Polynomial Kernels and Faster Algorithms for the Dominating Set Problem on Graphs with an Excluded Minor

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Abstract. The domination number of a graph $G = (V, E)$ is the minimum size of a dominating set $U \subseteq V$, which satisfies that every vertex in $V \setminus U$ is adjacent to at least one vertex in U. The notion of a problem kernel refers to a polynomial time algorithm that achieves some provable reduction of the input size. Given a graph G whose domination number is k, the objective is to design a polynomial time algorithm that produces a graph G' whose size depends only on k, and also has domination number equal to k. Note that the graph G' is constructed without knowing the value of k. Problem kernels can be used to obtain efficient approximation and exact algorithms for the domination number, and are also useful in practical settings.

In this paper, we present the first nontrivial result for the general case of graphs with an excluded minor, as follows. For every fixed h, given a graph G with n vertices that does not contain K_h as a topological minor, our $O(n^{3.5} + k^{O(1)})$ time algorithm constructs a subgraph G' of G, such that if the domination number of G is k, then the domination number of G' is also k and G' has at most k^c vertices, where c is a constant that depends only on h. This result is improved for graphs that do not contain $K_{3,h}$ as a topological minor, using a simpler algorithm that constructs a subgraph with at most ck vertices, where c is a constant that depends only on h.

Our results imply that there is a problem kernel of polynomial size for graphs with an excluded minor and a linear kernel for graphs that are $K_{3,h}$-minor-free. The only previous kernel results known for the dominating set problem are the existence of a linear kernel for the planar case as well as for graphs of bounded genus. Using the polynomial kernel construction, we give an $O(n^{3.5} + 2^{O(\sqrt{k})})$ time algorithm for finding a dominating set of size at most k in an H-minor-free graph with n vertices. This improves the running time of the previously best known algorithm.

Key words: H-minor-free graphs, degenerated graphs, dominating set problem, fixed-parameter tractable algorithms, problem kernel.

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1 Introduction

The input to a parameterized problem is a pair (x, k), where x is the problem instance, k is the parameter, and $n := |(x, k)|$ denotes the input size. A parameterized problem is fixed-parameter tractable if it can be solved in time $f(k) \cdot n^c$, for a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a constant c. A kernelization is a polynomial time computable function that given input (x, k) constructs an equivalent input (x', k'), such that $k' \leq k$ and $|x'| \leq g(k)$ for a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$. The image x' is called the problem kernel of x. In this paper, the notion of a kernel for the dominating set problem refers to a polynomial time algorithm that given a graph G whose domination number is k, constructs a graph G' whose size depends only on k, and also has domination number equal to k.

It is easy and known that a parameterized problem is kernelizable if and only if it is fixed-parameter tractable. Thus, a fixed-parameter algorithm for the dominating set problem gives a trivial kernel whose size is some function of k, not necessarily a polynomial. Problem kernels can be used to obtain efficient approximation and exact algorithms for the domination number, and are also useful in practical settings.

Our main result is a polynomial problem kernel for the case of graphs with an excluded minor. This is the most general class of graphs for which a polynomial problem kernel has been established. To the best of our knowledge, the only previous results are a linear kernel for the planar case as well as for graphs of bounded genus. For a general introduction to the field of parameterized complexity, the reader is referred to [13],[15], and [23].

Fixed-Parameter Algorithms for the Dominating Set Problem. The dominating set problem on general graphs is known to be $W[2]$-complete [13]. This means that most likely there is no $f(k) \cdot n^c$-algorithm for finding a dominating set of size at most k in a graph of size n for any computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ and constant c. This suggests the exploration of specific families of graphs for which this problem is fixed-parameter tractable.

The method of bounded search trees has been used to give an $O(8^k n)$ time algorithm for the dominating set problem in planar graphs [3] and an $O((4g + 40)^k n^2)$ time algorithm for the problem in graphs of bounded genus $g \geq 1$ [14]. The algorithms for planar graphs were improved to $O(4^{6\sqrt{34k}} n)$ [1], then to $O(2^{27\sqrt{k}} n)$ [20], and finally to $O(2^{15.13\sqrt{k}} k + n^3 + k^4)$ [17]. Fixed-parameter algorithms are now known also for map graphs [10] and for constant powers of H-minor-free graphs [11]. The running time given in [11] for finding a dominating set of size k in an H-minor-free graph G with n vertices is $2^{O(\sqrt{k})} n^c$, where c is a constant depending only on H. In a previous paper with Alon, we proved that the dominating set problem is fixed-parameter tractable for degenerated graphs, by establishing an algorithm with running time $k^{O(dk)} n$ for finding a dominating set of size k in a d-degenerated graph with n vertices [5].

Kernels for the Dominating Set Problem. The reduction rules introduced by Alber, Fellows, and Niedermeier were the first to establish a linear problem kernel for planar graphs [4]. The kernel obtained was of size $335k$, where k is the domination number of the graph. Fomin and Thilikos proved that the same rules of Alber et al. provide a linear kernel of size $O(k+g)$ for graphs of genus g [16]. Chen et al. improved the upper bound for the planar case to $67k$ [9]. They also gave the first lower bound, by proving that for any $\epsilon > 0$, there is no $(2-\epsilon)k$ kernel for the planar dominating set problem, unless $P = NP$. It is interesting to note that Alber, Dorn, and Niedermeier introduced a reduction rule that explores the joint neighborhood of l distinct vertices [2], but this general rule has been applied only for $l = 1$ and $l = 2$, in order to prove that the directed dominating set problem on planar graphs has a linear size kernel. Their reduction rule generates a constraint, which is encoded by a corresponding gadget in the graph. Thus, the kernel constructed is not necessarily a subgraph of the input graph.

