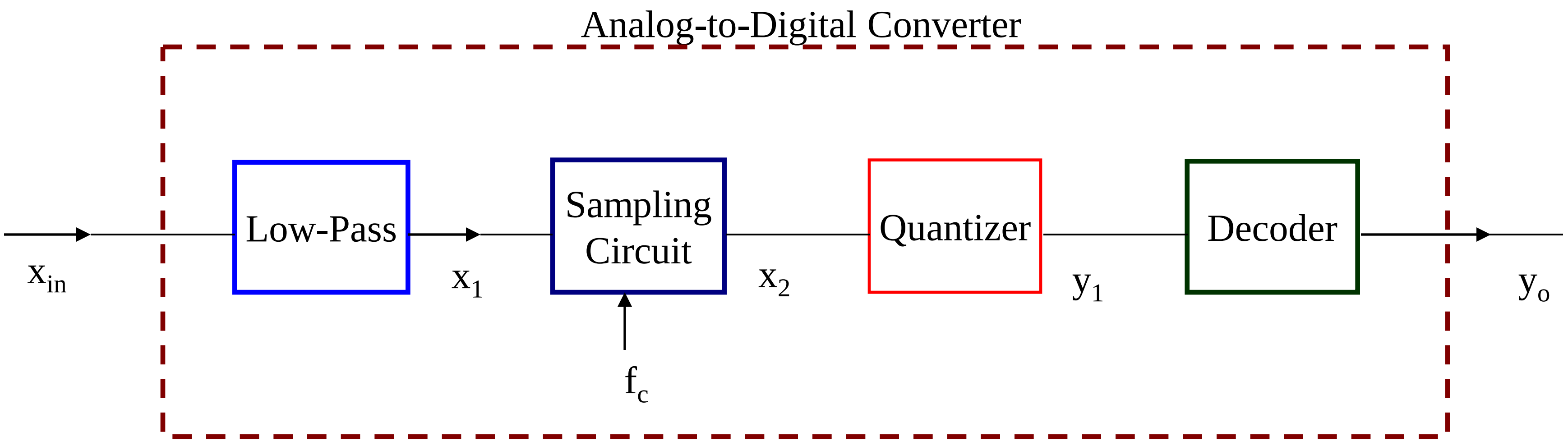
SWITCHED-CAPACITOR FUNDAMENTALS AND CIRCUITS

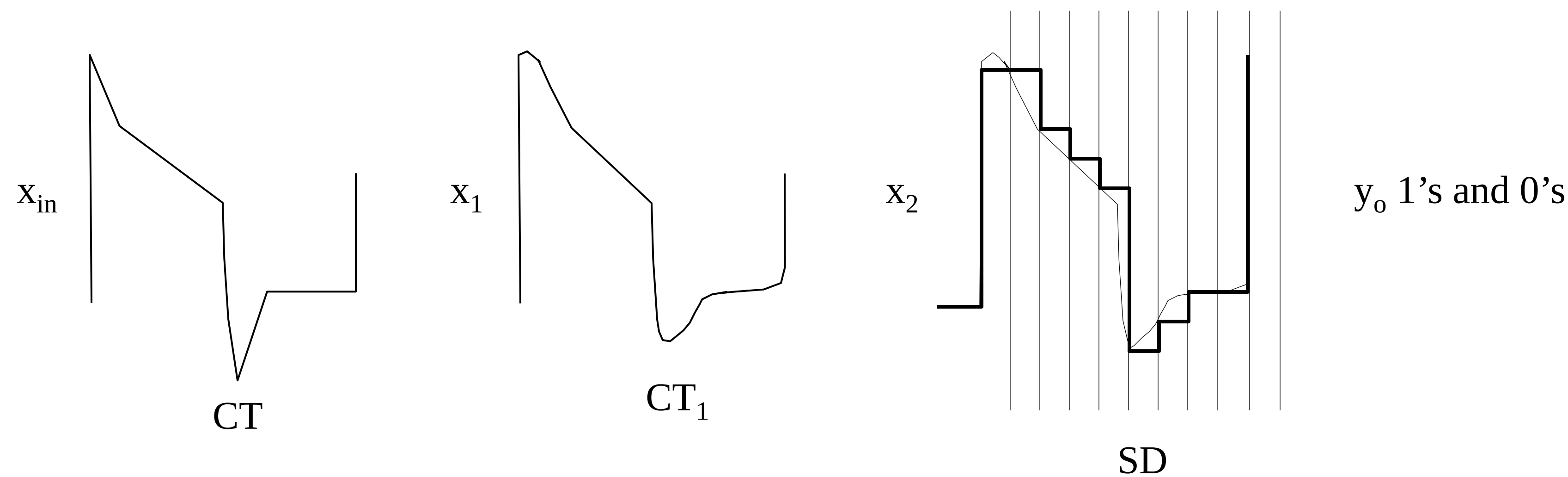
- Mathematical Background
- Basic Building Blocks
- SC Filters
- SC Oscillators

FUNDAMENTALS ON SC CIRCUITS

- $\tau = RC = R_{eq}C = \dfrac{T}{C_{eq}}C = \dfrac{1}{f_c}\left(\dfrac{C}{C_{eq}}\right)$ very accurate!
- Bottlenecks: Switches and High-Speed Op Amps. Non-idealities ⇨ enemies!
- The best approach in audio applications.
- Design is carried-out in the Z domain, same domain as for digital circuits.
- Conventional Programming is done via capacitor banks. Spread?
- New non-uniform sampling techniques opens practical application possibilities.

Examples of Analog and Digital Signals





Analog and Digital Signals and Systems Concepts

Types of signals	Mathematical Description
• Continuous-Time (CT) --Continuous values in time --Continuous values in magnitude	Laplace
• Sampled Data (SD) --Continuous values in magnitude --Discrete values in time	Z-Transform
• Digital --Discrete values in magnitude --Discrete values in time	Z-Transform

OPERATOR Z^{-1} ⇨ UNIT DELAY

$$Z^{-1}Y(Z) = Z^{-1}\sum_{n=0}^{\infty} y(n)Z^{-n}$$

$$= \sum_{n=0}^{\infty} y(n)Z^{-(n+1)} = \sum_{n=1}^{\infty} y(n-1)Z^{-n}$$

$$Z^{-1}Y(Z) = \sum_{n=0}^{\infty} y(n-1)Z^{-n} - y(-1) = Y[y(n-1)] - y(-1)$$

$$\text{If } y(nT) = 0 \text{ for } t < 0,\ y(-1) = 0$$

$$\boxed{Z^{-1}Y(Z) = Z^{-1}[y(nT)] = Y[y(n-1)T]}$$ ⇦

In general

$$Z^{-m}Y(Z) = Y[y(n-m)T]$$ ⇦

STABILITY OF A SECOND-ORDER
HOMOGENEOUS DIFFERENCE EQUATION

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = 0$$

$$Y(Z)\left(1 + a_1 Z^{-1} + a_2 Z^{-2}\right) = 0$$

The C.E.

$$Z^2 + a_1 Z + a_2 = (Z - Z_1)(Z - Z_2) = 0$$

COMPLEX ROOTS

$$Z_1, Z_2 = \frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}, \text{ if } \frac{a_1^2}{4} - a_2 < 0, \; Z_1, Z_2 = re^{\pm j\theta} = x \pm jy$$

$$r = \sqrt{x^2 + y^2} = \sqrt{a_2}, \; cos\,\theta = \frac{x}{r} = -\frac{a_1}{2\sqrt{a_2}}$$

The C. E. yields

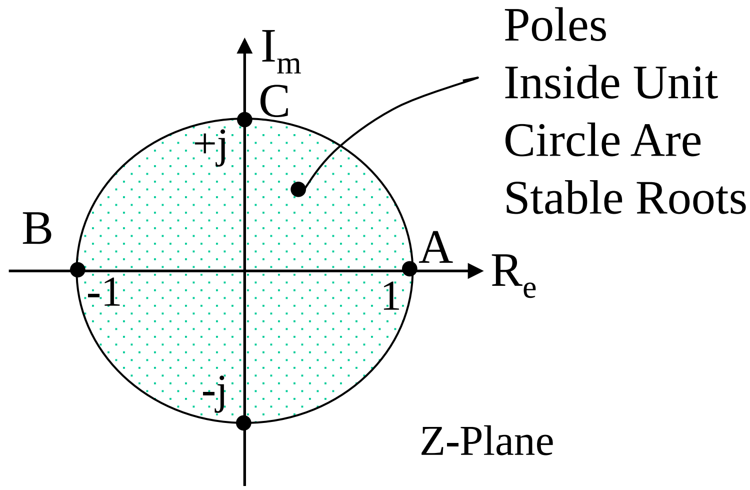
$$Z^2 - 2r\,cos\,\theta Z + r^2 = 0$$

Natural Response

$$y(n) = Kr^n\,cos\left(2\pi\frac{nf}{fs} + \phi\right)$$

$$a_1, \quad a_2 \leftrightarrow r, \quad \theta$$

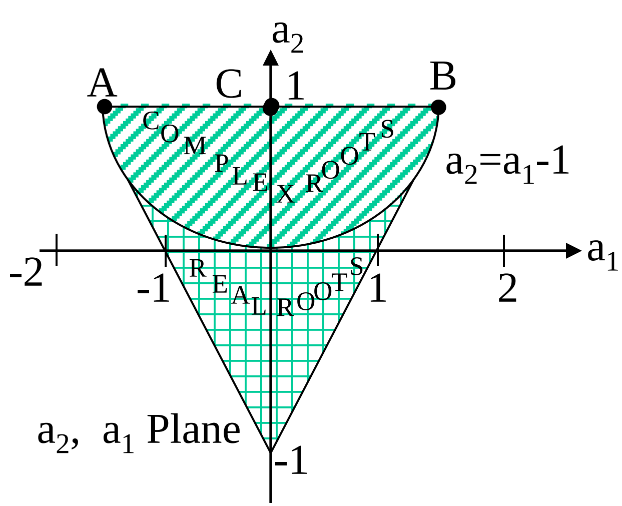
$$k, \quad \phi \leftarrow \text{Inital Conditions}$$



$$a_1^2 > 4a_2$$

for

Real Roots



$$\left.\begin{aligned} -a_1 &= 2r\cos\theta \\ a_2 &= r^2 \end{aligned}\right\}$$

For
Complex
Stable Roots

Relationship between the poles in the Z-Plane and coefficients a_1 and a_2.

MAPPING BETWEEN THE S- AND THE Z-PLANES FOR HIGH SAMPLING

Backward

$$Z_{p1} = \frac{1}{1 - s_p T}$$

Forward

$$Z_{p2} = 1 + s_p T$$

Bilinear

$$Z_{p3} = \frac{1 + s_p T/2}{1 - s_p T/2}$$

Impulse Invariant

$$Z_{p4} = e^{S_p T}$$

For small $s_p T$, $s_p T << 1$.

$$Z_{p1} \cong 1 + s_p T + (s_p T)^2 + (s_p T)^3 + \cdots$$

$$Z_{p2} = 1 + s_p T$$

$$Z_{p3} \cong 1 + (s_p T) + \frac{(s_p T)^2}{2} + \frac{(s_p T)^3}{4} + \cdots$$

$$Z_{p4} \cong 1 + s_p T + \frac{(s_p T)^2}{2} + \frac{(s_p T)^3}{6} + \cdots$$

Ignoring higher order effects

$$Z_{p1} \cong Z_{p2} \cong Z_{p3} \cong Z_{p4} \cong 1 + s_p T$$

For Very High Sampling Rate
The Approximation For The
Mapping is the Same!

