CSCE 222
Discrete Structures for Computing

Sequences, Sums, and Products

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Sequences

• A **sequence** is a function from a subset of the integers to a set $S$.
  • A discrete structure used to represent an ordered list
  • $\{a_n\}$ denotes the sequence $a_0$, $a_1$, $a_2$, ...
    • Sometimes the sequence will start at $a_1$
  • $a_n$ is a **term** of the sequence

• Example
  • $1, 2, 3, 5, 8$ is a sequence with five terms
  • $1, 3, 9, 27, ..., 3^n, ...$ is an infinite sequence
  • What is the function which generates the terms of the sequence $5, 7, 9, 11, ...$?
    • If $a_0 = 5$
      • $a_n = 5 + 2n$
    • If if $a_1 = 5$
      • $a_n = 3 + 2n$
Geometric Progression

• A **geometric progression** is a sequence of the form
  \[a, ar, ar^2, ar^3, \ldots\]
  where the **initial term** \(a\) and the **common ratio** \(r\) are real numbers.

• Example
  • \(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\)
  • What is the initial term \(a\) and common ratio \(r\)?
    • \(a = 1\)
    • \(r = \frac{1}{2}\)
    • \(a_n = ar^n\)
    • \(a_n = \left(\frac{1}{2}\right)^n\)
Arithmetic Progression

• An arithmetic progression is a sequence of the form
  \[ a, a + d, a + 2d, \ldots \]
  where the initial term \( a \) and the common difference \( d \) are real numbers.

• Example
  • \(-1, 3, 7, 11, \ldots\)
  • What is the initial term \( a \) and common difference \( d \)?
    • \( a = -1 \)
    • \( d = 4 \)
    • \( a_n = a + dn \)
    • \( a_n = -1 + 4n \)
Exercises

List the first several terms of these sequences:

• the sequence \( \{a_n\} \), where \( a_n = (n + 1)^{n+1} \).
  • 1, 4, 27, 256, 3125, ...

• the sequence that begins with 2 and in which each successive term is 3 more than the preceding term.
  • 2, 5, 8, 11, 14, ...
  • \( a_n = 2 + 3n \)

• the sequence that begins with 3, where each succeeding term is twice the preceding term.
  • 3, 6, 12 ,24, 48, ...
  • \( a_n = 3 \cdot 2^n \)

• the sequence where the \( n \)th term is the number of letters in the English word for the index \( n \).
  • 4, 3, 3, 5, 4, 4, 3, 5, 4, 3, 6, ...
Recurrence Relations

• A **recurrence relation** for the sequence \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more of the previous terms of the sequence.
  
  • Geometric progression: \( a_n = ar^n \)
    
    • Recurrence relation: \( a_n = ra_{n-1}, \ a_0 = a \) \quad (a_0 \text{ is the initial condition})
  
  • Arithmetic progression: \( a_n = a + dn \)
    
    • Recurrence relation: \( a_n = a_{n-1} + d, \ a_0 = a \)
  
  • Fibonacci Sequence: \( \{f_n\} = 0, 1, 1, 2, 3, 5, 8, 13, ... \)
    
    • \( f_n = f_{n-1} + f_{n-2}, \ f_0 = 0, \ f_1 = 1 \)
  
  • Factorials: 1, 1, 2, 6, 24, 120, 720, 5040, ...
    
    • \( n! = n(n-1)(n-2)\cdots 2 \cdot 1 \)
    
    • \( a_n = na_{n-1}, \ a_0 = 1 \)
Solving Recurrence Relations

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.
- To *solve* a recurrence relation, find a **closed formula** (explicit formula) for the terms of the sequence.
  - Closed formulae do not involve previous terms of the sequence.

- Example
  - Relations of the form $a_n = ra_{n-1}$
    - Geometric progression
    - Closed form: $a_n = ar^n$
  - Relations of the form $a_n = a_{n-1} + d$
    - Arithmetic progression
    - Closed form: $a_n = a + dn$
Technique for Solving Recurrence Relations

• **Iteration**
  - Forward: work forward from the initial term until a pattern emerges. Then guess the form of the solution.
  - Backward: work backward from \( a_n \) toward \( a_0 \) until a pattern emerges. The guess the form of the solution.

• Example: \( a_n = a_{n-1} + 3, \quad a_0 = 2 \)
  - Forward
    - \( a_1 = 2 + 3 \)
    - \( a_2 = (2 + 3) + 3 = 2 + 2 \cdot 3 \)
    - \( a_3 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \)
    - ...
    - \( a_n = 2 + 3n \)
  - Backward
    - \( a_n = (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3 \)
    - \( a_n = (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3 \)
    - \( a_n = (a_{n-4} + 3) + 3 \cdot 3 = a_{n-4} + 4 \cdot 3 \)
    - ...
    - \( a_n = a_{n-n} + n \cdot 3 = a_0 + 3n = 2 + 3n \)
Exercises

• Find the first few terms of a solution to the recurrence relation
  \[ a_n = a_{n-1} + 2n + 3, \quad a_0 = 4 \]
  • \[ a_1 = 4 + 2(1) + 3 = 9 \]
  • \[ a_2 = 9 + 2(2) + 3 = 16 \]
  • \[ a_3 = 16 + 2(3) + 3 = 25 \]
  • ...

