CSCE 222
Discrete Structures for Computing

Sequences, Sums, and Products

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Sequences

- A **sequence** is a function from a subset of the integers to a set S.
  - A discrete structure used to represent an ordered list
  - $\{a_n\}$ denotes the sequence $a_0, a_1, a_2, \ldots$
    - Sometimes the sequence will start at a_1
  - a_n is a **term** of the sequence

- Example
  - $1, 2, 3, 5, 8$ is a sequence with five terms
  - $1, 3, 9, 27, \ldots, 3^n, \ldots$ is an infinite sequence
  - What is the function which generates the terms of the sequence $5, 7, 9, 11, \ldots$?
    - If $a_0 = 5$
      - $a_n = 5 + 2n$
    - If if $a_1 = 5$
      - $a_n = 3 + 2n$

Geometric Progression

- A **geometric progression** is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots$$

where the **initial term** a and the **common ratio** r are real numbers.

- Example
  - $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
  - What is the initial term a and common ratio r?
    - $a = 1$
    - $r = \frac{1}{2}$
    - $a_n = ar^n$
    - $a_n = \left(\frac{1}{2}\right)^n$

Arithmetic Progression

- An **arithmetic progression** is a sequence of the form

$$a, a + d, a + 2d, \ldots$$

where the **initial term** a and the **common difference** d are real numbers.

- Example
  - $-1, 3, 7, 11, \ldots$
  - What is the initial term a and common difference d?
    - $a = -1$
    - $d = 4$
    - $a_n = a + dn$
    - $a_n = -1 + 4n$

Exercises

List the first several terms of these sequences:

- the sequence $\{a_n\}$, where $a_n = (n+1)^{n+1}$.
  - 1, 4, 27, 256, 3125,…
- the sequence that begins with 2 and in which each successive term is 3 more than the preceding term.
  - 2, 5, 8, 11, 14, …
  - $a_n = 2 + 3n$
- the sequence that begins with 3, where each succeeding term is twice the preceding term.
  - 3, 6, 12 ,24, 48, …
  - $a_n = 3 \cdot 2^n$
- the sequence where the nth term is the number of letters in the English word for the index n.
  - 4, 3, 3, 5, 4, 4, 3, 5, 5, 4, 3, 6, …

Recurrence Relations

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses $a_n$in terms of one or more of the previous terms of the sequence.
  - Geometric progression: $a_n = ar^n$
    - Recurrence relation: $a_n = ra_{n-1},\ \ a_0 = a$ (a_0 is the **initial condition**)
  - Arthmetic progression: $a_n = a + dn$
    - Recurrence relation: $a_n = a_{n-1} + d,\ \ a_0 = a$
  - Fibonacci Sequence: $\{f_n\} = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$
    - $f_n = f_{n-1} + f_{n-2},\quad f_0 = 0,\ \ f_1 = 1$
  - Factorials: $1, 1, 2, 6, 24, 120, 720, 5040, \ldots$
    - $n! = n(n-1)(n-2)\cdots 2 \cdot 1$
    - $a_n = na_{n-1},\quad a_0 = 1$

Solving Recurrence Relations

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.
- To *solve* a recurrence relation, find a **closed formula** (explicit formula) for the terms of the sequence.
  - Closed formulae do not involve previous terms of the sequence.

- Example
  - Relations of the form $a_n = ra_{n-1}$
    - Geometric progression
    - Closed form: $a_n = ar^n$
  - Relations of the form $a_n = a_{n-1} + d$
    - Arithmetic progression
    - Closed form: $a_n = a + dn$

Technique for Solving Recurrence Relations

- **Iteration**
  - Forward: work forward from the initial term until a pattern emerges. Then guess the form of the solution.
  - Backward: work backward from $\mathrm{a_n}$ toward a_0 until a pattern emerges. The guess the form of the solution.
- Example: $a_n = a_{n-1} + 3, \quad a_0 = 2$
  - Forward
    - $a_1 = 2 + 3$
    - $a_2 = (2 + 3) + 3 = 2 + 2 \cdot 3$
    - $a_3 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$
    - …
    - $a_n = 2 + 3n$
  - Backward
    - $a_n = (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3$
    - $a_n = (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3$
    - $a_n = (a_{n-4} + 3) + 3 \cdot 3 = a_{n-4} + 4 \cdot 3$
    - …
    - $a_n = a_{n-n} + n \cdot 3 = a_0 + 3n = 2 + 3n$

Exercises

- Find the first few terms of a solution to the recurrence relation
$$a_n = a_{n-1} + 2n + 3, \qquad a_0 = 4$$
    - $a_1 = 4 + 2(1) + 3 = 9$
    - $a_2 = 9 + 2(2) + 3 = 16$
    - $a_3 = 16 + 2(3) + 3 = 25$
    - …
- Solve the recurrence relation above.
    - From the first few terms, the pattern seems to be $a_n = (n+2)^2$
    - This can be proved, but we need techniques we haven’t learned yet.

