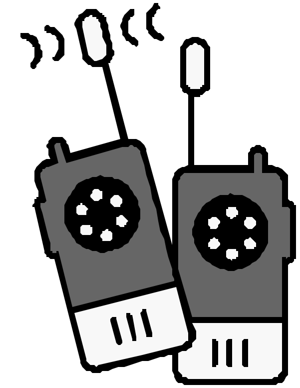


RADIO FREQUENCY - METRICS

Distortion

Consider a nonlinear system described by the following equation:

$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) \qquad \text{(A.1)}$$

Where y(t) and x(t) is the output and input of the system respectively.

Assume x(t)=Acos(ωt), then from equation (A.1) we get:

$$y(t) = a_0 + a_1 A\cos(wt) + a_2 A^2 \cos^2(wt) + a_3 A^3 \cos^3(wt) \qquad \text{(A.2a)}$$

$$y(t) = \left(\alpha_0 + \frac{\alpha_2 A^2}{2}\right) + \left(\alpha_1 A + \frac{3\alpha_3 A^3}{4}\right)cos(\omega t) + \left(\frac{\alpha_2 A^2}{2}\right)cos(2\omega t) + \left(\frac{\alpha_3 A^3}{4}\right)c0s(3\omega t) \qquad \text{(A.2b)}$$

Note that the DC (fundamental) magnitude is affected by the even (odd) harmonic components.

In equation (A.2b), the term with the input frequency is called the fundamental and

The higher order terms the harmonics. LNAs are typical examples of eq. (a.2). Harmonic distortion factors (HD_i) provide a measure for the distortion introduced by each harmonic for a given input signal level

(using a single tone at a given frequency). HD_i is defined as the ratio of the output

signal level of the i[th] harmonic to that of the fundamental. The THD is the geometric

mean of the distortion factors. Assuming $a_1 A >> 3a_3 A^3/4$ the second harmonic distortion HD_2 , the third harmonic distortion HD_3 and the total harmonic distortion THD are defined as:

$$HD_2 = \frac{a_2 A}{2a_1} \qquad \text{(A.3a)}$$

$$HD_3 = \frac{a_3 A^2}{4a_1} \qquad \text{(A.3b)}$$

$$THD = \sqrt{HD_2^2 + HD_3^2 + HD_4^2 + \ldots\ldots} \qquad \text{(A.3c)}$$

For fully differential systems, ideally even harmonics will vanish and only odd harmonics remain. In reality, however, mismatches corrupt the symmetry, yielding finite even order harmonics. In a fully differential system with $e\%$ mismatch and from equation (A.3a), HD_2 is given by:

$$HD_2 = e\,\frac{a_2 A}{2a_1} \tag{A.4}$$

1-dB compression point

The 1-dB compression point is defined as the point where the fundamental gain deviates from the ideal small signal gain by 1 dB, as shown in Fig. A.1. From the previous definition and from equation (A.2b), we have:

$$20\log\left(a_1 A_{1-dB} + \frac{3a_3 A_{1-dB}^3}{4}\right) = 20\log(a_1 A_{1-dB}) - 1 = 20\log(0.89125 a_1 A_{1-dB}) \tag{A.5}$$

Note that 20 log 0.89125=-1dB, |1-0.89125| = 0.10875

$$A_{1-dB}^{2} = 0.10875\frac{4}{3}\frac{|a_1|}{|a_3|} = k\frac{|a_1|}{|a_3|} \qquad \text{(A.6)}$$

Note that $A_{1\text{-dB}}$ often occurs for a certain V_{pp} value in a typical resistor R_s=50Ω.

Thus at the 1-dB compression point, the value of HD_3 can be calculated as:

$$HD_3 = \frac{1}{4}\frac{k}{A_{1-dB}}A^2 = \frac{0.145}{4}\frac{|a_3|}{|a_1|}\frac{|a_1|}{|a_3|} = \frac{0.145}{4}\times 100\% = 3.6\% \qquad \text{(A.7)}$$

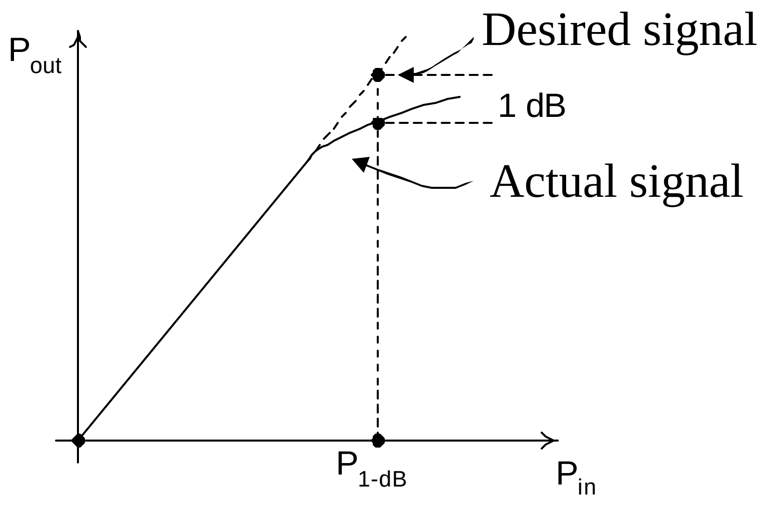


Fig. A.1 Definition of the 1-dB compression point

Intermodulation Distortion

Consider $x(t) = A\cos(w_1 t) + A\cos(w_2 t)$, then from equation (A.1) and trigonometric identities:

$$\begin{aligned} y(t) = {} & \left(a_0 + a_2 A^2\right) + \left(a_1 A + \frac{9a_3 A^3}{4}\right)\cos(w_1 t) + \left(a_1 A + \frac{9a_3 A^3}{4}\right)\cos(w_2 t) + \\ & \left(\frac{a_2 A^2}{2}\right)\cos(2w_1 t) + \left(\frac{a_2 A^2}{2}\right)\cos(2w_2 t) + \left(a_2 A^2\right)\cos((w_1 + w_2)t) + \\ & \left(a_2 A^2\right)\cos((w_1 - w_2)t) + \left(\frac{3a_3 A^3}{4}\right)\cos((2w_1 - w_2)t) + \left(\frac{3a_3 A^3}{4}\right)\cos((2w_2 - w_1)t) + \\ & \left(\frac{3a_3 A^3}{4}\right)\cos((2w_1 + w_2)t) + \left(\frac{3a_3 A^3}{4}\right)\cos((2w_2 + w_1)t) + \\ & \left(\frac{a_3 A^3}{4}\right)\cos(3w_1 t) + \left(\frac{a_3 A^3}{4}\right)\cos(3w_2 t) \end{aligned} \tag{A.8}$$

The third order input intercept point IIP_{3i} is defined as the intercept point of the fundamental component with the third order intermodulation component, as shown in Fig. A.2. From the previous definition and from equation (A.8), we have:

$$(\alpha_1 A_{IIP3i}) = \left(\frac{3\alpha_3 A_{IIP3i}^3}{4}\right) \Rightarrow A_{IIP3i}{}^2 = \frac{4}{3}\frac{|\alpha_1|}{|\alpha_3|} \quad \text{(A.9)}$$

The input IP_3 is $A_{IIP3} = \left[\frac{4}{3}\frac{|\alpha_1|}{|\alpha_3|}\right]^{\frac{1}{2}}$ and the output IP_3 is $\alpha_1 A_{IIP3}$

The third order intermodulation distortion IM_3 is defined as:

$$IM_3 = \frac{3}{4}\frac{|\alpha_3|}{|\alpha_1|} A^2 = 3HD_3 \quad \text{(A.10)}$$

