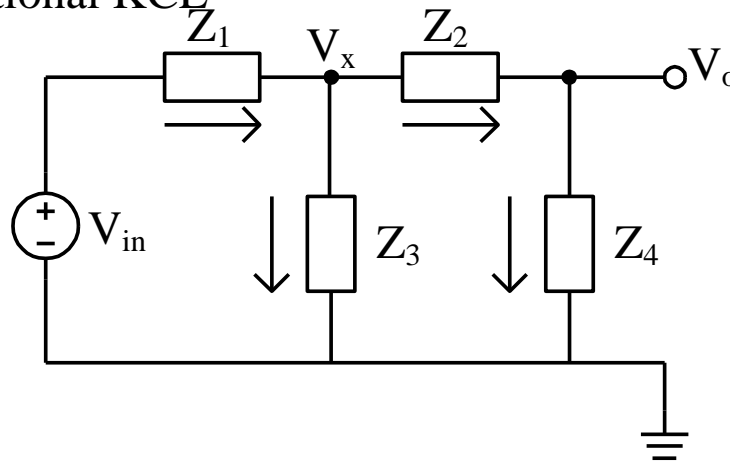


## NODAL EQUATIONS BY INSPECTION

Knowing how to write systematically network analysis equations is primordial to good circuit design. Here we will review one popular analysis technique partially used in SPICE.

Consider the following example

a) Using conventional KCL



$$\frac{V_{in} - V_x}{Z_1} = \frac{V_x}{Z_3} + \frac{V_x - V_o}{Z_2} = 0 \quad (1)$$

Re - writing (1) becomes

$$V_x \left( \frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_2} \right) - \frac{V_o}{Z_2} = \frac{V_{in}}{Z_1} \quad (1')$$

Conventional ,

$$\frac{V_x - V_o}{Z_2} = \frac{V_o}{Z_4} \quad (2)$$

Re - witting (2) becomes

$$-\frac{V_x}{Z_2} + V_o \left( \frac{1}{Z_4} + \frac{1}{Z_2} \right) = 0 \quad (2')$$

How can we write directly by inspection Eqs (1') and (2')?

We can do it using the indefinite admittance matrix which must be expressed as

$$\underline{Y}\underline{V} = \underline{I}$$

Where  $Y$  is the indefinite admittance matrix,  $V$  is the nodal voltage vector and  $I$  is the independent current source vector. In the example here presented we have two independent nodal voltage  $V_x$  and  $V_o$ . Thus we can write directly the equations (1') and (2').

$$V_x \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - \frac{V_o}{Z_2} - \frac{V_{in}}{Z_1} = 0$$

or equivalent

$$V_x \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - \frac{V_o}{Z_2} = \frac{V_{in}}{Z_1} \quad (1'')$$

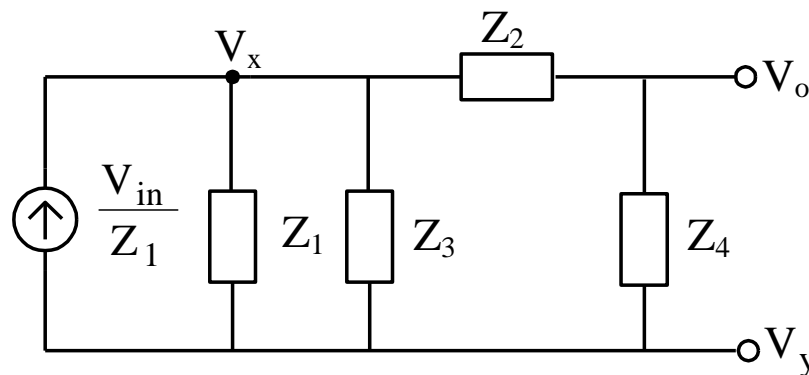
Note that for the node under analysis we multiply  $V_x$  by the sum of all admittances connected to node  $X$  minus the contributions of the neighbor connected nodes, in this case  $V_{in}$  and  $V_o$ . These (negative) contributions consist of the neighbor voltages multiplied by the sum of the admittances connected between  $V_x$  and  $V_{in}$  and  $V_o$ .

