L6: Short-time Fourier analysis and synthesis

- Overview
- Analysis: Fourier-transform view
- Analysis: filtering view
- Synthesis: filter bank summation (FBS) method
- Synthesis: overlap-add (OLA) method
- STFT magnitude

This lecture is based on chapter 7 of [Quatieri, 2002]

Overview

Recap from previous lectures

- Discrete time Fourier transform (DTFT)
  - Taking the expression of the Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, the DTFT can be derived by numerical integration

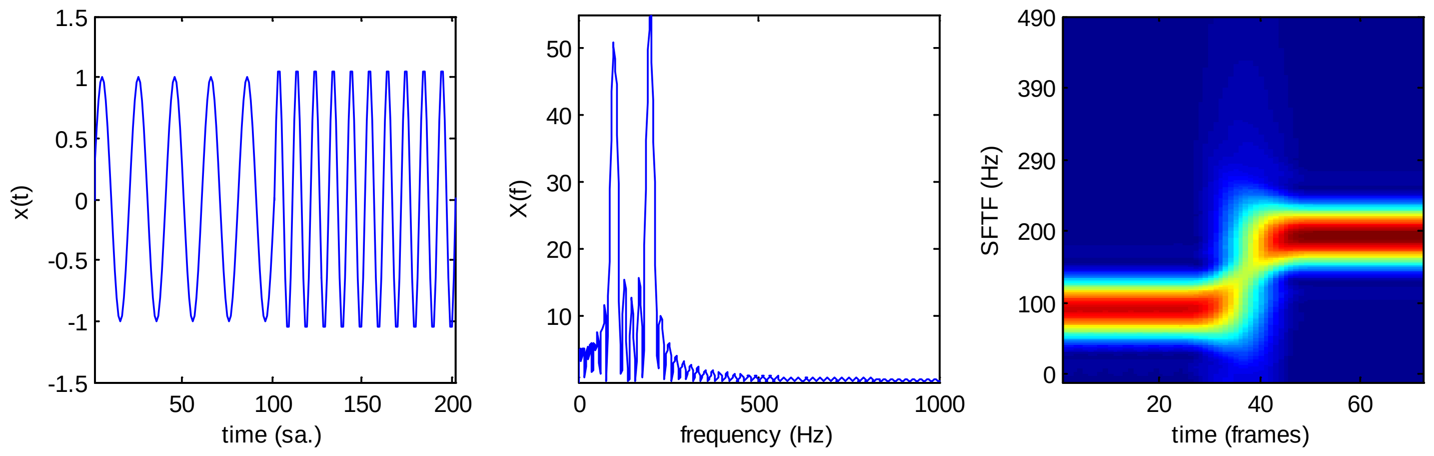
$$X\left(e^{j\hat{\omega}}\right) = \sum_{-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

    - where $x[n] = x(nT_S)$ and $\hat{\omega} = 2\pi F/F_S$
- Discrete Fourier transform (DFT)
  - The DFT is obtained by "sampling" the DTFT at N discrete frequencies $\omega_k = 2\pi F_s/N$, which yields the transform

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

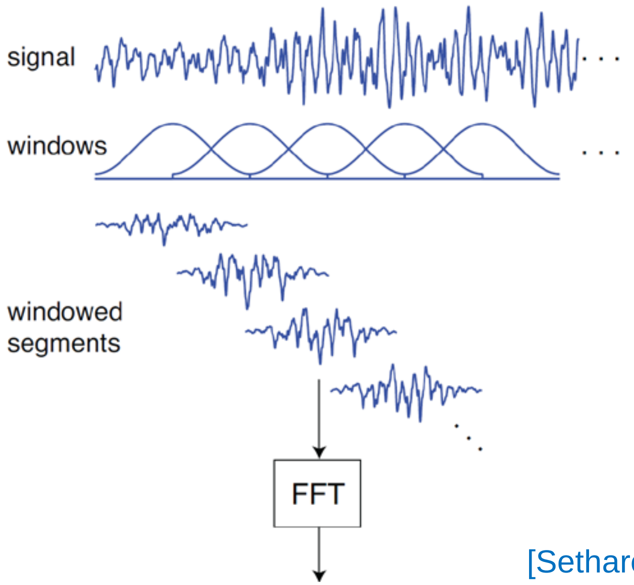
Why is another Fourier transform needed?

- The spectral content of speech changes over time (non stationary)
  - As an example, formants change as a function of the spoken phonemes
  - Applying the DFT over a long window does not reveal transitions in spectral content
- To avoid this issue, we apply the DFT over short periods of time
  - For short enough windows, speech can be considered to be stationary
  - Remember, though, that there is a time-frequency tradeoff here



The short-time Fourier transform in a nutshell

– Define analysis window (e.g., 30ms narrowband, 5 ms wideband)
– Define the amount of overlap between windows (e.g., 30%)
– Define a windowing function (e.g., Hann, Gaussian)
– Generate windowed segments (multiply signal by windowing function)
– Apply the FFT to each windowed segment



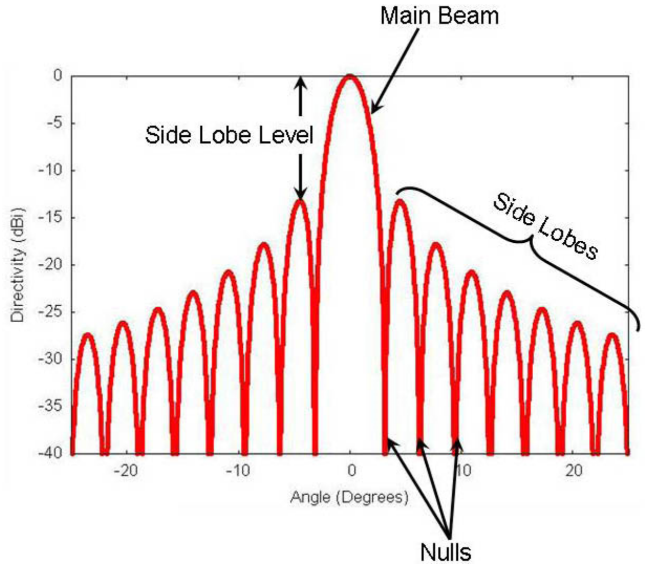
[Sethares, 2007]