Note from equations (A.6) and (A.9) that:

$$\frac{A_{IIP3i}{}^2}{A_{1-dB}{}^2} = 9.195 \Rightarrow A_{IIP3i}(dB) \cong A_{1-dB}(dB) + 10 \quad \text{(A.11)}$$

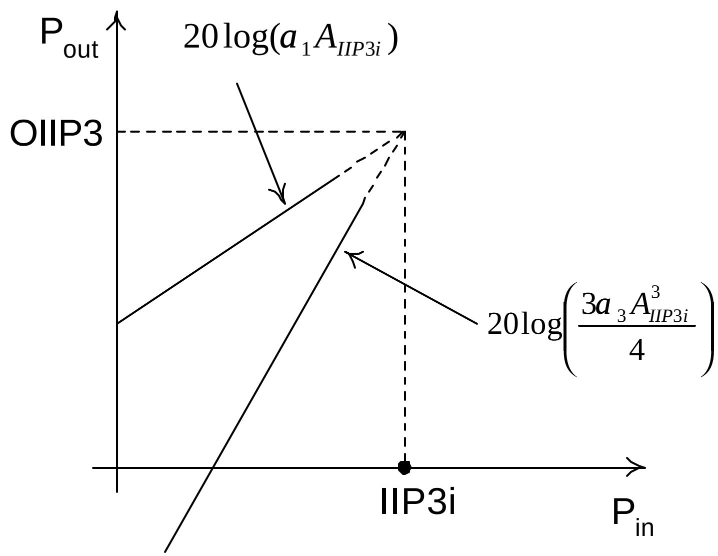


Fig. A.2 Definition of the third order intercept point

How to Measure IIP3

Let us assume that the input of a nonlinear device characterized by (A.2a) which consists of three components, the desired input signal with amplitude A_s at f_o, and two interferers at f_1 and f_2, where $f_1 = f_o + \Delta f$ and $f_2 = f_o + 2\Delta f$. The amplitude of the IM3 products are denoted by A_{IM3}. Then one can write from (A.10).

$$\frac{A_{1,2}}{A_{IM3}} = \frac{|\alpha_1| A_s}{3|\alpha_3| A_s^3 / 4} = \frac{4|\alpha_1|}{3|\alpha_3|}\frac{1}{A_s^2} \qquad \text{(B.1)}$$

Since

$$A_{IP3}^2 = \frac{4}{3}\frac{|\alpha_1|}{|\alpha_3|}$$

Then (B.1) can be written as

$$\frac{A_{1,2}}{A_{IM3}} = \frac{A_{IP3}^2}{A_s^2} \qquad \text{(B.2)}$$

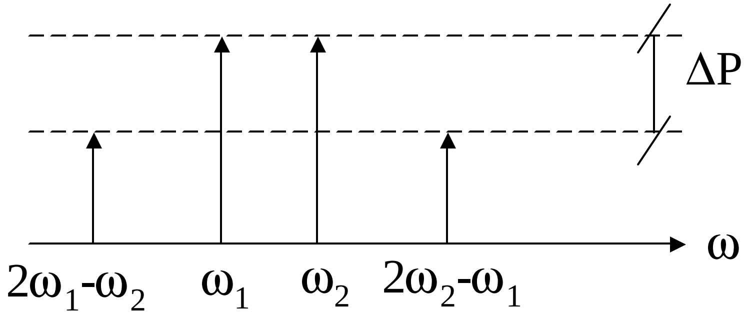
Thus by taking the log of (B.1) and (B.2) and equating them, yields

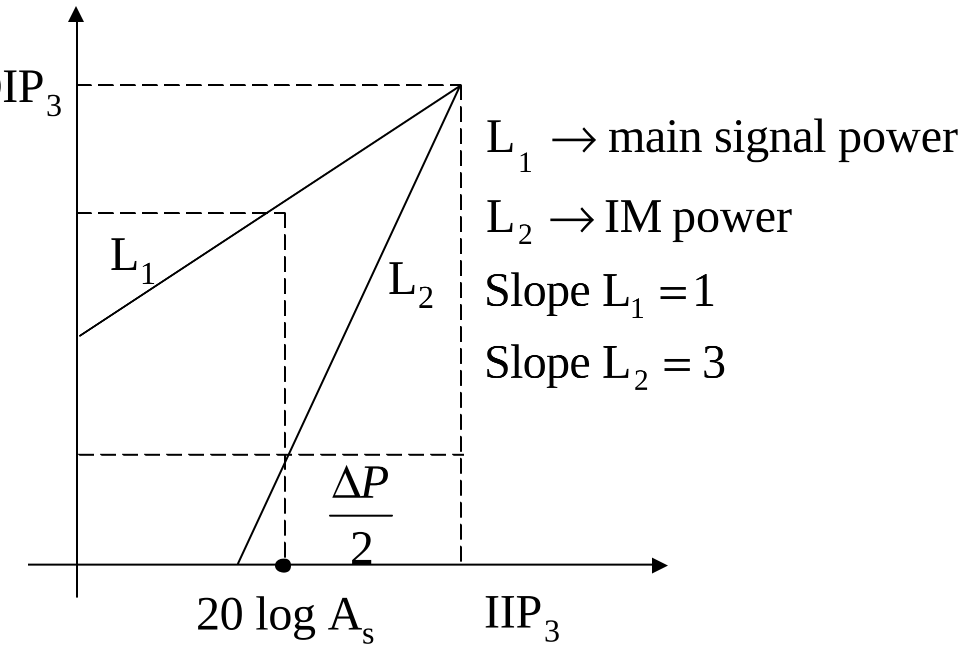
$$20\,log\,A_{1,2} - 20\,log\,A_{IM3} = 20\,log\,A_{IP3}^2 - 20\,log\,A_s^2$$

or equivalent

$$20\log A_{IP3} = \frac{1}{2}\left(20\log A_{1,2} - 20\log A_{IM3}\right) + 20\log A_s \qquad \text{(B.3)}$$

- Observe that A_{IP3} can be determined with only one input level, thus there is no need for extrapolation.





- This approach is not accurate, but it provides a good estimate of IP_3.

Dynamic Range

The are many definitions for the dynamic range. We define here the 1-dB compression dynamic range $DR_{1\text{-dB}}$ and the spurious free dynamic range (SFDR). The SFDR is the difference, in dB, between the fundamental tone and the highest spur, which could be an intermodulation harmonic, in the bandwidth of interest.

$$DR_{1-dB} = P_{i,1dB} - P_{i,mds} \tag{A.12}$$

$$SFDR = \frac{2}{3}(IIP_{3i} - P_{i,mds}) \tag{A.13}$$

where , $P_{i,mds} = -174dBm + 10\log B + NF$ $\quad P_{o,mds} = P_{i,mds} + a_1\big|_{dB}$

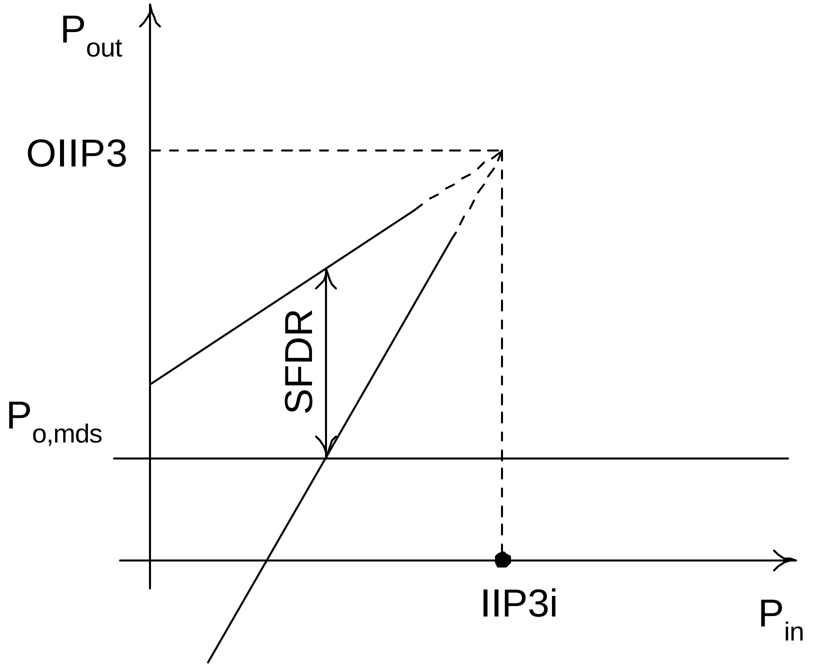


Fig. A.3 Definition of SFDR

Remarks on Dynamic Range.