Let us practice with node at  $V_o$ , thus

$$V_o \left( \frac{1}{Z_2} + \frac{1}{Z_4} \right) - \frac{V_x}{Z_2} = 0 \quad (2'')$$

Consider next a more general example.

Let us assume a Norton transformation and that the ground node becomes different from zero and equal to  $V_y$  i.e.



where

$$I_{in} = \frac{V_{in}}{Z_1}$$

Now we will have 3 nodal equations

$$V_x \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - V_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} \right) - \frac{V_o}{Z_2} = I_{in} \quad (3)$$

$$-V_x \left( \frac{1}{Z_1} + \frac{1}{Z_3} \right) + V_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4} \right) - \frac{V_o}{Z_4} = -I_{in} \quad (4)$$

$$-\frac{V_x}{Z_2} - \frac{V_y}{Z_4} + V_o \left( \frac{1}{Z_2} + \frac{1}{Z_4} \right) = 0 \quad (5)$$

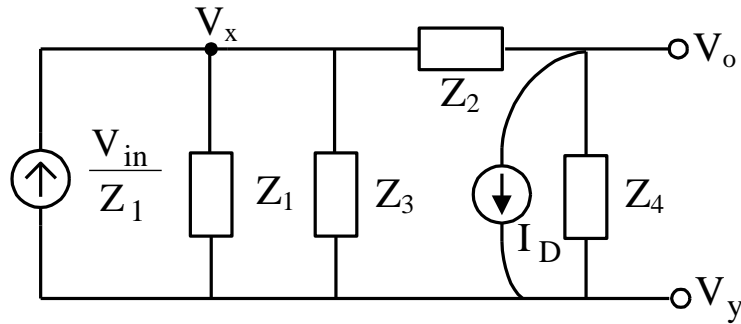
Notice that if you sum all the coefficients of  $V_x$ ,  $V_y$  or  $V_o$  the sum is zero, which is a consequence of KCL.

Observe that you can write equations (3), (4), and (5) in a matrix form.

$$\begin{bmatrix} \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} & -\frac{1}{Z_1} - \frac{1}{Z_3} & -\frac{1}{Z_2} \\ -\frac{1}{Z_1} - \frac{1}{Z_3} & \frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4} & -\frac{1}{Z_4} \\ -\frac{1}{Z_2} & -\frac{1}{Z_4} & \frac{1}{Z_2} + \frac{1}{Z_4} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_o \end{bmatrix} = \begin{bmatrix} I_n \\ -I_n \\ 0 \end{bmatrix}$$

Here you can verify that the sum of any column or row of Y is zero. Also observe the symmetry of the matrix Y due to the fact that we do not have depended sources.

Next let us consider another example to illustrate how the dependent sources can be incorporated in the analysis.



where

$$I_D = G_m (V_x - V_y)$$

We can go directly to our previous matrix and incorporate  $I_D$  or writing the 3 nodal equations.

$$V_x \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - V_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} \right) - \frac{V_o}{Z_2} = \frac{V_{in}}{Z_1}$$

$$-V_x \left( \frac{1}{Z_1} + \frac{1}{Z_3} \right) + V_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4} \right) - \frac{V_o}{Z_4} = -\frac{V_{in}}{Z_1} + G_m (V_x - V_y)$$

$$-\frac{V_x}{Z_2} \quad -V_y \frac{1}{Z_4} \quad + V_o \left( \frac{1}{Z_2} + \frac{1}{Z_4} \right) = -G_m (V_x - V_y)$$

Note that we must have only independent sources on the right side of the equations. Thus we will move the dependent sources into the left side of the equations.

$$\mathbf{V}_x \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - \mathbf{V}_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} \right) - \frac{\mathbf{V}_o}{Z_2} = \frac{\mathbf{V}_{in}}{Z_1}$$

$$- \mathbf{V}_x \left( \frac{1}{Z_1} + \frac{1}{Z_3} + \mathbf{G}_m \right) + \mathbf{V}_y \left( \frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4} + \mathbf{G}_m \right) - \frac{\mathbf{V}_o}{Z_4} = - \frac{\mathbf{V}_{in}}{Z_1}$$

