

Mathematical Induction

Let $P(n)$ be a predicate that takes any positive integer as an argument.

Suppose $P(1)$ is true

$P(k) \rightarrow P(k+1)$ is true

Then, $P(2)$ is true

b/c $P(1) \wedge P(1) \rightarrow P(2)$

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Then $P(3)$ is true

$$\text{b/c } P(2) \wedge P(2) \rightarrow P(3)$$

...

Therefore

$P(n)$ is true

for all $n \geq 1$

By the principle
of mathematical
induction.

Example: prove $\sum_{i=0}^n i = \frac{n(n+1)}{2}$

Basis Step: $n=0$
 $0 = \frac{0(0+1)}{2}$

$$\sum_{i=0}^0 i = 0 \quad \frac{0(0+1)}{2} = 0$$

$$0=0 \checkmark$$

$P(0)$ is true

$$P(n) := \sum_{i=0}^n i = \frac{n(n+1)}{2}$$

Inductive Step

Assume $P(k)$ for some k

$$\sum_{i=0}^k i = \frac{k(k+1)}{2} \quad \text{for some } k \geq 0$$

Show $P(k+1)$ holds

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$P(k) \rightarrow P(k+1)$

$$\sum_{i=0}^{k+1} i = (k+1) + \sum_{i=0}^k i$$

$$= (k+1) + \frac{k(k+1)}{2}$$

by
Ind.
Hyp.

$$= (k+1) \left(1 + \frac{k}{2}\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore P(n)$ is true for
all $n \geq 0$

□

Example 2

Prove $\sum_{i=0}^{n-1} 2i+1 = n^2$

Basis step: $n=1 \cdot P(1)$

$$\sum_{i=0}^{1-1} 2i+1 = 2(0)+1 = 1 \quad 1^2 = 1 \quad \checkmark$$

Inductive step: $P(k) \rightarrow P(k+1)$

Assume $\sum_{i=0}^{k-1} 2^{i+1} = k^2$ for some $k \geq 1$

Show $\sum_{i=0}^k 2^{i+1} = (k+1)^2$

$$\begin{aligned}\sum_{i=0}^k 2^{i+1} &= 2(k)+1 + \underbrace{\sum_{i=0}^{k-1} 2^{i+1}}_{= k^2 \text{ by IH}} \\ &= 2k+1 + k^2\end{aligned}$$

$$= (k+1)^2$$

$$\therefore \sum_{i=0}^{n-1} 2^{i+1} = n^2 \quad \forall n \geq 1$$

Prove $\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1}$ \square

$$P(n) := \sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1}$$

$$P(n) = \sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1}$$

$$1 + \sum_{i=0}^{a-1} a^i = a^a - 1$$

$$a^0 = \frac{a-1}{a-1}$$

$$1 = 1 \quad \checkmark$$

Assume $\sum_{i=0}^{k-1} a^i = \frac{a^k - 1}{a-1}$ for some $k > 1$

Prove $P(k+1)$

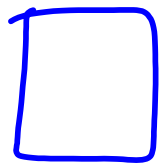
$$\sum_{i=0}^k a^i = \frac{a^{k+1} - 1}{a-1}$$

$$a^k + \sum_{i=0}^{k-1} a^i = \frac{a^{k+1} - 1}{a-1}$$

$$a^k + \frac{a^k - 1}{a-1} = \frac{a^{k+1} - 1}{a-1} \quad \text{by I.H.}$$

$$\frac{(a-1)a^k + a^k - 1}{a-1} = \frac{a^{k+1} - 1}{a-1}$$

$$a^{k+1} = \frac{a^{k+1}}{a^1} = a^k$$



Mathematically

Prove $n < 2^n$ for $n > 0$

Base Step $n=1$ $P(1)$

$$1 < 2 \quad \checkmark$$

Inductive step $P(k) \rightarrow P(k+1)$

Assume $P(k) := \underline{k} < 2^k$ some $k > 0$

Show $P(k+1) := k+1 < 2^{k+1}$

$$2^{k+1} = \underline{2^k} \cdot \underline{2} \quad (\supset) \quad (\underline{k}) \geq \underline{2} \quad \text{by IH}$$

$$\begin{matrix} 2k & (\supset) & k+1 \\ k & (\supset) & 1 \end{matrix}$$

true when $k > 1$

$P(2) := 2 < 2^2 = 4 \checkmark \quad \forall n > 1$
 $\underbrace{P(1)} \wedge \underbrace{P(2) \wedge \dots \wedge P(k) \rightarrow P(k+1)}_{\text{Inductive Step}} \rightarrow \dots$
 $\therefore P(n)$ is true for
 all $n > 1$

Base Case $n=2$

$$2 < 2^2$$

$$2 < 4 \checkmark$$

Ind Hyp

Assume $P(k)$
 $k < 2^k$

Show $P(k+1)$

$$k+1 < 2^{k+1}$$

$$2^{k+1} - 2 \cdot 2^k > 2 \cdot k$$

$$2^{k+1} > \underline{2k} > k+1$$

by IH
 $k > 1$

$$\therefore \forall n > 1 P(n)$$