$$Z_p \cong 1 + s_p T$$

RELATIONS BETWEEN POLES IN THE S- AND Z- DOMAIN

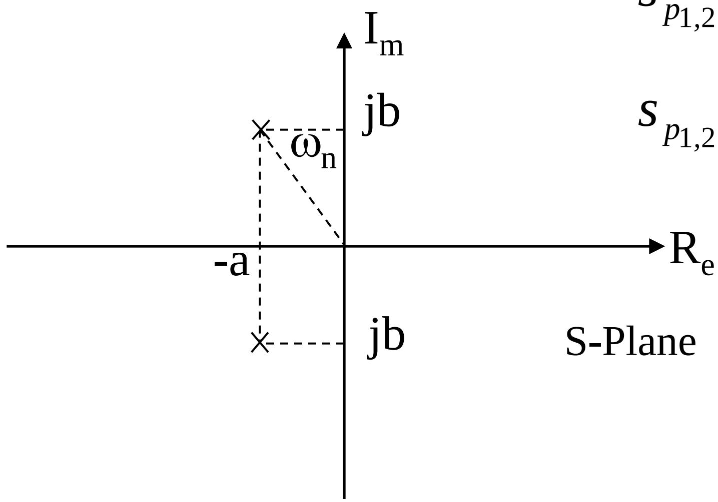
(High Sampling)

$$s^2 + \frac{\omega_n}{Q} s + \omega_n^2$$

$$= (s+a)^2 + b^2 = 0$$

$$s_{p1,2} = -\frac{\omega_n}{2Q} \pm j\frac{\omega_n}{2Q}\sqrt{4Q^2 - 1}$$

$$s_{p1,2} = -a \pm jb$$



S-Plane

$$a = \frac{\omega_n}{2Q}$$

$$b = \frac{\omega_n}{2Q}\sqrt{4Q^2 - 1}$$

$$Z_{p1,2} = 1 + S_{p1,2}T$$

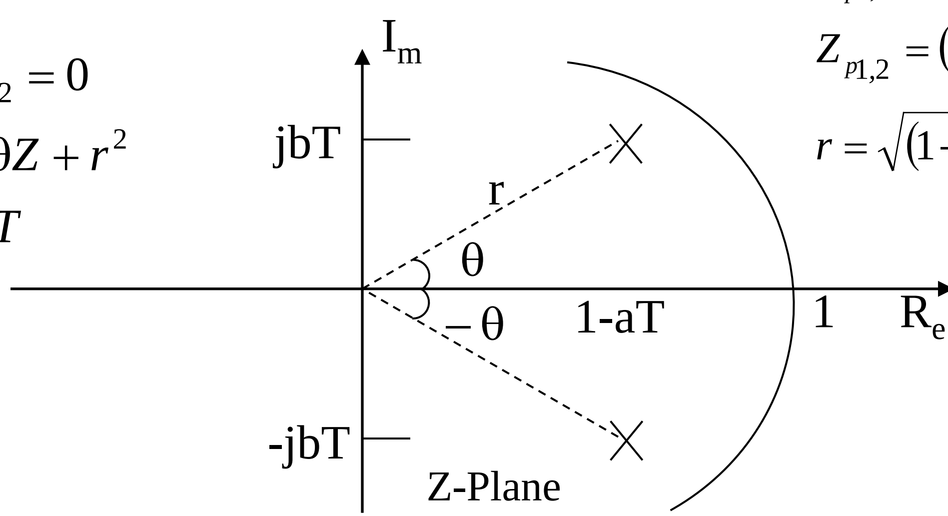
$$Z_{p1,2} = (1 - aT) \pm jbT = re^{\pm j\theta}$$

$$r = \sqrt{(1 - aT)^2 + (bT)^2}$$

$$Z^2 + b_1 Z + b_2 = 0$$

$$Z^2 - 2r\cos\theta Z + r^2$$

$$Z_{pi} = 1 + s_{pi}T$$



$$r = \sqrt{\left(1 - \frac{\omega_n T}{2Q}\right)^2 + (\omega_n T)^2} \cong \sqrt{1 - \frac{\omega_n T}{Q} + (\omega_n T)^2}$$

OR

$$Q = \frac{\omega_n T}{1 - r^2 + (\omega_n T)^2} = \frac{\omega_n T}{(1+r)(1-r) + (\omega_n T)^2}$$

High Q's

$$\mathrm{Q}\Big|_{\mathrm{r}\to 1} \cong \left.\frac{\omega_n T}{2(1-r) + (\omega_n T)^2}\right|_{\omega_n T \ll 1} \cong \frac{\omega_n T}{2(1-r)}$$

then

$$r = \frac{2Q - \theta}{2Q} = 1 - \frac{\theta}{2Q}$$

Also by a series expansion

$$cos\,\theta \cong 1 - \frac{\theta^2}{2}$$

Thus

$$b_1 = 2r\,cos\,\theta \cong -2\left(1-\frac{\theta}{2Q}\right)\left(1-\frac{\theta^2}{2}\right)$$

$$b_1 \cong \left(2-\theta^2-\frac{\theta}{Q}\right)$$

$$b_2 = r^2$$

$$b_2 \cong \left(1-\frac{\theta}{2Q}\right)^2 = 1+\frac{\theta^2}{4Q^2}-\frac{\theta}{Q}$$

Also see Table of mappings.

In fact if we let,

$$b_2 \cong 1 - \frac{\theta}{Q}$$

then

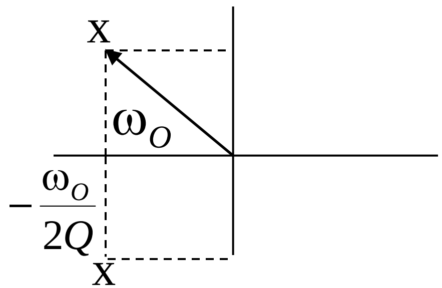
$$b_1 = (b_2 + 1) + \theta^2$$

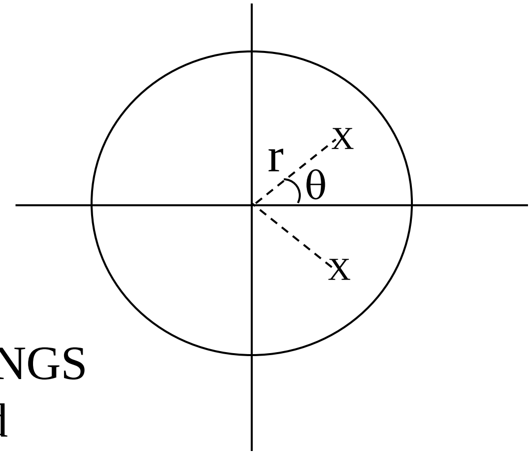
Recall that high Q and high sampling rate conditions have been used. Thus

and

$$\boxed{\theta^2 = 1 + b_2 + b_1}$$

$$Q \cong \frac{\theta}{1 - b_2}$$





MAPPINGS
Forward
Backward
Bilinear
Impulse Invariant
Mixed

$$H(s) = \frac{N(S)}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} \rightarrow H(Z) = \frac{N(Z)}{1 + b_1 Z^{-1} + b_2 Z^{-2}}$$

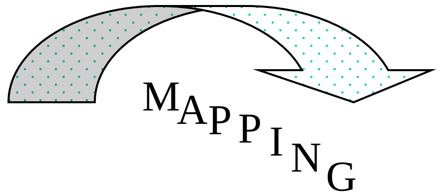
where

$$b_1 = 2r\,cos\,\theta$$

$$b_2 = r^2$$

(See Table of Quadratic Mappings)

$$H(s)=\frac{k}{s^2+\frac{\omega_o}{Q}s+\omega_o^2}$$



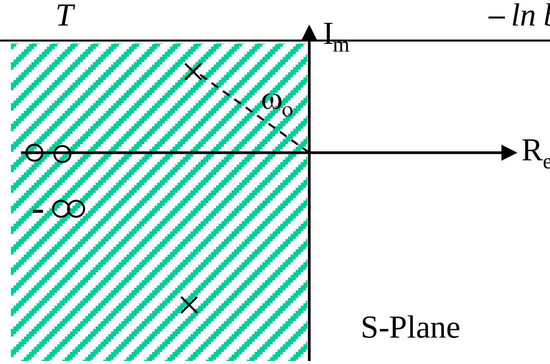
$$H(Z)=\frac{\text{Numerator}}{1+b_1Z^{-1}+b_2Z^{-2}}$$

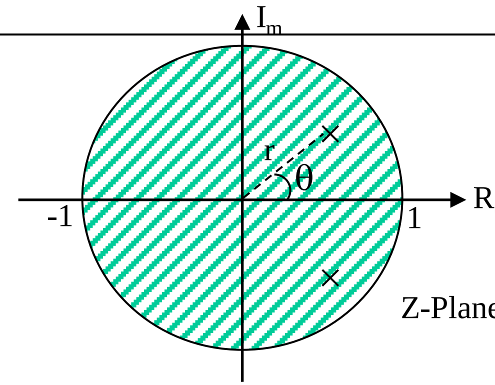
Table 2

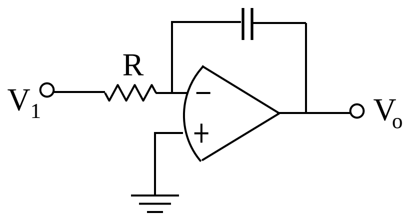
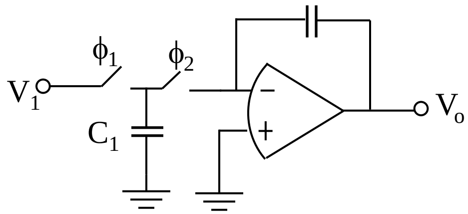
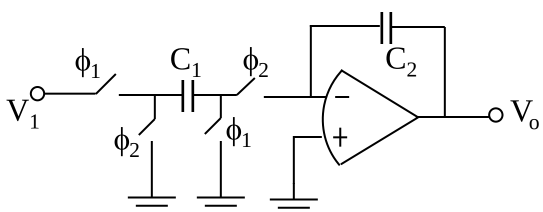
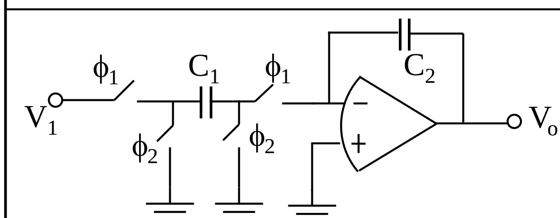
Type of Mapping	$f_o(H_Z$	Q	b_1	b_2
Forward	$\frac{\sqrt{(1+b_1+b_2)}}{T}$	$\frac{\sqrt{(1+b_1+b_2)}}{b_1+2}$	$-2+\frac{\omega_0 T}{Q}$	$1-\frac{\omega_o T}{Q}+(\omega_o T)^2$
Backward	$\frac{\sqrt{\frac{1+b_1+b_2}{b_2}}}{T}$	$-\frac{\sqrt{b_2(1+b_1+b_2)}}{b_1+2b_2}$	$-\frac{2+\frac{\omega_o T}{Q}}{1+\frac{\omega_o T}{Q}+(\omega_o T)^2}$	$\frac{1}{1+\frac{\omega_o T}{Q}+(\omega_o T)^2}$
Bilinear $a=2/T$ $a\cong\omega_o\,cot(\omega_o T/2)$	$a\sqrt{\frac{1+b_1+b_2}{1-b_1+b_2}}$	$\frac{\sqrt{(1+b_1+b_2)(1-b_1+b_2)}}{2(1-b_2)}$	$\frac{2(\omega_o^2-a^2)}{a^2+\frac{\omega_o}{Q}a+\omega_o^2}$	$\frac{a^2-\frac{\omega_o}{Q}a+\omega_o^2}{a^2+\frac{\omega_o}{Q}a+\omega_o^2}$
Impluse invariant	$\frac{\sqrt{\left(cos^{-1}\frac{-b_1}{2\sqrt{b_2}}\right)^2+\frac{1}{4}(ln\,b_2)^2}}{T}$	$\frac{\sqrt{\left(cos^{-1}\frac{-b_1}{2\sqrt{b_2}}\right)^2+\frac{1}{4}(ln\,b_2)^2}}{-ln\,b_2}$	$-2e^{\frac{\omega_o T}{-2Q}}cos\left(\frac{\omega_o T}{2Q}\sqrt{4Q^2-1}\right)$	$e^{\frac{-\omega_o T}{Q}}$

$$b_1=-2r\cos\theta$$

$$b_2=r^2$$

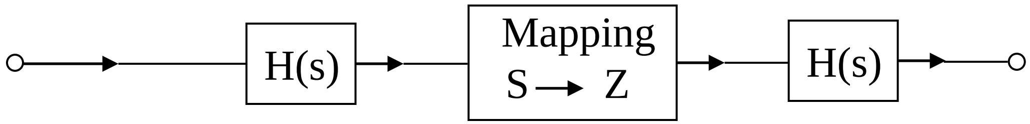




Type of Integrator	Magnitude, $\lvert H(e^{j\omega T})$	Phase, Arg $H(e^{j\omega T})$	Mapping (Equivalent)	Transfer Function
	$\frac{\omega_o}{\omega}$	$\frac{\pi}{2}$	In the S-Plane	i.e. $H(s) = -\frac{1}{sR_1C_2} = -\frac{\omega_o}{s}$
	For V_o at ϕ_2 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{\pi}{2}$	LDI	$H(z) = -\frac{C_1}{C_2}\frac{z^{-\frac{1}{2}}}{1-z^{-1}}$
Inverting (Forward)	For V_o at ϕ_1 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{\pi}{2} - \frac{\omega T}{2}$	Forward	$H(z) = -\frac{C_1}{C_2}\frac{z^{-1}}{1-z^{-1}}$
	For V_o at ϕ_2 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{-\pi}{2}$	LDI	$H(z) = \frac{C_1}{C_2}\frac{z^{-\frac{1}{2}}}{1-z^{-1}}$
Non-Inverting	For V_o at ϕ_1 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{-\pi}{2} - \frac{\omega T}{2}$	Forward	$H(z) = \frac{C_1}{C_2}\frac{z^{-1}}{1-z^{-1}}$
	For V_o at ϕ_1 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{\pi}{2} + \frac{\omega T}{2}$	Backward	$H(z) = -\frac{C_1}{C_2}\frac{z^{-\frac{1}{2}}}{1-z^{-1}}$
Inverting (Backward)	For V_o at ϕ_2 $\frac{\omega_o}{\omega}\frac{\omega T/2}{sin(\omega T/2)}$	$\frac{\pi}{2}$	LDI	$H(z) = -\frac{C_1}{C_2}\frac{z^{-\frac{1}{2}}}{1-z^{-1}}$