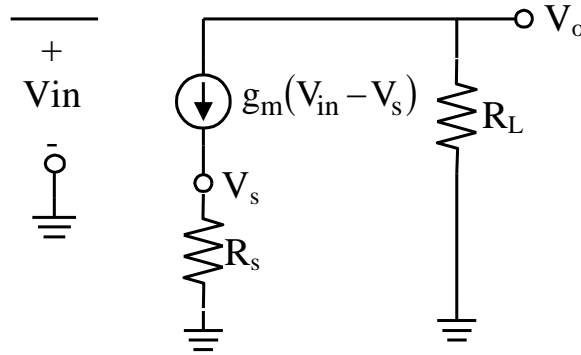
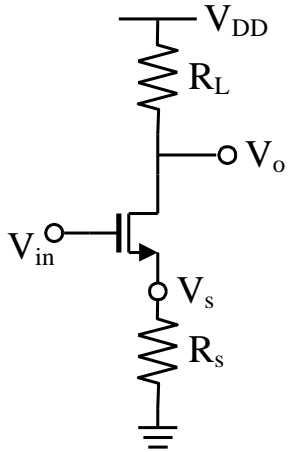
$$- \mathbf{V}_x \left( \frac{1}{Z_2} - \mathbf{G}_m \right) - \mathbf{V}_y \left( \frac{1}{Z_4} + \mathbf{G}_m \right) + \mathbf{V}_o \left( \frac{1}{Z_2} + \frac{1}{Z_4} \right) = 0$$

Observe that for all floating nodes the sum of the coefficients of all nodal voltages is zero.



# EXAMPLES TO ILLUSTRATE NODAL ANALYSIS

## EXAMPLE 1



Small Signal Circuit

- Two Nodal Equations for  $V_o$  and  $V_s$ .

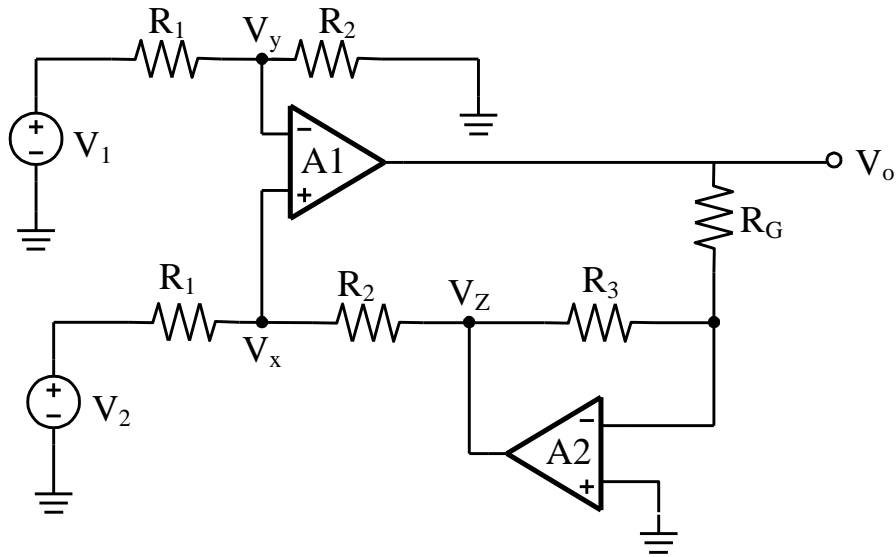
$$\frac{V_o}{R_L} = -g_m(V_{in} - V_s)$$

$$\frac{V_s}{R_s} = g_m(V_{in} - V_s)$$

Solving these two equations yields

$$\frac{V_o}{V_{in}} = \frac{-g_m R_L}{1 + g_m R_s}$$

## Example 2



Assume first ideal op amps. (Note  $V_x = V_y$ )

$$V_x \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{V_1}{R_1} = 0 \quad (1a)$$

$$V_x \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{V_z}{R_2} = \frac{V_2}{R_1} \quad (1b)$$

