Prove $7^n - 2^n$ is divisible by 5

Basis Step $n=0$

Show $P(0)$

$7^0 - 2^0 = 5(k) \quad k \in \mathbb{Z}$

$1 - 1 = 0 \quad 0 = 5(0)$ ✓

Inductive Step

Assume $P(k)$ for some $k \geq 0$

$7^k - 2^k$ is divisible by 5

Show $P(k+1)$

$7^{k+1} - 2^{k+1}$ is divisible by 5

$$= 7\cdot 7^k - 2\cdot 2^k$$

$$= (5+2)7^k - 2\cdot 2^k$$

$$= 5\cdot 7^k + 2\cdot 7^k - 2\cdot 2^k$$

$$= \underbrace{5\cdot 7^k}_{\text{divisible by 5}} + \underbrace{2(7^k - 2^k)}_{\text{divisible by 5 by IH}}$$

$$= 5\cdot 7^k + 2\cdot 5b \qquad b \in \mathbb{Z}$$

$$= 5(7^k + 2\cdot b)$$

is divisible by 5

$\therefore$ $7^n - 2^n$ is divisible by 5 $\forall n \geq 0$

Prove $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B$

$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$

Basis Step $n = 1$

$A_1 \cap B = A_1 \cap B$ ✓

Inductive Step

Assume $(A_1 \cup \cdots \cup A_k) \cap B$
$= (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$

Show $(A_1 \cup \cdots \cup A_{k+1}) \cap B$
$= \underline{(A_1 \cap B) \cup \cdots \cup (A_{k+1} \cap B)}$

$(A_1 \cup \cdots \cup A_{k+1}) \cap B$

$= \left(\underbrace{(A_1 \cup \cdots \cup A_k)} \cup \underbrace{A_{k+1}} \right) \cap B$

$= \left((A_1 \cup \cdots \cup A_k) \cap B \right) \cup (A_{k+1} \cap B)$

by distributivity law

$$= \underbrace{(A_1 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B)}_{\text{by associativity}}$$

$\therefore$ the generalized distributivity law holds for $n \geq 1$

$\square$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cup B \cup C) \cap D = (A \cap D) \cup \cdots (C \cap D)$$

Strong Induction

Let $P(n)$ be a predicate that takes any positive integer as an argument.

Suppose we know:

1) $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$

2) $(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)$

Then $P(n)$ holds for all $n \geq 1$

by Strong Induction

2^{nd} principle of Mathematical Induction

<u>Claim:</u> Let n be an integer greater than 1. Then n can be written as a product of primes.

Proof: by (strong) induction on n

Basis Step: $n=2$

$n=2$ is prime therefore can be written as a product of primes

Inductive Step:

Assume: $P(i)$ holds for all $i \leq k$, for some $k \geq 2$

Show: $P(k+1)$ holds

$k+1$ can be written as a product of primes

case $k+1$ is prime:

then $k+1$ is a product

of a single prime: $k+1$.

Case $k+1$ is composite:

then $k+1 = a \cdot b$

$2 \le a, b \le k$

By Ind. Hyp, a and b can be written as a product of primes.

then $k+1$ can be written as a product of a's prime factors and b's prime factors

$\therefore$ by strong induction n can be written as a product of primes, $n \geq 2$