Relations

Chapter 9

Overview

- Relations and Their Properties
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

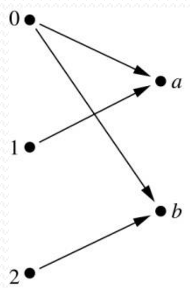
Section 9.1

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b),(a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are

  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4, and R_6: $(1,2)$ is in R_1 and R_6: $(2,1)$ is in R_2, R_5, and R_6: $(1, -1)$ is in R_2, R_3, and R_6 : $(2,2)$ is in R_1, R_3, and R_4.

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x\, (x,x) \in R$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),
$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition:A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$

- **Example**: The following relations on the integers are antisymmetric:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$.

For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

The following relations are not antisymmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \land (y,z) \in R \longrightarrow (x,z) \in R]$$

- **Example**: The following relations on the integers are transitive:

  $R_1 = \{(a,b) \mid a \leq b\}$, ⟵

  For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.

  $R_2 = \{(a,b) \mid a > b\}$,

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

  $R_4 = \{(a,b) \mid a = b\}$.

  The following are not transitive:

  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

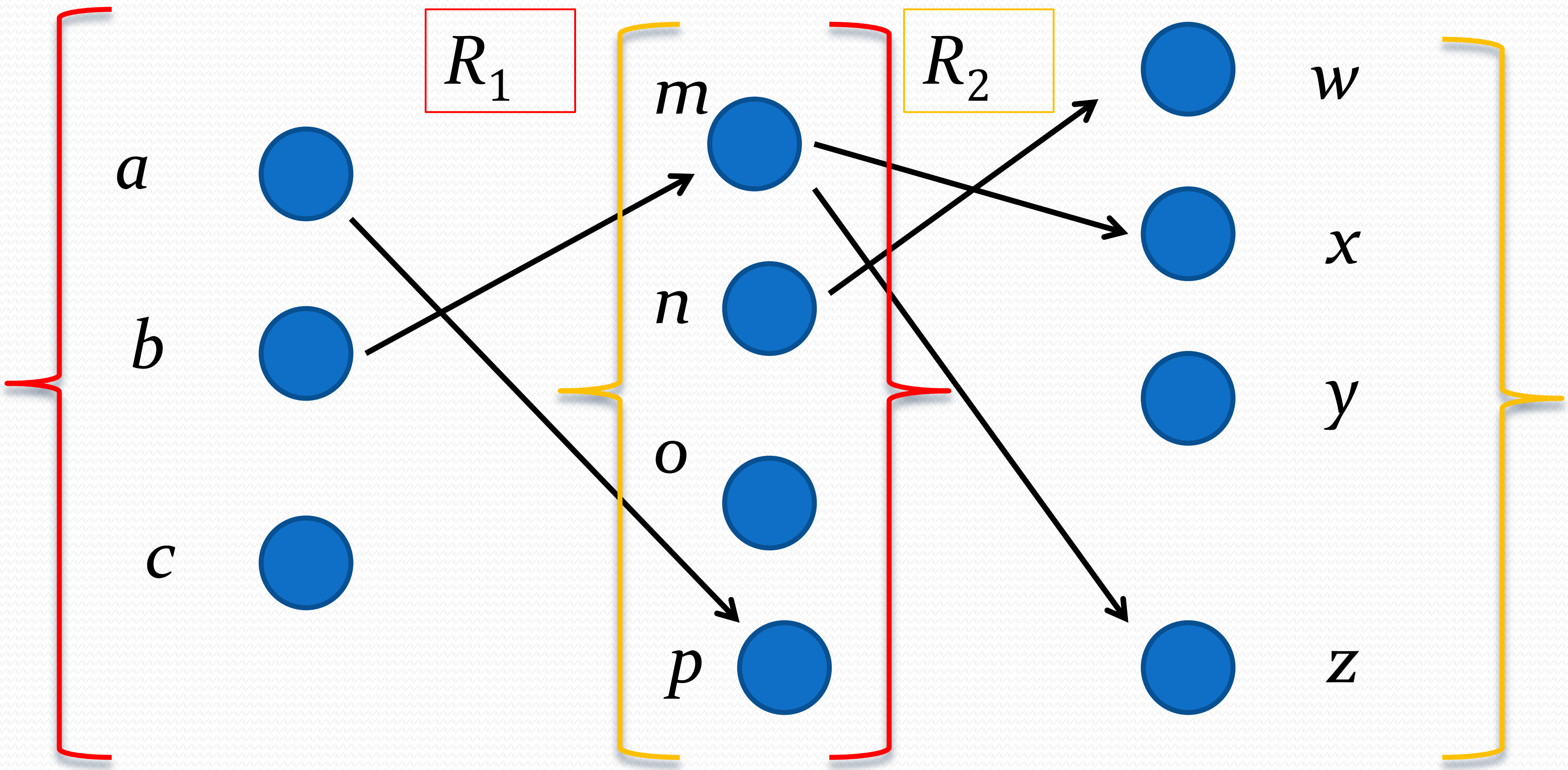
Then the *composition* (or *composite*), $R_2 \circ R_1$, of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