Type of Filters in the Z-Domain

Specifications



i.e.,

consider

Use prewarping

$$H(s) = \frac{\pm H_o \omega_p^2}{s^2 + \frac{\omega_p}{Q} s + \omega_p^2}$$

LP

Apply: A backward transformation

$$s \rightarrow \left(1 - Z^{-1}\right)/T \quad , \quad \text{then}$$

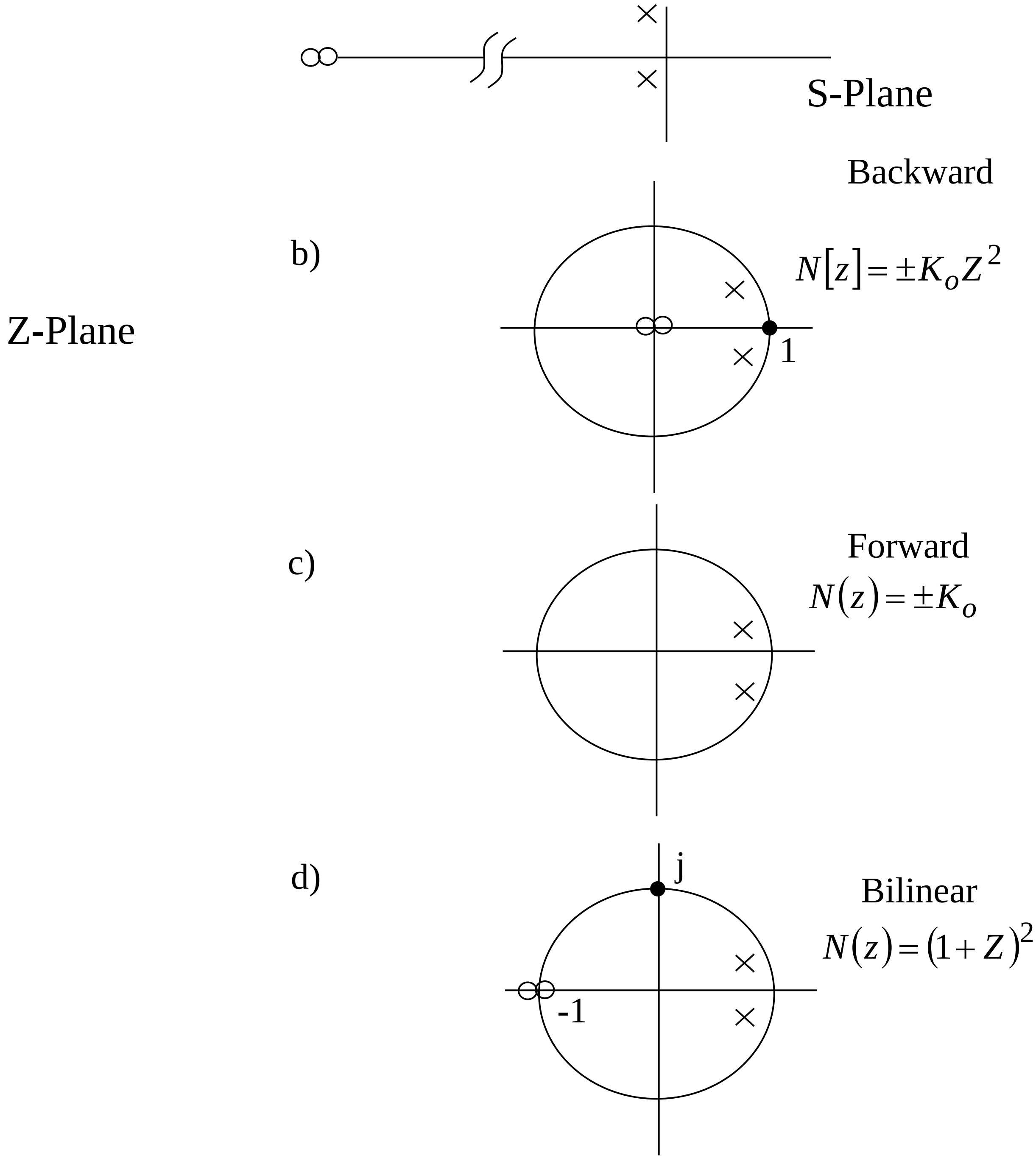
$$H(z) = \frac{\pm K_o Z^2}{r^2 - 2r\cos\theta Z + Z^2}$$

: A forward Transformation

$$H(z) = \frac{\pm K_o}{r^2 - 2r\cos\theta Z + Z^2}$$

: A Bilinear Transformation

$$H(z) = \frac{K_o (1 + Z)^2}{r^2 - 2r\cos\theta Z + Z^2}$$


S-Plane
Z-Plane
b)
Backward
$N[z] = \pm K_o Z^2$
1
c)
Forward
$N(z) = \pm K_o$
d)
j
Bilinear
$N(z) = (1 + Z)^2$
-1

Another Example.

Bandpass

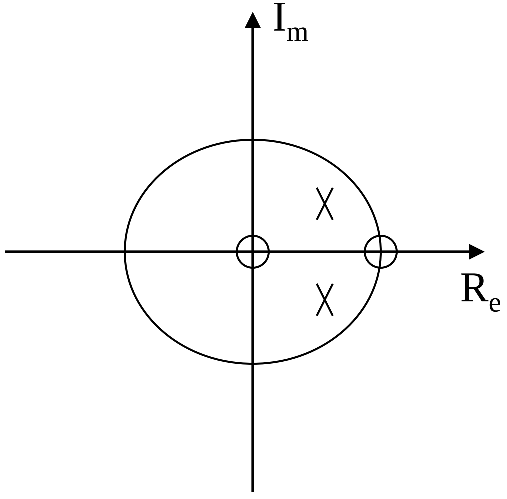
$$H(s) = \frac{\pm H_o \frac{\omega_p}{Q}}{s^2 + \frac{\omega_p}{Q} s + \omega_p^2}$$

i) Backward Tansformation

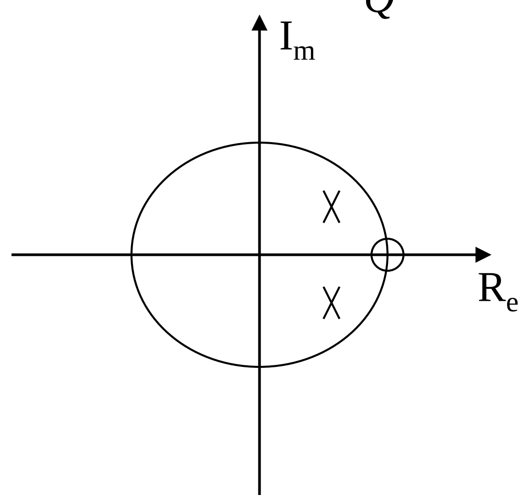
$$s \rightarrow \left(1 - Z^{-1}\right) / T$$

$$H(z) = \frac{\pm H_o \frac{\omega_p}{QT}\left(1 - Z^{-1}\right)}{\frac{\left(1 - Z^{-1}\right)^2}{T^2}} + \frac{\omega_{pi}}{Q_i}\frac{1 - Z^{-1}}{T} + \omega_{pi}^2$$

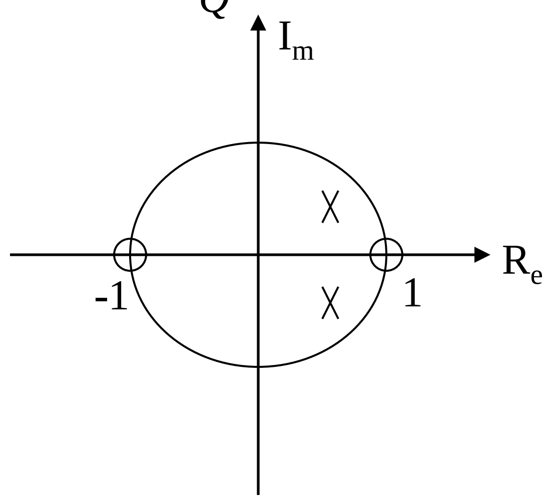
$$H(z) = \frac{H_o \frac{\omega_p T Z (Z - 1)}{Q\left(1 + \omega_p T / Q + (\omega_p T)^2\right)}}{\frac{1}{1 + \frac{\omega_p T}{Q}} + (\omega_p T)^2 - Z\frac{Z + \omega_p T}{1 + \frac{\omega_p T}{Q} + (\omega_p T)^2} + Z^2}$$



BACKWARD



FORWARD



BILINEAR

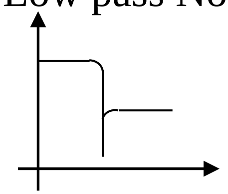
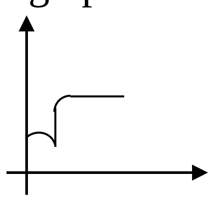
TYPE OF FILTER	NUMERATOR
LP 20 (bilinear)	$K_O\left(1+z^{-1}\right)^2$
LP 02 (forward	$K_O z^{-2}$
LP11	$K_O z^{-1}\left(1+z^{-1}\right)$
LP 10	$K_O\left(1+z^{-1}\right)$
LP 01	$K_O z^{-1}$
BP 20 (bilinear)	$K_O\left(1-z^{-1}\right)\left(1+z^{-1}\right)$
BP 01 (forward)	$K_O z^{-1}\left(1-z^{-1}\right)$
BP 00 (backward)	$K_O\left(1-z^{-1}\right)$
HP	$K_O\left(1-z^{-1}\right)^2$
Notch (symmetric)	$K_O\left[1+2\left(1-cos\ \Theta_o\right)z^{-1}+z^{-2}\right]$
Low pass Notch 	where $cos\,\theta_O = \frac{2r\,cos\,\theta}{1+r^2}$ θ_O and θ are the angles of the zero and pole locations, respectively
High pass Notch	Same as above with $cos\ \theta_O > \frac{2r\ cos\ \theta}{1+r^2}$
	Same as above with $cos\ \theta_O < \frac{2r\ cos\ \theta}{1+r^2}$
All pass	$K_O\left(r^2-2r\,cos\,\theta z^{-1}+z^{-2}\right)$

Table Numerators of second-order transfer functions in the z-plane

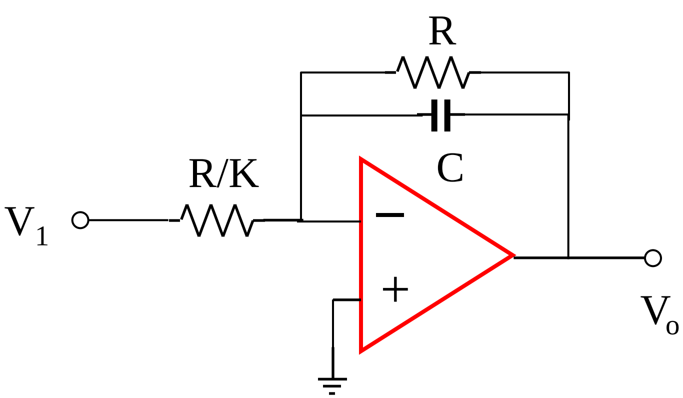
$D(s)=1-2r\,cos\,\theta z^{-1}+r^2 z^{-2}$

Examples of Basic Implementations

- First-order Low Pass (Continuous-Time)