Windowing function

- To "localize" the speech signal in time, we define a windowing function $w[n,\tau]$, which is generally tapered at its ends to avoid unnatural discontinuities in the speech segment
- Any window affects the spectral estimate computed on it
  - The window is selected to trade off the width of its main lobe and attenuation of its side lobes
- The most common are the Hann and Hamming windows (raised cosines)

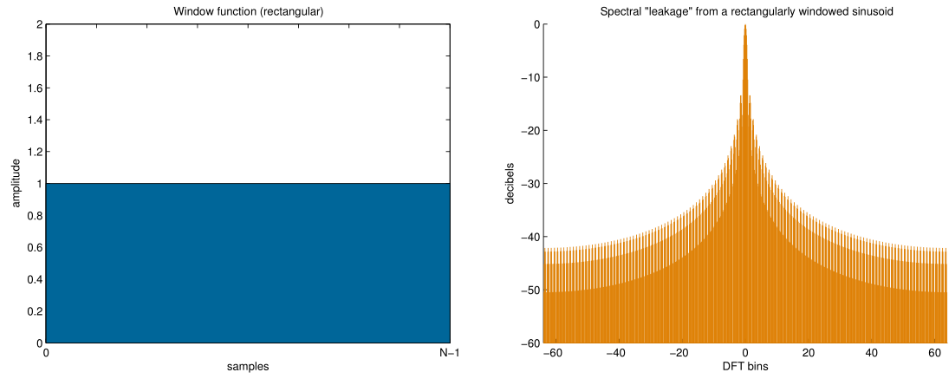
$$w[n,\tau] = 0.54 - 0.4\cos\left[\frac{2\pi(n-\tau)}{N_w - 1}\right]$$

