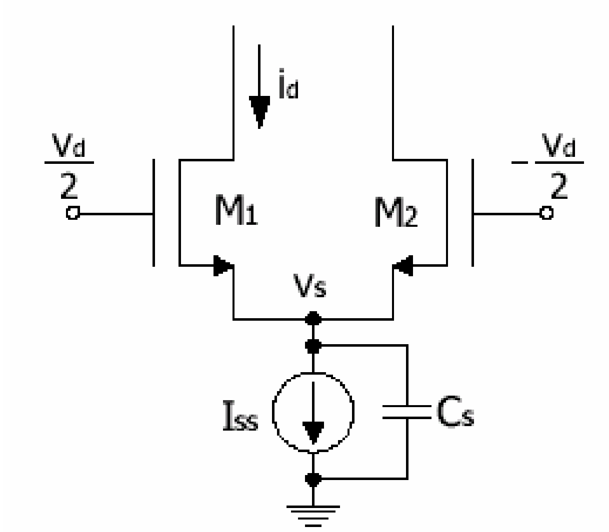
	Volterra Analysis	Cadence Simulation
HD_2	-33.7dB	-38dB
HD_3	-49dB	-58dB
IM3	-34dB	-42.9dB

Simulation Results





Example 2: Differential Pair



- Assume the only memory effect is introduced by the parasitic capacitance (Cs) at node Vs, all other capacitance are ignored.
- DC bias current: $$I_{ss} = K\left(V_{GS} - V_t\right)^2 \quad (1a)$$
- The small signal voltages applied to the gates of M1 and M2 are $V_d/2$ and $-V_d/2$ respectively
- The goal is to obtain the Volterra series expansion for the small signal drain current id of M1 (or M2) in terms of input differential voltage vd up to 3rd order

Procedure:

1. Determine the Volterra series expansion of vs:

$$\begin{aligned} v_s &= G_1 \circ v_d + G_2 \circ v_d^2 + G_3 \circ v_d^3 + \cdots \\ &= v_{s1} + v_{s2} + v_{s3} + \cdots \end{aligned} \quad (1b)$$

$v_{sk} = G_k \circ v_d^k$ is the k-th order term of the Volterra series of vs

2. Apply KCL law at the common source node:

$$C_s \frac{dV_s}{dt} + I_{ss} = \frac{K}{2}\left[(V_{gs1} - V_t)^2 + (V_{gs2} - V_t)^2\right] \quad (2a)$$

where Vs, Vgs1 and Vgs2 can be written into DC and signal small signal terms:

$$V_s = V_S + v_s$$

$$V_{gs1} = V_{GS} + v_{gs1} \qquad V_{gs2} = V_{GS} + v_{gs2}$$

$$v_{gs1} = \frac{v_d}{2} - v_s \qquad v_{gs2} = -\frac{v_d}{2} - v_s \quad (2b)$$

Procedure:

3. Combining equations (1) & (2) yields:

$$C_s \frac{dv_s}{dt} + 2g_m v_s - K v_s^2 = \frac{K}{4} v_d^2 \qquad (3)$$

where $g_m = K\left(V_{GS} - V_t\right)$

Substituting (1b) into (3) and taking phasor form of $\frac{dv_s}{dt}$:

$$\left(j\omega C_s + 2g_m\right)\left(v_{s1} + v_{s2} + v_{s3} + \cdots\right) - K\left(v_{s1} + v_{s2} + v_{s3} + \cdots\right)^2 = \frac{K}{4} v_d^2 \qquad (4)$$

Procedure:

4. Keeping only the 1st order terms of (4):

$$\left(j\omega C_s + 2g_m\right) G_1(\omega) \circ v_d = 0 \qquad (5)$$

$$\longrightarrow \qquad G_1(\omega) = 0$$

$$\longrightarrow \qquad v_{s1} = 0 \qquad (6)$$

5. Substituting (6) into (4), keeping only the 2nd order terms:

$$\left[j\left(\omega_1 + \omega_2\right) C_s + 2g_m\right] G_2(\omega_1, \omega_2) \circ v_d^2 = \frac{K}{4} v_d^2 \qquad (7)$$

$$\longrightarrow \qquad G_2(\omega_1, \omega_2) = \frac{\frac{K}{4}}{j\left(\omega_1 + \omega_2\right) C_s + 2g_m} \qquad (8)$$

Notice that w becomes w_1+w_2 because we are interested in the 2nd order terms

Because of the operator $\circ$, (8) consists of four equations for four different cases:

$\omega_1, \omega_2 = \pm\omega_a, \pm\omega_b$ Where w_1 and w_2 are symbols and w_a, w_b are the frequency components of input.