Also note that

$$V_z = -\frac{R_3}{R_G} V_o \quad (1c)$$

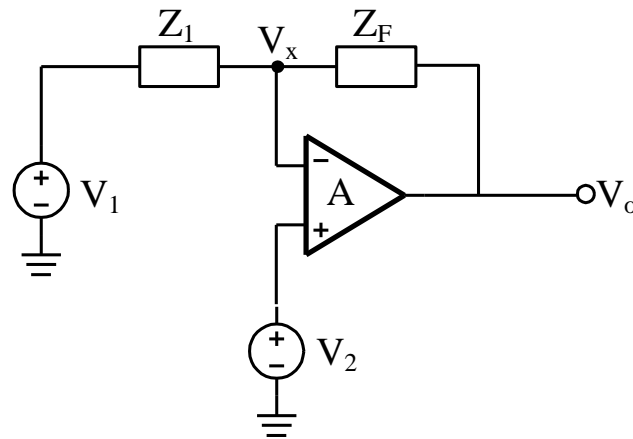
(1b) - (1a) yields

$$-\frac{V_z}{R_2} + \frac{V_1}{R_1} = \frac{V_2}{R_1} \Rightarrow -\left(-\frac{R_3}{R_G}\right) \frac{V_o}{R_2} + \frac{V_1}{R_1} = \frac{V_2}{R_1}$$

$$V_o = \frac{(V_2 - V_1) R_G R_2}{R_1 R_3} = \frac{R_2 R_G}{R_1 R_3} (V_2 - V_1)$$

$$V_o = (V_2 - V_1) \frac{R_2 R_G}{R_1 R_3}$$

### Example 3



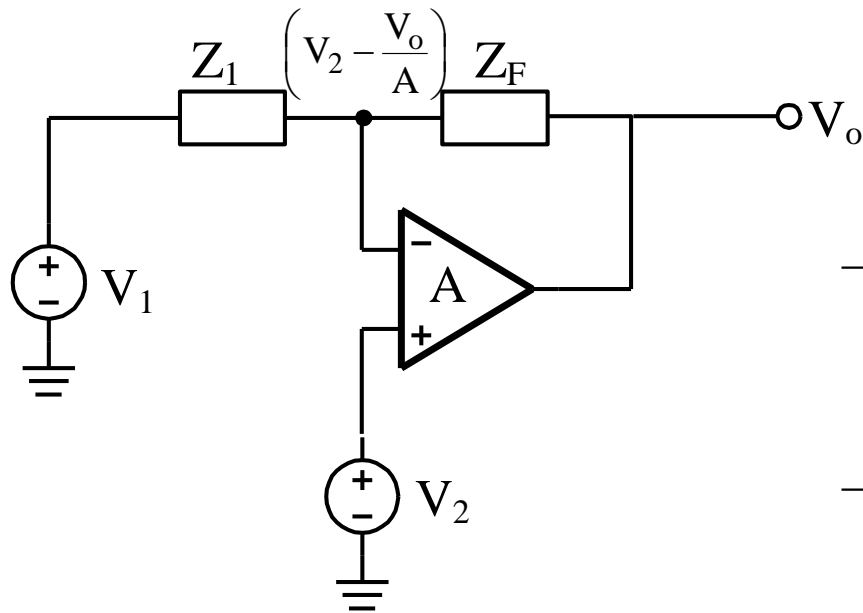
If A is not ideal we can write

$$V_o = A(V_2 - V_x)$$

or

$$V_x = V_2 - \frac{V_o}{A}$$

Thus we can solve the following equivalent circuit replacing  $V_x$ .