If R is a relation on set A, then

- $R^2 = R \circ R$
- $R^3 = R^2 \circ R$
- …
- $R^n = R^{n-1} \circ R$

Representing the Composition of a Relation



$$R_1 \circ R_2 = \{(b,x),(b,z)\}$$

Equivalence Relations

Section 9.5

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is **reflexive**, **symmetric**, and **transitive**.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: $a \equiv b\ (mod\ m)$ means $a \,\%\, m = b \,\%\, m$ where % is the modulus operator (computes remainder of $a \div b$)

- *Reflexivity*: $a \,\%\, m = a \,\%\, m$ is true.
- *Symmetry*: if $a \,\%\, m = b \,\%\, m$, then $b \,\%\, m = a \,\%\, m$ is true.
- *Transitivity*: if $a \,\%\, m = b \,\%\, m$ and $b \,\%\, m = c \,\%\, m$, then $a \,\%\, m = c \,\%\, m$ is true.

Divides

Example: Show that the "divides" relation on the set of positive integers is **not** an equivalence relation.

Solution: "divides" is not symmetric and is therefore not an equivalence relation.

- *Symmetry*: Counterexample: 2 divides 4, but 4 does not divide 2. Hence, the relation is not symmetric.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

$[a]_R = \{s \mid (a,s) \in R\}$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$

Proof: We show that (i) implies (ii).

Assume that aRb.

Now suppose that $c \in [a]$.

Then aRc.

Because aRb and R is symmetric, bRa.

Because R is transitive and bRa and aRc, it follows that bRc.

Hence, $c \in [b]$.

Therefore, $[a] \subseteq [b]$.

A similar argument (omitted here) shows that $[b] \subseteq [a]$.

Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

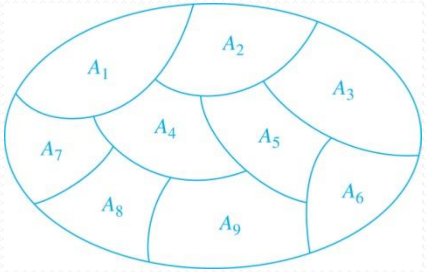
(*see text for proof that* (ii) *implies* (iii) *and* (iii) *implies* (i))

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i forms a partition of S if and only if

- $A_i \neq \emptyset$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and

$$\bigcup_{i \in I} A_i = S.$$



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

.

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Section 9.6

Partial Orderings

Definition 1: A relation *R* on a set S is called a *partial ordering,* or *partial order,* if it is **reflexive**, **antisymmetric**, and **transitive**. A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the "divides" relation is a partial ordering on the set of integers.