Procedure:

6. For the 3[rd] order term:

$$\left[j\left(\omega_1+\omega_2+\omega_3\right)C_s+2g_m\right]G_3\left(\omega_1,\,\omega_2,\,\omega_3\right)\circ v_d^3=0 \qquad (9)$$

$$\longrightarrow \qquad G_3\left(\omega_1,\,\omega_2,\,\omega_3\right)=0$$

Replace jw by j(w1+w2+w3) since we are interested in the 3rd order terms

Notice that G_1 always consists of one frequency component (w); G2 always consists of two (w1, w2), and G3 consists of three(w1, w2, w3).

7. Higher order kernels can be calculated by repeating the above steps.

So: $$v_s = v_{s1}+v_{s2}+v_{s3}+\cdots = G_1\circ v_d+G_2\circ v_d^2+G_3\circ v_d^3+\cdots$$

$$= v_{s2}+\cdots = G_2\left(\omega_1,\,\omega_2\right)\circ v_d^2+\cdots$$

Procedure:

- 8. Our final goal is to obtain the Volterra series expansion for id in terms of vd:

$$\begin{aligned} i_d &= H_1 \circ v_d + H_2 \circ v_d^2 + H_3 \circ v_d^3 + \cdots \\ &= i_{d1} + i_{d2} + i_{d3} + \cdots \end{aligned}$$

Using the MOS device equation containing DC and time varying components:

$$I_d + i_d = \frac{K}{2}\left(V_{GS} + \frac{v_d}{2} - v_s - V_t\right)^2 = K\left(V_{GS} - V_t\right)\cdot\left(\frac{v_d}{2} - v_s\right) + \frac{K}{2}\left(\frac{v_d}{2} - v_s\right)^2 + \frac{K}{2}\left(V_{GS} - V_t\right)^2$$

→

$$i_d = g_m\left(\frac{v_d}{2} - v_s\right) + \frac{K}{2}\left(\frac{v_d}{2} - v_s\right)^2$$

Volterra series expansion of id becomes:

$$i_{d1} + i_{d2} + i_{d3} + \ldots = g_m\left(\frac{1}{2}v_d - v_{s2} - \ldots\right) + \frac{K}{8}v_d^{\,2} + \frac{K}{2}\left(v_{s2} + \ldots\right)^2 - \frac{K}{2}v_d\left(v_{s2} + \ldots\right) \qquad (11)$$

Procedure:

- 9. Keeping the 1st order term of v_d coefficients in (11):

$$i_{d1} = \frac{1}{2} g_m v_d$$

$$\rightarrow \qquad H_1(\omega) = \frac{1}{2} g_m$$

10. Isolating the 2nd order terms of v_d coefficients in (11):

$$i_{d2}(\omega_1, \omega_2) = \frac{K}{8} v_d^2 - g_m G_2(\omega_1, \omega_2) \circ v_d^2$$

$$\rightarrow \qquad H_2(\omega_1, \omega_2) \quad \approx j(\omega_1 + \omega_2) \frac{K}{16} \frac{C_s}{g_m}$$

11. Separating the 3rd order terms of v_d coefficients from (11):

$$i_{d3}(\omega_1, \omega_2, \omega_3) \quad = \quad -\tfrac{K}{2} v_d v_{s2}$$

$$H_3(\omega_1, \omega_2, \omega_3) \approx -\frac{K^2}{16 g_m} \left[1 - j(\omega_1 + \omega_2 + \omega_3) \frac{C_s}{3 g_m} \right]$$

Summary of steps to determine Volterra Series:

- **Step1.** Determine the Volterra series expansion of the intermediate variable vs in terms of input signal v_d

$$v_s = G_1 \circ v_d + G_2 \circ v_d^2 + G_3 \circ v_d^3 + \cdots \qquad (1)$$

- **Step2.** Using KCL and MOS device equation to express output signal i_d in terms of v_d and v_s

$$i_d = \frac{K}{2}\left(\frac{v_d}{2} - v_s\right)^2 + g_m\left(\frac{v_d}{2} - v_s\right) \qquad (2)$$

- **Step3.** Substitute vs determined in(1) into (2) and derive:

$$i_d = H_1 \circ v_d + H_2 \circ v_d^2 + H_3 \circ v_d^3 + \cdots$$

Hn becomes a function of Gn, and Gn has been determined in step1.

