L5: Digital filters

Linear time invariant systems

Impulse response

Transfer function

Digital filter analysis

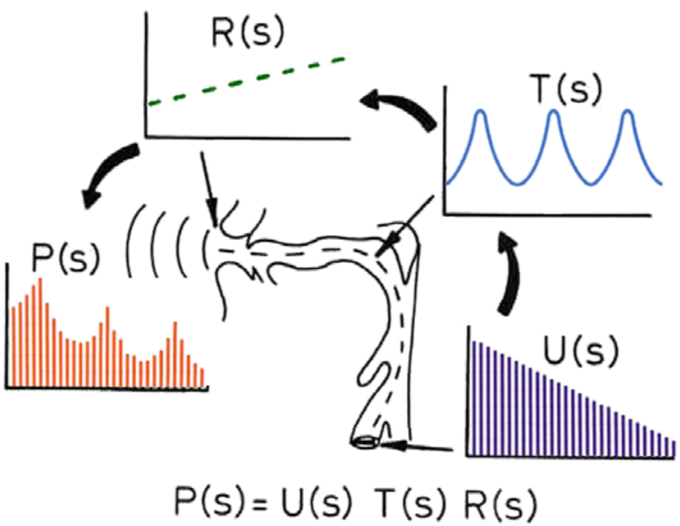
Example: speech synthesis

This lecture is based on chapter 10 of [Taylor, TTS synthesis, 2009]

Filters

A filter is a mathematical model of a system used for modifying signals

- In some applications, one is interested in “filtering out” unwanted portions of a signal
- Our interest in filters here comes from the acoustic theory of speech
  - According to the “source-filter” model, speech is a process by which a glottal source is modified by a vocal tract filter



Linear time invariant (LTI) filters

- A class of linear filters whose behavior does not change over time
  - Linearity implies that the filter meets the scaling and superposition properties

$$x[n] \mapsto y[n] \Longrightarrow \alpha x_1[n] + \beta x_2[n] \mapsto \alpha y_1[n] + \beta y_2[n]$$