$$w[n,\tau] = 0.5\left(1 - \cos\left(\frac{2\pi(n-\tau)}{N-1}\right)\right)$$

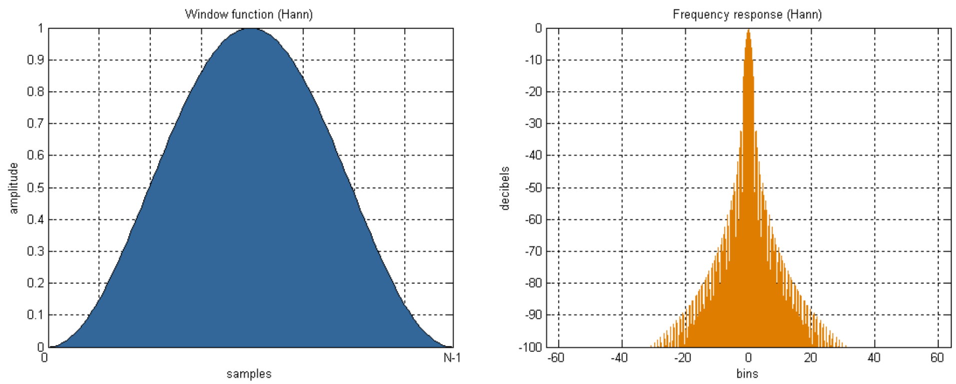


http://en.wikipedia.org/wiki/Window_function

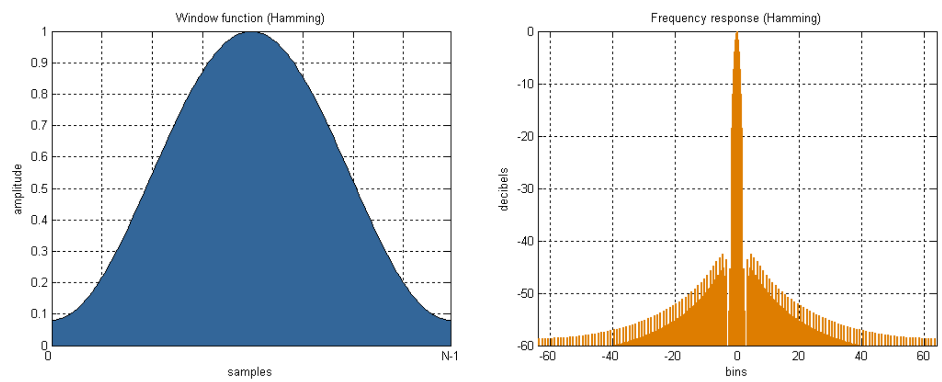
Rectangular



Hann



Hamming



http://en.wikipedia.org/wiki/Window_function

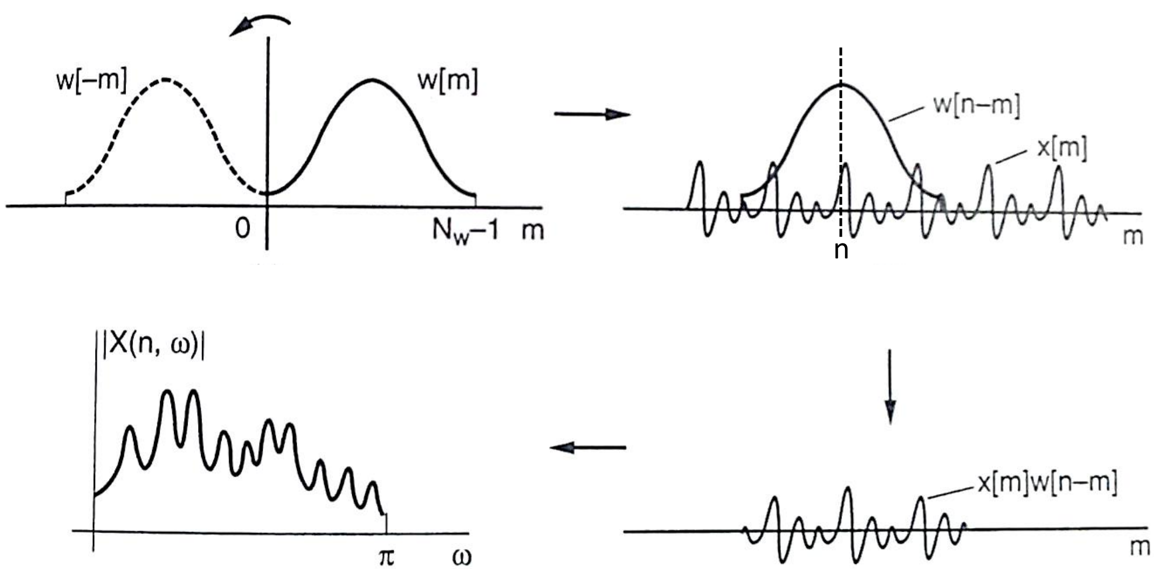
STFT: Fourier analysis view

Discrete-time Short-time Fourier transform

– The Fourier transform of the windowed speech waveform is defined as

$$X(n,\omega)=\sum_{m=-\infty}^{\infty}x[m]w[n-m]e^{-j\omega m}=\sum_{m=-\infty}^{\infty}f_n[m]e^{-j\omega m}$$

- where the sequence $f_n[m]=x[m]w[n-m]$ is a short-time section of $x[m]$ at time n, and $w[n]$ is non-zero only in the interval $[0,N_w-1]$



[Quatieri, 2002]

Discrete STFT

- By analogy with the DTFT/DFT, the discrete STFT is defined as

$$X(n,k) = X(n,\omega)\Big|_{\omega=\frac{2\pi}{N}k}$$

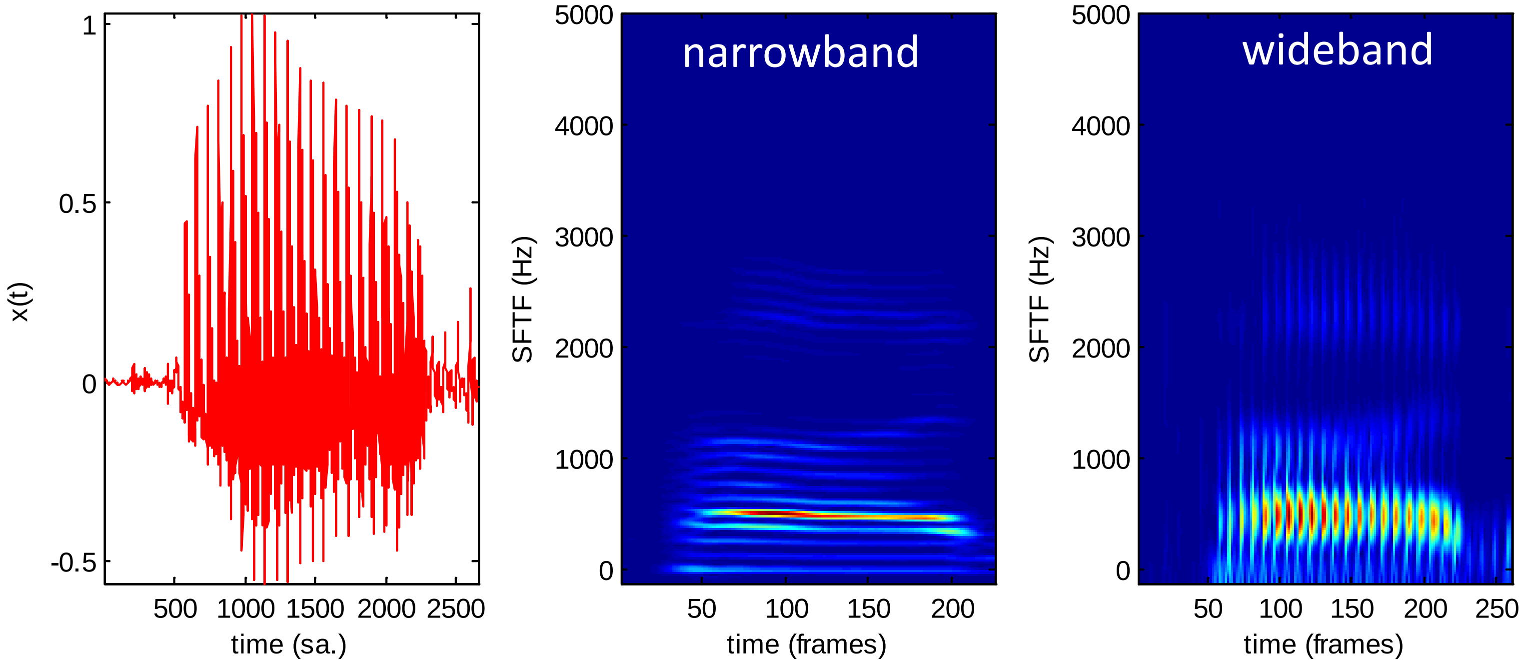
- The spectrogram we saw in previous lectures is a graphical display of the magnitude of the discrete STFT, generally in log scale

$$S(n,k) = \log|X(n,k)|^2$$

  - This can be thought of as a 2D plot of the relative energy content in frequency at different time locations