The upper end of the dynamic range is defined as the maximum input level in a two tone test for which the IM_3 products do not exceed the noise floor.

Recall (B.3):

$$20\,log\,A_{IP3} = \frac{1}{2}\left(20\,log\,A_{1,2} - 20\,log\,A_{IM3}\right) + 20\,log\,A_s$$

This can be written, assuming all the parameters in dBm, as follows:

$$P_{IIP3} = P_s + \frac{P_{out} - P_{IM,out}}{2}$$

Where $P_{IM,out}$ is the power of IM_3 components at the output. G is the circuit power gain in dB, and $P_{IM,in}$ is the input referred level of the Im_s products. Thus

$$P_{out} = P_s + G$$

$$P_{IM,out} = P_{IM,in} + G$$

$$P_{IIP3} = P_s + \frac{P_s - P_{IM,in}}{2} = \frac{3P_s - P_{IM,in}}{2}$$

Whereby

$$P_s = P_{in} = \frac{2P_{IIP3} + P_{IM,in}}{3}$$

Observe that the input level for which the IM products become equal to the noise floor yields:

$$P_{in,max} = \frac{2P_{IP3} + NF_{total}}{3}$$

Where $NF_{total} = -174dBm + NF + 10\,log\,B$

Therefore

$$SFDR = \frac{2P_{IIP3} + NF_{total}}{3} - \left(NF_{total} + SNR_{min}\right)$$

$$SFDR = \frac{2\left(P_{IIP3} - NF_{total}\right)}{3} - SNR_{min}$$

Example

For a simple differential pair as shown in Fig. A.4.

$$I_0 = I_1 - I_2,\;\; I_{DC} = I_1 + I_2 \qquad \text{(A.14)}$$

where , $I_1 = \frac{I_{DC}}{2} + \frac{I_0}{2},\ I_2 = \frac{I_{DC}}{2} - \frac{I_0}{2}$

$$I_1 = \frac{b}{2}\left(V_{GS1} - V_T\right)^2,\ I_2 = \frac{b}{2}\left(V_{GS2} - V_T\right)^2,\ \text{where } b = K_n \frac{W}{L} \tag{A.15}$$

$$V_{GS1} = V_T + \sqrt{\frac{2I_1}{b}}, \quad V_{GS2} = V_T + \sqrt{\frac{2I_2}{b}} \tag{A.16}$$

$$V_{GS1} - V_{GS2} = V_{in}^{+} - V_{in}^{-} = v_d \tag{A.17}$$

,

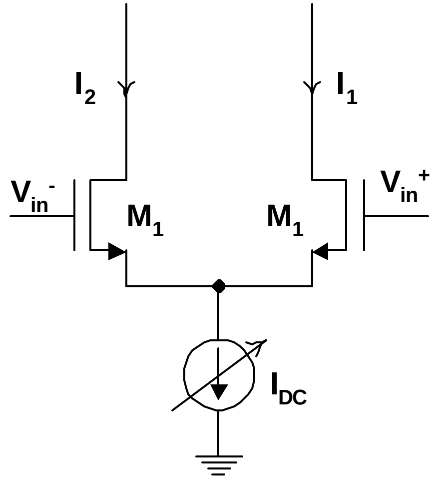


Fig. A.4 Simple differential pair

Substituting equation (A.16) in (A.17), we have:

$$v_d = \sqrt{\frac{2}{b}}\left(\sqrt{I_1} - \sqrt{I_2}\right) \tag{A.18}$$

Substituting equation (A.14) in (A.18), we have:

$$v_d = \sqrt{\frac{2}{b}}\left(\sqrt{\frac{I_{DC}}{2} + \frac{I_0}{2}} - \sqrt{\frac{I_{DC}}{2} - \frac{I_0}{2}}\right) \tag{A.19}$$

Squaring both sides and after some algebraic manipulation, we can write:

$$I_0 = v_d \sqrt{bI_{DC}}\left(1 - \frac{bv_d^2}{4I_{DC}}\right)^{1/2} \tag{A.20}$$

Expanding I_o in a power series in terms of v_d , we have:

$$I_0 = a_1 v_d + a_3 v_d^3 + O(v_d^5) \tag{A.21}$$

where $a_1 = \sqrt{bI_{DC}} = G_{in}$, and

$$a_3 = \frac{1}{8}\sqrt{bI_{DC}}\,\frac{b}{I_{DC}} = \frac{1}{8}\frac{G_{in}}{\left(V_{GS} - V_T\right)^2} = \frac{1}{8}\frac{G_{in}}{V_{DSAT}^2}$$

According to (A.3b), HD_3 is given by:

$$HD_3 = \frac{1}{32}\frac{A^2}{V_{DSAT}^2} \qquad \text{(A.22)}$$

According to (A.6), the 1-dB compression point A_{1-dB}^2 is given by:

$$A_{1-dB}^2 = k\frac{a_1}{a_3} = 8kV_{DSAT}^2 \Rightarrow A_{1-dB} = 1.077\times V_{DSAT} \qquad \text{(A.22)}$$

According to (A.9), the third order intercept point A_{IIP3i}^2 is given by:

$$A_{IIP3i}^2 = \frac{4}{3}\frac{a_1}{a_3} = 10.66\times V_{DSAT}^2 \Rightarrow A_{IP3i} = 3.266\times V_{DSAT} \qquad \text{(A.23)}$$

According to (A.10), the third order intermodulation distortion IM_3 is given by:

$$IM_3 = \frac{3}{4}\frac{a_3}{a_1}A^2 = \frac{3}{32}\left(\frac{A}{V_{DSAT}}\right)^2 \qquad \text{(A.24)}$$

According to the derivation in section 2.3.3, an expression for IM_3 of a psuedo-differential pair (removing the tail current source of Fig. A.4) can be obtained as:

$$IM_3 = \frac{3}{16} \frac{A^2 q}{V_{DSAT}(1+qV_{DSAT})^2(2+qV_{DSAT})} \qquad \text{(A.25)}$$

Cascaded Nonlinear Stages:

Consider two nonlinear stages in cascade as shown in Fig. A.5. It can be shown that the overall third order intercept point A_{IIP3} is given by:

$$\frac{1}{A_{IIP3}^2} \cong \frac{1}{A_{IIP3,1}^2} + \frac{G_1^2}{A_{IIP3,2}^2} \qquad \text{(A.26)}$$

where $A_{IIP3,i}$ is the input IIP3 point of the i^{th} stage, and G_i is the gain of the i^{th} stage.