- LTI filters are generally described in terms of difference equations

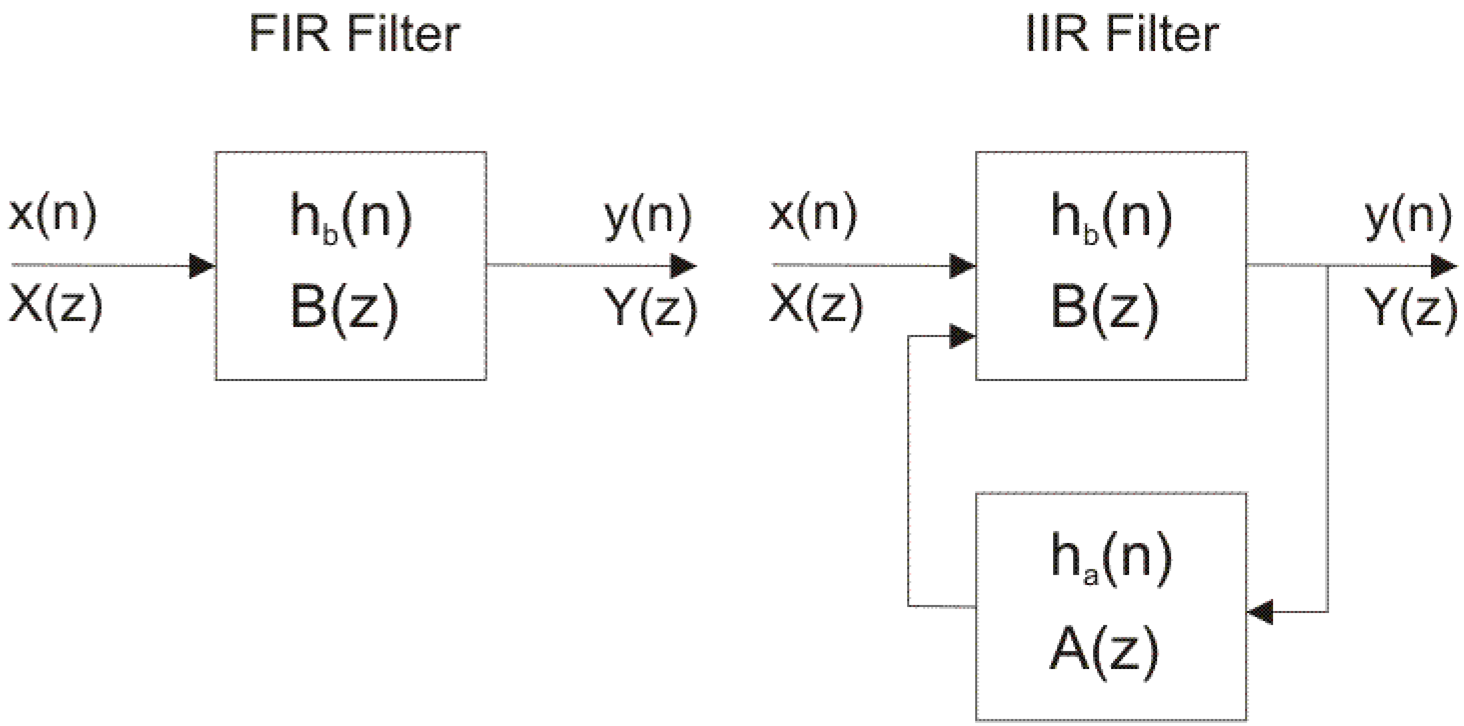
Types of LTI filters

- Finite impulse response (FIR)
  - Operate only on previous values of the input

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- Infinite impulse response (IIR)
  - Operate as well on previous values of the output

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] + \sum_{l=0}^{N} a_l y[n-l]$$



http://www.mikroe.com/eng/chapters/view/73/chapter-3-iir-filters/

The impulse response

- The properties of a filter in the time domain can be described by its response when the input is an impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- Consider the IIR filter defined by $y[n] = x[n] + 0.8y[n-1]$
  - Impulse response has no fixed duration (it is infinite, hence the name)
  - The response is an exponential decay controlled by $a_1 = 0.8$
    - For $a_1 > 1$, output grows exponentially, and the filter is said to be unstable
- Now consider the IIR filter $y[n] = -1.8y[n-1] + y[n-2]$
  - In this case, the response has the shape of a sine wave
- Finally, consider the IIR filter $y[n] = -1.78y[n-1] + 0.9y[n-2]$
  - In this case, the response has the shape of a decaying sine wave, a mix of the previous two signals
- Thus, the response characteristics are entirely defined by the parameters of the filter

Example

ex5p1.m

- Generate example of IIR and FIR filters
- Show how the impulse response is infinite for IIR but finite for FIR (examples from Taylor §10.4.1-2)

The filter convolution sum

- If we know the impulse response $h[n]$ of a filter, its response to any input sequence $x[n]$ can be computed as

$$y[n] = \sum_k x[k]h[n-k]$$

The filter transfer function

- The impulse response describes the filter properties in the time domain
- We will now see how to describe the filter in the frequency domain
- Consider the generic IIR filter

$$y[n] = b_0x[n] + b_1x[n-1] + \cdots + b_Mx[n-M] + a_1y[n-1] + a_2y[n-2] + a_Ny[n-N]$$

- And let's apply the Z transform

$$Y(z) = b_0X(z) + b_1X(z)z^{-1} + \cdots + b_MX(z)z^{-M} + a_1Y(z)z^{-1} + \cdots + a_NY(z)z^{-N}$$

– which, grouping terms, can be expressed as

$$Y(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{M-1}}{1 - a_1 z^{-1} - \cdots - a_N z^{N-1}} X(z)$$

– from which the transfer function of the filter can be defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{l=0}^{N} a_l z^{-l}}$$

NOTES

– As we will see in the next few slides, the transfer function $H(z)$ fully defines the filter's characteristics in the frequency domain

– It can be shown that the transfer function is the Z-transform of the impulse response $H(z) = \sum h[k] z^{-k}$

– The transfer function is a ratio of two polynomials whose coefficients are those of the difference equation