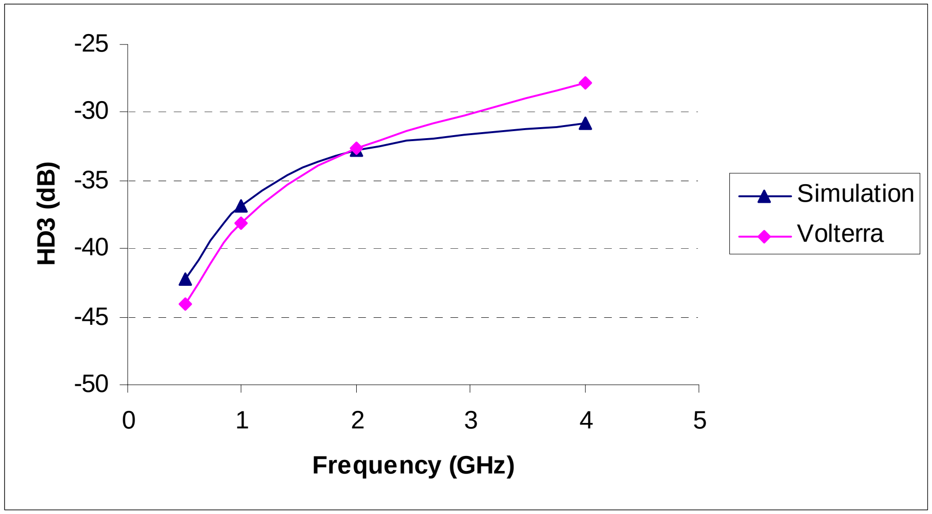
Volterra series versus Taylor series of a MOS differential pair

	Volterra Kernel	Taylor Coefficient
1st order	$\frac{1}{2}g_m$	$\frac{1}{2}g_m$
2nd order	$j\left(\omega_1+\omega_2\right)\frac{K}{16}\frac{C_s}{g_m}$	0
3rd order	$-\frac{K^2}{16g_m}\left[1-j\left(\omega_1+\omega_2+\omega_3\right)\frac{C_s}{3g_m}\right]$	$-\frac{K^2}{16g_m}$

Note that for Taylor expansion, second order distortion is 0, while Volterra series reveals the high frequency second order distortion for a differential pair.

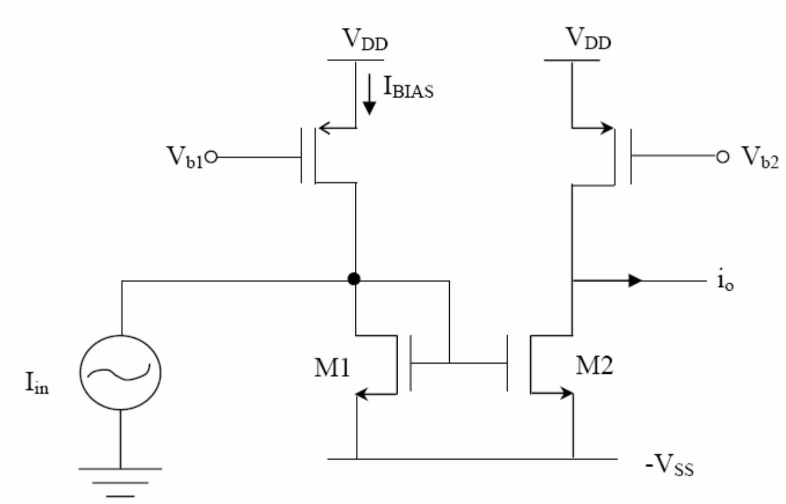
Verification of Volterra Series

For MOS transistors with W/L=50um/0.6um, Volterra series and simulated HD3 result comparison.



$$HD3 = \frac{{V_d}^2 |H_3|}{4H_1}$$

Example 3: Current Mirror



- At high frequency, the parasitic capacitance of the transistors cannot be neglected. Therefore, Volterra Series is used to model this non-linear system with memory.
- The parasitic capacitance Cp seen at the gate of M1, M2 affects the Vgs, which is related to the current mirror accuracy.
- The goal is to obtain the Volterra series expansion for the small signal drain current io of M2 in terms of input current Iin up to 3rd order.