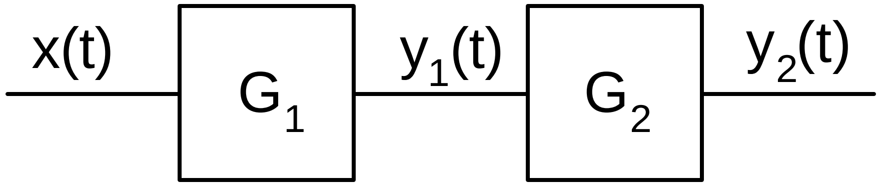
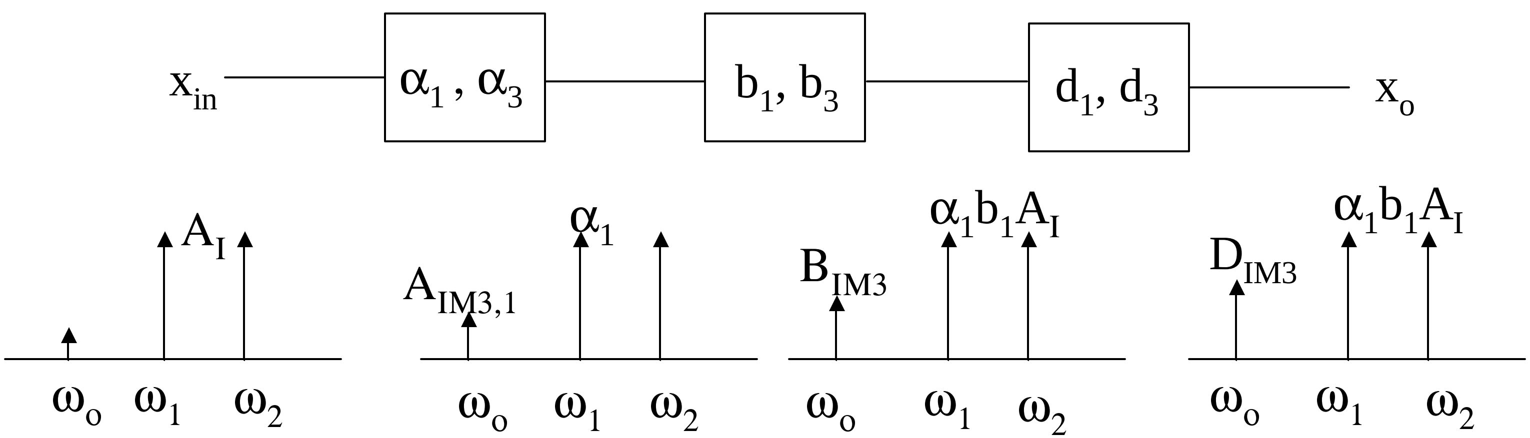


Fig. A.5 Cascaded nonlinear stages

Equation (A.26) can be generalized for more stages as:

$$\frac{1}{A_{IIP3}^2} \cong \frac{1}{A_{IIP3,1}^2} + \frac{G_1^2}{A_{IIP3,2}^2} + \frac{G_1^2 G_2^2}{A_{IIP3,3}^2} \qquad \text{(A.27)}$$

Let us consider the case of three blocks.



Where

$$A_{IM3,1} = \frac{\alpha_1 A_I^3}{A_{IP3,1}^2} \qquad ; \quad B_{IM3} = b_1 A_{IM3,1} + A_{IM3,2}$$

$$D_{IM3} = d_1 b_1 A_{IM3,1} + d_1 A_{IM3,2} + A_{IM3,3}$$

$$A_{IM3,2} = \frac{(\alpha_1 A_I)^3 b_1}{A_{IP3,2}^2} \qquad ; \quad A_{IM3,2} = \frac{(\alpha_1 b_1 A_I)^3 d_1}{A_{IP3,3}}$$

Total A_{IM3} becomes:

$$A_{IM3,total} = d_1 b_1 A_{IM3,1} + d_1 A_{IM3,2} + A_{IM3,3}$$

$$A_{IM3,total} = \frac{\alpha_1 b_1 d_1 A_I^3}{A_{IP3,casc}^2}$$

Then

$$\frac{1}{A_{IP3,casc}^2} = \frac{1}{A_{IP3,1}^2} + \frac{\alpha_1^2}{A_{IP3,2}^2} + \frac{b_1^2\alpha_1^2}{A_{IP3,3}^2}$$

$$A_{IP3,cas}^2 = \frac{1}{\dfrac{1}{A_{IP3,1}^2} + \dfrac{\alpha_1^2}{A_{IP3,2}^2} + \dfrac{b_1^2\alpha_1^2}{A_{IP3,3}}}$$

$$IIP_{3,i}\left(dBV_{rms}\right) = 20 log\, A_{IP3,i}$$

i.e., $\alpha_1^2 = G\,R_o/R_s \Rightarrow \alpha_1(\,dB\,) = G_{dB} + 10\,log\left(R_o/R_s\right)$

Cadence Simulation

To simulate the 1-dB compression point or the two-tone intermodulation distortion of the differential amplifier, the setup shown in Fig. A.6 is used. Swept Periodic Steady State (SPSS) analysis simulation of SpectreRF is chosen. The input is applied through **PORT0**. It is a power source and is called “**psin**” in “**analogLib**” library. The output resistance of the source is set as 50Ω. A physical resistance of 50Ω (not shown in Fig.

A.5) should be placed in parallel with the source for matching purposes, since the resistance seen form the gate of the MOSFET transistor is infinity. The source type should be "**sine**" as shown in Fig. A.7. The input consists of two relatively close frequencies ($F_1=F_{in}=10MHz, F_2=F_{in}+1MHz=11MHz$) and their power levels are set equal to a design variable called $\mathbf{P_{in}}$ (make sure that the Amplitude and Amplitude2 fields are left empty). This is the variable that will be swept in the SPSS simulation.