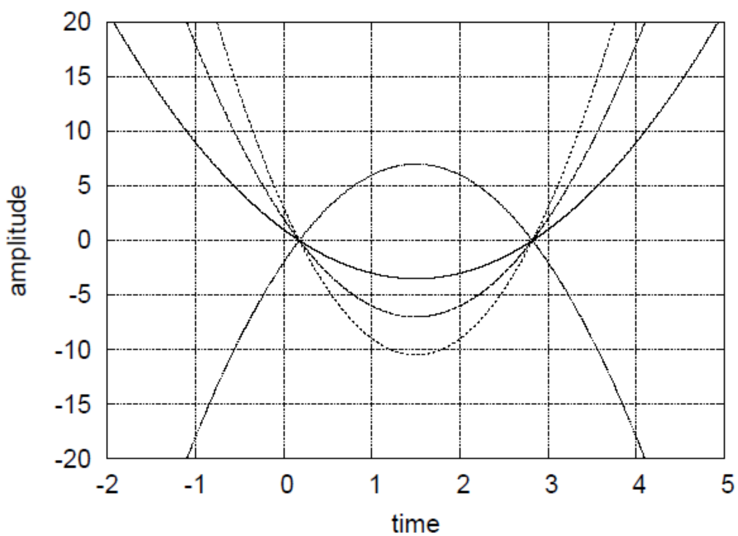
Filter analysis and design

Filter analysis

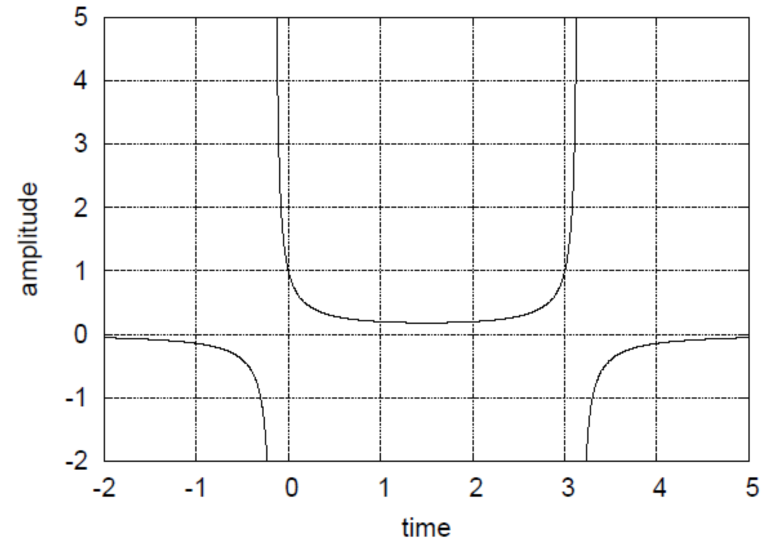
- The coefficients of first order filters are readily interpretable, for example as the rates of decay of exponentials
- For higher-order filters, interpretation of the coefficients is very hard
- Instead, we employ polynomial analysis to produce an easier interpretation of the transfer function

Polynomial analysis and design

- Consider the quadratic expression $f(x) = 2x^2 - 6x + 1$
  - This equation can be factorized as $f(x) = G(x - q_1)(x - q_2)$, where (q_1, q_2) are the roots of the expression and G is the gain
  - The roots (q_1, q_2) are called the zeros because $f(q_i) = 0$
- Now consider the inverse filter function $f(x) = \frac{1}{2x^2-6x+1}$
  - This curve is very different, and the function "blows up" at $x = \{q_1, q_2\}$
  - The roots (q_1, q_2) are called the poles … maybe because they create a pole-like effect on the curve?



(b) plot of $g \times (2x^2 - 6x + 1)$ for different values of g



(a) plot of $1/(2x^2 - 6x + 1)$

[Taylor, 2009]

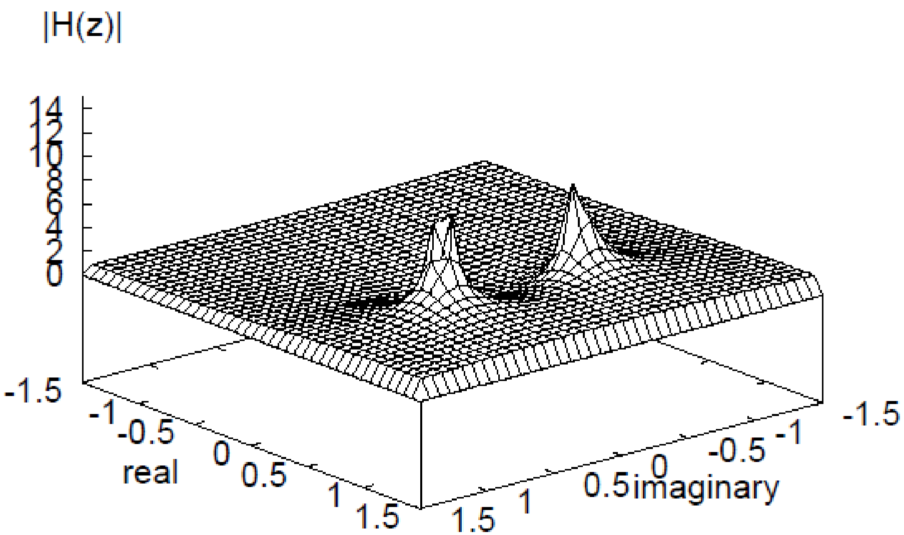
- We can now use polynomials to analyze our filter's transfer function
- Consider the transfer function