- *Reflexivity*: *a* divides *a* for all integers *a*. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If *a* and *b* are positive integers such that *a* divides *b* and *b* divides *a*, then *a* = *b*. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that *a* divides *b* and *b* divides *c*. Then there are positive integers *k* and *l* such that *b* = *ak* and *c* = *bl*. Hence, *c* = *a*(*kl*), so *a* divides *c*. Therefore, the relation is transitive.

- ($\mathbf{Z}^+$, |) is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on $\mathcal{P}(S)$, the power set of a set S.

- *Reflexivity*: $A \subseteq A$ always.
- *Antisymmetry*: If A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.*

Ex 1: is $\leq$ a total order on $\mathbb{N}$?

Ex 2: is $\subseteq$ a total order on $\mathcal{P}(S)$?

Definition 4: $(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Ex: is $(\mathbb{N}, \leq)$ a well-ordered set?

Lexicographic Order

Definition: Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is,

$$(a_1, a_2) \prec (b_1,b_2),$$

if $a_1 \prec_1 b_1$ or ($a_1 = b_1$ and $a_2 \prec_2 b_2$).

- This definition can be easily extended to a lexicographic ordering on strings.

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

Representing Relations

Section 9.3

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1,a_2, a_3\}$ and $B = \{b_1,b_2, b_3,b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

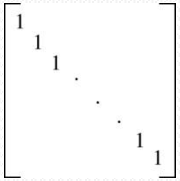
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i,b_j) with $m_{ij} = 1$, it follows that:

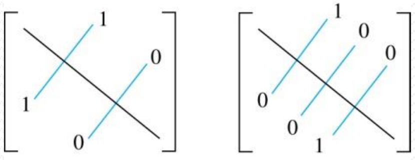
$R = \{(a_1, b_2), (a_2, b_1),(a_2, b_3), (a_2, b_4),(a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

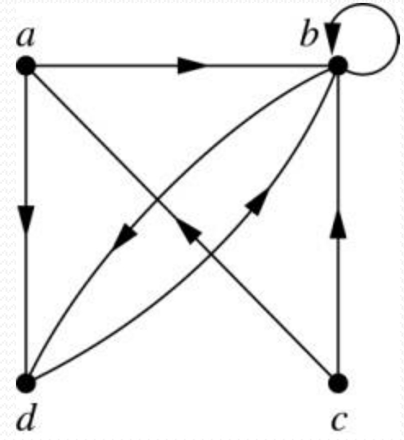
Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

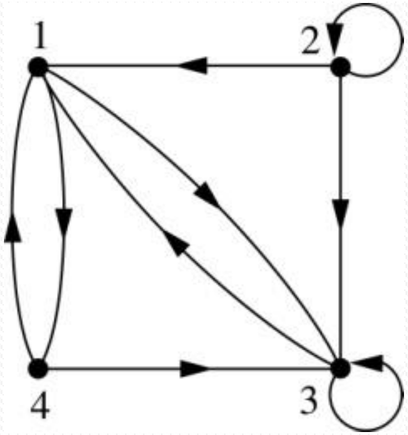
- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?