Procedure:

Follow the same steps as described in Example 1:

1. Determine the Volterra series of the gate voltage vi first:

$$vi = G_1 \circ I_{in} + G_2 \circ I_{2in} + G_3 \circ I_{3in} + \ldots = vi_1 + vi_2 + vi_3 + \ldots \quad (1)$$

where $vi_k = G_k \circ I_{kin}$ is the k-th order term of the Volterra series of vi

2. Appling KCL at the gate node of M1:

$$C_p \frac{dV_i}{dt} + \frac{K}{2}(V_i - V_t)^2 = I_{in} + I_{BIAS} \quad (2)$$

where V_i can be written into DC and small signal term:

$$V_i = V_I + v_i \quad (3)$$

Also,

$$\frac{K}{2}(V_I - V_t)^2 = I_{BIAS} \quad (4)$$

Procedure:

3. Substituting (3) and (4) into (2) and simplifying it:

$$C_p \frac{dv_i}{dt} + g_m v_i + \frac{K}{2} v_i^{\,2} = I_{in} \qquad (5)$$

where gm = K(VI – Vt) is the transconductance of M1

Substituting (5) into (1) and taking the phasor form of $\frac{dv_i}{dt}$

$$(jwC_p + g_m)(v_{i1} + v_{i2} + v_{i3} + \ldots) + \frac{K}{2}\left(v_{i1} + v_{i2} + v_{i3} + \ldots\right)^2 = I_{in} \qquad (6)$$

Procedure:

4. The Volterra kernel Gk can be obtained by equating the same order term of I_{in} at both sides of (6). Keeping only the first order terms:

$$(jwC_p + g_m)G_1(w) \circ I_{in} = I_{in} \tag{7}$$

→ $$G_1(w) = \frac{1}{jwC_p + g_m} \tag{8}$$

5. Substituting into (6), and keeping only the second order terms:

$$[j(w_1 + w_2)C_p + g_m]G_2(w_1, w_2) \circ I^2_{in} + \frac{K}{2}G_1^{\;2}(w) \circ I^2_{in} = 0 \tag{9}$$

→ $$G_2(w_1, w_2) = -\frac{\frac{K}{2}G_1^{\;2}(w)}{j(w_1 + w_2)C_p + g_m} = -\frac{\frac{K}{2}\frac{1}{(jwC_p + g_m)^2}}{j(w_1 + w_2)C_p + g_m} \tag{10}$$

$$\approx -\frac{K}{2g^3{}_m}[1 - j(w_1 + w_2)\frac{C_p}{g_m}](1 - \frac{2jw_1C_p}{g_m})$$

Procedure:

6. Factoring out the third order terms from (6):

$$[j(w_1 + w_2 + w_3)C_p + g_m]G_3(w_1, w_2, w_3) \circ I_{3in} + K\,G_2(w_1, w_2)G_1(w_1) \circ I_{3in} = 0$$

$$\rightarrow \quad G_3(w_1, w_2, w_3) = -\frac{KG_2(w_1, w_2)G_1(w_1)}{j(w_1 + w_2 + w_3)C_p + g_m}$$

$$\approx \frac{K^2}{2g^4_m}[1 - j(w_1 + w_2)\frac{C_p}{g_m}](1 - \frac{2jw_1C_p}{g_m})[1 - j(w_1 + w_2 + w_3)\frac{C_p}{g_m}] \qquad (11)$$

7. Now, vi has been expanded into Volterra series up to the third order. Our final goal is to obtain the Volterra series expansion for i_o in terms of I_{in}:

$$i_o = H_1 \circ I_{in} + H_2 \circ I_{2in} + H_3 \circ I_{3in} + \ldots = i_{o1} + i_{o2} + i_{o3} + \ldots \qquad (12)$$

Procedure:

8. The small signal output current io can be related to vi as:

$$i_o = g_{m2} v_i \qquad (13)$$

Substituting (1) and (12) into (13):

$$io1 + io2 + io3 = g_{m2}G1\,(w1) \circ I_{in} + g_{m2}G2(w1,w2) \circ I_{2in} + g_{m2}G3(w1,w2,w3) \circ I_{3in}$$

→

$$H_1(w) = g_{m2}G_1(w_1) \approx \frac{g_{m2}}{g_{m1}}(1 - jw_1\frac{C_p}{g_{m1}})$$

$$H_2(w_1, w_2) = g_{m2}G_2(w_1, w_2) \approx -\frac{Kg_{m2}}{2g^3{}_{m1}}[1 - j(w_1 + w_2)\frac{C_p}{g_{m1}}](1 - \frac{2jw_1C_p}{g_{m1}})$$

$$H_3(w_1, w_2, w_3) = g_{m2}G_3(w_1, w_2, w_3)$$

$$\approx \frac{K^2 g_{m2}}{2g^4{}_{m1}}[1 - j(w_1 + w_2)\frac{C_p}{g_{m1}}](1 - \frac{2jw_1C_p}{g_{m1}})[1 - j(w_1 + w_2 + w_3)\frac{C_p}{g_{m1}}]$$

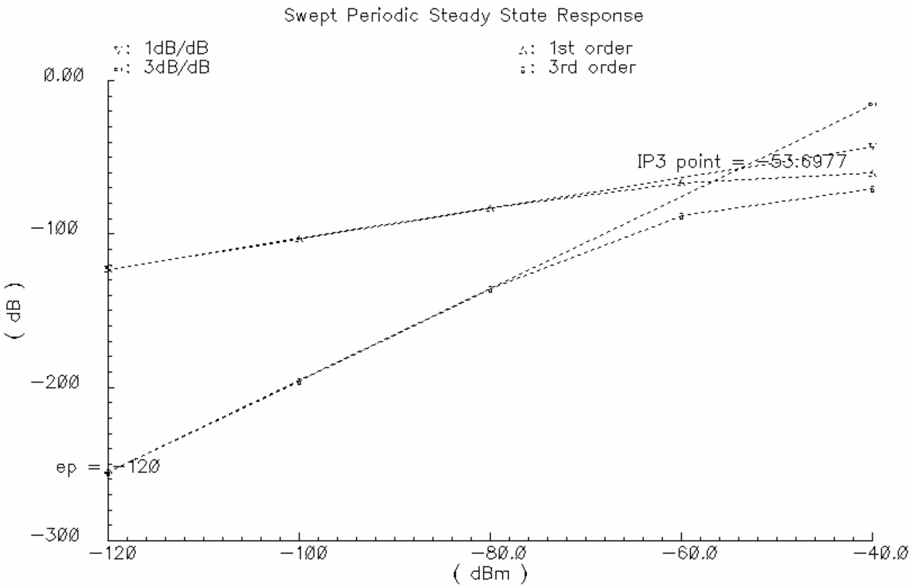
Verification of Volterra Series

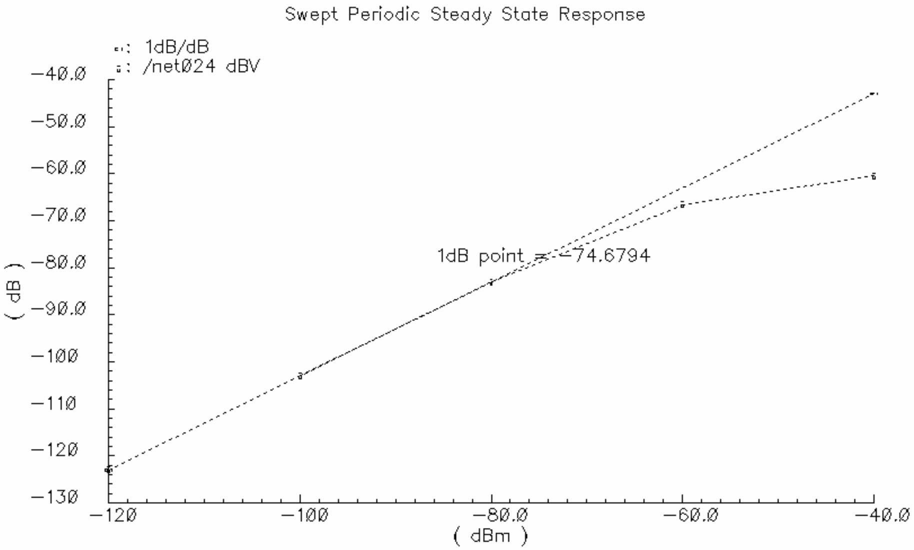
Assume a current gain of 4, use long channel length to minimize channel length modulation effect. Choose M1 = 2um/1um, M2 = 4*2um/1um, I_{BIAS} = 84.4uA, Cp~90fF, gm1 = 176μA/V, gm2 = 705μA/V, $I_{in} = A_1 \cos w_1 t + A_2 \cos w_2 t$, while w_1 = 910 * 2pi MHz, w_2 = 900 * 2pi MHz, $A_1 = A_2 = I_{BIAS}/10$ = 8.441μA, we can get:

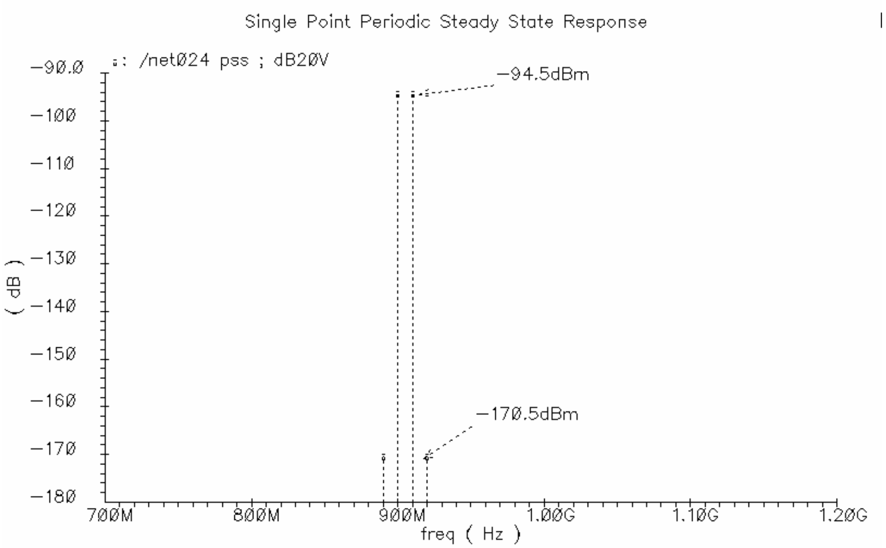
$$IM3 = \frac{3A_2^{\ 2}}{4}\left|\frac{H_3(jw_1,-jw_2,-jw_2)}{H_1(jw_1)}\right| = 62dB$$

IIP3 = P_{in} + IM_3/2 = -91.5dBm + 62/2 = - 60.5dBm
1-dB compression point = IIP3 – 10dB = -70.5dBm