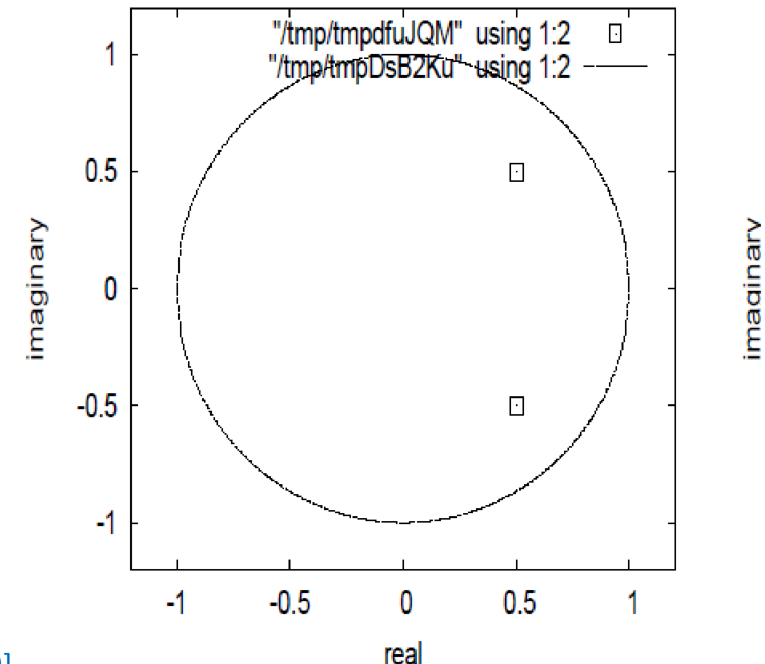
$$H(z) = \frac{1}{z^2 - a_1 z - a_2}$$

- Since transfer functions are generally expressed in terms of z^{-1}, we multiply numerator and denominator by z^{-2} to obtain

$$H(z) = \frac{z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = G \frac{z^{-2}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

- The figures in the next slide show the shape of the transfer function for $a_1 = 1, a_2 = -0.5$
  - In this case the roots of the denominator are complex $0.5 \pm j0.5$
  - Note how the shape of the filter can be described by the position of the poles in the Z plane; we do not need to plot $|H(z)|$


|H(z)|
real
imaginary
-1.5
-1
-0.5
0
0.5
1
1.5


"/tmp/tmpdfuJQM" using 1:2
"/tmp/tmpDsB2Ku" using 1:2
imaginary
real
1
0.5
0
-0.5
-1

[Taylor, 2009]

– The same analysis can be extended to any LTI filter

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^M}{1 - a_1 z^{-1} - \cdots - a_N z^N}$$

– By expressing it in terms of its factors

$$H(z) = \frac{(1 - q_1 z^{-1})(1 - q_2 z^{-1}) \ldots (1 - q_M z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1}) \ldots (1 - p_N z^{-1})}$$

– And then analyzing the position of its poles and zeros in the Z plane

Frequency interpretation of H(z)

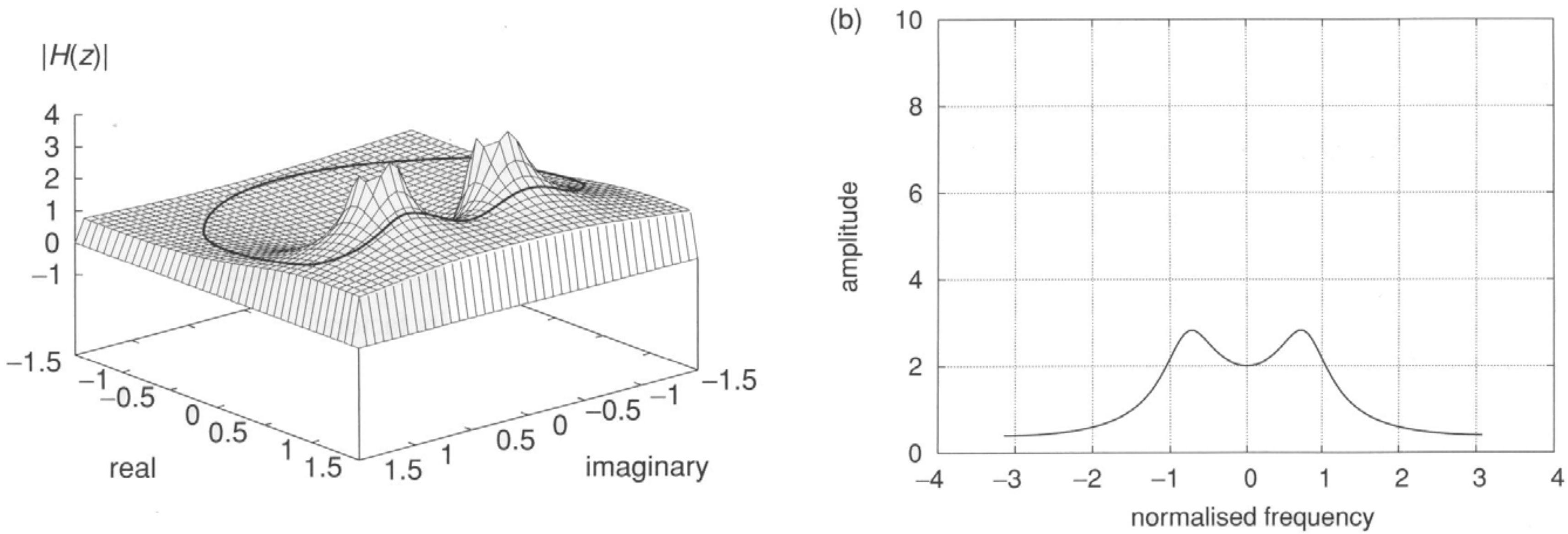
- Recall that the z transform for the digital signal $x[n]$is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- And that its Fourier transform is obtained by making $z = e^{j\hat{\omega}}$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