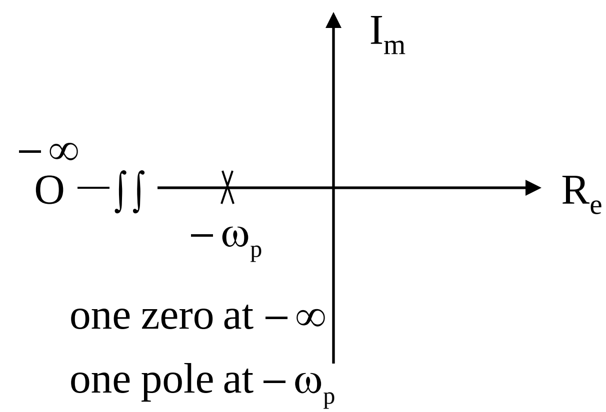
$$H(s) = \frac{K}{1 + s/\omega_p} \quad ; \quad \frac{V_o(s)}{V_{in}(s)} = \frac{K\omega_p}{s + \omega_p} \quad ; \quad \frac{dv_o(t)}{dt} + \omega_p v_o(t) = K\omega_p\ v_{in}(t)$$

Continuous-time

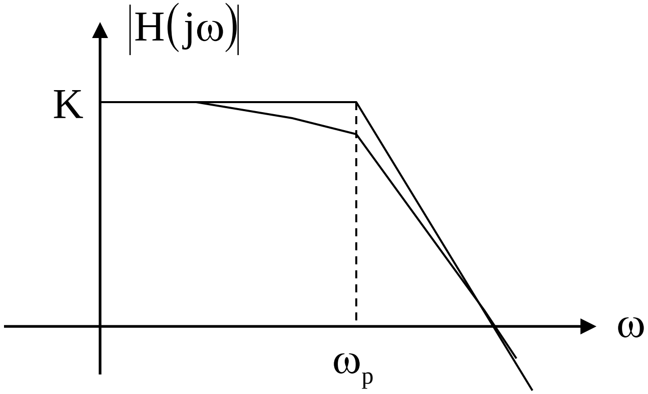


$$C = \frac{1}{R\omega_p}$$

s-plane

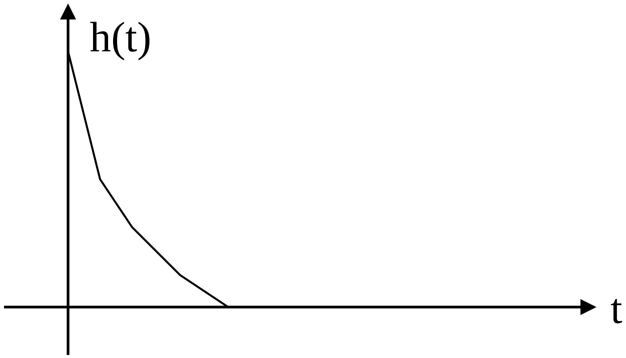


Stability implies to have poles in the left-half plane (LHP)



$$H(s)\Big|_{s=j\omega} = H(j\omega) = \frac{-K}{1+j\omega/\omega_p} = \frac{K}{\left[1+\left(\omega/\omega_p\right)^2\right]^{1/2}} \angle -\tan \omega/\omega_p$$

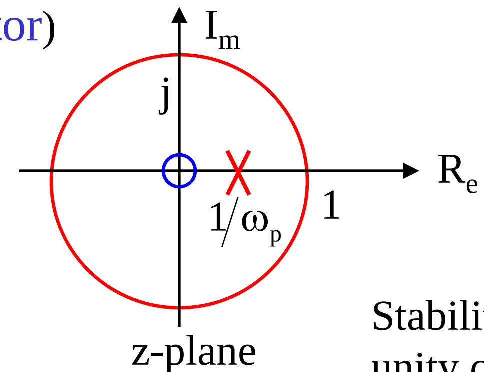
Frequency Response



$$h(t) = K\omega_p e^{-\omega_p t}$$

Impulse Response

• First-Order Low-Pass (Discrete-Time) Sampled-Data (i.e., Switched-Capacitor)



$$H(z)=\frac{V_o(z)}{V_{in}(z)}=\frac{K_1}{1-z^{-1}/\omega_{p1}}=\frac{K_1\omega_{p1}}{\omega_{p1}-z^{-1}}\ ;$$

Stability implies poles inside unity circle

$$\left[\omega_{p1}-z^{-1}\right]V_o(z)=K_1\omega_{p1}V_{in}(z)$$

$$\omega_{p1}V_o(z)-z^{-1}V_o(z)=K_1\omega_{p1}V_{in}(z)$$

Taking the inverse z-transform

$$\omega_{p1}v_o(nT)-v_o(n-1)T=K_1\omega_{p1}v_{in}(nT)$$

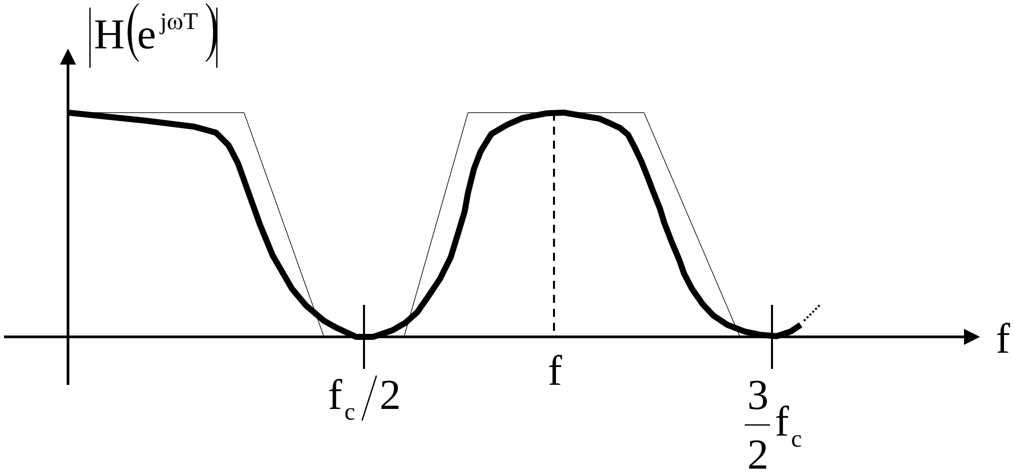
$$v_o(n-1)T-\omega_p v_o(nT)=-K_1\omega_{p1}v_{in}(nT)$$

This difference equation represents the first-order low pass in the Z-domain.

Note that

$$Z^{-1}X_o(z)\Rightarrow x_o(n-1)T$$

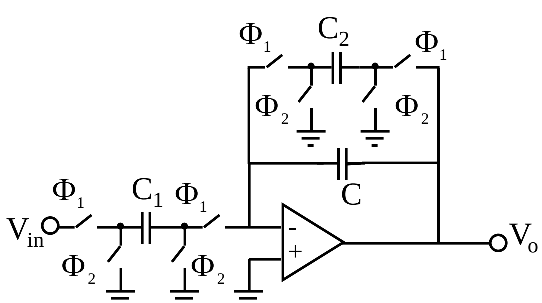
$$Z^{-b}X_o(z)\Rightarrow x_o(n-b)T$$



Frequency Response (A periodic transfer function!)

$$H(z) = \frac{K_1\omega_{p1}Z}{\omega_{p1}Z - 1}\Bigg|_{z=e^{j\omega}} = \frac{K_1\omega_{p1}(\cos\omega T + j\sin\omega T)}{(\omega_{p1}\cos\omega T - 1) + j\sin\omega T}$$

$$H(e^{j\omega}) = \frac{K_1\omega_{p1}}{\left\{(\omega_{p1}\cos\omega T - 1)^2 + \sin^2\omega T\right\}^{1/2}}\overbrace{\left[\omega T - \tan^{-1}\frac{\sin\omega T}{\omega_{p1}\cos\omega T - 1}\right]}^{\text{phase}}$$



where

$$K_1\omega_{p1} = \frac{C_1}{C}$$

$$\omega_{p1} = 1 + \frac{C_2}{C}$$

What are the relationships between the s-plane and z-plane?

- There are a number of mappings between the two planes
- The most popular and exact is the bilinear mapping.

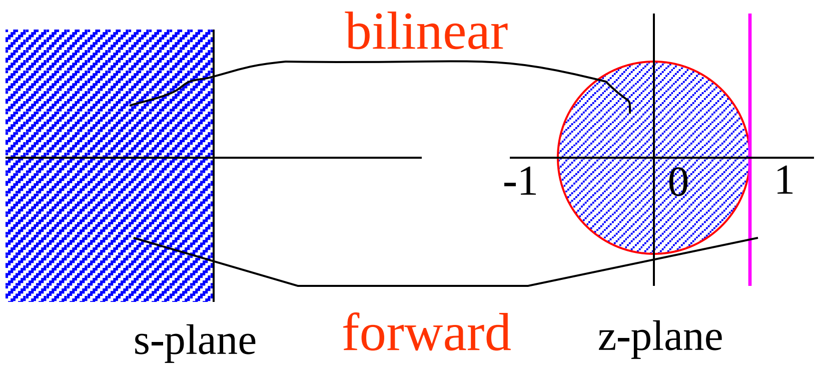
$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{T}\frac{z - 1}{z + 1} \quad \text{or} \quad z = \frac{1 + (T/2)s}{1 - (T/2)s}$$

- The commonly used for high-sampling rate is:

$$s = \frac{1}{T}\frac{1 - z^{-1}}{z^{-1}} = \frac{1}{T}\frac{z - 1}{1}$$

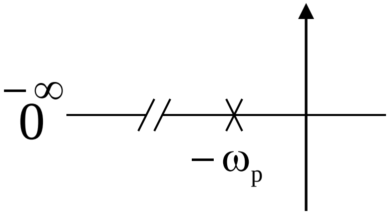
or

$$z = sT + 1$$

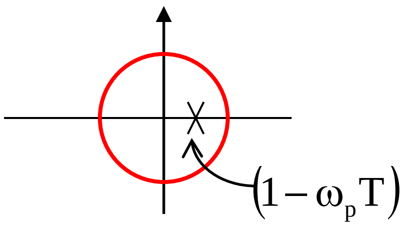


Example. Approximate a first-order low-pass continuous-time to a discrete-time low-pass under high-sampling conditions. $(fs/f >> 1)$

$$H(s) = \frac{K}{1 + s/\omega_p} = \frac{K\omega_p}{s + \omega_p} \Bigg|_{s = \frac{1}{T}(z-1)}$$



$$H(z) = \frac{K\omega_p T}{z - 1 + \omega_p T} = \frac{K\omega_p T}{z - (1 - \omega_p T)}$$



What is the 3dB cut-off frequency in both domains?

$f_{3dB} = \omega_p$ CT

For H(z) is more complex than the computation of $f_{3dB.}$

$$H(e^{j\omega T}) = \frac{K\omega_p T}{\cos\omega T + j\sin\omega T - (1 - \omega_p T)}$$

$$\left|H(e^{j\omega T})\right|_{\omega = \omega_{3dB}} = \left|H(e^{j\omega_{3dB} T})\right| / \sqrt{2}$$

$$\omega_{3dB} = \frac{1}{T}\cos^{-1}\frac{2 - 2\omega_p T - (\omega_p T)^2}{2(1 - \omega_p T)}$$

Numerical example. Low-Pass (First Order)

$$\omega_p = 2\pi \times 10^3 \, r/s$$

$$f_c = f_s = 20KHz \quad ; \quad T = 1/f_s \quad ; \quad \omega_p T = \frac{2\pi \times 10^3}{20 \times 10^3} = 0.1\pi$$

$$K = 2$$

For the continuous-time

$$f_{3dB} = 1KHz$$

$$H(j\omega)\Big|_{\omega = \omega_{3dB}} = \frac{2}{\sqrt{2}} = 1.4142$$

For the sample-data

$$f_{3dB} = \frac{f_s}{2\pi} \cos^{-1} \frac{2 - 2\omega_p T - (\omega_p T)^2}{2(1 - \omega_p T)}$$

$$f_{3dB} \cong 1.16KHz$$

Switched - Capacitor Filters

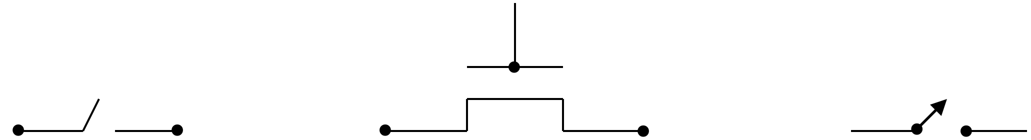
- Use Z-transform mathematics
- Are described by difference equations
- Time constants are proportional to capacitor ratios
- Best implementation for audio applications
- Originally the basic goal was to replace resistors by switches and capacitors
- This design approach is one of the most popular in the industry