• Solve the recurrence relation above.
  • From the first few terms, the pattern seems to be \( a_n = (n + 2)^2 \)
  • This can be proved, but we need techniques we haven’t learned yet.
Exercises Continued

• Find and solve a recurrence relation for the sequence 0, 1, 3, 6, 10, 15,...
  • Take differences: 1 - 0 = 1, 3 - 1 = 2, 6 - 3 = 3, 10 - 6 = 4, 15 - 10 = 5, ...
  • Write down the relation: $a_n = a_{n-1} + n, \quad a_0 = 0$
  • Use backwards iteration
    • $a_n = a_{n-1} + n$
      • $= a_{n-2} + (n - 1) + n$
      • $= a_{n-3} + (n - 2) + (n - 1) + n$
      • ...
      • $= a_0 + (1 + 2 + \cdots + n)$
      • $= 0 + 1 + 2 + \cdots + n$
    • The sum of the integers from 1 to $n$ is $\frac{n(n+1)}{2}$
  • $a_n = \frac{n(n+1)}{2}$
Summations

• The sum of the $m$ through $n$-th terms of sequence $\{a_k\}$

$$a_m + a_{m+1} + \cdots + a_n$$

is denoted as

$$\sum_{i=m}^{n} a_i$$

where $i$ is the **index of summation**, $m$ is the **lower limit**, and $n$ is the **upper limit**.
Examples

• Use summation notation to express the sum from 1 to 100 of $\frac{1}{x}$:
  \[ \sum_{i=1}^{100} \frac{1}{i} \]

• Evaluate $\sum_{i=0}^{3}(2i + 1)$
  \[ \sum_{i=0}^{3} (2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) = 1 + 3 + 5 + 7 = 16 \]
Manipulations of Summation Notation

\[ \sum_{i} ca_i = c \sum_{i} a_i \]

\[ \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} (a_i + b_i) \]

\[ \sum_{i=m}^{n} a_i + c = \sum_{i=m}^{n} a_i + c(n - m + 1) \]

\[ \sum_{i=m}^{n} a_i = \sum_{i=m}^{l} a_i + \sum_{i=l+1}^{n} a_i \]

\[ \sum_{i=m}^{n} a_i = \sum_{j=0}^{n-m} a_{j+m} = \sum_{k=0}^{n-m} a_{n-k} \]

\[ \sum_{i=m}^{n} a_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{m-1} a_i \]
Exercises

• Evaluate

  • $3+3+3+3+3+3+3+3+3+3 = 30$

• Let $S = \{1,3,5,7\}$. Evaluate

  • $1+9+25+49 = 84$

• Evaluate

  • $3 \sum_{i=0}^{8} 2^i = 3(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256) = 1533$
Summation Identities

• There are **MANY**, but here are two you should know:
  • Geometric Series

\[
\sum_{i=0}^{n-1} r^i = \begin{cases} 
  n & r = 1 \\
  \frac{r^n - 1}{r - 1} & r \neq 1
\end{cases}
\]

• Arithmetic Series

\[
\sum_{i=0}^{n-1} i = \frac{n(n - 1)}{2}
\]
Geometric Series Identity Proof \((r \neq 1)\)

Let \( S = \sum_{i=0}^{n-1} r^i \)

\[
\begin{align*}
    rS &= r \sum_{i=0}^{n-1} r^i \\
    &= \sum_{i=0}^{n-1} r^{i+1} \\
    &= \sum_{j=1}^{n} r^j \\
    &= \left( \sum_{j=0}^{n-1} r^j \right) + r^n - r^0 \\
    &= S + r^n - 1 \\
    S(r - 1) &= r^n - 1 \\
    S &= \frac{r^n - 1}{r - 1}
\end{align*}
\]

\(\square\)
Products

• The product of the $m$ through $n$-th terms of sequence $\{a_k\}$

$$a_m \cdot a_{m+1} \cdots a_n$$

is denoted as

$$\prod_{i=m}^{n} a_i$$

Example:

$$n! = \prod_{i=1}^{n} i$$
A Product Identity

\[
\prod_{i=m}^{n} ca_i = c^{n-m+1} \left( \prod_{i=m}^{n} a_i \right) = (c)(c) \cdots (c) \cdot (a_m)(a_{m+1}) \cdots (a_n)
\]

\[
= (c^{\sum_{i=m}^{n}}) \left( \prod_{i=m}^{n} a_i \right) = c^{n-m+1} \left( \prod_{i=m}^{n} a_i \right)
\]