- Therefore, you can find the frequency response by substituting $\hat{\omega}$ with the frequency of interest
  - Since $e^{j\hat{\omega}}$ is unit length, this can be thought of as sweeping out a circle of radius 1 in the z-domain
  - This is consistent with the fact that the spectrum $X(e^{j\hat{\omega}})$ is periodic with period $\hat{\omega} = 2\pi$


|H(z)|
4
3
2
1
0
−1
−1.5
−1
−0.5
0
0.5
1
1.5
real
1.5
1
0.5
0
−0.5
−1
−1.5
imaginary
(b)
10
8
6
4
2
0
amplitude
−4
−3
−2
−1
0
1
2
3
4
normalised frequency

[Taylor, 2009]

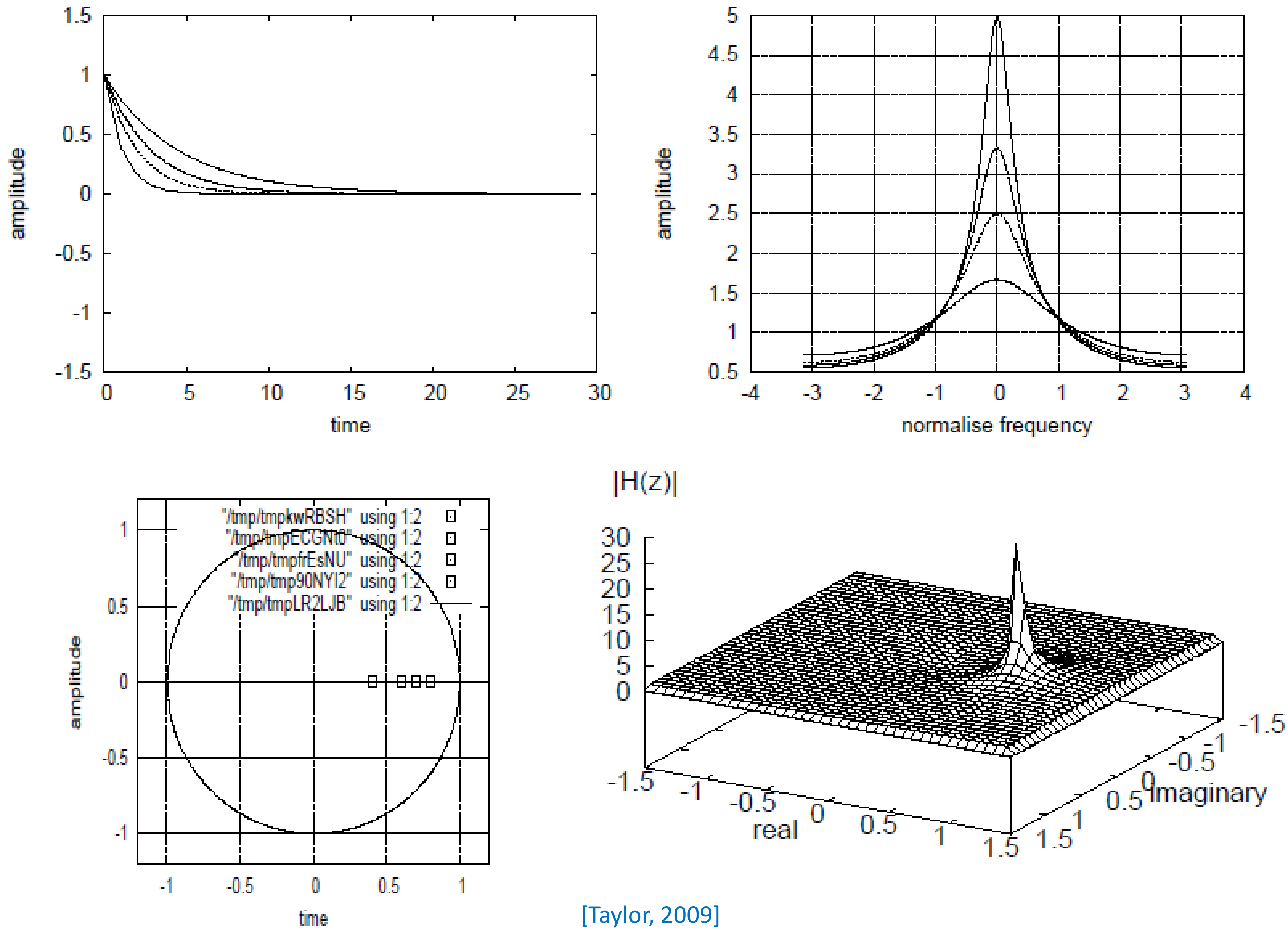
Filter characteristics

- Consider the following first-order IIR filter

$$h[n] = b_0 x[n] - a_1 y[n-1]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} = \frac{b_0}{1 - a_1 e^{-j\omega}}$$