– For a long window $w[n]$, the result is the narrowband spectrogram, which exhibits the harmonic structure in the form of horizontal striations

– For a short window $w[n]$, the result is the wideband spectrogram, which exhibits periodic temporal structure in the form of vertical striations



STFT: filtering view

The STFT can also be interpreted as a filtering operation

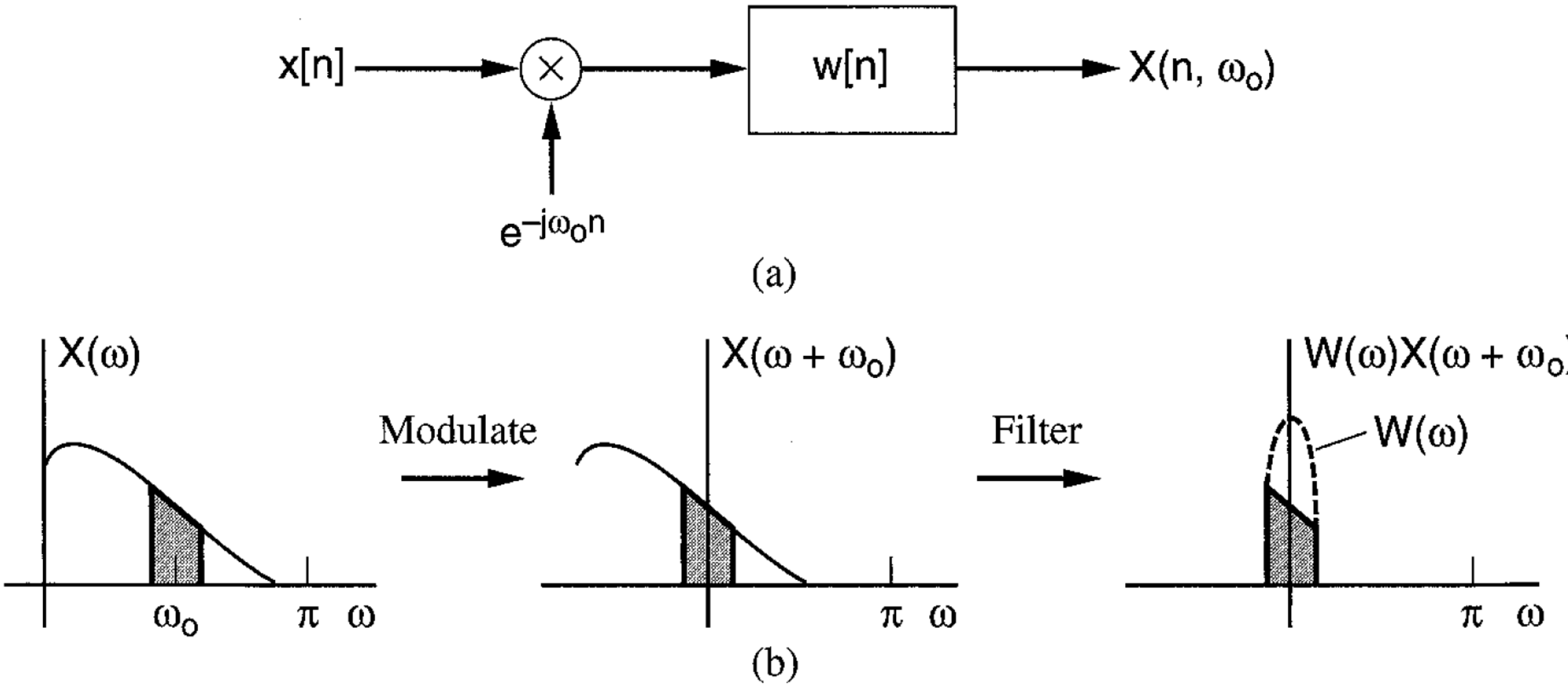
- In this case, the analysis window $w[n]$ plays the role of the filter impulse response
- To illustrate this view, we fix the value of ω at ω_0, and rewrite

$$X(n,\omega_0) = \sum_{m=-\infty}^{\infty} \left(x[m]e^{-j\omega_0 m}\right) w[n-m]$$

  - which can be interpreted as the convolution of the signal $\left(x[n]e^{-j\omega_0 n}\right)$ with the sequence $w[n]$:

$$X(n,\omega_0) = \left(x[n]e^{-j\omega_0 n}\right) * w[n]$$

  - and the product $x[n]e^{-j\omega_0 n}$ can be interpreted as the modulation of $x[n]$ up to frequency ω_0 (i.e., per the frequency shift property of the FT)