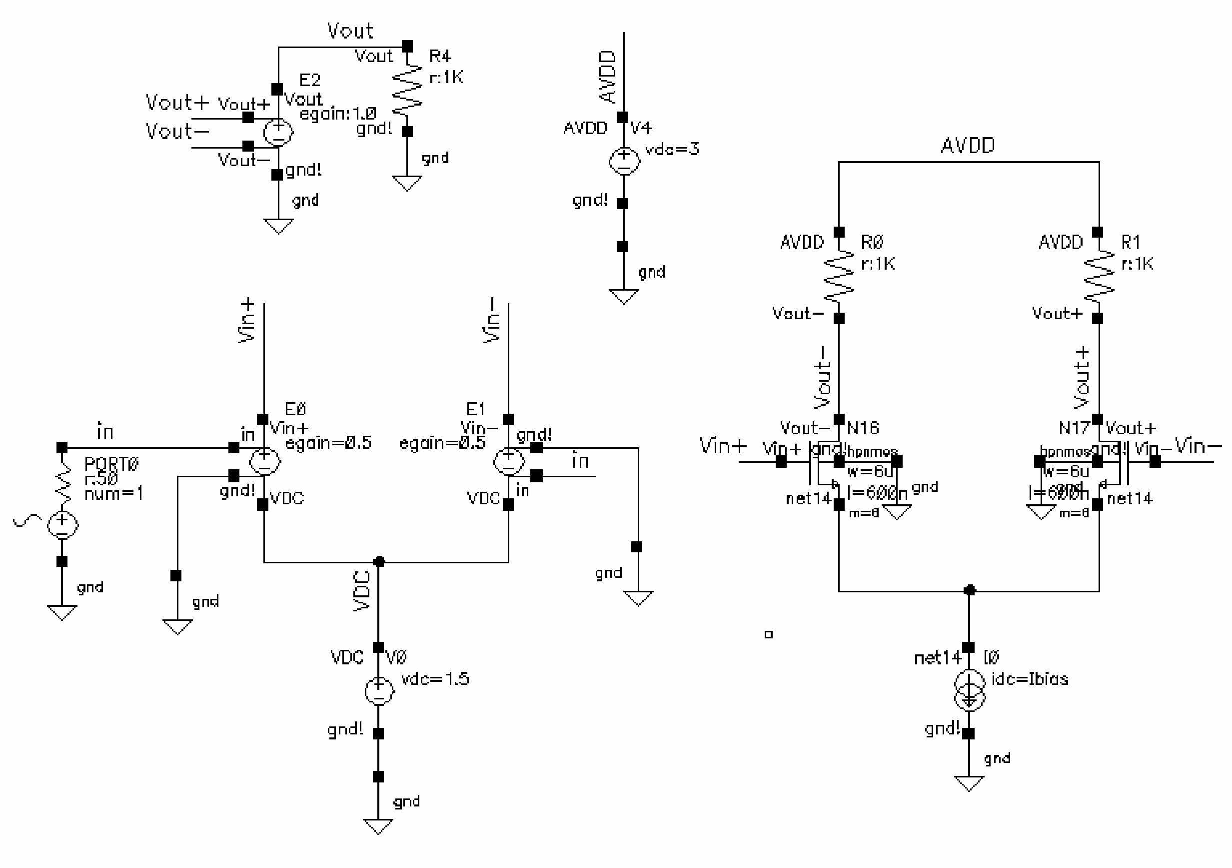


Fig. A.6 Swept periodic steady state (SPSS) simulation setup

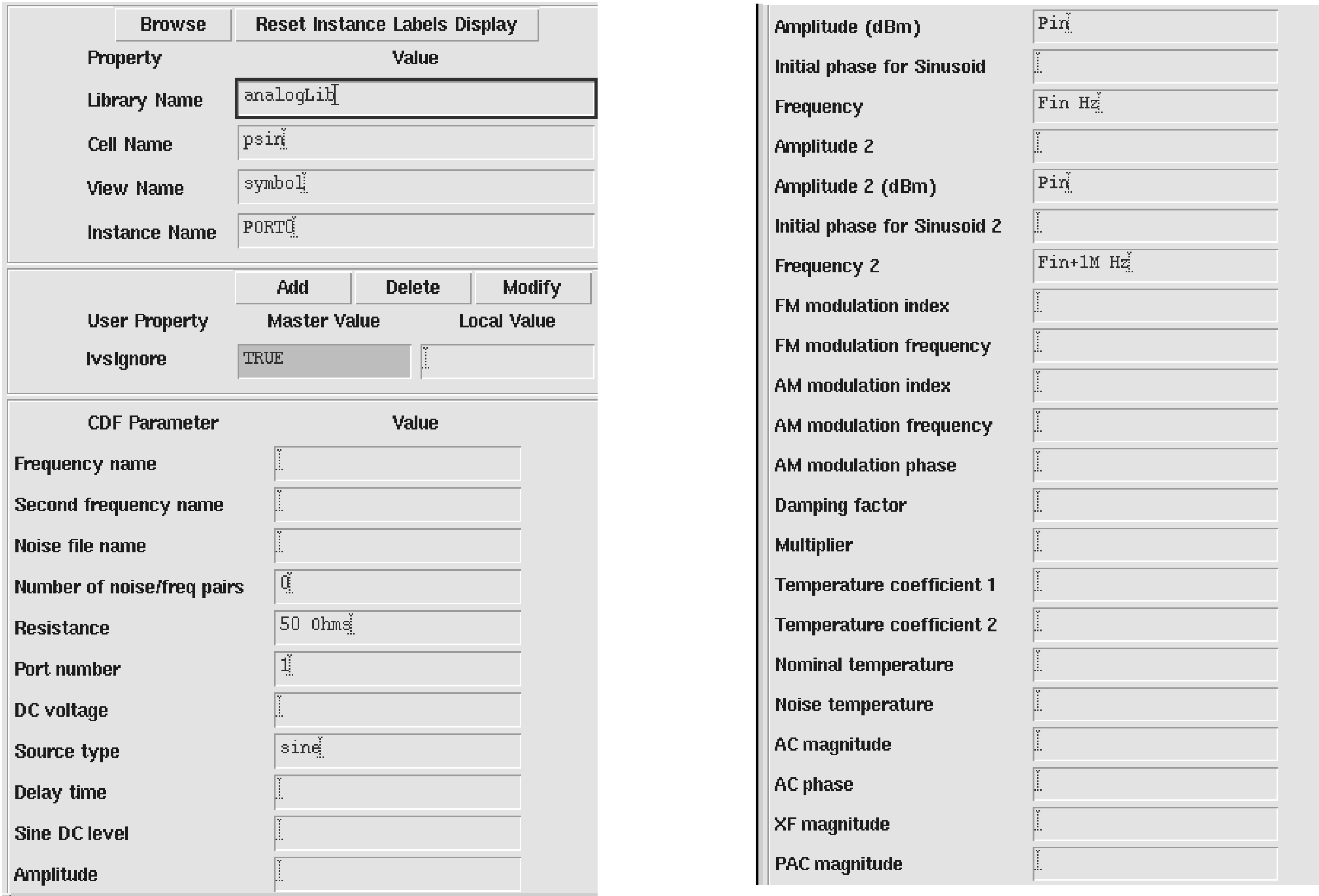
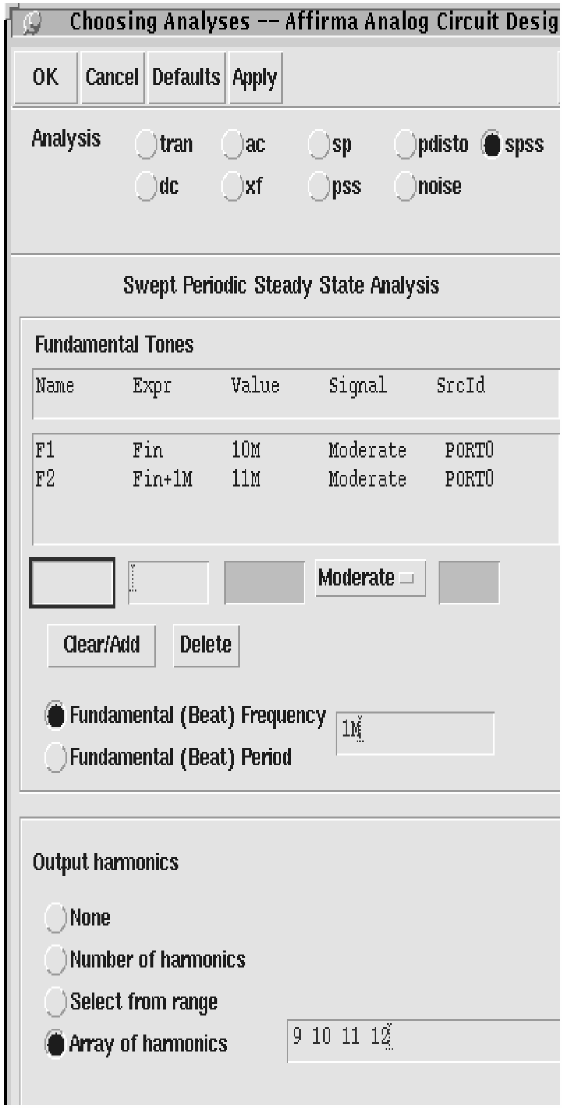


Fig. A.7 PORTO setup

In the simulation window, select SPSS analysis. The **Fundamental (Beat)** frequency is the highest frequency common to all inputs shown in the **Fundamental Tones** section. In the sweep section, select "**variable**" to sweep $\mathbf{P_{in}}$ from –30dBm to 10dBm. Only the harmonics of interest (9,10,11,12) are saved to reduce the disk area required for saving, as shown in Fig. A.8.