Using Nodal Equations

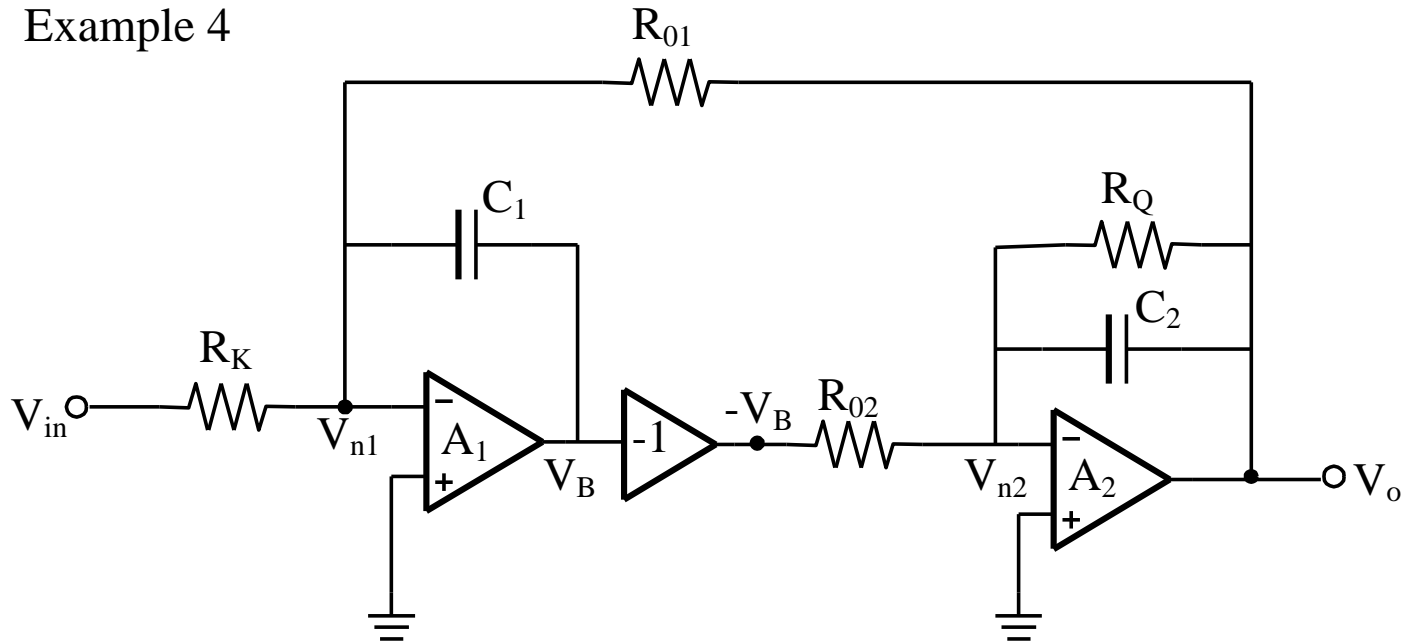
$$-\frac{V_o}{Z_F} + \left( V_2 - \frac{V_o}{A} \right) \left( \frac{1}{Z_1} + \frac{1}{Z_F} \right) - \frac{V_1}{Z_1} = 0$$

$$-\frac{V_o}{Z_F} - \frac{V_o}{A} \left( \frac{1}{Z_1} + \frac{1}{Z_F} \right) = \frac{V_1}{Z_1} - V_2 \left( \frac{1}{Z_1} + \frac{1}{Z_F} \right)$$

Thus (multiplying both sides by  $Z_F$ )

$$V_o = \frac{-\frac{Z_F}{Z_1} V_1}{1 + \frac{1}{A} \left( 1 + \frac{Z_F}{Z_1} \right)} + \frac{\left( 1 + \frac{Z_F}{Z_1} \right) V_2}{1 + \frac{1}{A} \left( 1 + \frac{Z_F}{Z_1} \right)}$$

### Example 4



$$V_{n1} \left( \frac{1}{R_K} + \frac{1}{R_{o1}} + sC_1 \right) - sC_1 V_B - \frac{V_o}{R_{o1}} = \frac{V_{in}}{R_K} \quad (1)$$

$$V_{n2} \left( \frac{1}{R_{o2}} + sC_2 + \frac{1}{R_Q} \right) + \frac{V_B}{R_{o2}} - V_o \left( sC_2 + \frac{1}{R_Q} \right) = 0 \quad (2)$$

Note that

$$V_{n_{1,2}} = \begin{cases} 0 & \text{for ideal } A_{1,2} \\ -\frac{V_{B,O}}{A} & \text{for finite } A_{1,2} \end{cases}$$

