The Pigeonhole Principle

If $k+1$ objects (pigeons) are placed into $k>0$ boxes (holes), then at least 1 box has more than 1 object.

Proof: by contradiction

Assume $k+1$ objects placed into k boxes and no box has more than 1 object.

Then, at most there are k objects.

This contradicts assumption of $k+1$ objects. $\square$

Corollary: if $f: X \to Y$ and $|X| > |Y|$
then f is <u>not</u> one-to-one.

Example: Birthday Problem

16 people before 2 w/ same birthday

Example: Every integer n has a multiple which is solely composed of 0s and 1s in its decimal expansion.

<u>Proof</u>: Consider the $n+1$ integers

$$1, 11, 111, \ldots, \underbrace{111\cdots1}_{n+1 \text{ 1s}}$$

$$\begin{array}{r} L = q_L n + r \\ - \quad S = q_S n + r \\ \hline L - S = (q_L - q_S) n \end{array}$$

There are n different remainders possible (mod n).

At least 2 integers have same remainder (larger − smaller) is a multiple of n. □

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then at least one box has at least $\lceil N/k \rceil$ objects.

Example: Birth months. How many people are required so that at least 3 have the same birth month?

$$\left\lceil \frac{N}{12} \right\rceil = 3 \implies N = 25$$

Example: How many cards must be drawn from a standard 52-card, 4-suit deck to guarantee that at least 3 cards of the same suit are drawn?

$$\left\lceil \frac{N}{4} \right\rceil = 3 \qquad N = 9$$

Permutations

ordered arrangements of distinct objects.

Ex: How many ways can a deck of cards be ordered?

$$\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48} \cdots \underline{2}\ \underline{1} = 52!$$

Ex: How many permutations of 5 cards (selected from a deck) are there?

$$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$$

__r-permutation__: ordering of r elements from a set of n.

$$P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)) \qquad 0 \le r \le n$$

$$= \frac{n!}{(n-r)!}$$ r-permutations of N objects

$$\underbrace{\_\_\ \_\_\ \_\_\ \_\_\ \_\_}_{r}\ \underbrace{\cdots\cdots\ \_\_\ \_\_}_{n-r} \qquad \frac{n!}{(n-r)!}$$

Combinations

unordered selection of distinct objects.

r-combination: unordered selection of r objects from a set of n.

{A, B, C, D, E}

ABC ACD BCD CDE
ABD ACE BCE
ABE ADE BDE

Ex: How many different 5-card hands can be dealt from a 52-card deck? (Order of cards in hand does not matter)

$\underline{}\ \underline{}\ \underline{}\ \underline{}\ \underline{}$
$52\ 51\ 50\ 49\ 48 = \dfrac{52!}{47!\,5!}$

52! ways to order deck

5! ways to order the parts where order doesn't matter

$$C(n,r) = \frac{n!}{r!(n-r)!}, \quad 0 \le r \le n$$

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,r) = \binom{n}{r}$$ "n choose r" ${n \choose r}$

$$C(n,r) = \frac{P(n,r)}{r!}$$ $$P(n,r) = r!\,C(n,r)$$

$$C(n,r) = C(n,n-r)$$