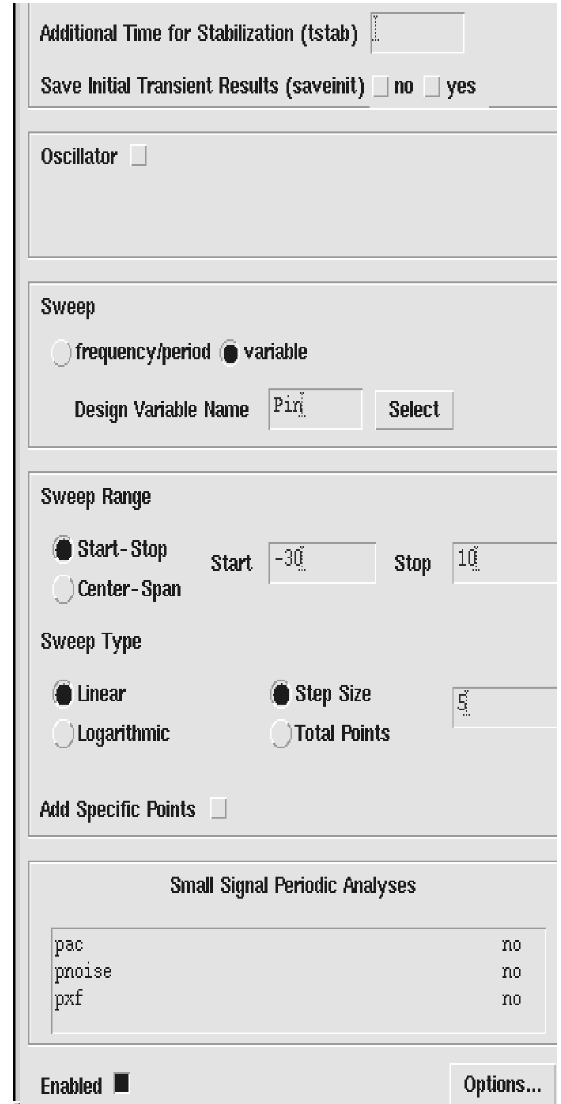


Fig. A.8 SPSS analysis setup

From “**Options**” button choose “**gear2only**” as the integration method. It is important to switch to “**flat**” netlisting to run SPSS. Select **Setup->Environment.** Set the netlist type as “**flat**”. After running the simulation, you can display the IIP_3 plot from Analog Artist, select **Results->Direct Display->SPSS**. Setup the form as shown in Fig. A.9. The 1st order harmonic is at 10MHz and the 3rd order harmonic is at 9MHz.

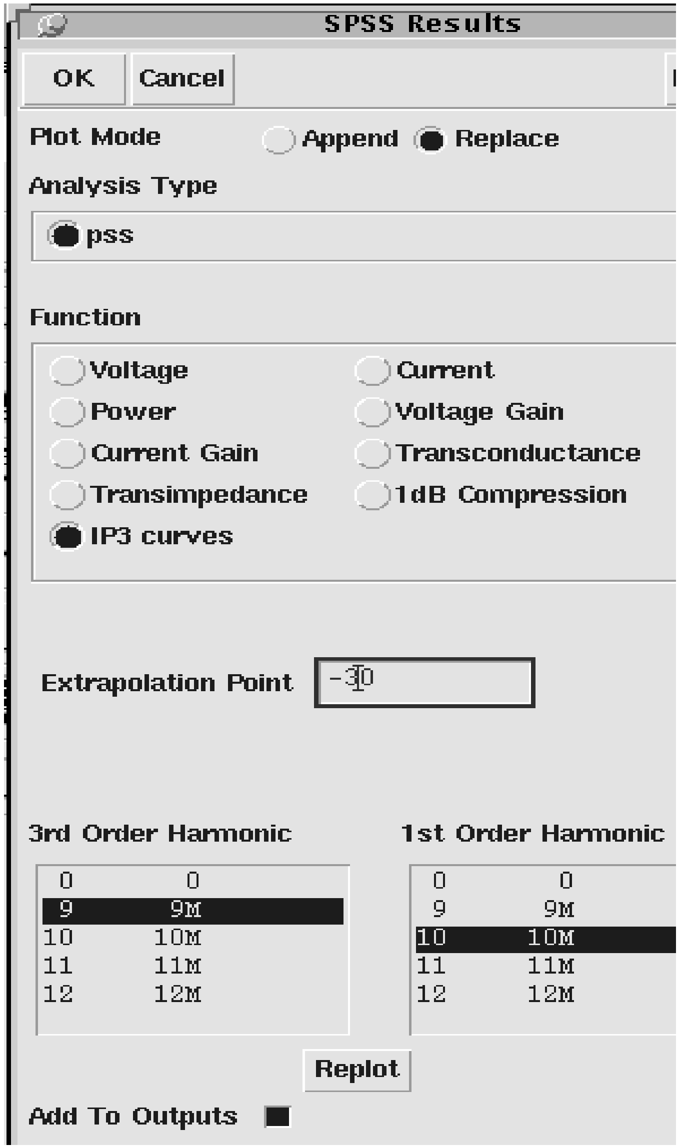


Fig. A.9 IIP3 results setup

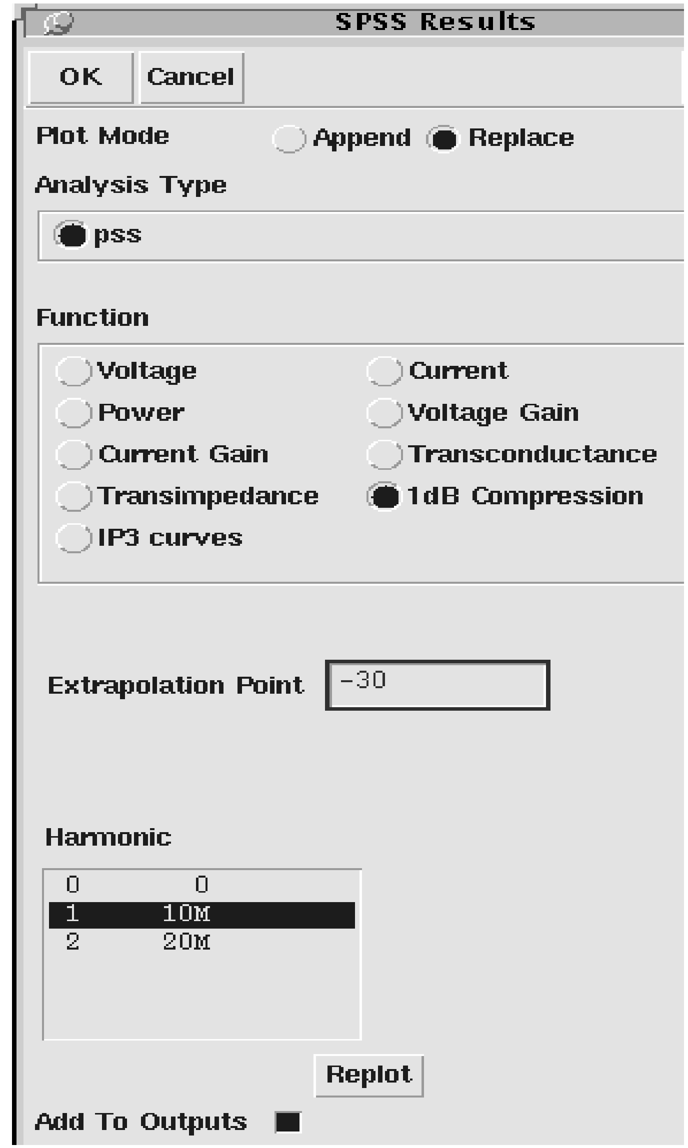


Fig. A.10 1-dB Compression results setup

For the 1-dB compression point simulation only one frequency F_{in} is specified and the N**umber of harmonics** is set to 2. $\mathbf{P_{in}}$ is swept from –30dBm to 10dBm. The form of SPSS results is set as shown in Fig. A.10. The 1st order harmonic is at 10MHz. Select node V_{out} (of Fig. A.5). The 1-dB compression point plot should look like Fig. A.11. The 1-dB compression point is about –4.9dBm, compared to the theoretical value calculated according to (A.22), that is –3.3dBm for our case of V_{DSAT}=200mV. The IIP_3 plot should look like Fig. A.12. IIP_3 is about 6.1dBm, which is very close to the theoretical value calculated according to equation (A.23), that is 6.3dBm for our case of V_{DSAT}=200mV.