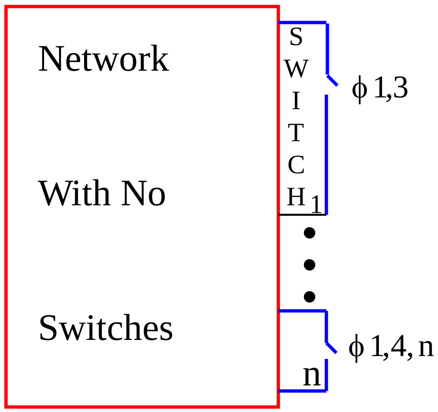
SC Advantages

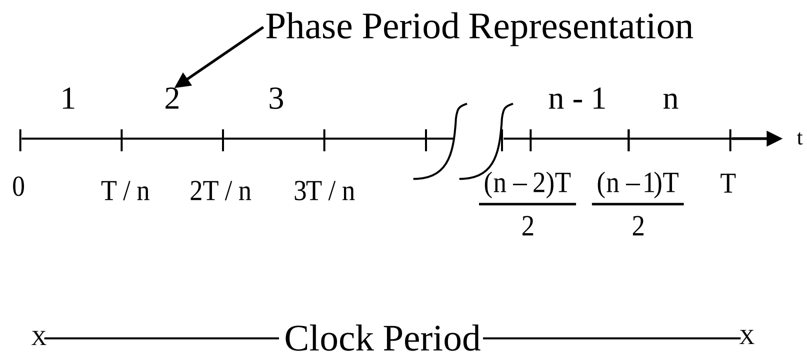
- Reduced silicon area
- Good accuracy. Time constants are implemented with capacitor ratios (~0.1%)
- Don’t require a low-impedance output stage (OTA’s could be used)
- Could be implemented using digital circuit process technology
- Very useful in the audio range

NOTATION

Switches Representation

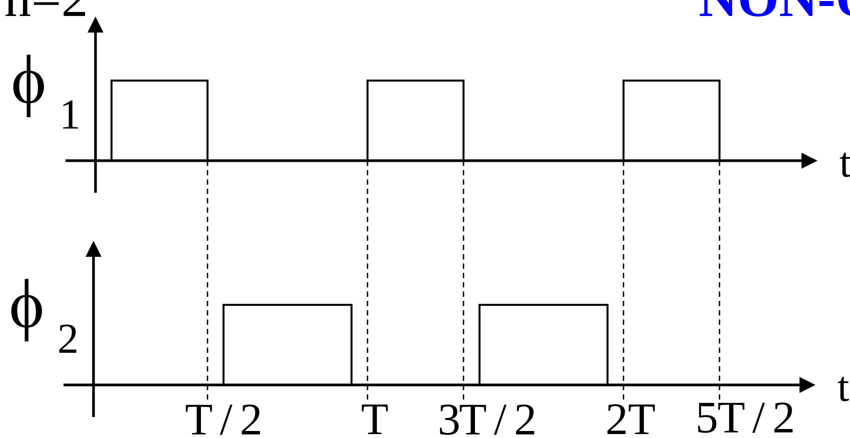




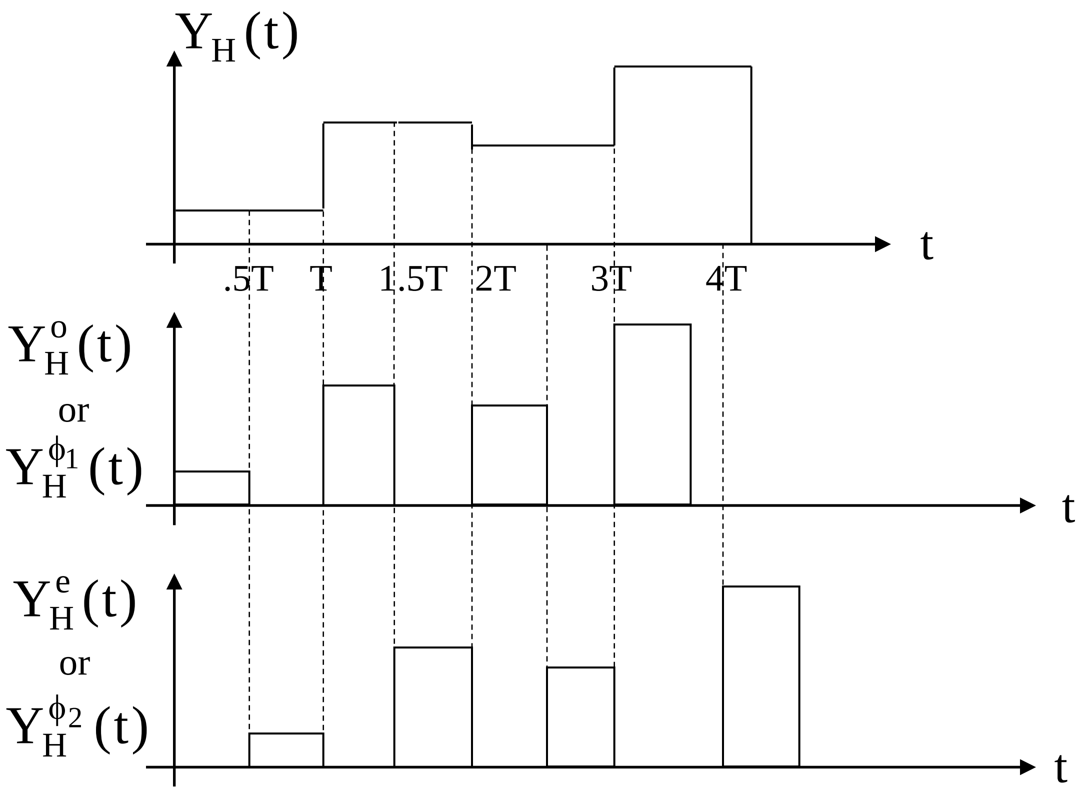


EXAMPLE n=2

NON-OVERLAPPING



PHASE PERIODS OF A CONVENTIONAL CLOCK SEQUENCE

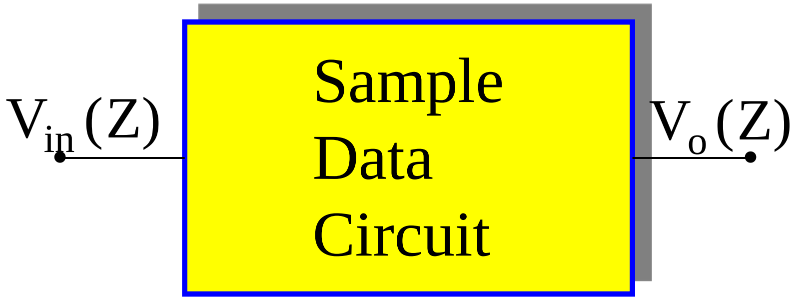


S/H and its respective odd and even components

$$y_H(t) = y_H^o(t) + y_H^e(t)$$

or

$$Y_H(Z) = Y_H^o(Z) + Y_H^e(Z)$$



$$V_{in}(Z) = V_{in}^{e}(Z) + V_{in}^{o}(Z)$$

$$V_{o}(Z) = V_{o}^{e}(Z) + V_{o}^{o}(Z)$$

$$H(Z) = \frac{V_o(Z)}{V_{in}(Z)}$$

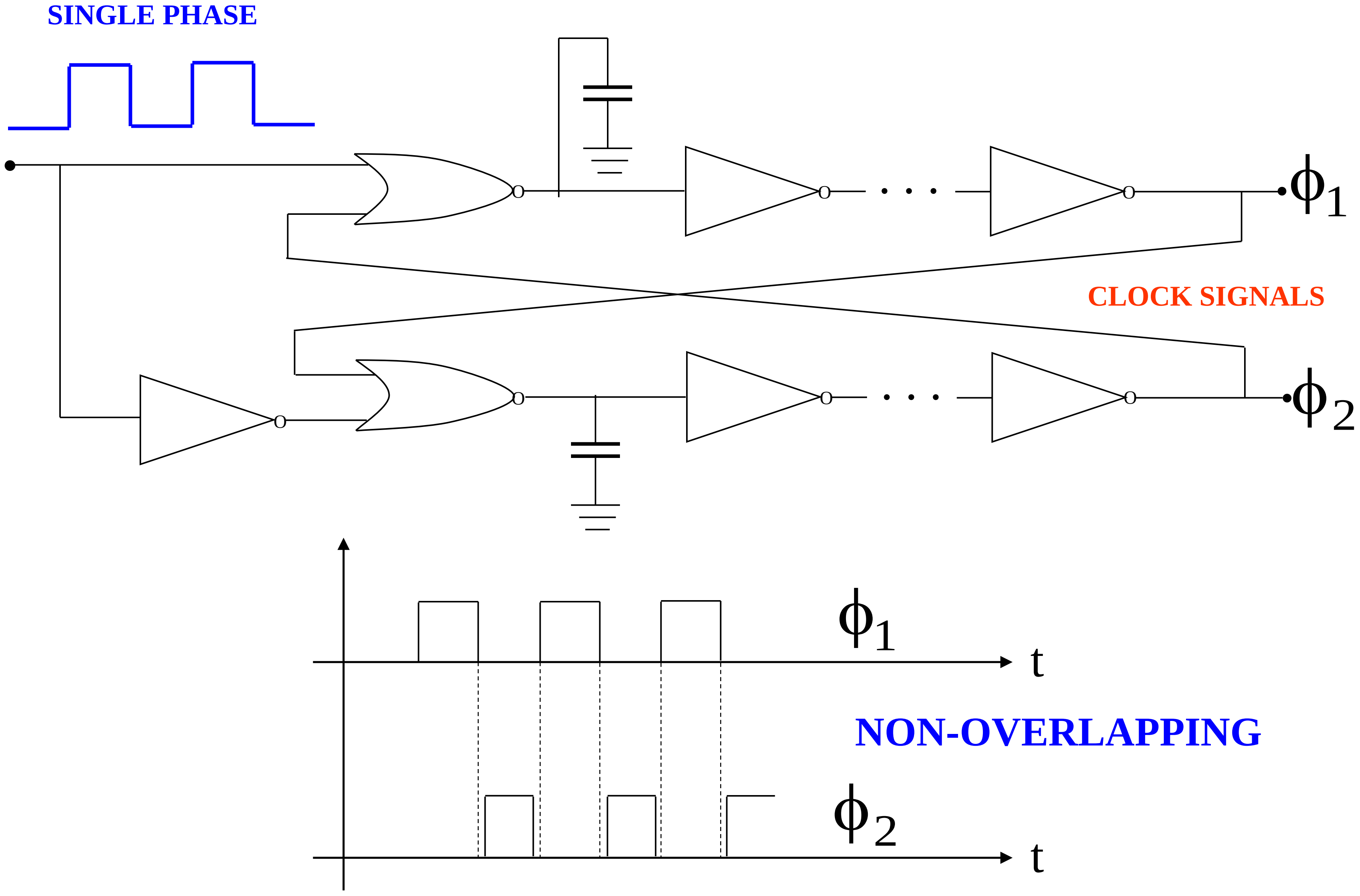
CONVENTIONAL NOTATION FOR TRANSFER FUNCTION IS:

$$H^{ij}(Z) = \frac{V_o^{j}(Z)}{V_{in}^{i}(Z)}$$

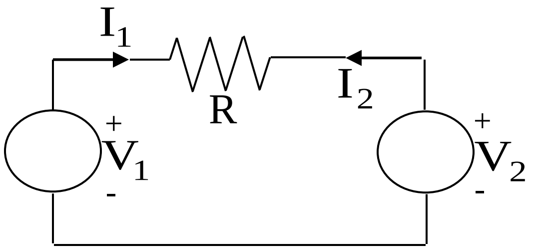
i and *j* can be either “e” or “o”. i,e.,

$$H^{eo}(Z) = \frac{V_o^{o}(Z)}{V_{in}^{e}(Z)}$$

Two-Phase Clock Generator



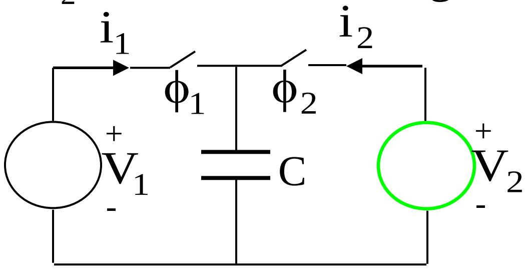
Switched-Capacitor Equivalent Resistor

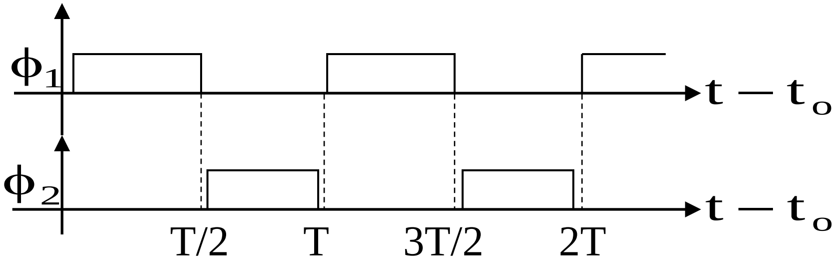


Continuous (Conventional) Resistor

$$R = \frac{V_1 - V_2}{I_1} = \frac{V_2 - V_1}{I_2} \Rightarrow I_2 = \frac{V_2 - V_1}{R}$$