- The figures in the next page show the time- and frequency domain response, pole locations and pole locations in the z-domain for $b_0 = 1$ and $a_1 = \{0.8, 0.7, 0.6, 0.4\}$
  - This type of filter is known as a resonator, and the peak is known as a resonance because frequencies near that peak are amplified by the filter
- Analysis
  - As the length of the decay increases, the peak becomes sharper
  - Large a_1 corresponds to slow decays and narrow bandwidths
  - Small a_1 corresponds to fast decays and broad bandwidths


amplitude
time
amplitude
normalise frequency
amplitude
time
"/tmp/tmpkwRBSH" using 1:2
"/tmp/tmpECGNt0" using 1:2
"/tmp/tmpfrEsNU" using 1:2
"/tmp/tmp90NYI2" using 1:2
"/tmp/tmpLR2LJB" using 1:2
|H(z)|
real
imaginary

[Taylor, 2009]

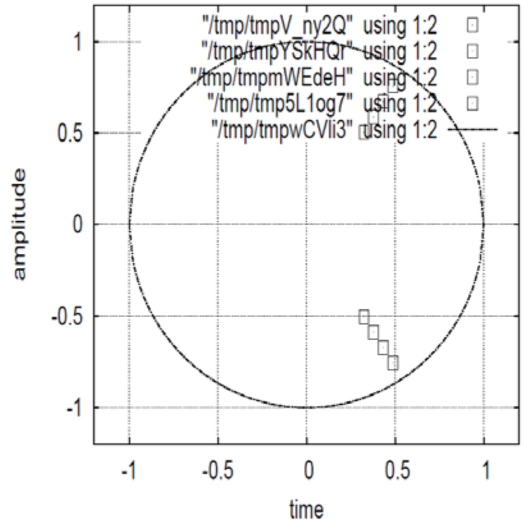
- Resonances are generally described by three properties: amplitude, frequency, and bandwidth
  - The radius of the pole controls the amplitude and bandwidth
  - The angle of the pole controls the frequency; in this case $\hat{\omega} = 0$ since the pole lies on the real line
- In order to model speech resonances at non-zero frequencies, we then move the pole away from the real axis
  - This will result in a complex pole $p_1 = re^{j\theta} = \alpha + j\beta$, which leads to a complex filter coefficient a_1; see next slide
  - For this reason, we introduce complex-conjugate pairs of poles $re^{\pm j\theta}$

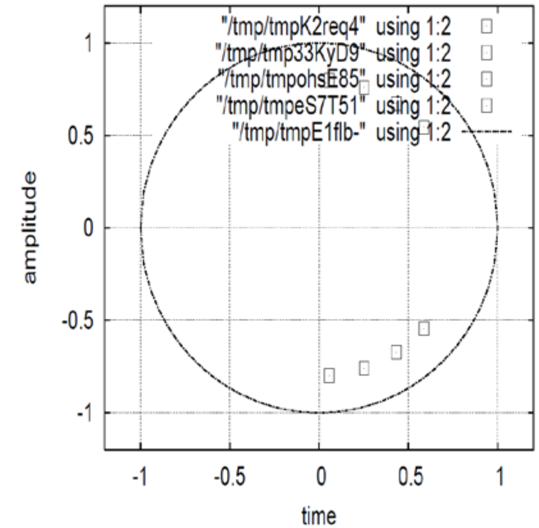
$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2rcos(\theta)z^{-1} + r^2z^{-1}}$$