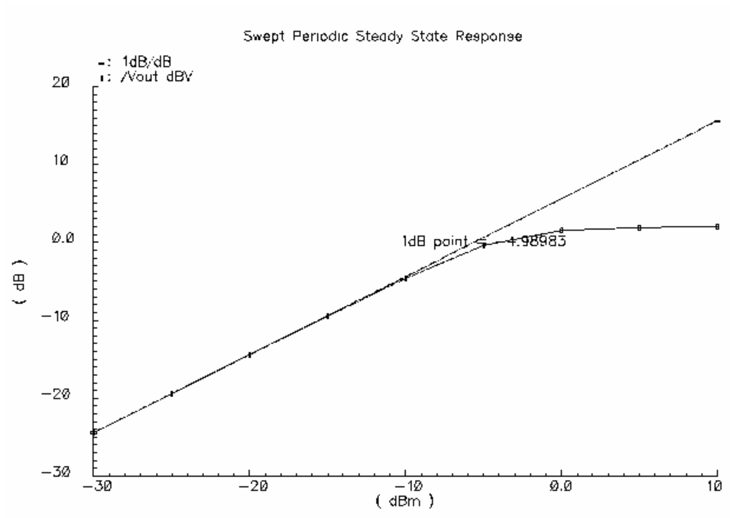


Fig. A.11 1-dB compression point plot

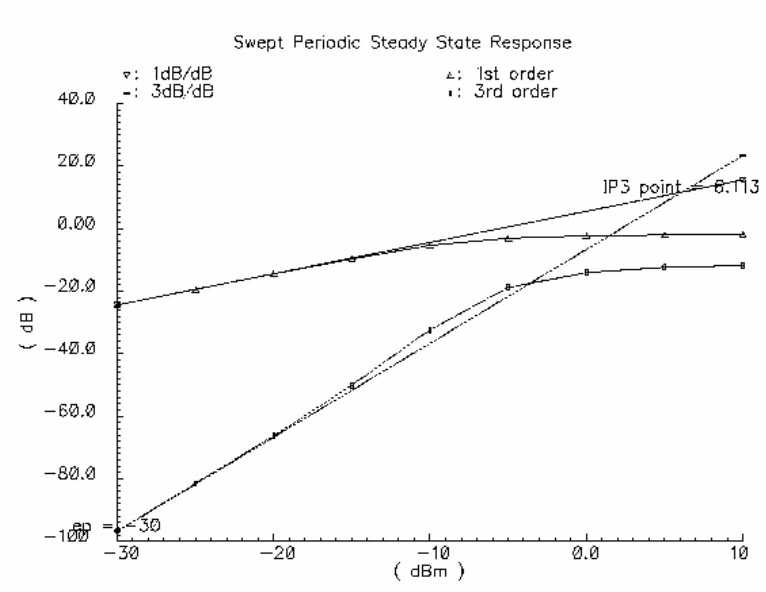
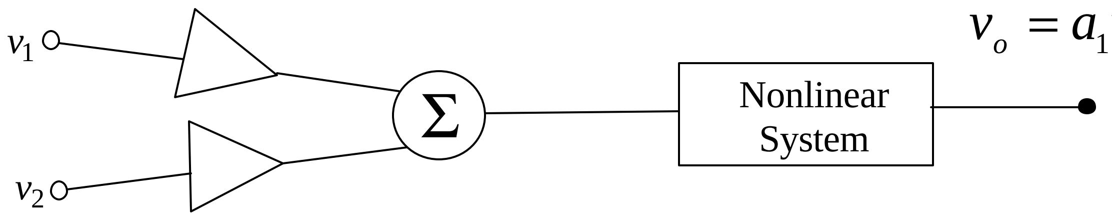


Fig. A12 IIP3 plot

Intermodulation Distortion



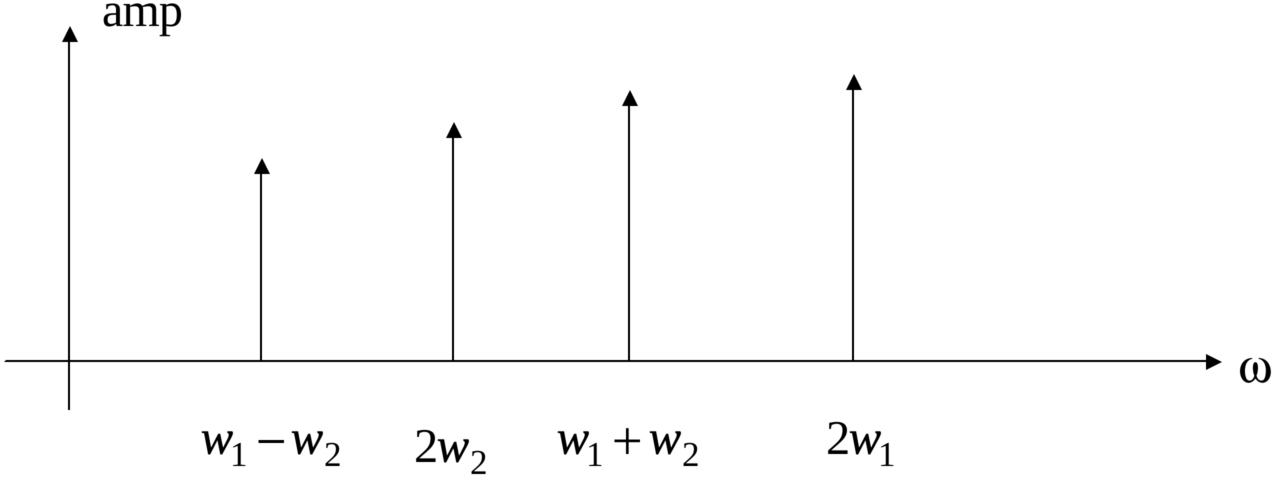
$$v_o = a_1 v + a_2 v^2 + a_3 v^3 + \cdots$$

$$v = v_1 + v_2 = V_{1A}\cos w_1 t + V_{2A}\cos w_2 t$$

$$\begin{aligned} v_o = {} & a_1\left[V_{1A}\cos w_1 t + V_{2A}\cos w_2 t\right] + \\ & a_2\left[V_{1A}\cos w_1 t + V_{2A}\cos w_2 t\right]^2 + \\ & a_3\left[V_{1A}\cos w_1 t + V_{2A}\cos w_2 t\right]^3 + \\ & \cdots \end{aligned}$$

Let us first consider only the second-order term

$$
\begin{aligned}
a_2 v^2 &= a_2\left[V_{1A}\cos w_1 t + V_{2A}\cos w_2 t\right]^2 \\
&\quad a_2 V_{1A}^2 \cos^2 w_1 t + a_2 V_{2A}^2 \cos^2 w_2 t \\
&\quad + 2a_2 V_{1A} V_{2A}\left[\cos w_1 t + \cos w_2 t\right] \\
&= \frac{1}{2} a_2 \left[V_{1A}^2 + V_{2A}^2 + \frac{a_2 V_{1A}^2}{2}\cos 2w_1 t \right] \\
&\quad + \frac{1}{2} a_2 V_{2A}^2 \cos 2w_2 t + a_2 V_{1A} V_{2A} \cos(w_1 + w_2)t \\
&\quad + a_2 V_{1A} V_{2A} \cos(w_1 - w_2)t
\end{aligned}
$$



