CSCE 222
Discrete Structures for Computing

Proofs Methods and Strategies

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Exhaustive Proofs

- Prove for every element in the domain
- Ex: $(n+1)^3 \geq 3^n$ for $n \in \{0,1,2,3,4\}$
  - Exhaustive Proof:
  - $(0+1)^3 \geq 3^0$
    - $1 \geq 1$
  - $(1+1)^3 \geq 3^1$
    - $8 \geq 3$
  - $(2+1)^3 \geq 3^2$
    - $27 \geq 9$
  - $(3+1)^3 \geq 3^3$
    - $64 \geq 27$
  - $(4+1)^3 \geq 3^4$
    - $125 \geq 81$
  - □

Proof by Cases

- Prove for every case in the theorem
- Ex: if n is an integer, then $n^2 \geq n$
  - Proof by cases:
  - Case $n \geq 1$:
    - $n \cdot n \geq 1 \cdot n$
    - $n^2 \geq n$
  - Case $n \leq 1$:
    - $n^2 \geq n$, since n^2 is positive and n is negative
  - Case $n = 0$:
    - $0^2 \geq 0$
  - The claim holds in all cases □

Leveraging Proof by Cases

- When you can't consider every case all at once
- When there's no obvious way to start, but extra information in each case helps
- Ex: Formulate and prove a conjecture about the final digit of perfect squares
  - List some perfect squares
  - Look at the final digit $(13^2 = 16\mathbf{9})$
  - See a pattern?

Leveraging Proof by Cases

- Theorem: The final digit of a perfect square is 0,1,4,5,6, or 9.
- Proof:
  - $n = 10a + b,\ \ b \in \{0,1,2,\dots,9\}$
  - $n^2 = (10a + b)^2 = 10(10a^2 + 2ab) + b^2$
    - n^2 and b^2 have the same final digit
  - Case $b = 0$:
    - $0^2 = \mathbf{0}$, n^2 ends in 0
  - Case $b \in \{1,9\}$:
    - $1^2 = \mathbf{1}, 9^2 = 8\mathbf{1}$, n^2 ends in 1
  - Case $b \in \{2,8\}$:
    - $2^2 = \mathbf{4}, 8^2 = 6\mathbf{4}$, n^2 ends in 4
  - Case $b \in \{3,7\}$:
    - $3^2 = \mathbf{9}, 7^2 = 4\mathbf{9}$, n^2 ends in 9
  - Case $b \in \{4,6\}$:
    - $4^2 = 1\mathbf{6}, 6^2 = 3\mathbf{6}$, n^2 ends in 6
  - Case $b = 5$:
    - $5^2 = 2\mathbf{5}$, n^2 ends in 5
  - $\square$

Leveraging Proof by Cases

- Theorem: $x^2 + 3y^2 = 8$ has no integer solutions
- Proof by Cases:
  - $x^2 \leq 8 \quad |x| < 3$
  - $3y^2 \leq 8 \quad |y| < 2$
  - 15 cases
    - $x \in \{-2, -1, 0, 1, 2\}$
    - $y \in \{-1, 0, 1\}$
  - 6 cases
    - $x^2 \in \{0, 1, 4\}$
    - $y^2 \in \{0, 1\}$
  - $\leq$6 cases
    - $x^2 \in \{0, 1, \mathbf{4}\}$
    - $3y^2 \in \{0, \mathbf{3}\}$
    - $4 + 3 = 7 \neq 8$
  - $\therefore$ no integer solutions $\square$

Without Loss of Generality (wlog)

- Assert that the proof for one case can be reapplied with only straightforward changes to prove other specified cases.
- Ex: Let x, y be integers. If xy and $x + y$ are both even, then x and y are both even.
  - Proof
  - Use contraposition
    - $\big((x\ is\ odd) \vee (y\ is\ odd)\big) \rightarrow \big((xy\ is\ odd) \vee (x + y\ is\ odd)\big)$
  - Assume $(x\ is\ odd) \vee (y\ is\ odd)$
  - Wlog, assume x is odd.
  - Case y even:
    - $x + y = (odd) + (even) = odd$ ✓
  - Case y odd:
    - $xy = (odd)(odd) = odd$ ✓
  - $\therefore \big((x\ is\ odd) \vee (y\ is\ odd)\big) \rightarrow \big((xy\ is\ odd) \vee (x + y\ is\ odd)\big)$
  - It follows that $\big((xy\ is\ even) \wedge (x + y\ is\ even)\big) \rightarrow \big((x\ is\ even) \wedge (y\ is\ even)\big)$