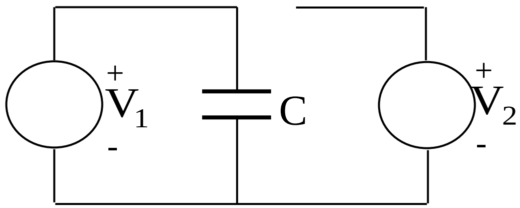
V_1 and V_2 are constant voltage sources





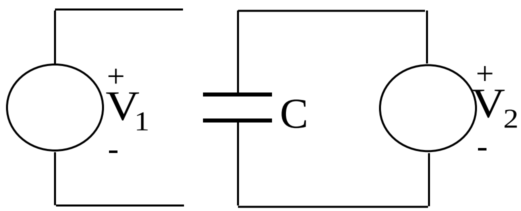
At time $t = t_o$ we apply the clocks.

At ϕ_1



$$Q(t_o + T/2) = CV_1$$

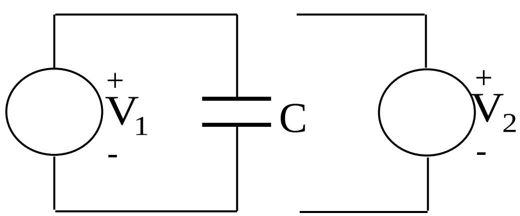
At ϕ_2



$$Q(t_o + T) = CV_2 - CV_1$$

$$Q(t_o + T) = C(V_2 - V_1)$$

For the next period, at ϕ_1



$$Q(t_o + \frac{3T}{2}) = C(V_1 - V_2)$$

$$i = \frac{dQ}{dt}$$

$$Q_1 = \int_{t_o + T}^{t_o + 3T/2} i_1(t)dt = \int_{t_o + T/2}^{t_o + 3T/2} i_1(t)dt$$

The average current $I_{1\,(aver)}$ becomes

$$I_{1\,(aver)} = \frac{Q(t_o + \frac{3T}{2})}{T}$$

$$I_{1\,(aver)} = \frac{1}{T}\int_{t_o + 3T/2}^{t_o + 3T/2} i_1(t)dt$$

$$I_{1\,(aver)} = \frac{C(V_1 - V_2)}{T} = \frac{\Delta Q}{\Delta t}$$

or

$$\frac{T}{C} = \frac{V_1 - V_2}{I_{1\,(aver)}}$$

Comparing with the continuous time resistor

$$R_{eq} = \frac{T}{C} = \frac{1}{f_C C}$$

EXAMPLE. $R = 250K\Omega, \quad f_C = 128KHz$

$$\frac{A_R}{A_C} \cong 32$$

$$C = \frac{1}{R_{eq} f_C} = \frac{1}{250 \times 128 \times 10^6} = 31.25pF$$

	Continuous R	"SC-R"	
AREA	5,776	178.57	$mils^2$

ACCURACY OF TIME CONSTANTS

⊙ CONTINUOUS TIME:

$$\tau = R_1 C_2$$

$$\frac{d\tau}{\tau} = \frac{dR_1}{R_1} + \frac{dC_2}{C_2} \rightarrow \pm 40\% \rightarrow \pm 65\%$$

where $\frac{d\tau}{\tau}$ is interpreted as the accuracy of τ.

TEMPERATURE DEPENDANT!

⊙ DISCRETE TIME:

$$\tau = \frac{1}{f_C C_1} \cdot C_2 = T(\frac{C_2}{C_1})$$

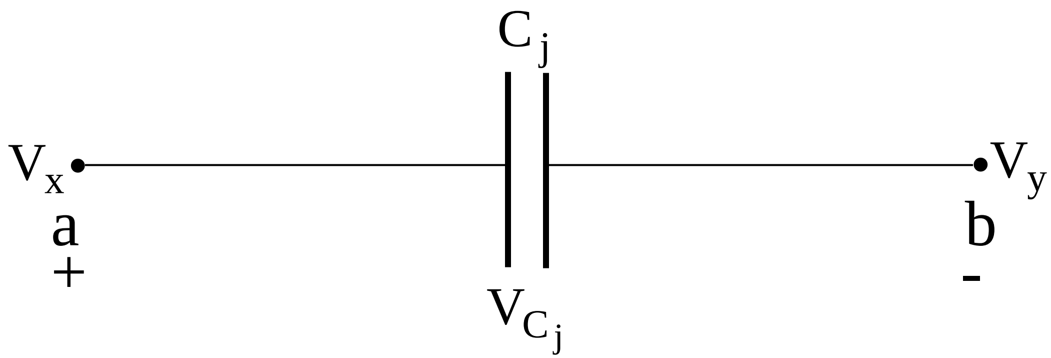
$$\frac{d\tau}{\tau} = \frac{dT}{T} + \frac{dC_2}{C_2} - \frac{dC_1}{C_1}$$

$$\frac{d\tau}{\tau} \cong \frac{dC_2}{C_2} - \frac{dC_1}{C_1} \rightarrow 0.1\%$$

KC_HL KIRCHOFF "CHARGE" LAW

$$\sum_{i=1}^{n} Q_i = 0$$

$$Q_j = C_j V_{C_j}$$



$$t_1 > 0$$

$$t_2 > t_1$$

at $t = t_2$

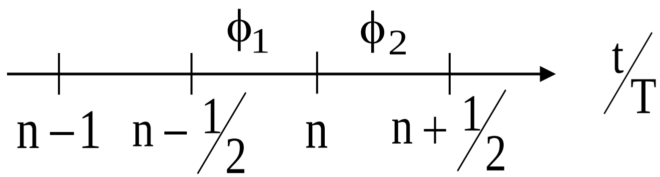
$$V_{C_j} = V_x(t_2) - V_y(t_2) - (V_x(t_1) - V_y(t_1))$$

Voltage across the capacitor $V_{C_j} =$ voltage difference (at present time) across the capacitor minus initial condition (at past time).

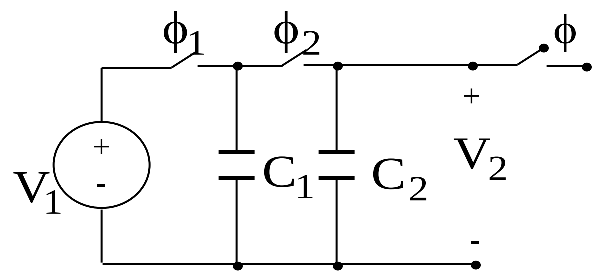
CHARGE CONSERVATION

ANALYSIS METHOD

$$q_L(n) = q_M(n-1) + \hat{q}_C(n)$$



EXAMPLE



for ϕ_2

$$C_2[V_2^e(n) - V_2^o(n - \tfrac{1}{2})] + C_1[V_2^e(n) - V_1^o(n - \tfrac{1}{2})] = 0$$

for ϕ_1

$$C_2[V_2^o(n - \tfrac{1}{2}) - V_2^e(n-1)] = 0 \Rightarrow V_2^o(n - \tfrac{1}{2}) = V_2^e(n-1)$$

THEN

$$C_1 v_2^o(n) + C_2 v_2^o(n) = C_2 v_2^o(n-1) + C_1 v_1^o(n-1)$$

$$(C_1 + C_2) V_2^o(Z) - C_2 Z^{-1} V_2^o(Z) = C_1 V_1^o(Z) Z^{-1}$$

$$H^{oo}(Z) = \frac{\frac{C_1}{C_1} Z^{-1}}{\frac{C_1 + C_2}{C_1} - \frac{C_2}{C_1} Z^{-1}} = \frac{Z^{-1}}{1 + \alpha - \alpha Z^{-1}}$$

$$H^{oo}(Z) = \frac{\frac{C_1}{C_1} Z^{-1}}{\frac{C_1 + C_2}{C_1} - \frac{C_2}{C_1} Z^{-1}} = \frac{Z^{-1}}{1 + \alpha - \alpha Z^{-1}}$$

$$H^{oo}(Z) = \frac{V_o^o(Z)}{V_{in}^o(Z)} = \frac{V_o^e(Z) Z^{-1/2}}{V_{in}^o(Z)}$$

$$V_2^e(n) = V_2^e(n + 1/2)$$

$$C_2[V_2^o(n + 1/2) - V_2^o(n - 1/2)] + C_1[V_2^o(n + 1/2) - V_1^o(n - 1/2)]$$

$$C_2[V_2^o(Z)][Z^{1/2} - Z^{-1/2}] + C_1 V_2^o(Z) Z^{1/2} = C_1 V_1^o(Z) Z^{-1/2}$$

$$C_2(1 - Z^{-1}) V_2^o(Z) + C_1 V_2^o(Z) = C_1 V_1^o(Z) Z^{-1/2}$$

$$\frac{V_2^o(Z)}{V_1^o(Z)} = \frac{C_1 Z^{-1}}{C_2(1 - Z^{-1}) + C_1} = \frac{C_1 Z^{-1}}{C_2 + C_1 - C_2 Z^{-1}}$$

$$\frac{V_2^o(Z)}{V_1^o(Z)} = \frac{C_1}{(C_2 + C_1)Z - C_2}\Bigg|_{Z \cong 1 + ST} = \frac{C_1}{(C_2 + C_1) + (C_2 + C_1)ST - C_2}$$

For high-sampling rate $\omega T << 1$

$$\frac{V_2^o(Z)}{V_1^o(Z)} = \frac{C_1}{C_1 + (C_2 + C_1)5T} = \frac{1}{1 + (\frac{C_2 + C_1}{C_1})5T} = \frac{1}{1 + 5 \Big/ \frac{C_1}{(C_2 + C_1)T}}$$

$$Z \cong 1 + ST$$

$$f_{3d\beta} \cong \frac{1}{2\pi} \cdot \frac{C_1}{(C_2 + C_1)T} = \frac{1}{2\pi} \cdot \frac{f_c}{1 + \frac{C_2}{C_1}}$$