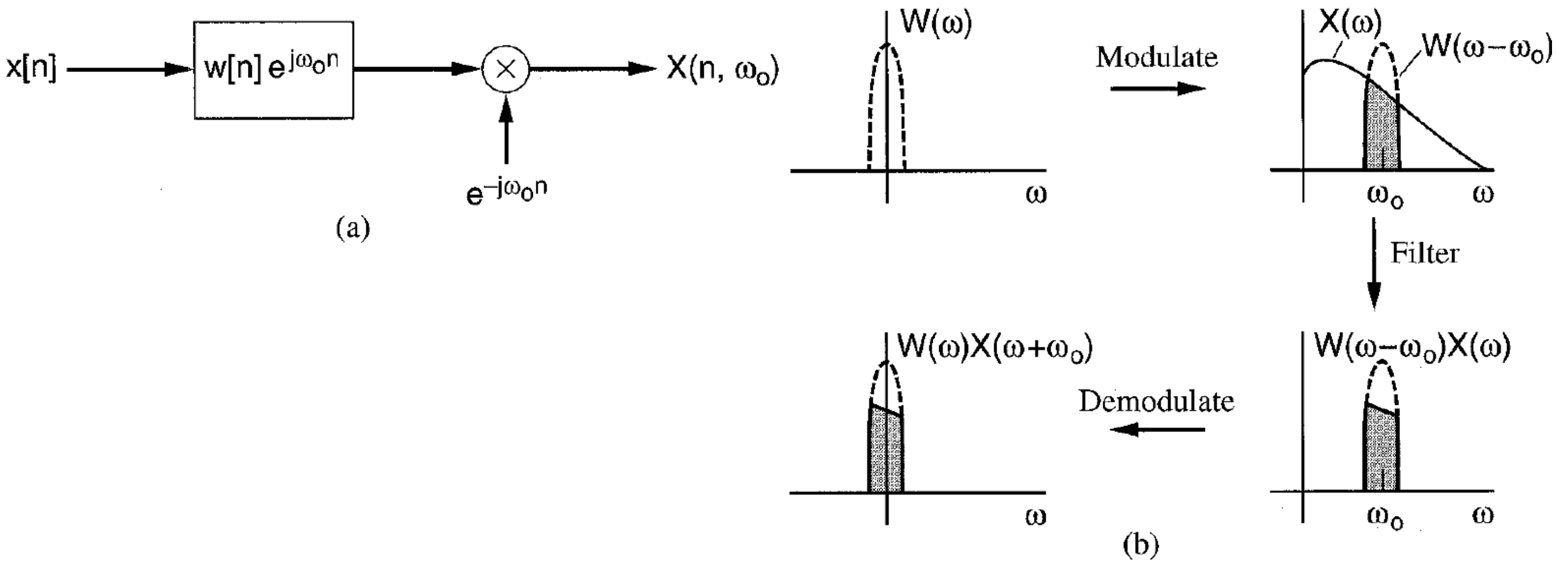


[Quatieri, 2002]

– Alternatively, we can rearrange as [Quatieri, 2002]

$$X(n,\omega_0) = e^{-j\omega_0 n}\left(x[n] * w[n]e^{j\omega_0 n}\right)$$

- In this case, the sequence $x[n]$ is first passed through the same filter (with a linear phase factor $e^{j\omega_0 n}$), and the filter output is demodulated by $e^{-j\omega_0 n}$



[Quatieri, 2002]

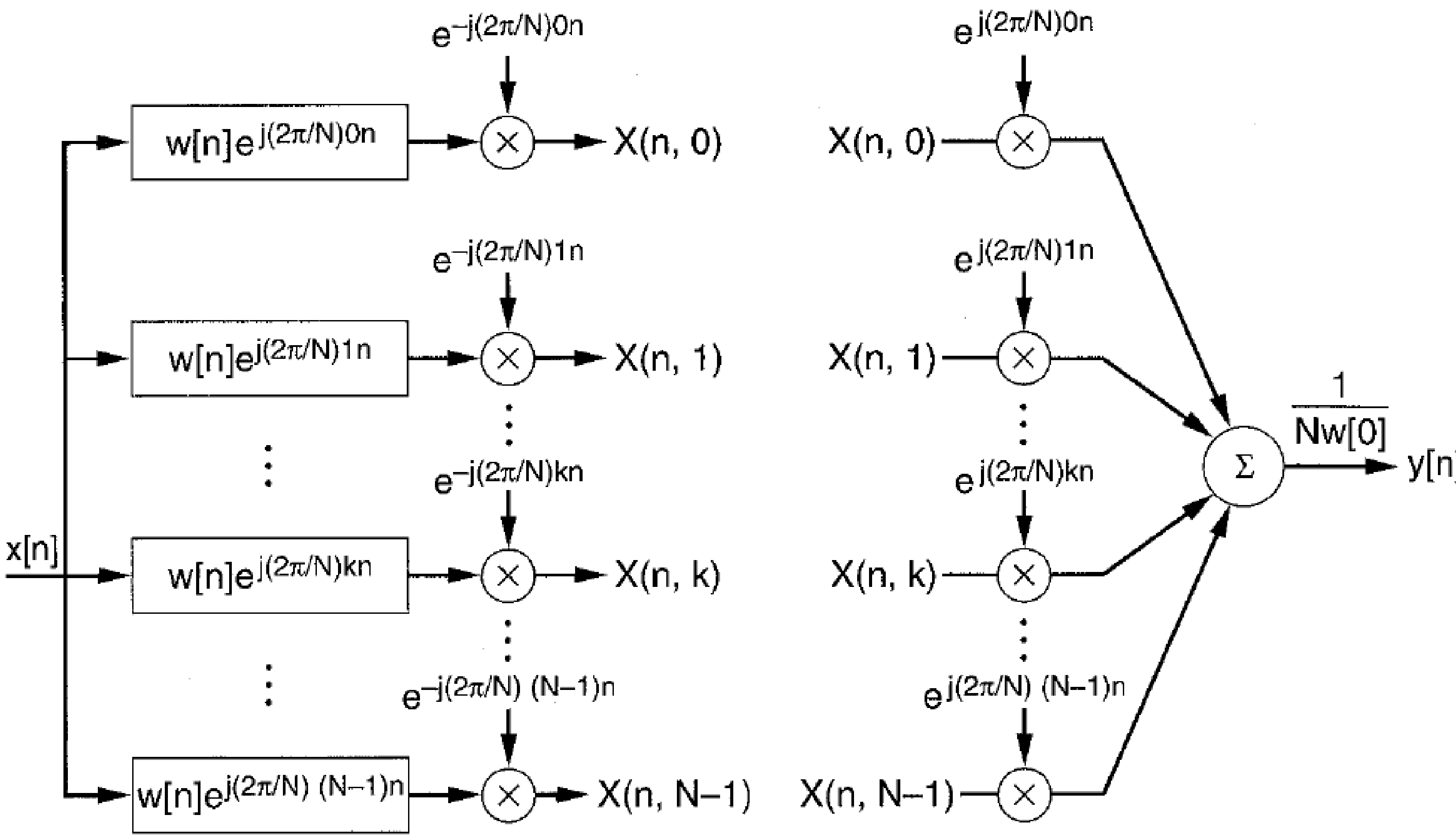
– This later rearrangement allows us to interpret the discrete STFT as the output of a filter bank