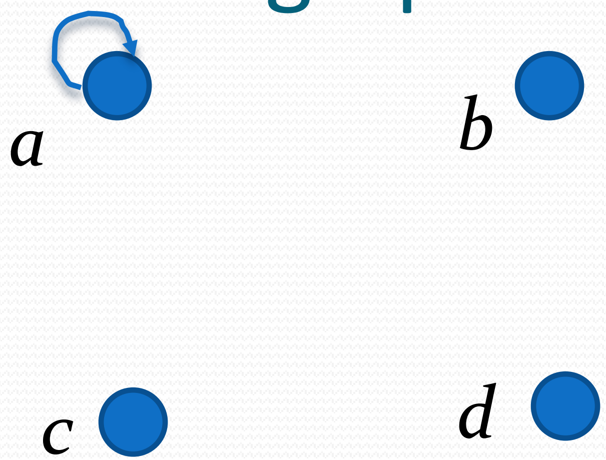


Solution: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

Determining which Properties a Relation has from its Digraph

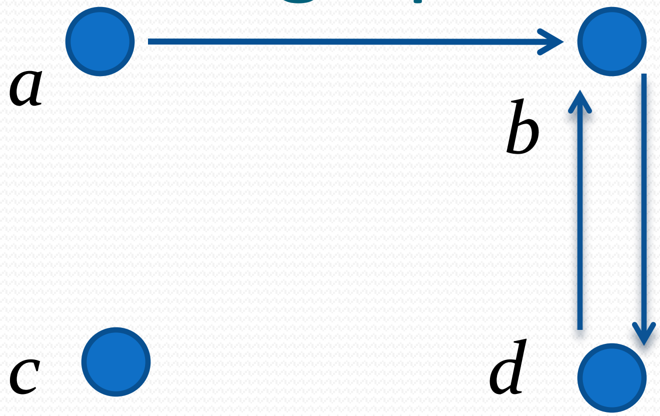
- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

Determining which Properties a Relation has from its Digraph – Example 1



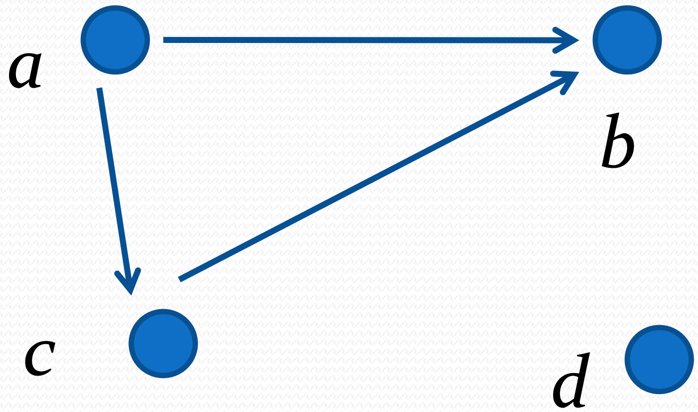
- *Reflexive?*
  - No, not every vertex has a loop
- *Symmetric?*
  - Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?*
  - Yes (trivially), there is no edge from one vertex to another
- *Transitive?*
  - Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



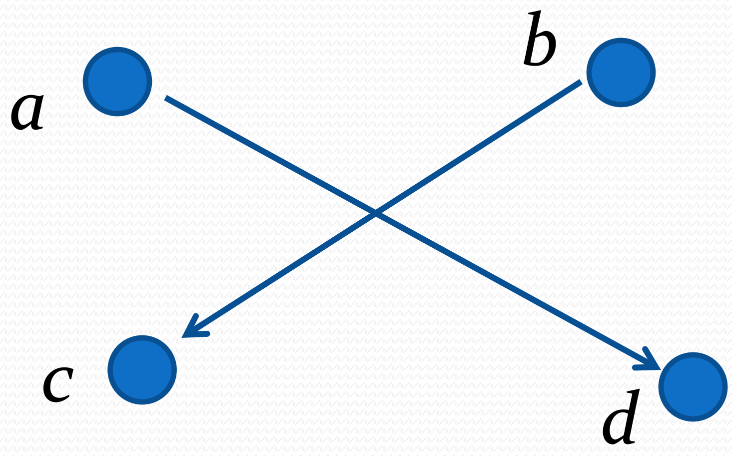
- *Reflexive?*
  - No, there are no loops
- *Symmetric?*
  - No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?*
  - No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?*
  - No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*