Exercises Continued

- Find and solve a recurrence relation for the sequence 0, 1, 3, 6, 10, 15,…
  - Take differences: 1 - 0 = **1**, 3 - 1 = **2**, 6 – 3 = **3**, 10 – 6 = **4**, 15 – 10 = **5**, …
  - Write down the relation: $a_n = a_{n-1} + n, \quad a_0 = 0$
  - Use backwards iteration
  - $a_n = a_{n-1} + n$
    - $= a_{n-2} + (n-1) + n$
    - $= a_{n-3} + (n-2) + (n-1) + n$
    - …
    - $= a_0 + (1 + 2 + \cdots + n)$
    - $= 0 + 1 + 2 + \cdots + n$
    - The sum of the integers from 1 to n is $\frac{n(n+1)}{2}$
  - $a_n = \frac{n(n+1)}{2}$

Summations

- The sum of the m through n-th terms of sequence $\{a_k\}$

$$a_m + a_{m+1} + \cdots + a_n$$

is denoted as

$$\sum_{i=m}^{n} a_i$$

where i is the **index of summation**, m is the **lower limit**, and n is the **upper limit**.

Examples

- Use summation notation to express the sum from 1 to 100 of $\frac{1}{x}$:

$$\sum_{i=1}^{100} \frac{1}{i}$$

- Evaluate $\sum_{i=0}^{3}(2i + 1)$

$$\sum_{i=0}^{3}(2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1)$$

$$= 1 + 3 + 5 + 7 = 16$$

Manipulations of Summation Notation

$$\sum_{i} ca_i = c\sum_{i} a_i$$

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{l} a_i + \sum_{i=l+1}^{n} a_i$$

$$\sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} (a_i + b_i)$$

$$\sum_{i=m}^{n} a_i = \sum_{j=0}^{n-m} a_{j+m} = \sum_{k=0}^{n-m} a_{n-k}$$

$$\sum_{i=m}^{n} a_i + c = \sum_{i=m}^{n} a_i + c(n - m + 1)$$

$$\sum_{i=m}^{n} a_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{m-1} a_i$$

Exercises

- Evaluate

$$\sum_{i=1}^{10} 3$$

  - 3+3+3+3+3+3+3+3+3+3 = 30
- Let $S = \{1,3,5,7\}$. Evaluate

$$\sum_{j \in S} j^2$$

  - 1+9+25+49 = 84
- Evaluate

$$\sum_{i=0}^{8} 3 \cdot 2^i$$

  - $3\sum_{i=0}^{8} 2^i = 3(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256) = 1533$

Summation Identities

- There are **MANY**, but here are two you should know:
  - Geometric Series

$$\sum_{i=0}^{n-1} r^i = \begin{cases} n & r = 1 \\ \dfrac{r^n - 1}{r - 1} & r \neq 1 \end{cases}$$

  - Arithmetic Series

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

Geometric Series Identity Proof $(r \neq 1)$

Let $S = \sum_{i=0}^{n-1} r^i$

$$rS = r \sum_{i=0}^{n-1} r^i$$

$$= \sum_{i=0}^{n-1} r^{i+1}$$

$$= \sum_{j=1}^{n} r^j$$

$$= \left(\sum_{j=0}^{n-1} r^j \right) + r^n - r^0$$

$$rS = S + r^n - 1$$

$$S(r-1) = r^n - 1$$

$$S = \frac{r^n - 1}{r - 1}$$

$\square$

Products

- The product of the m through n-th terms of sequence $\{a_k\}$

$$a_m \cdot a_{m+1} \cdot \cdots \cdot a_n$$

is denoted as

$$\prod_{i=m}^{n} a_i$$

Example:

$$n! = \prod_{i=1}^{n} i$$

A Product Identity

$$\prod_{i=m}^{n} ca_i = c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right)$$

$$\prod_{i=m}^{n} ca_i = (ca_m)(ca_{m+1})\cdots(ca_n)$$

$$= (c)(c)\cdots(c)\cdot(a_m)(a_{m+1})\cdots(a_n)$$

$$= \left(c^{\sum_{i=m}^{n} 1}\right)\left(\prod_{i=m}^{n} a_i\right)$$

$$= c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right)$$