$\square$

- Exhaustive proof and proof by cases are only valid when you prove **every** case
- Ex: every positive integer is the sum of 18 fourth powers of integers.
  - 79 is the counterexample
- Ex: if x is a real number, then x^2 is positive
  - Case $x\ is\ positive$:
    - $(positive)(positive) = (positive)$ ✓
  - Case $x\ is\ negative$:
    - $(negative)(negative) = (positive)$ ✓
  - Case $x\ is\ zero$:
    - $(zero)(zero) = zero$ X

Existence Proofs

- To prove $\exists x P(x)$
  - Constructive: find a witness
  - Nonconstructive: shown without witness
- Ex: Show that there exists some integer that is expressible as the sum of 2 cubes in 2 different ways
  - Constructive proof: $1729 = 10^3 + 9^3 = 12^3 + 1^3$

Existence Proofs

- Ex: Show that there exists 2 irrational numbers x, y such that x^y is rational
  - Nonconstructive proof:
    - Known: $\sqrt{2}$ is irrational
    - Consider $x = y = \sqrt{2}$, so $x^y = \sqrt{2}^{\sqrt{2}}$
    - If $\sqrt{2}^{\sqrt{2}}$ is rational, then $x = y = \sqrt{2}$ are the witnesses
    - If $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$
      - $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$
      - $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ are the witnesses
    - Either way, we found irrational x, y such that x^y is rational
      - We just don't know which value to use for x

Uniqueness Proofs

- "**exactly one** element satisfies $P(x)$"
  - **Existence:** $\exists x\, P(x)$
  - **Uniqueness:** $\forall y\, P(y) \rightarrow (y = x)$
  - $\exists x \forall y\, P(x) \wedge \left(P(y) \rightarrow (y = x)\right)$
  - $\exists x \forall y\, P(y) \leftrightarrow (y = x)$

Uniqueness Proofs

- $ar + b = 0$ has a unique solution when a, b are real and $a \neq 0$
  - Proof:
  - Existence
    - $ar + b = 0$
    - $ar = -b$
    - $r = -\frac{b}{a}$
  - Uniqueness
    - Assume $\exists s \; as + b = 0$
    - Then, $ar + b = as + b$
    - $ar + b - b = as + b - b$
    - $\frac{ar}{a} = \frac{as}{a}$
    - $r = s$
  - $\therefore r = -\frac{b}{a}$ is **the** solution to $ar + b = 0$

Proof Strategies

- Forward
  - Start with premises, plug and chug to the conclusion.
    - Direct proof
  - Start with negation of conclusion, plug and chug to negation of premises
    - Indirect proof
- Backward
  - Work backwards from the conclusion to find the correct steps for a direct proof

Backward Reasoning

- Prove that $\frac{(x+y)}{2} \geq \sqrt{xy}$ for positive reals x, y

– $\frac{(x+y)}{2} \geq \sqrt{xy}$	$\sqrt{}$ both sides
– $\frac{(x+y)^2}{4} \geq xy$	divide by 4
– $(x+y)^2 \geq 4xy$	factor LHS
– $x^2 + 2xy + y^2 \geq 4xy$	add $4xy$
– $x^2 - 2xy + y^2 \geq 0$	expand LHS
– $(x-y)^2 \geq 0$	Tautology

$\therefore \frac{(x+y)}{2} \geq \sqrt{xy}$

Adapting Existing Proofs

- Theorem: $\sqrt{3}$ is irrational
  - Proof: $\sqrt{2}$ is irrational, use that proof
- $\sqrt{n}$ is irrational when n is not a perfect square.
  - Same kind of proof: contradiction

Tilings

- Can a standard checkerboard (8×8) be tiled by dominoes?
  - Yes.
- Can a standard checkerboard with one corner missing be tiled by dominoes?
  - No.
  - 63 squares is not an even number of squares
- Can a standard checkerboard with two opposite corners missing be tiled by dominoes?
  - No.
  - Opposite corners have the same color.
    - 32 black + 30 white cannot be tiled because each domino covers 1 white and 1 black square