Determining which Properties a Relation has from its Digraph – Example 3



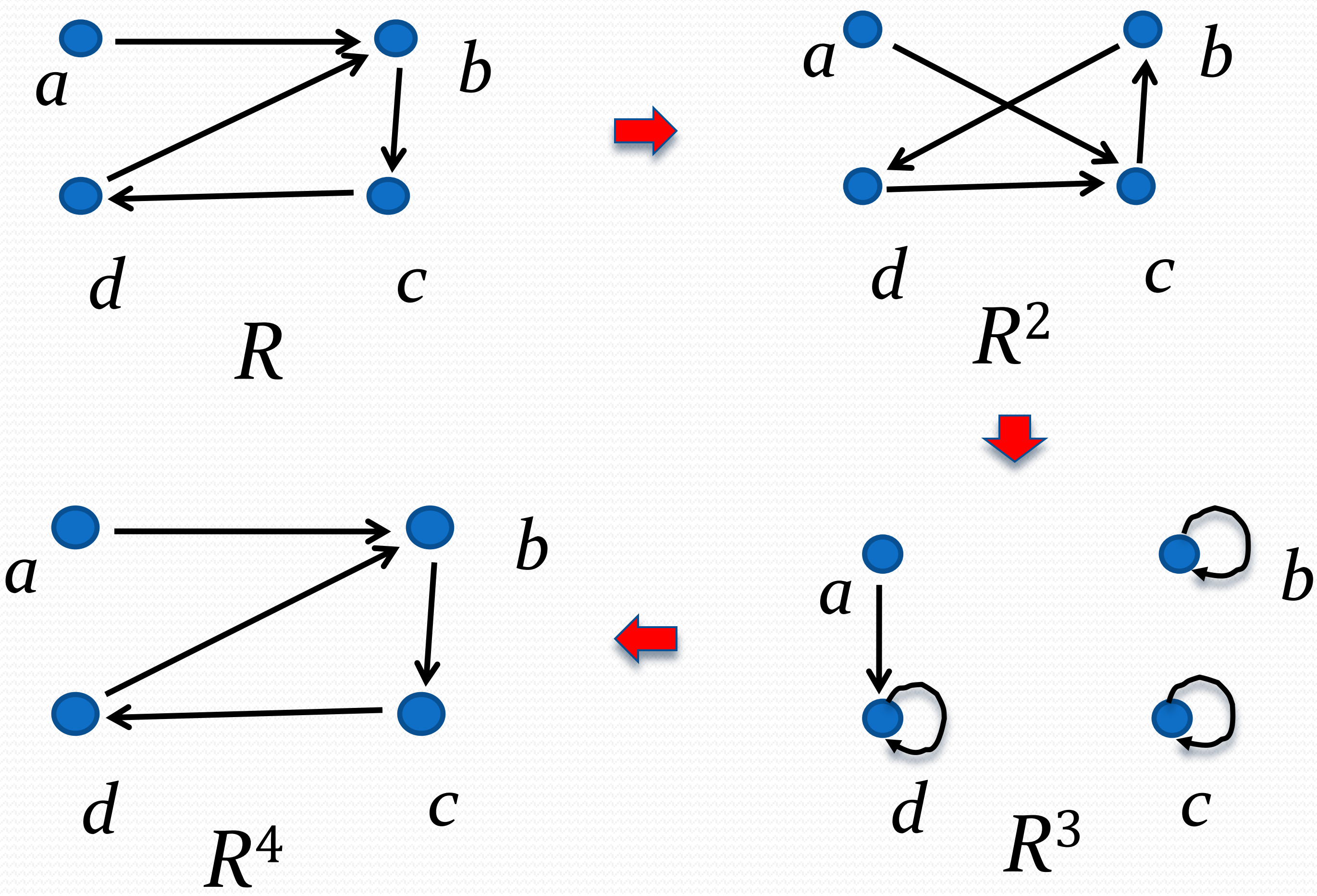
- *Reflexive?*
  - No, there are no loops
- *Symmetric?*
  - No, for example, there is no edge from *c* to *a*
- *Antisymmetric?*
  - Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?*
  - No, there is no edge from *a* to *b*

Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive?*
  - No, there are no loops
- *Symmetric?*
  - No, for example, there is no edge from d to a
- *Antisymmetric?*
  - Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?*
  - Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).