$$X(n,k) = e^{-j\frac{2\pi}{N}kn}\left(x[n] * w[n]e^{j\frac{2\pi}{N}kn}\right)$$

  - Note that each filter is acting as a bandpass filter centered around its selected frequency

– Thus, the discrete STFT can be viewed as a collection of sequences, each corresponding to the frequency components of $x[n]$ falling within a particular frequency band

  - This filtering view is shown in the next slide, both from the analysis side and from the synthesis (reconstruction) side



(a) analysis

(b) synthesis

[Quatieri, 2002]

Examples

ex6p1.m

Generate STFT using Matlab functions

ex6p2.m

Generate filterbank outputs using the filtering view of the STFT

ex6p3.m

Time-frequency resolution tradeoff (Quatieri fig 7.8)

Short-time synthesis

Under what conditions is the STFT invertible?

- The discrete-time STFT $X(n,\omega)$ is generally invertible
  - Recall that

$$X(n,\omega)=\sum_{m=-\infty}^{\infty} f_n[m]e^{-j\omega n}$$

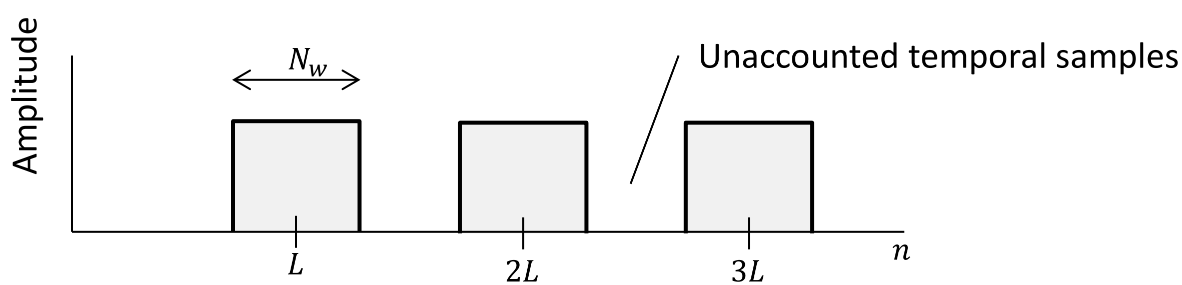
with $f_n[m]=x[m]w[n-m]$

  - Evaluating $f_n[m]$ at $m=n$ we obtain $f_n[n]=x[n]w[0]$
  - So assuming that $w[0]\neq 0$, we can estimate $x[n]$ as

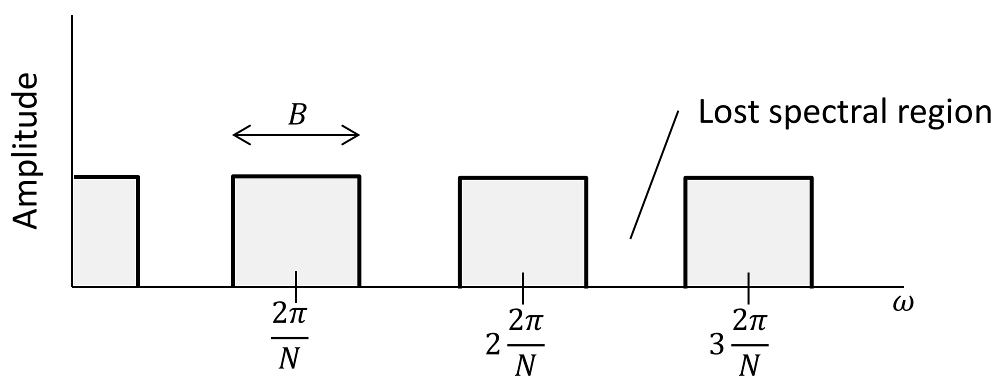
$$x[n]=\frac{1}{2\pi w[0]}\int_{-\pi}^{\pi} X(n,\omega)e^{j\omega n}d\omega$$

- This is known as a *synthesis equation* for the DT STFT

- Redundancy of the discrete-time STFT
  - There are many synthesis equations that map $X(n, \omega)$ uniquely to $x[n]$
  - Therefore, the STFT is very redundant if we move the analysis window one sample at a time $(n = 1,2,3 \dots)$
  - For this reason, the STFT is generally computed by decimating over time, that is, at integer multiples $(n = L, 2L, 3L \dots)$
- For large L, however, the DT STFT may become non-invertible
  - As an example, assume that $w[n]$ is nonzero over its length N_w
  - In this case, when $L > N_w$, there are some samples of $x[n]$ that are not included in the computation of $X(n, \omega)$
  - Thus, these samples can have arbitrary values yet yield the same $X(kL, \omega)$
  - Since $X(kL, \omega)$ is not uniquely defined, it is not invertible



