CSCE 222
Discrete Structures for Computing

Functions

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Functions

- Let A and B be non-empty sets.
- A **function** from A to B is an assignment of exactly one element of B to each element of A.
  - $f(a) = b$
  - $f: A \to B$
- Ex:
  - $A =$ students at TAMU, B$= \mathbb{N}$
  - $f(a) =$ the number of credit hours for student a

Terminology

- $f\colon A \rightarrow B$
  - A is the **domain** of f
  - B is to **codomain** of f
  - $f(a) = b$ is the **image** of a
  - a is the **preimage** of $f(a) = b$
  - $\{x \mid x = f(a)\ for\ a \in A\}$ is the **range** *of* f

Examples of Functions

- $floor: \mathbb{R} \rightarrow \mathbb{Z}$ "greatest integer"
  - $floor(4.2) = \lfloor 4.2 \rfloor = 4$
- $ceil: \mathbb{R} \rightarrow \mathbb{Z}$ "least integer"
  - $ceil(4.2) = \lceil 4.2 \rceil = 5.2$
- $pow: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$
  - $pow(x, y) = x^y$
  - $pow(3,5) = 3^5 = 243$
- $max: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$
  - $max(3,5) = 5$

Properties of Functions

- **One-to-one** (injective)
  - $\forall a \forall b \left(\left(f(a) = f(b) \right) \to a = b \right)$
- **Onto** (surjective)
  - $\forall y \exists x \; f(x) = y$
- **Bijective**
  - One-to-one **and** Onto
  - If $f: A \to \mathrm{B}$ is bijective, then there exists $f^{-1}\colon\ B \to A$, the **inverse function**.

Caesar’s Cipher

- Let A be the Roman alphabet
- Let $C: A \to A$
  - $C(a) = d$
  - $C(b) = e$
  - $C(c) = f$
  - …
  - $C(w) = z$
  - $C(x) = a$
  - $C(y) = b$
  - $C(z) = c$
- Is Caesar’s Cipher a bijection?
  - Yes.
  - VFLHQFH!

One Way Functions

- Let S be the set of strings of alphabetic characters.
- Let $J = \mathbb{N}_{<d}$ be the non-negative integers less than d.
- Let h: $S \rightarrow \mathrm{J}$.
  - h takes the ASCII code for each character in a string, adds them together, divides by d, and returns the remainder.
  - $h(s) = (\sum_{c \in S} c)\ mod\ d$
- Is h one-to-one?
  - No. $\therefore$ h is not invertible.

Exercise

- $f: \mathbb{R} \to \mathbb{R}$
- $f(x) = 2x^2 - 5$
- Is f
  - One-to-one?
    - No. $f(x) = f(-x)$
  - Onto?
    - No. $\neg\exists x\ f(x) < -5$

Compositions of Functions

- Let $g: A \to B$ and $f: B \to C$
- $f \circ g: A \to C$
  - $(f \circ g)(x) = f(g(x))$
- Ex: $s: \mathbb{R}^+ \to \mathbb{R}^+$ and $f: \mathbb{R}^+ \to \mathbb{Z}$
  - $s(x) = \sqrt{x}$
  - $f(x) = \lfloor x \rfloor$
  - $(f \circ s)(x) = \lfloor \sqrt{x} \rfloor$

Exercise

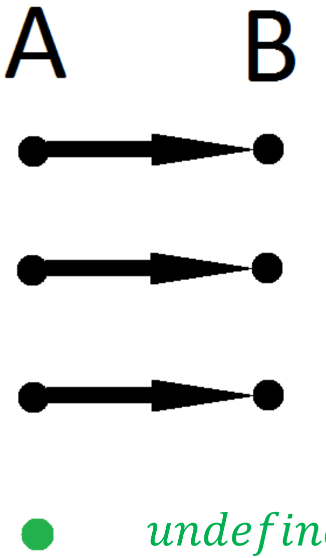
- Let $f\colon \mathbb{N} \to \mathbb{N}$ and $l\colon \mathbb{R}^+ \to \mathbb{R}$
  - $f(n) = n! = n \cdot (n-1) \cdot \cdots \cdot 1$
  - $l(x) = \log_2 x$
    - $\log_2 x = y \leftrightarrow 2^y = x$
- How to write $\log_2(n!)$?
  - $(l \circ f)(n)$
  - $\mathbb{N} \subset \mathbb{R}^+$

Partial Functions

- A **partial function** $f\colon A \to B$ is an assignment to each element in a subset of A *(domain of definition of* f*)* to a unique element in B.
- f is **undefined** for $A - D$.
  - D is the domain of definition of f.
- – If $D = A$, then f is a **total function.**
- – Ex: Let $f\colon \mathbb{R} \to \mathbb{R}$
  - $f(x) = \frac{1}{x}$
    - – undefined at $x = 0$
  - $f(x) = \sqrt{x}$
    - – undefined at $x < 0$
  - $f(x) = \pm\sqrt{x^2 + 1}$
    - – not a function from $\mathbb{R}$ to $\mathbb{R}$ because of $\pm$

Summary

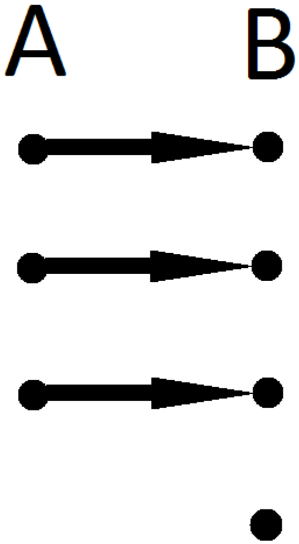
- **Function** maps elements of A to elements of B.



$undefined \Rightarrow partial\ function$

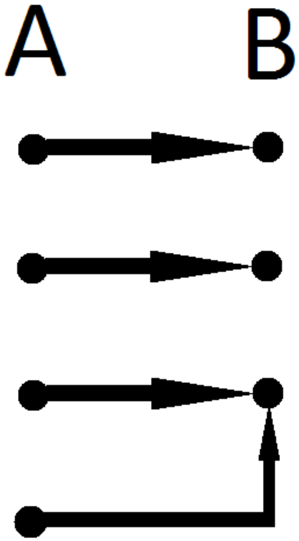
Summary

- **One-to-one**: every element in A maps to a unique element in B.



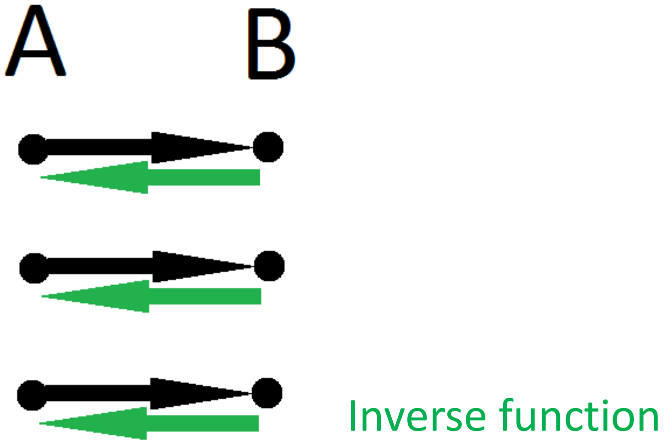
Summary

- **Onto**: every element in B is ampped to by some element in A.



Summary

- **Bijection**: one-to-one and onto.



Summary

- **Composition:** $(f \circ g)(x) = f\big(g(x)\big)$