The terms with $(\omega_1 + \omega_2)$ and $(\omega_1 - \omega_2)$ are used to determine the second-order intermodulation (IM_2).

$$IM_2 = \frac{a_2 V_{1A} V_{2A}}{a_1 V_{1A}} \Bigg|_{V_{1A} = V_{2A}} = \frac{a_2}{a_1} V_{2A} = 2HD_2$$

For small distortion, assuming that the amplitude of the fundamental component is V_{oA} , $V_{oA} \cong a_1 V_{1A}$ Then

$$IM_2 = \frac{a_2}{a_1^2} V_{oA}$$

Now let us consider the a_3v^3 term:

$$
\begin{aligned}
a_3v^3 &= a_3\left[V_{1A}\cos w_1t + V_{2A}\cos w_2t\right]^3 \\
&= a_3V_{1A}^3\cos^3 w_1t + 3a_3V_{1A}V_{2A}^2\cos w_1t\cos^2 w_2t \\
&+ 3a_3V_{1A}^2V_{2A}\cos^2 w_1t\cos w_2t + a_3V_{2A}^3\cos^3 w_2t \\
&= \cdots + a_{32}\cos\left(w_1 \pm 2w_2t\right) + a_{33}\cos\left(2w_1t \pm w_2t\right)
\end{aligned}
$$

Where

$$
a_{32} = \frac{3}{4}a_3V_{1A}V_{2A}^2
$$

$$
a_{33} = 0.75a_3V_{1A}^2V_{2A}
$$

Broad-Band amplifiers $V_{1A}=V_{2A}$

$$V_o(IM_3) = 0.75 a_3 V_{1A}^3$$

$$IM_3 = \frac{0.75 a_3 V_{1A}^3}{a_1 V_{1A}} = \frac{0.75 a_3}{a_1} V_{1A}^2$$

For small distortion

$$IM_3 = \frac{0.75 a_3}{a_1^3} V_{oA}^2 = 3HD_3$$

For receivers

$$V_{1A} << V_{2A} \Rightarrow Yielding\ 1\% IM_3$$

$$IM_3^{'} = \frac{3 a_3 V_{1A} V_{2A}^2}{a_1 V_{1A}} = \frac{0.75 a_3 V_{2A}^2}{a_1}$$

$$IM_3 = \frac{3 a_3 V_{1A}^2 V_{2A}}{a_1 V_{1A}} = \frac{0.75 a_3 V_{1A} V_{2A}}{a_1}$$

Other important definitions are:

- Cross modulation i.e.,

$$v = v_1 + v_2 = V_{1A}\cos w_1 t + V_{2A}\left(1 + m\cos w_1 t\right)\cos w_2 t$$
$$CM = 4IM_3 = 12HD_3$$

- Third-order intercept point (IP3)

  Let us recall that

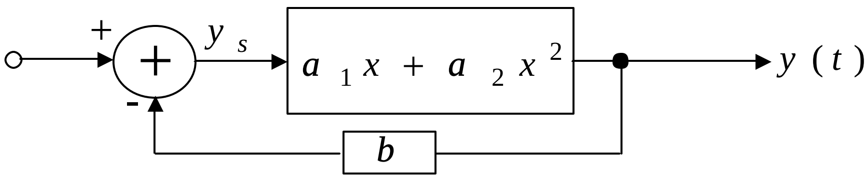
$$v_o(t) = a_1 v + a_2 v^2 + a_3 v^3 + \cdots$$

In fully differential amplifiers

$$v_o(t) \cong a_1 v(t) + a_3 v^3(t)$$

$$v_o(t)\Big|_{v(t)=V_A\cos wt}$$

Feedback effect on nonlinearity



Feedback system incorporating a nonlinear feedforward amplifier.

The output can be approximated as *y » acos wt + b cos 2wt*. Our objective is to determine *a* and *b*. The output of the subtractor can be written as

$$y_S = x(t) - by(t) \quad (1)$$

$$= V_m \cos wt - b\left(a\cos wt + b\cos 2wt\right) \quad (2)$$

$$= \left(V_m - ba\right)\cos wt - bb\cos 2wt. \quad (3)$$

This signal experiences the nonlinearity of the feedforward amplifier, thereby producing an output given by:

$$y(t) = a_1\left[(V_m - b)\cos wt - bb\cos 2wt\right] + a_2\left[(V_m - b)\cos wt - bb2wt\right]^2 \qquad (4)$$

$$= \left[a_1(V_m - ba) - a_2(V_m - ba)bb\right]\cos 2wt + \left[-a_1 bb + \frac{a_2(V_m - ba)^2}{2}\right]\cos 2wt + \cdots \qquad (5)$$

The coefficients of cos 2ω*t* in (5) must be equal to *a* and *b*, respectively:

$$a = (a_1 - a_2 bb)(V_m - ba) \qquad (6)$$

$$b = -a_1 bb + \frac{a_2 (V_m - ba)^2}{2} \qquad (7)$$

The assumption of small nonlinearity implies that both a_2 and b are small Quantities, yielding and hence

$$a = \frac{a_1}{1 + ba_1} V_m, \qquad (8)$$

which is to be expected because $\beta\alpha_1$ is the loop gain. To calculate *b*, we write

$$V_m - ba \approx \frac{a}{a_1} \qquad (9)$$

Thus expressing (7) as

$$b = -a_1 bb + \frac{1}{2} a_2 \left(\frac{a}{a1} \right)^2 \qquad (10)$$

That is,

$$b(1 + a_1 b) = \frac{a_2}{2} \left(\frac{a}{a1} \right)^2 \qquad (11)$$

$$= \frac{a_2}{2a_1^2} \frac{a_1^2}{(1 + b a_1)^2} V_m^2 . \qquad (12)$$

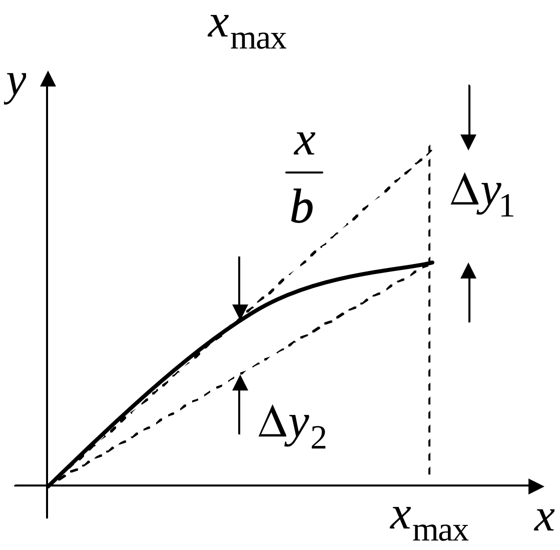
It follows that

$$b = \frac{a_2 V_m^2}{2} \frac{1}{\left(1 + b a_1\right)^3}. \qquad (13)$$

For a meaningful comparison, we normalize the amplitude of the second harmonic to that of the fundamental:

$$\frac{b}{a} = \frac{a_2 V_m}{2} \frac{1}{a_1} \frac{1}{\left(1 + b a_1\right)^2}. \qquad (4)$$

Without feedback, on the other hand, such a ratio would be equal to $\left(\alpha_2 V_m^2 / 2\right) / \alpha_1 V_m = \alpha_2 V_m / (2\alpha_1)$. Thus, the relative magnitude of the second harmonic has dropped by a factor of $(1 + b a_1)^2$.



Gain error and nonlinearity in a feedback system.