– Likewise, the discrete STFT $x(n,k)$ is not always invertible
  - Consider the case where $w[n]$ is band-limited with bandwidth B
  - If the sampling interval $2\pi/N$ is greater than B, some of the frequency components in $x[n]$ do not pass through any of the filters of the STFT
  - Thus, those frequency components can have any arbitrary values yet produce the same discrete STFT
  - In consequence, depending on the frequency sampling resolution, the discrete STFT may become non invertible



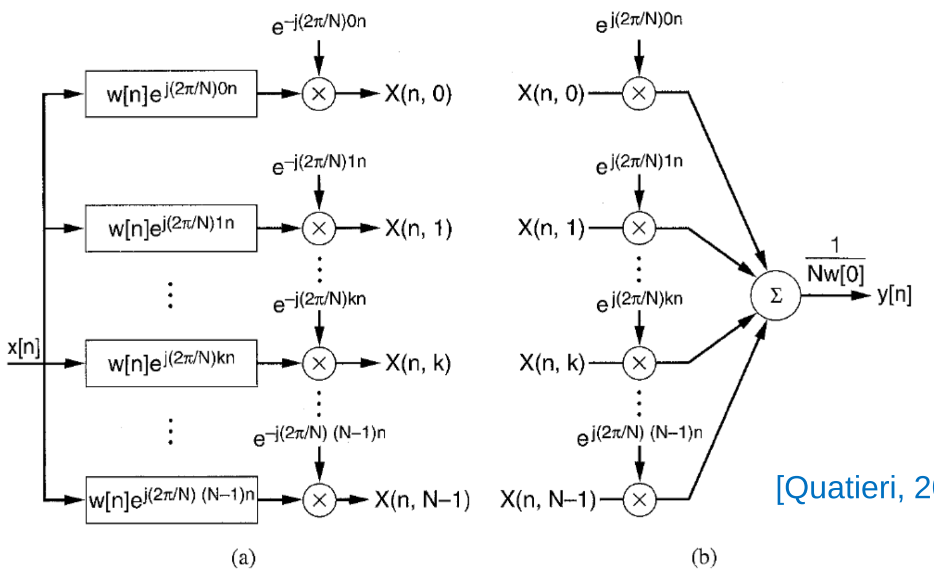
[Quatieri, 2002]

Synthesis: filter bank summation

FBS is based on the filtering interpretation of the STFT

- As we saw earlier, according to this interpretation the discrete STFT is considered to be the set of outputs from a bank of filters
- In the FSB method, the output of each filter is modulated with a complex exponential, and these outputs are summed to recover the original signal

$$y(n) = \frac{1}{Nw[0]} \sum_{m=-\infty}^{\infty} X(n,k) e^{j\frac{2\pi}{N}nk}$$



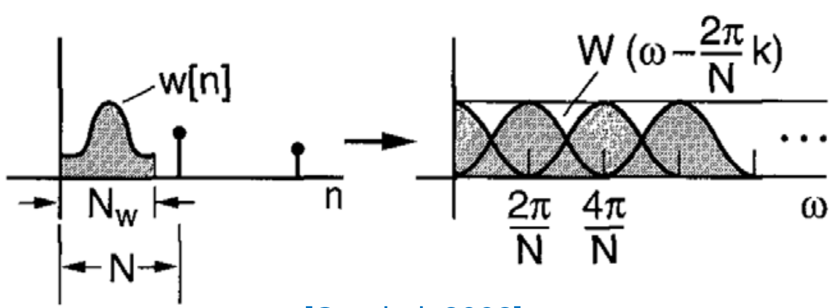
[Quatieri, 2002]

– Under which conditions does FBS yield exact synthesis?

- It can be shown that $y[n] = x[n]$ if either
  1. The length of $w[n]$ is less than or equal to the no. of filters ($N_w < N$), or
  2. For $N_w > N$:

$$\sum_{k=0}^{N-1} W\left(\omega - \frac{2\pi}{N}k\right) = Nw[0]$$

- The latter is known as the *BFS constraint*, and states that the frequency response of the analysis filters should sum to a constant across the entire bandwidth



[Quatieri, 2002]

Synthesis: Overlap-add

OLA is based on the Fourier transform view of the STFT

- In the OLA method, we take the inverse DFT for each fixed time in the discrete STFT
- In principle, we could then divide by the analysis window
  - This method is not used, however, as small perturbations in the STFT can become amplified in the estimated signal $y[n]$
- Instead, we perform an OLA operation between the sections
  - This works provided that $w[n]$ is designed such that the OLA effectively eliminates the analysis windows from the synthesized sequence
  - The intuition is that the redundancy within overlapping segments and the averaging of the redundant samples averages out the effect of windowing
- Thus, the OLA method can be expressed as

$$y[n] = \frac{1}{W(0)} \sum_{p=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} X(p,k) e^{j\frac{2\pi}{N}kn} \right]$$

  - where the term inside the square brackets is the IDFT

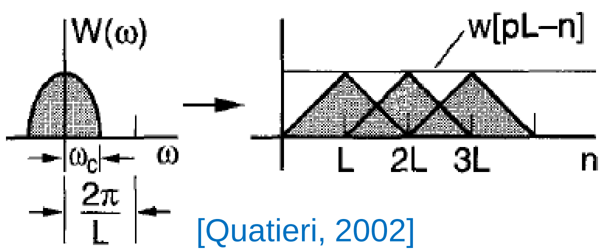
– Under which conditions does OLA yield exact synthesis?