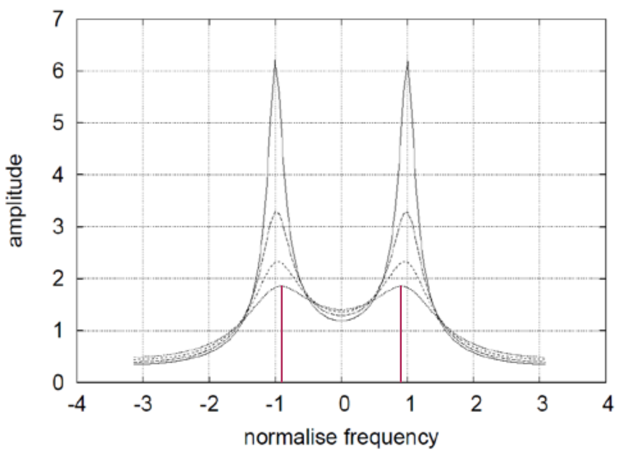
- Examples for various pole positions are shown in the next slide
  - For constant θ, the filter becomes sharper as $r \rightarrow 1$
    - For small r, the skirts of the two poles overlap and shift the resonance
  - For constant r, resonances move away from $\hat{\omega} = 0$ as $\theta \rightarrow 1$

r	θ	p_1	p_2	a_1	a_2
0.9	1.0	0.48 + 0.75j	0.48 - 0.75j	0.97	-0.81
0.8	1.0	0.43 + 0.67j	0.43 - 0.67j	0.86	-0.64
0.7	1.0	0.38 + 0.59j	0.38 - 0.59j	0.75	-0.48
0.6	1.0	0.32 + 0.51j	0.32 - 0.51j	0.65	-0.36

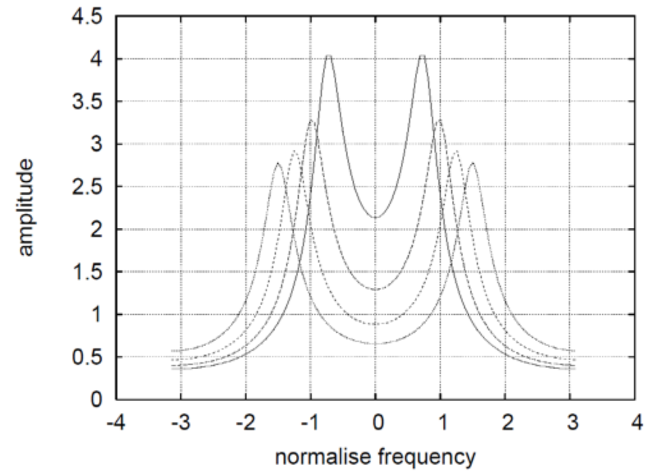
r	θ	p_1	p_2	a_1	a_2
0.8	0.75	0.58+0.54j	0.58-0.54j	1.17,	-0.64
0.8	1.0	0.43+0.67j	0.43-0.67j	0.86	-0.64
0.8	1.25	0.25+0.76j	0.25-0.76j	0.50	-0.64
0.8	1.5	0.056+0.78j	0.05-0.78j	0.11	-0.64







[Taylor, 2009]

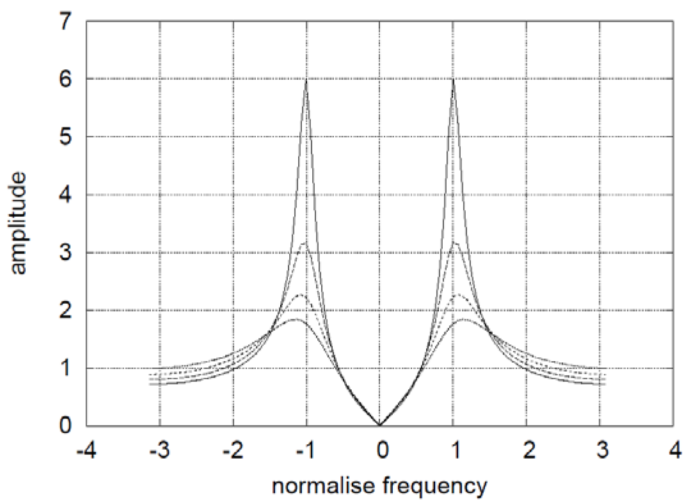


– Effect of zeros

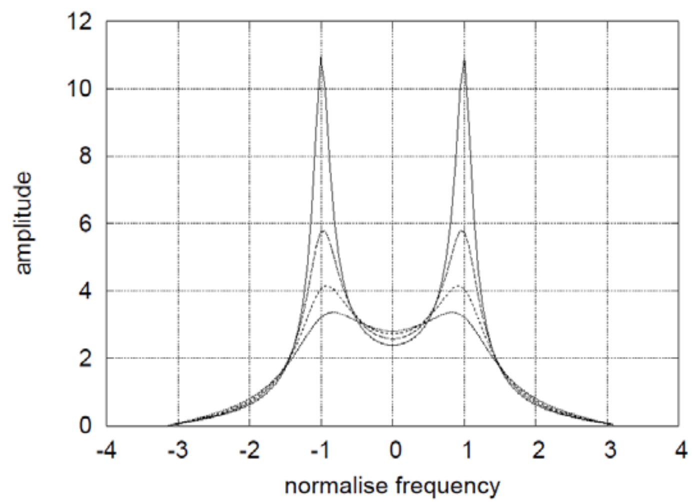
- Adding a term $b_1 = 1$ places a zero at the origin
- Adding a term $b_1 = -1$ places a zero at the ends of the spectrum