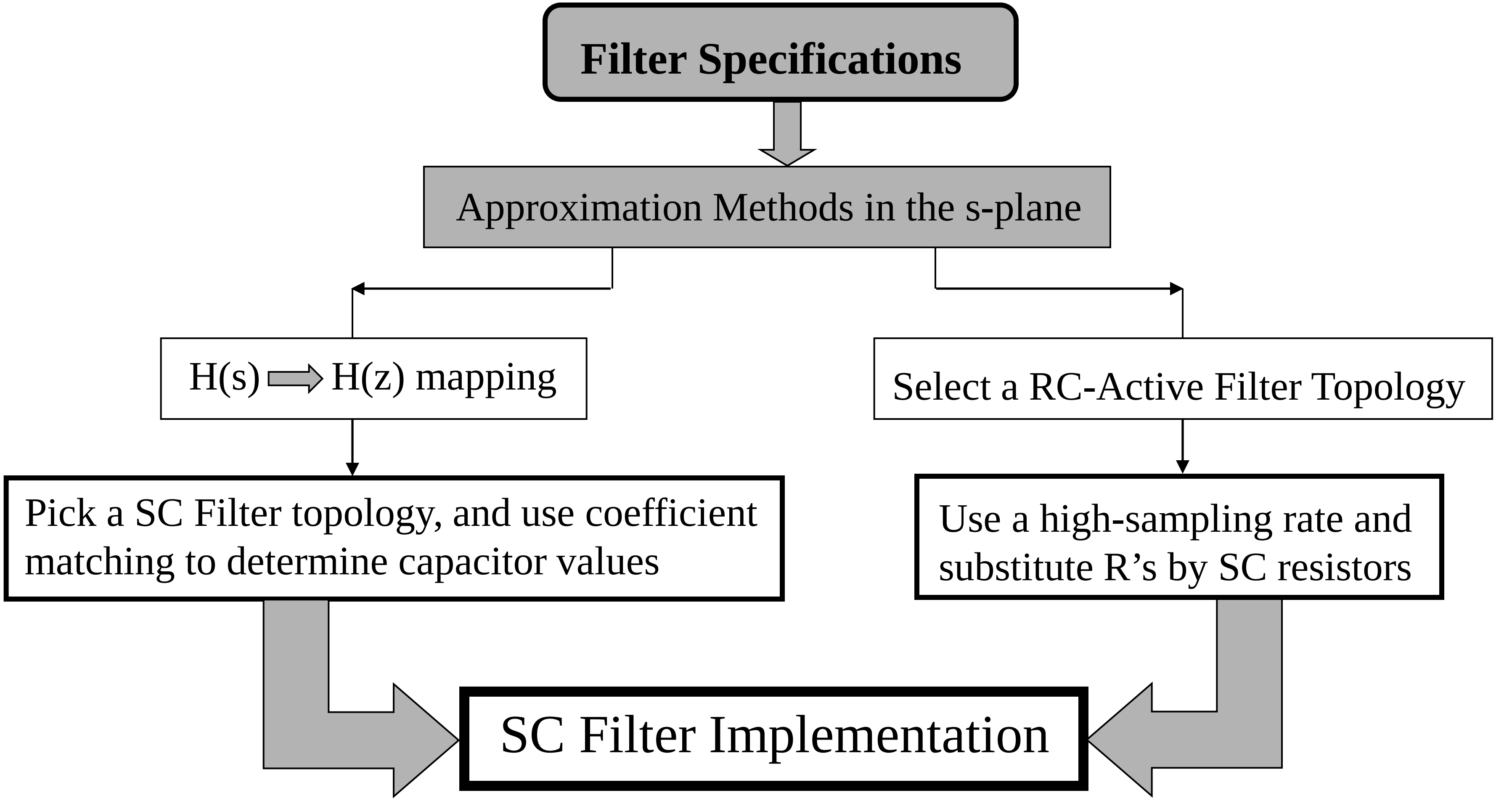
Aside:

$$f_{3d\beta} \cong \frac{1}{2\pi} \cdot \frac{1}{R_1 C_2} \cong \frac{1}{2\pi} \cdot \frac{1}{\frac{TC_2}{C_1}} = \frac{1}{2\pi} \cdot \frac{f_c}{\frac{C_2}{C_1}}$$

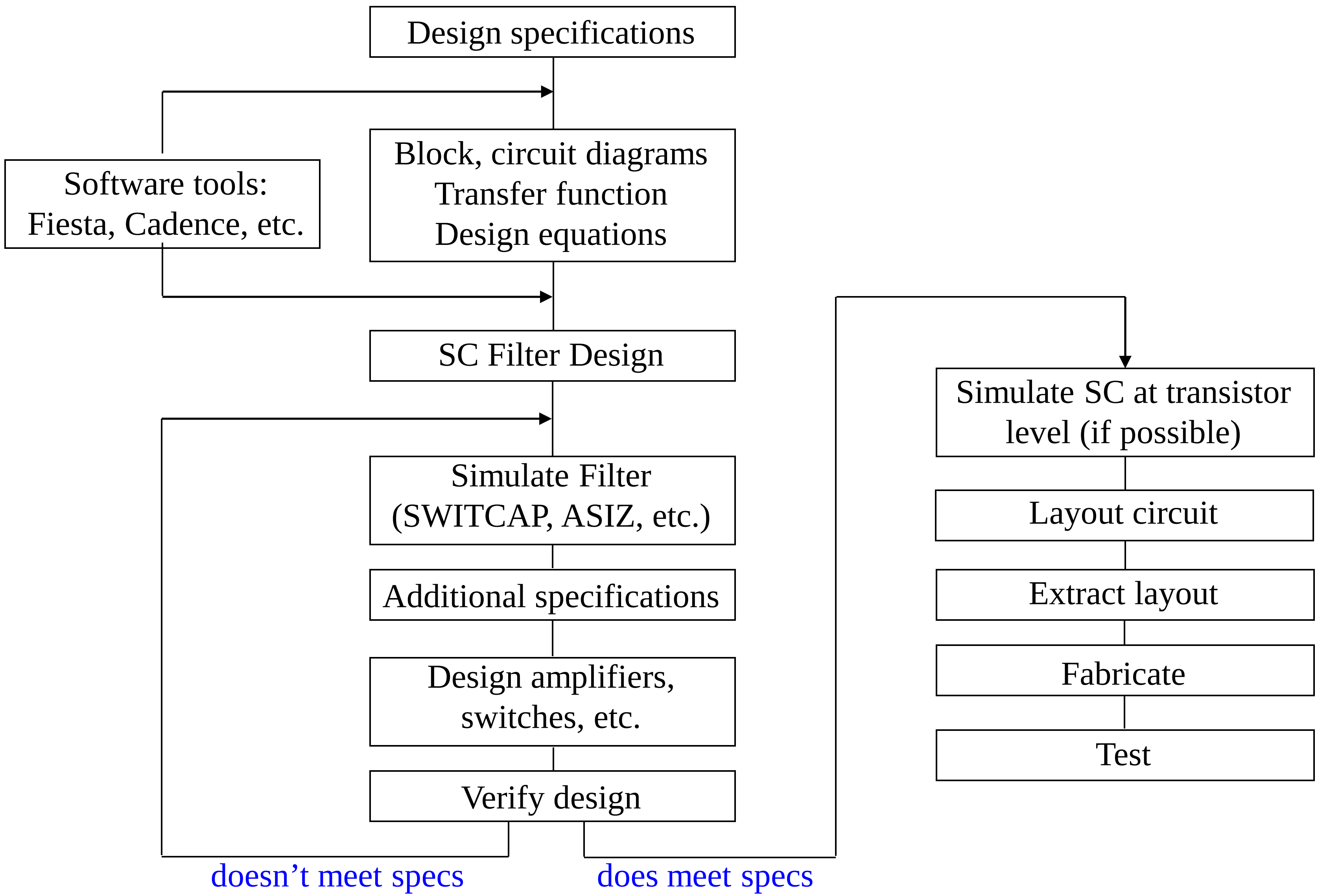
Switched-Capacitor (SC) Filters

How to design SC Filters ?

- Two basic approaches



Systematic SC Filter Design



What are the advantages and disadvantages of the two filter design procedures ?

- Mapping Techniques
  - + Systematic and well documented (see Matlab)
  - + It can use any sampling rate, including the (minimum) Nyquist rate.
  - - Difficult to implement by hand calculation

- Transforming R to SC resistors
  - + It is, conceptually, easier to follow for analog designer
  - + Its design is straightforward
  - - Yields not an optimal design for area, imposed high sampling rate involves larger capacitor ratios.

Switched-Capacitor Filters Components

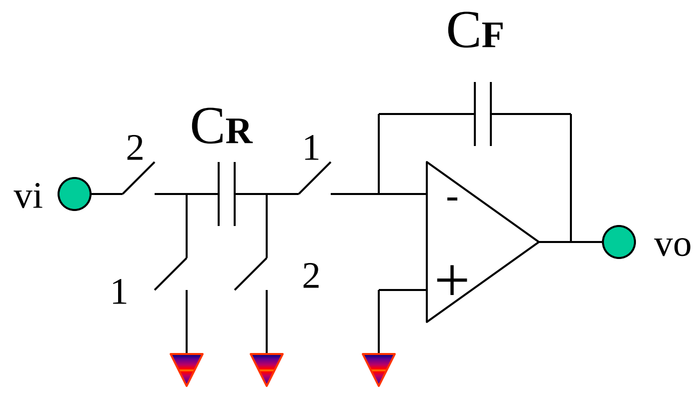
Basic Elements

Capacitors
- polysilicon
- metal1-metal2
- parasitics, clock-feedthrough

Switches
- N-MOS
- Transmission gates
- Noise and on-resistance

OTAS
- DC-gain
- settling time (GBW, phase margin)
- noise

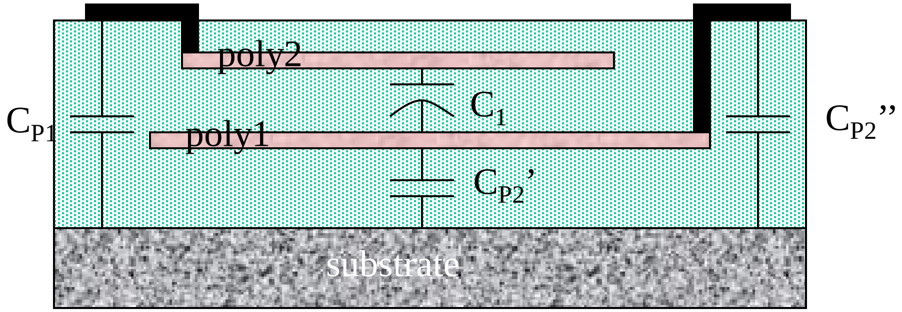
Non-overlapping clock phases



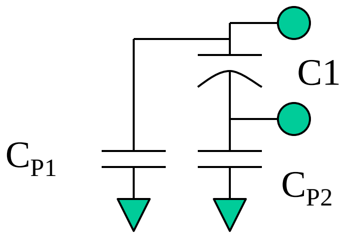
Typical switched-capacitor integrator

Switched-Capacitor Filters: CAPACITORS

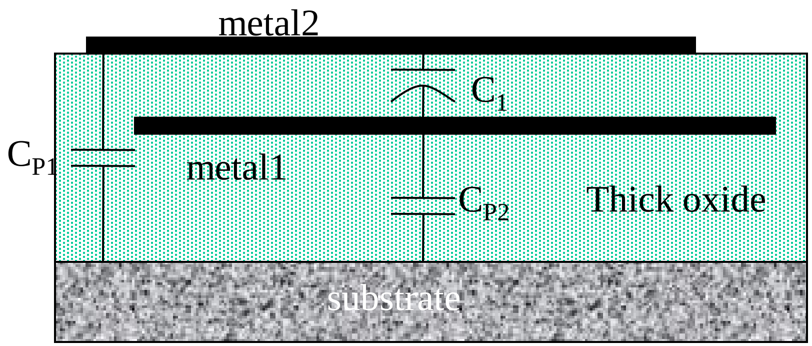
polysilicon



CP1, CP2'' are very small (1-5 % of C1)
CP2' is around 10-30 % of C1



metal1-metal2

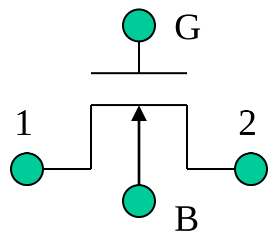


SWITCHES

MOS transistors biased in triode region, ~100-100 kΩ

Off resistances ~ G Ω

N-MOS

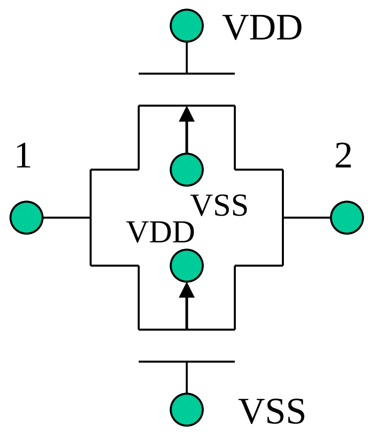


• V_1, $V_2 < V_G - V_T$ ($V_T \sim 1$ V)

• Resistance $R_S \cong \frac{L}{\mu_n C_{ox} W (V_{GS} - V_T)}$

• Small resistance for low voltages but high resistance for large voltages

• Transmission gates



• Rail to rail operation, provided that VDD and IVSSI > VT

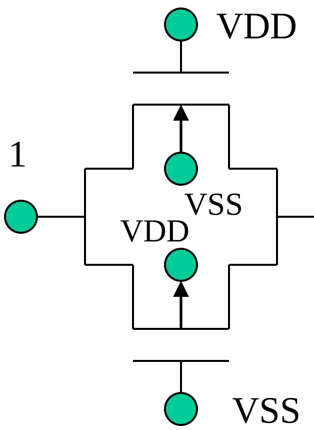
• Smaller resistance (fast response)