- It can be shown that if the discrete STFT has been decimated by a factor L, the condition $y[n] = x[n]$ is met when

$$\sum_{p=-\infty}^{\infty} w[pL - n] = \frac{W(0)}{L}$$

- which holds when either
  1. The analysis window has finite bandwidth with maximum frequency ω_c less than $2\pi/L$, or
  2. The sum of all the analysis windows (obtained by sliding $w[n]$ with L-point increments) adds up to a constant
- In this case, $x[n]$ can then be resynthesized as

$$x[n] = \frac{L}{W(0)} \sum_{p=-\infty}^{\infty} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(pL, k) e^{j\frac{2\pi}{N}kn} \right]$$



[Quatieri, 2002]

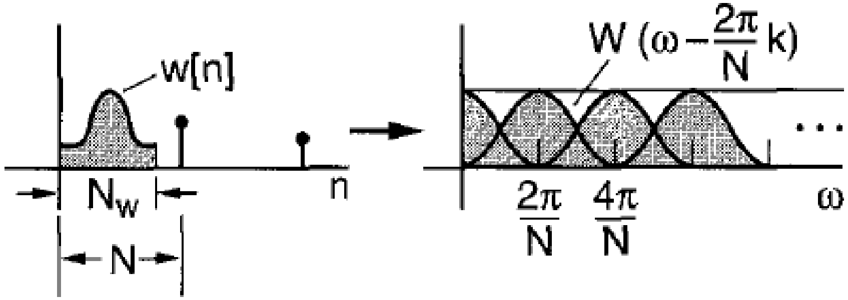
FBS Method

$$y[n] = \left[\frac{1}{Nw[0]}\right] \underbrace{\sum_{k=0}^{N-1} X(n,k)\, e^{j\frac{2\pi}{N}kn}}$$

Adding Frequency Components For Each n

FBS Constraint: $\displaystyle\sum_{k=0}^{N-1} W\left(\omega - \frac{2\pi}{N}k\right) = Nw[0]$

For $N_w < N \rightarrow y[n] = x[n]$



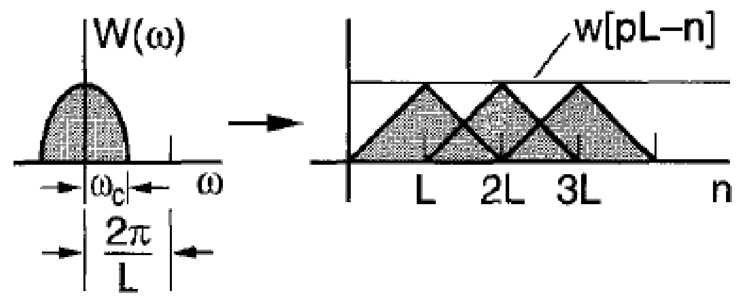
OLA Method

$$y[n] = \left[\frac{L}{W(0)}\right] \underbrace{\sum_{p=-\infty}^{\infty} x[n]\, w[pL-n]}$$

Adding Time Components For Each n

OLA Constraint: $\displaystyle\sum_{p=-\infty}^{\infty} w[pL-n] = \frac{W(0)}{L}$

For $\omega_c < \frac{2\pi}{L} \rightarrow y[n] = x[n]$



[Quatieri, 2002]

STFT magnitude

The spectrogram (STFT magnitude) is widely used in speech

- For one, evidence suggests that the human ear extracts information strictly from a spectrogram representation of the speech signal
- Likewise, trained researchers can visually “read” spectrograms, which further indicates that the spectrogram retains most of the information in the speech signal (at least at the phonetic level)
- Hence, one may question whether the original signal $x[n]$ can be recovered from $|X(n,\omega)|$, that is, by ignoring phase information

Inversion of the STFTM

- Several methods may be used to estimate $x[n]$ from the STFTM
- Here we focus on a fairly intuitive least-squares approximation

Least-squares estimation from the STFT magnitude

- In this approach, we seek to estimate a sequence $x_e[n]$ whose STFT magnitude $|X_e(n,\omega)|$ is "closest" (in a least-squared-error sense) to the known STFT magnitude $|X(n,\omega)|$
- The iteration takes place as follows
  - An arbitrary sequence (usually white noise) is selected as the first estimate $x_e^1[n]$
  - We then compute the STFT of $x_e^1[n]$ and modify it by replacing its magnitude by that of $|X(n,\omega)|$

$$X^1(m,\omega) = |X(m,\omega)| \frac{X_e^i(m,\omega)}{\left|X_e^i(m,\omega)\right|}$$

  - From this, we obtain a new signal estimate as

$$x_e^i[n] = \frac{\sum_{m=-\infty}^{\infty} w[m-n] g_m^{i-1}[n]}{\sum_{m=-\infty}^{\infty} w^2[m-n]}$$

where $g_m^{i-1}[n]$ is the inverse DFT of $X^{i-1}(m,\omega)$

  - And the process continues iteratively until convergence or a stopping criterion is met

- It can be shown that this process reduces the distance between $|X_e(n,\omega)|$ and $|X(n,\omega)|$ at each iteration
- Thus, the process converges to a local minimum, though not necessarily a global minimum

– All steps in the iteration can be summarized as (Quatieri, 2002; p. 342)

$$x_e^{i+1}[n] = \frac{\sum_{m=-\infty}^{\infty} w[m-n] \frac{1}{2\pi} \int_{-\pi}^{\pi} X^i(m,\omega) e^{j\omega n} d\omega}{\sum_{m=-\infty}^{\infty} w^2[m-n]}$$

where $X^i(m,\omega) = |X(m,\omega)| \frac{X_e^i(m,\omega)}{|X_e^i(m,\omega)|}$

Example

<u>ex6p4.m</u>

Estimate a signal from its STFT magnitude