Let us assume the ideal case then eliminating  $V_B$  from (1) and (2) yields.

$$-V_B = \frac{V_{in}}{sR_K C_1} + \frac{V_o}{sR_{o1} C_1}$$

$$\frac{V_o}{V_{in}} = \frac{-\frac{1}{R_K R_{o2} C_1 C_2}}{s^2 + s \frac{1}{R_Q C_2} + \frac{1}{R_{o1} R_{o2} C_1 C_2}}$$

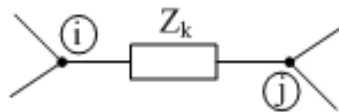
Reader. Solve for  $V_{n_1} = -\frac{V_B}{A_1}$  and  $V_{n_2} = -\frac{V_o}{A_2}$

# SUMMARY OF NODE METHOD

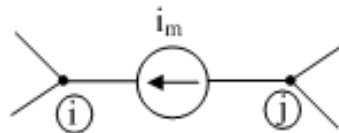
$$YV = I_s$$

Where  $Y$  is a  $n-1 \times n-1$  array,  $V$  is a vector, representing the unknown node voltages, of dimension  $n-1$ .  $I_s$  is also a vector of  $n-1$  dimension of the known current sources.

How to generate systematically the admittance matrix  $Y$ :



$$Y = \begin{matrix} & \begin{matrix} i & j \end{matrix} \\ \begin{matrix} i \\ j \end{matrix} & \begin{bmatrix} \dots & \frac{1}{Z_k} & \dots & -\frac{1}{Z_k} \\ \dots & -\frac{1}{Z_k} & \dots & \frac{1}{Z_k} \end{bmatrix} \end{matrix}$$

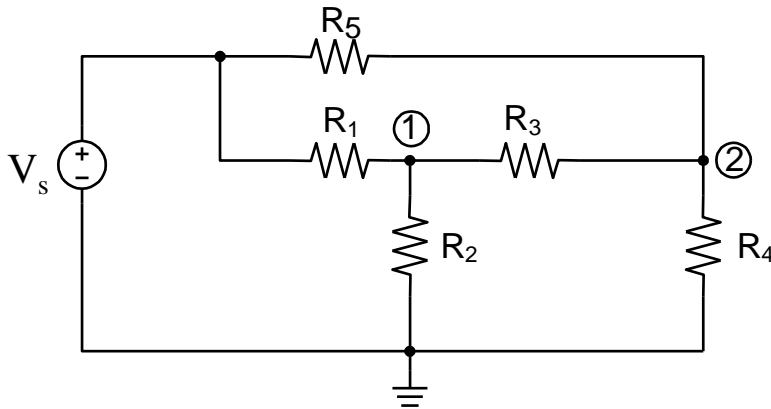


$$I_s = \begin{matrix} \begin{matrix} i \\ j \end{matrix} & \begin{bmatrix} i_m \\ \vdots \\ -i_m \end{bmatrix} \end{matrix}$$



For voltage sources we can consider two illustrative cases:

### GOUNDED VOLTAGE SOURCES



$$\begin{bmatrix} G_1 + G_3 + G_2 & -G_3 \\ -G_1 & G_3 + G_4 + G_5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{V_s}{R_1} \\ \frac{V_s}{R_5} \end{bmatrix}$$

$$\textcircled{1} \quad G_1 V_1 = -i_v$$

$$\textcircled{2} \quad (G_2 + G_3)V_2 - G_3 V_3 = i_v$$

$$\textcircled{3} \quad -G_3 V_2 + (G_3 + G_4)V_3 = 0$$

but

$$V_1 - V_2 = V_s$$

$$\text{or } V_1 = V_s + V_2$$

By adding equations  $\textcircled{1}$  and  $\textcircled{2}$  and substituting  $V_1$ , we can obtain



$$\begin{aligned} (G_1 + G_2 + G_3)V_2 - G_3 V_3 &= -G_1 V_s \\ -G_3 V_2 + (G_3 + G_4)V_3 &= 0 \end{aligned}$$

### FLOATING VOLTGE SOURCES