Simulation Results



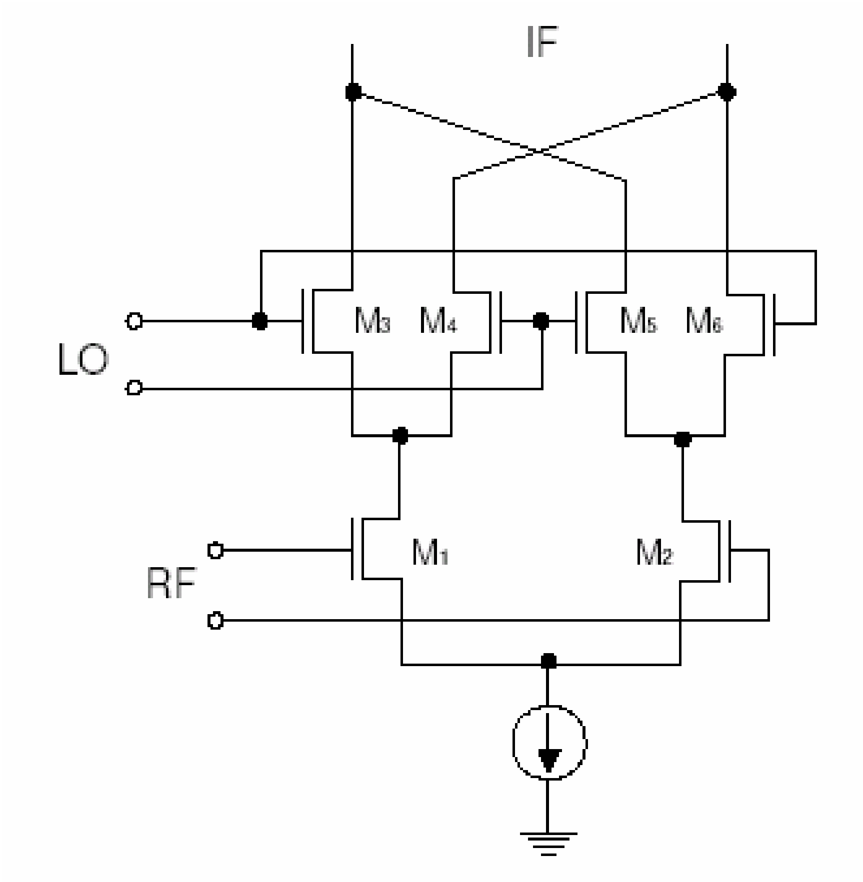




Comparison between Volterra analytical results and Cadence simulation

	Volterra analysis	Cadence simulation
IIP3	-60.5dBm	-53.7dBm
IM3	62dB	76dB
1-dB point	-70.5dBm	-74.7dBm

Example 4:Gilbert Mixer



- The high frequency linearity of Gilbert mixer is dominated by the V-I conversion.

- Transistors M3-M6 works as large signal current switches and are not limiting factor for linearity.

- Assume V_{LO} is not switching ans so the mixer is a NLTI system

Volterra kernels:

- The Volterra series analysis of Gilbert mixer follows the Volterra series analysis of a differential pair
- The Volterra kernels for the V-I conversion transistors are:

$$H_1(\omega) = \frac{1}{2} g_m$$

$$H_2(\omega_1, \omega_2) \approx j(\omega_1 + \omega_2) \frac{K}{16} \frac{C_s}{g_m}$$

$$H_3(\omega_1, \omega_2, \omega_3) \approx -\frac{K^2}{16 g_m} \left[1 - j(\omega_1 + \omega_2 + \omega_3) \frac{C_s}{3 g_m} \right]$$

Finding IM3 and HD3

- the linearity expression for Gilbert mixer cell is:

$$HD_3 = \frac{1}{4}\frac{|H_3(j\omega_1, j\omega_1, j\omega_1)|}{|H_1(j\omega_1)|}A_{rf}^{\ 2}$$

$$IM_3 = \frac{3}{4}\frac{|H_3(j\omega_1, j\omega_1, j\omega_2)|}{|H_1(j\omega_1)|}A_{rf}^{\ 2}$$

- IM3 is caused by adjacent channel interference
- w_2 is defined to be at $-w_1-\Delta w$ such that the 3rd order nonlinearity term $2w_1+w_2$ will generate a term at $w_1-\Delta w$
- A_{rf} should be replaced by $A_{interference}$

Low frequency approximation

$$H_3(\omega_1, \omega_2, \omega_3) \approx -\frac{K^2}{16 g_m}\left[1 - j(\omega_1 + \omega_2 + \omega_3)\frac{C_s}{3 g_m}\right]$$

$$\xrightarrow{w_1 = w_2 = w_3 = 0} \; = \frac{-K}{16(V_{GS} - V_t)}$$

$$H_1(\omega) = \frac{1}{2} g_m = \frac{K}{2}(V_{GS} - V_t)$$

$$HD_3 = \frac{1}{4}\frac{|H_3(\omega_1, \omega_1, \omega_1)|}{|H_1(j\omega_1)|} A_{rf}{}^2 = \frac{A_{rf}{}^2}{32}\frac{K}{I_{SS}}$$

- For low frequency (no memory), HD_3 agrees with Taylor series expansion
- Do the same for IM3 and it would agree with Taylor series expansion