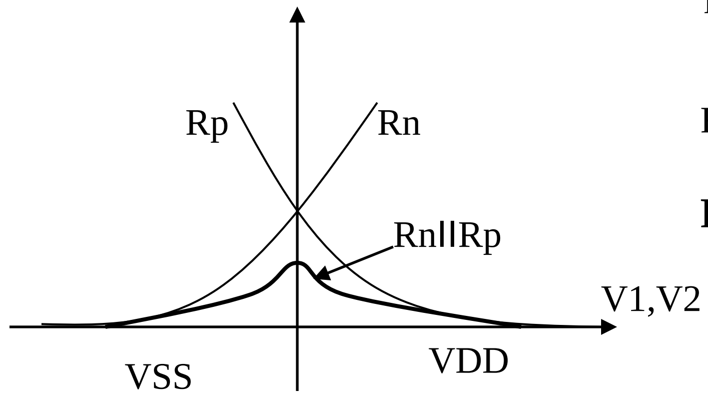
• 2 clock phases

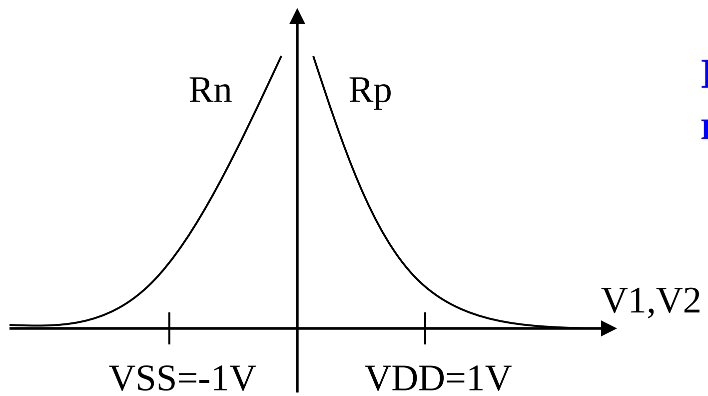
• More parasitics

Switched-Capacitor Filters: SWITCHES

Transmission gates







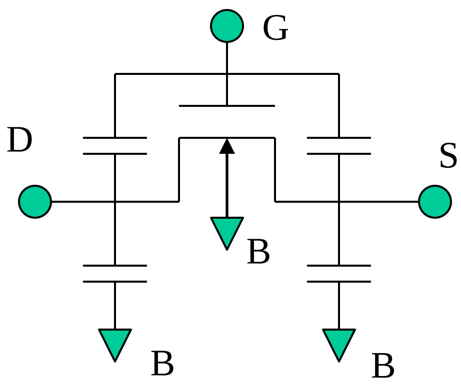
$$R_n \cong \frac{L}{\mu_n C_{ox} W(VDD - V1 - V_T)}$$

For V1, V2=0, W/L=1, $\mu_n C_{OX} = 10^{-4}$

$\mathbf{R_n = 10k/(VDD-1)}$

If VDD, |VSS|<V_T, the resistance is extremely high!!!

Switches (continues)



$$C_{GS,GD} = C_{OX} W L_D + \frac{1}{2} C_{OX} W L$$

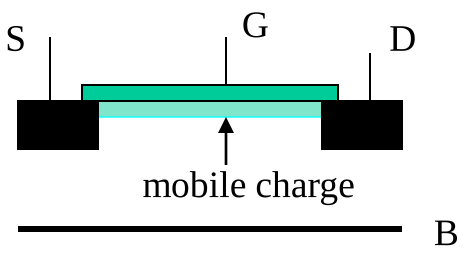
$$C_{BS,BD} = \frac{1}{2} \frac{C_{j0} W L}{\sqrt{1 + \frac{2\phi_F}{VSB}}} + C_{jBS}$$

$$\text{mobile-charge} = C_{OX} W L (V_{GS} - V_T)$$

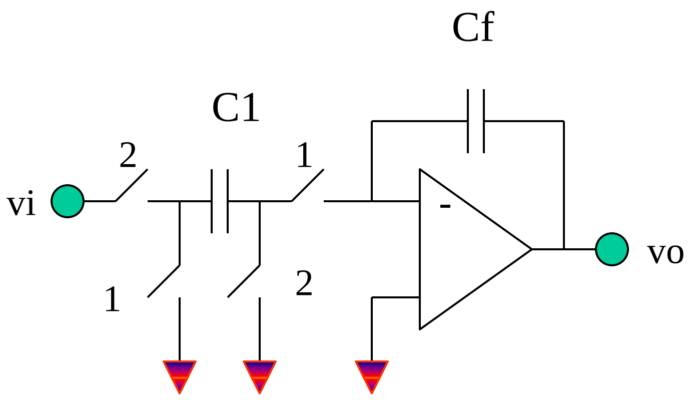
•CGS and CGD are Linear polysilicon capacitors, introduce offset voltage.

•$C_{SB,DB}$ are non-linear capacitors, introduce harmonic distortion components.

•Mobile charge introduces gain errors and harmonic distortion components.



Operational Transconductance Amplifier



PRACTICAL CONSIDERATIONS:

- **DC-gain**. The inverting input is not a real virtual ground. $v_{invert}=v_o/A_{dc}$

- **Settling time**. C1 must be discharged during phase 1. The main limitation is due to limited output current and phase margin.

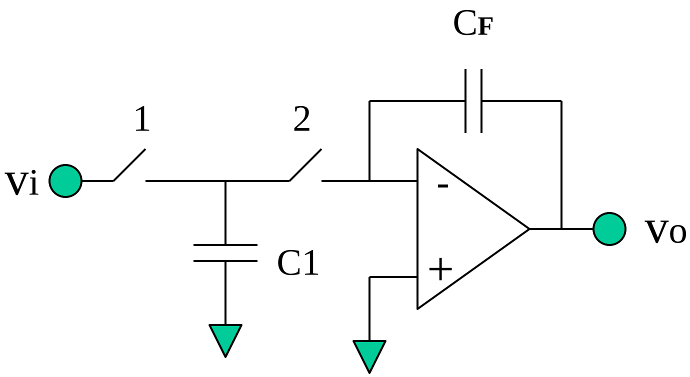
- **Clock feedthrough.** Can be alleviated by using especial clocking schemes.

- **Noise.** In most of the practical cases the dominant noise components are due to the Switches!!!

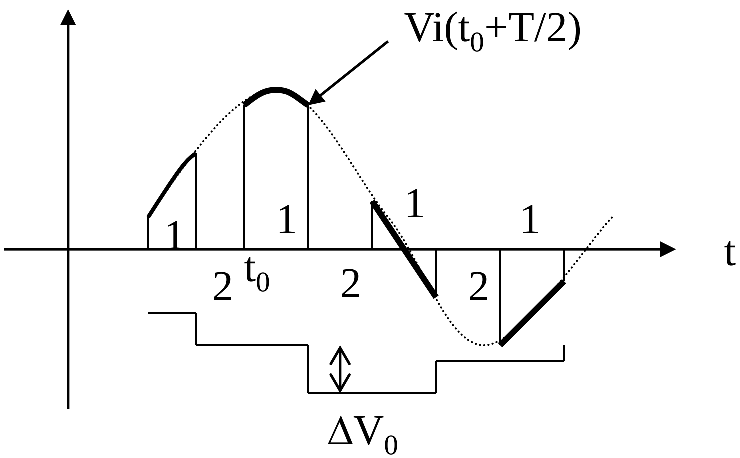
Switched-Capacitor Integrator Analysis

Phase 1 $t_0 < t \le t_0 + T/2$

Phase 2 $t_0 + T/2 < t \le t_0 + T$



Basic Stray-Sensistive Integrator



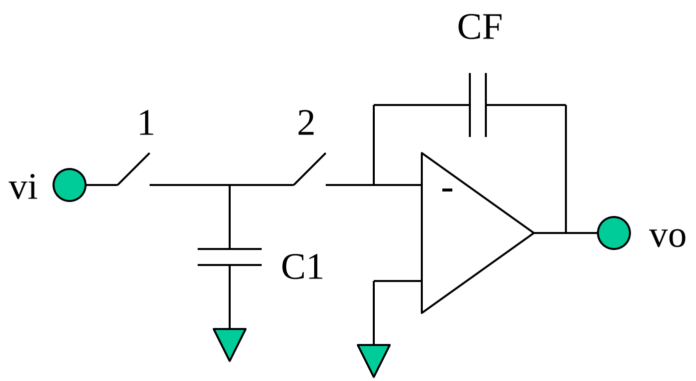
Charge conservation Principle:

Charge injected by C1 is equal to the charge absorbed by CF ($\Delta Q_{CI}=\Delta Q_{CF}$)

$$\Delta V_0 CF = -\Delta Q_{C1}$$

$$\Delta v_0 = -\frac{C1}{CF} v_i(t_0 + NT/2)$$

Switched-Capacitor Stray-Sensitive Integrator Analysis



Basic Integrator

Phase 1 $\quad t_0 < t \le t_0 + T/2$

$$v_{C1}(t) = v_i(t) \qquad v_{CF}(t) = v_{CF}(t_0)$$

$$v_{C1}(t_0 + NT/2) = v_i(t_0 + NT/2) \qquad v_{CF}(t_0 + NT/2) = v_{CF}(t_0)$$

Phase 2 $\quad t_0 + T/2 < t \le t_0 + T$

Charge conservation Principle:

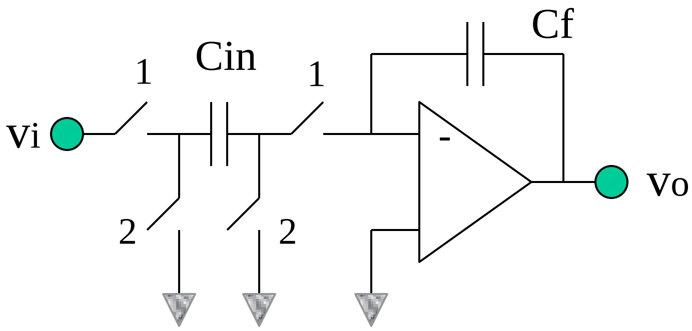
Charge injected by C1 is equal to the charge absorbed by CF (ΔQ_{CI}=ΔQ_{CF})

$$\left[(0-0)C_1 - \left(v_i(t_0 + T/2) - 0\right)C_1\right] + \left[\left(0 - v_0(t)\right)C_f - \left(0 - v_0(t_0 + T/2)\right)C_f\right] = 0$$

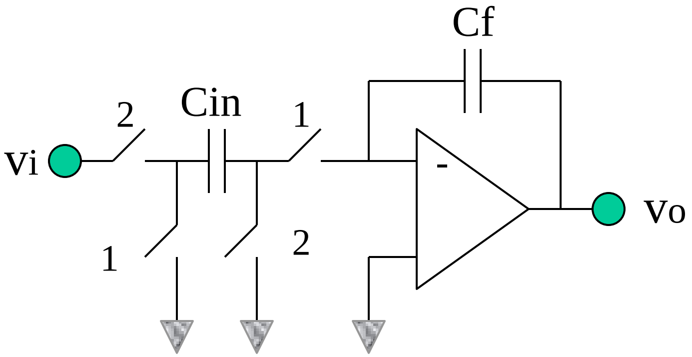
Solving for phase 2 (t=t0+T)

$$H(z) = -\frac{C_1}{C_f}\frac{z^{-1/2}}{1-z^{-1}}$$

Switched-Capacitor Integrators : Stray-Insensitive Integrators



Backward Integrator



Forward Integrator

Charge conservation ==> Phase 1

$$\left(v_i(nT)-0\right)C_{in}+\left[\left(v_0(nT)-0\right)C_f-\left(v_0(nT-T/2)-0\right)C_f\right]=0$$

Phase 2

$$0=\left(v_0(nT-T/2)-0\right)C_f-\left(v_0(nT-T)-0\right)C_f$$

Solving for phase 1

$$v_i(nT)C_{in}-\left[v_0(nT)-v_0(nT-T)\right]C_f=0$$

$$H(z)=-\frac{C_{in}}{C_f}\frac{1}{1-z^{-1}}$$

Charge conservation ==> Phase 1

$$\left(0-v_i(NT-T/2)\right)C_{in}+\left[\left(v_0(nT)\right)C_f-\left(v_0(nT-T/2)\right)C_f\right]=0$$

Phase 2

$$0=C_f v_0(nT-T/2)-C_f v_0(nT-T)$$

$$v_{cin}(NT-T/2)=v_i(NT-T/2)$$

$$v_i(nT-T/2)C_{in}-\left[v_0(nT)-v_0(nT-T)\right]C_f=0$$

$$H(z)=\frac{C_{in}}{C_f}\frac{z^{-1/2}}{1-z^{-1}}$$