– Thus, zeros add anti-resonances to the spectrum

$b_1 = 1$



$b_1 = -1$



- Thus, we can build any transfer function by placing poles and zeros at the appropriate locations and then multiplying their transfer functions
- Note, though, that poles that are close together will interact, so the final resonances of a system cannot always be predicted from their poles

Example

Let's now use an LTI filter to synthesize English vowel [ih]

- Remember that normal frequency F (Hz) can be converted into normalized frequency $\widehat{\omega} = 2\pi F/F_S$
- From this expression we can calculate pole positions as

$$\theta = 2\pi F/F_S$$
$$r = e^{-\pi B/F_S}$$

- From acoustic phonetics, we can estimate formant values for [ih] to be $\{F_1, F_2, F_3\} = \{300Hz, 2200Hz, 3000Hz\}$
- Formant bandwidths are harder to measure, so we assume all three to be equal to $B = 250Hz$
- Assuming a sampling frequency of $F_S = 16kHz$, this results in

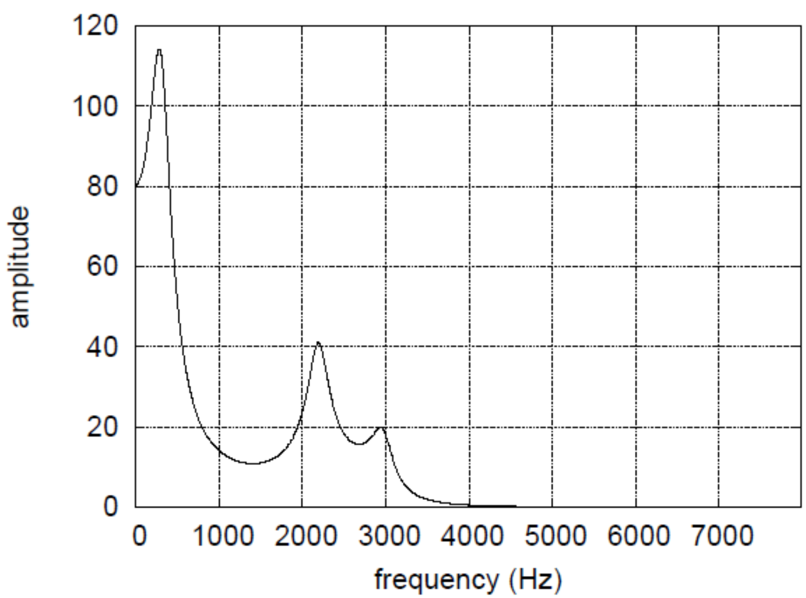
Formant	Frequency (Hz)	Bandwidth (Hz)	r	θ (normalised angular frequency)	pole
F1	300	250	0.95	0.12	0.963 + 0.116j
F2	2200	250	0.95	0.86	0.619 + 0.719j
F3	3000	250	0.95	1.17	0.370 + 0.874j

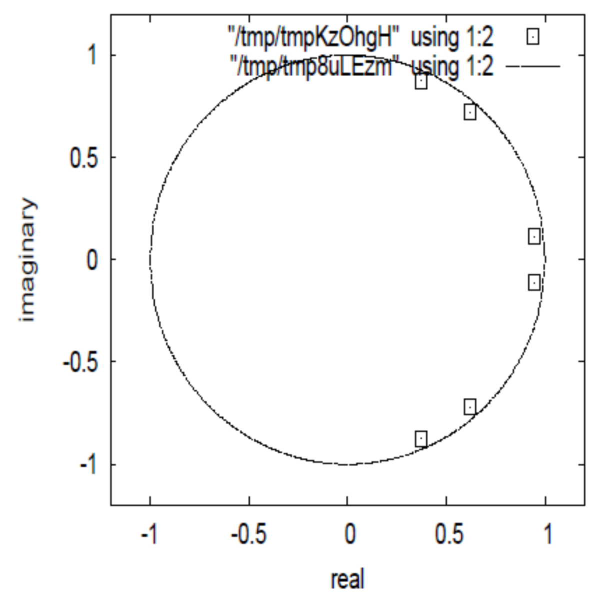
– The transfer function for each formant can be estimated as

$$H_n(z) = \frac{1}{(1 - p_n z^{-1})(1 - p_n{}^* z^{-1})}$$

– And the complete vocal tract TF can be estimated by multiplication

$$H(z) = H_1(z)H_2(z)H_3(z)$$





Example

<u>ex5p2.m</u>

Synthesize speech sample using the previous vocal tract filter and a pulse train as glottal excitation