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Mathematical Induction

Let $P(n)$ be a predicate that takes any positive integer as an argument.

Suppose $P(1)$ is true

$P(k) \rightarrow P(k+1)$ is true

Then, $P(2)$ is true b/c $P(1) \wedge P(1) \rightarrow P(2)$

Then $P(3)$ is true

b/c $P(2) \wedge P(2) \to P(3)$

...

Therefore

$P(n)$ is true

for all $n \geq 1$

By <u>the principle</u>

<u>of mathematical</u>

<u>induction.</u>

Example:

prove $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

<u>Basis Step:</u> $n = 0$

0 $\quad n(n+1)$

$$\sum_{i=0}^{0} i = 0 \qquad \frac{0(0+1)}{2} = 0$$

$$0 = 0 \checkmark$$

P(0) is true

$$P(n) := \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Inductive Step

Assume P(k) for some k

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \text{ for some } k \geq 0$$

Show P(k+1) holds

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$P(k) \rightarrow P(k+1)$$

$$\sum_{i=0}^{k+1} i = (k+1) + \sum_{i=0}^{k} i$$

$$= (k+1) + \frac{k(k+1)}{2}$$ by Ind. Hyp.

$$= (k+1)\left(1 + \frac{k}{2}\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ $P(n)$ is true for all $n \geq 0$

$\square$

Example 2

Prove $\sum_{i=0}^{n-1} 2i+1 = n^2$

Basis step: $n=1$: $P(1)$

$\sum_{i=0}^{1-1} 2i+1 = 2(0)+1 = 1$ $\quad 1^2 = 1$ ✓

Inductive step : $P(k) \rightarrow P(k+1)$

Assume $\underline{\sum_{i=0}^{k-1} 2i+1 = k^2}$ for some $k \geq 1$

Show $\sum_{i=0}^{k} 2i+1 = (k+1)^2$

$$\sum_{i=0}^{k} 2i+1 = 2(k)+1 + \underbrace{\sum_{i=0}^{k-1} 2i+1}_{= k^2 \text{ by IH}}$$

$$= 2k+1+k^2$$

$$= (k+1)^2$$

$$\therefore \sum_{i=0}^{n-1} 2i+1 = n^2 \quad \forall n \geq 1$$

$\square$

Prove $\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a-1}$

$$P(n) := \sum_{i=0}^{n-1} a^i = \frac{a^n-1}{a-1}$$

$$P(1) = \sum_{i=0}^{1-1} a^i = \frac{a^1-1}{\quad}$$

$$P(0) = \sum_{i=0} a^{i} = \frac{\;}{a-1}$$

$$a^0 = \frac{a-1}{a-1}$$

$$1 = 1 \checkmark$$

Assume $\displaystyle\sum_{i=0}^{k-1} a^i = \frac{a^k - 1}{a-1}$ for some $k \geq 1$

Prove $P(k+1)$

$$\sum_{i=0}^{k} a^i = \frac{a^{k+1} - 1}{a-1}$$

$$a^k + \sum_{i=0}^{k-1} a^i = \frac{a^{k+1} - 1}{a-1}$$

$$a^k + \frac{a^k - 1}{a-1} = \frac{a^{k+1} - 1}{a-1} \quad \text{by IH}$$

$$\frac{(a-1)a^k}{a-1} + \frac{a^k - 1}{a-1} = \frac{a^{k+1} - 1}{a-1}$$

$a^{k+1} \cancel{- a + a^{k}} - 1 = a^{k+1} - 1$

$\square$

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Prove $n < 2^n$ for $n > 0$

Base Step $n = 1$ $P(1)$

$1 < 2^1$ ✓

Inductive step $P(k) \to P(k+1)$

Assume $P(k) := k < 2^k$ some $k > 0$

Show $P(k+1) := k+1 < 2^{k+1}$

$2^{k+1} = 2^k \cdot 2 > k \cdot 2$ by IH

$2k > k+1$

$k > 1$

true when $k > 1$

$\cdot$ $P(2) := 2 < 2^2 = 4$ ✓ $\forall n > 1$

$\left(P(1) \wedge P(2) \wedge P(k) \to P(k+1) \right) \; k > 1$

$\therefore$ $P(n)$ is true for all $n > 0$

Base Case $n = 2$

$2 < 2^2$

$2 < 4$ ✓

Ind Hyp

Assume $P(\cancel{x}\, k)$

$k < 2^k$

Show $P(k+1)$

$k+1 < 2^{k+1}$

$2^{k+1} = 2 \cdot 2^k > 2 \cdot k$

by IH

$2^{k+1} > 2k > k+1 \quad k > 1$

$\therefore \forall n > 1 \, P(n)$