IM3 and HD3

- For intermediate frequency: $j(w_1+w_2)Cs \ll 2K(V_{GS}-V_t)$

$$HD3 = \sqrt{\frac{K}{I_{SS}}} \frac{{A_{rf}}^2}{32 \cdot (V_{GS} - V_t)} \left| 1 - \frac{j(2w_1)Cs}{2K(V_{GS} - V_t)} \right|$$

$$IM3 = \frac{3{A_{interference}}^2}{32(V_{GS} - V_t)^2} \left[\left| 1 - \frac{2}{3} \frac{j(w_1)Cs}{2K(V_{GS} - V_t)} \right| \right]$$

$$IM_3 \neq 3HD_3$$

When V_{LO} is switching…

- The mixer is no longer a time-invariant system
- The high-frequency distortion still dominated by V-I conversion
- To incorporate switching effect, multiply the output by the Fourier series representation of a square wave
- Frequency of the 1st and 3rd order products is shifted by f_{LO}, Amplitude are reduced by 1/pi.
- HD3 and IM3 remain unchanged

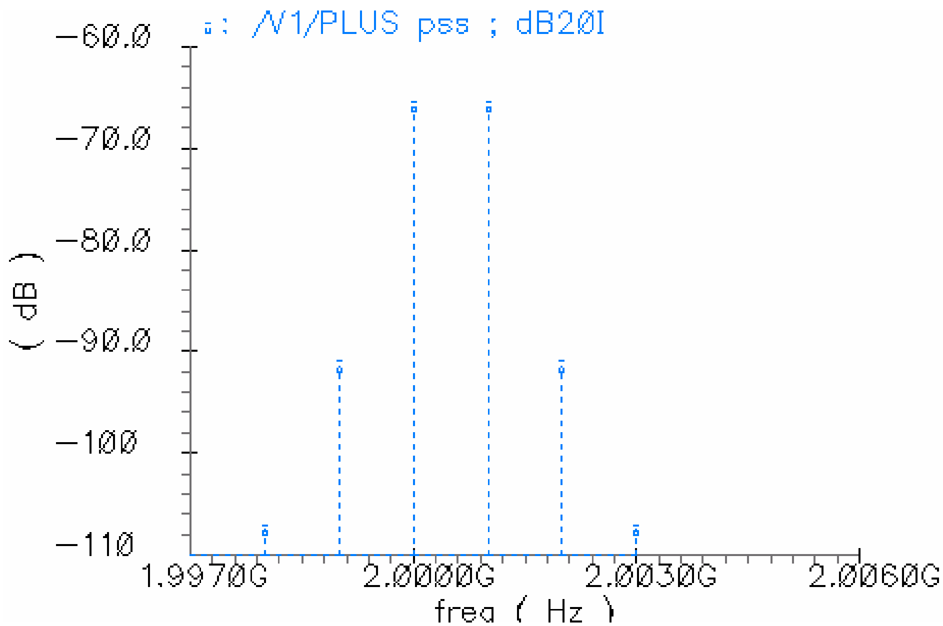
Gilbert Mixer IM_3 Verification

Assume a Gilbert Mixer has V-I conversion transistor M1=M2=50um/0.6um. Let f_{rf}=2GHz, two interference signal with 0dBm power (0.316V) at 2GHz(w_1) and 2.001GHz(w_2). Cs ~= 0.1pF. Make V_{GS}-V_t = 0.387V, K=6250 uA/$V^{2,}$ thus gm = 2.42mA/V

Theoretically: $$IM_3 = \frac{3}{4}\frac{|H_3(\omega_1,\omega_1,\omega_2)|}{|H_1(j\omega_1)|}A_{rf}^{\ 2} = -24.0dB$$

Simulation result

- Gilbert cell simulation result: with two tones at 2GHz and 2.001GHz, the input amplitude of each tone is 0dBm.



IM3 is around -25.5 dB from simulation. The deviation error is 1.5dB(5.9%)

HD_2

Incorporating Memory: $$HD_2 = \frac{A_{rf}\left|H_2\right|}{2\left|H_1\right|} = -32dB$$

Ignoring memory: $$HD_2 = \frac{A_{rf}a_2}{2a_1} = 0$$

Volterra series analysis predicts that RF feedthrough will still be Present in a balance mixer with zero mismatch!

Conclusions

- Taylor series analysis cannot calculate the distortion correctly at high frequency due to memory effect.
- Volterra series can calculate the high-frequency-low-distortion terms for any weakly non-linear time-invariant system with memory effect.
- Volterra series may diverge when nonlinearity is strong
- Volterra series are applied to four basic circuit examples and theoretical results agrees with simulation

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