Dynamic Programming

F[] an array that contains answers to fibo(n)

F[n] = fibo(n) // n^{th} fibonacci #

F[n] = null if fibo(n) not yet computed.

fibo(n):

if F[n] is not null, return F[n]

F[n] = fibo(n-1) + fibo(n-2)

return F[n]

The fibonacci sequence is a solution to the linear homogeneous recurrence relation of degree 2 with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

$c_1 = 1$, $c_2 = 1$, $a_0 = 0$, $a_1 = 1$

A linear homogenoous recurrence relation of degree k with constant coefficionts has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

c_i are real #s, $c_k \neq 0$

Basic approach: try $a_n = r^n$ where r is constant

$a_n = r^n$ is a solution iff

$$r^n = c_1 r^{n-1} + \cdots + c_k r^{n-k}$$

$$r^k = c_1 r^{k-1} + \cdots + c_k$$

$$\underbrace{r^k - c_1 r^{k-1} - \cdots - c_k = 0}_{\text{characteristic equation}}$$

characteristic roots

given roots, find explicit formula

$$a_n = 1a_{n-1} + 1a_{n-2}$$ $\leftarrow$ fibonacci

$$a_n = r^n$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$r^n = r^{n-1} + r^{n-2}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$ $\leftarrow$ characteristic eq.

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

Solve for α_1, α_2

use base cases/initial conditions

$a_0 = 0 \quad a_1 = 1$

$$a_0 = 0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 = \alpha_1 + \alpha_2 = 0$$

$$\boxed{\alpha_2 = -\alpha_1 = -\frac{1}{\sqrt{5}}}$$

$$a_1 = 1 = \alpha_1 r_1^1 + \alpha_2 r_2^1 = \alpha_1 r_1 + (-\alpha_1) r_2$$

$r_1 = \frac{1+\sqrt{5}}{2} \quad r_2 = \frac{1-\sqrt{5}}{2}$

$$= \alpha_1 (r_1 - r_2) = 1$$

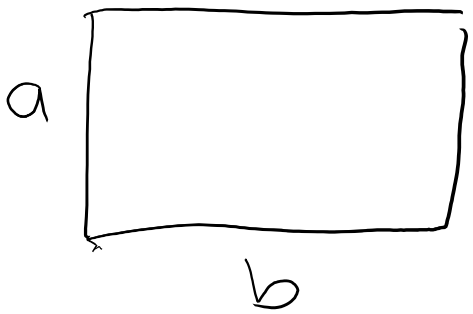
$$\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

$$\alpha_1 = \frac{1}{r_1 - r_2} = \boxed{\frac{1}{\sqrt{5}} = \alpha_1}$$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad -\frac{1}{\varphi} = \bar{\Phi} = \frac{1-\sqrt{5}}{2}$$



$$\frac{b}{a} = \frac{a+b}{b}$$ golden ratio

$$\varphi^2 = \varphi + 1$$

$$a_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \qquad \varphi = \frac{1+\sqrt{5}}{2} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

$O(\log n)$ to evaluate

Theorem 1 (LHRR of degree 2)

$\{a_n\}$ is a solution to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
iff $\underline{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$ for $n \geq 0$, where
α_1, α_2 are constants
and r_1, r_2 are the characteristic roots
(of $\underbrace{r^2 - c_1 r - c_2 = 0}_{\text{characteristic eq.}}$)

$a_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ is the solution to

$\varphi = \frac{1+\sqrt{5}}{2}$ P(n) $a_n = a_{n-1} + a_{n-2}$ $a_0 = 0$, $a_1 = 1$

$\psi = \frac{1-\sqrt{5}}{2}$

Proof by Induction

Basis step: P(0) and P(1)

$a_0 = 0$ $\frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$ ✓

$a_1 = 1$ $\frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1$ ✓

Inductive Step: $\left(\bigwedge_{j=0}^{k} P(j)\right) \to P(k+1)$

Assume $a_j = \dfrac{\varphi^j - \psi^j}{\sqrt{5}}$ for $j \leq k$, some $k \geq 1$

Show $a_{k+1} = \dfrac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$

$a_{k+1} = a_k + a_{k-1}$ by rec. def.

$= \dfrac{\varphi^k - \psi^k}{\surd}$

$$\frac{1}{\sqrt{5}}\left(\underbrace{\varphi^{k}+\varphi^{k-1}}_{\varphi^{k-1}(\underbrace{\varphi+1}_{\varphi^2})}-\left(\underbrace{\psi^{k}+\psi^{k-1}}_{\psi^{k-1}(\underbrace{\psi+1}_{\psi^2})}\right)\right)$$

remember

$r^2 = r + 1$

$r_1 = \varphi$

$r_2 = \psi$

$$a_{k+1} = \frac{1}{\sqrt{5}}\left(\varphi^{k+1} - \psi^{k+1}\right)$$

✓

$\square$

Next Time:
Divide and Conquer!