Our Results. We present the first nontrivial result that provides a kernel for the dominating set problem on the general class of H-minor-free graphs. The proofs that our new reduction rules bring us to a polynomial kernel are based on deep and new combinatorial results on the structure of dominating sets in graphs. This gives an $O(n^{3.5} + 2^{O(\sqrt{k})})$ time algorithm for finding a dominating set of size at most k in an H-minor-free graph with n vertices. For graphs that are $K_{3,h}$-minor-free, the reduction rules of Alber, Fellows, and Niedermeier [4] are shown to give a linear problem kernel. Due to space limitations, the proof that there is a linear kernel in this case appears in the appendix. All the reduction rules described in this paper have the property that the only modifications made to an input graph are the removal of vertices and edges. This implies that the graph obtained, as a result of applying the rules, is a subgraph of the input graph. The advantages of this approach are its simplicity and the fact that it preserves monotone properties, like planarity, being H-minor-free, and degeneracy. We show that the rules of Alber et al. can also be described in such a way.

2 Preliminaries

The paper deals with undirected and simple graphs. Generally speaking, we will follow the notation used in [8] and [12]. For a graph $G = (V, E)$ and a vertex $v \in V$, $N(v)$ denotes the set of all vertices adjacent to v (not including v itself), whereas $N[v]$ denotes $N(v) \cup \{v\}$. This is generalized to the neighborhood of arbitrary sets by defining $N(A) := \left(\bigcup_{v\in A} N(v)\right) \setminus A$ and $N[A] := \bigcup_{v\in A} N[v]$. The graph obtained from G by deleting a vertex v is denoted $G-v$. The subgraph of G induced by some set $V' \subseteq V$ is denoted by $G[V']$.

A dominating set of a graph $G = (V, E)$ is a subset of vertices $U \subseteq V$, such that every vertex in $V \setminus U$ is adjacent to at least one vertex in U. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set. For a set of vertices A, if $U \subseteq N[A]$, then we say that A *dominates* U.

A graph G is *d-degenerated* if every induced subgraph of G has a vertex of degree at most d. A d-degenerated graph with n vertices has less than dn edges. An edge is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex. A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. If a subdivision of a graph H is the subgraph of another graph G, then H is a *topological minor* of G. A graph H is called a *minor* of a graph G if it can be obtained from a subgraph of G by a series of edge contractions.

In this paper, we consider only simple paths, that is, paths of the form $x_0 - x_1 - \cdots - x_k$, where the x_i are all distinct. The vertices $x_1, \ldots, x_{k-1}$ are the inner vertices of the path. The number of edges of a path is its length. Suppose that $G = (V, E)$ is a graph, $U \subseteq V$, and r and l are two integers. We denote by $\widehat{U}_{r,l}$ the set of all vertices $v \in V \setminus U$ for which there are r vertex disjoint paths of length at most l from v to r different vertices of U. To avoid confusion, we stress the fact that v is the starting vertex of all the paths, but any other vertex belongs to at most one of the paths. The vertices of $\widehat{U}_{r,l}$ are called *central vertices*, and when the values of r and l are clear from the context, the simpler notation $\widehat{U}$ will be used.

3 Dominating Sets in Degenerated Graphs

Graphs with either an excluded minor or with no topological minor are known to be degenerated. We will apply the following useful propositions.

Proposition 1. *[7, 21] There exists a constant c such that, for every h, every graph that does not contain K_h as a topological minor is ch^2-degenerated.*

Proposition 2. *[22, 25, 26] There exists a constant c such that, for every h, every graph with no K_h minor is $ch\sqrt{\log h}$-degenerated.*

Some of our results for graphs with no topological K_h use the constant from Proposition 1. The results can be improved for graphs that are K_h-minor-free using Proposition 2.

A major part of Rule 2, described in section 5, involves getting a succinct representation of all sets of some bounded size that dominate a specific set of vertices in a degenerated graph. This useful representation is achieved by applying a $k^{O(dk)}n$ time algorithm from [5] for finding a dominating set of size at most k in a d-degenerated graph with n vertices. We need the following combinatorial lemma proved in that paper.

Lemma 1. *Let $G = (V, E)$ be a d-degenerated graph, and assume that $B \subseteq V$. If $|B| > (4d + 2)k$, then there are at most $(4d + 2)k$ vertices in G that dominate at least $|B|/k$ vertices of B.*

This gives the following useful characterization of dominating sets in degenerated graphs.

Theorem 1. *Suppose that $G = (V, E)$ is a d-degenerated graph with n vertices, $B \subseteq V$, and $k \geq 1$. There is a $k^{O(dk)}n$ time algorithm for finding a family*

$\mathcal{F}$ of $t \leq (4d+2)^k k!$ pairs (D_i, B_i) of subsets of V, such that $|D_i| \leq k$ and $|B_i| \leq (4d+2)k$ for every $1 \leq i \leq t$, for which the following holds. If $D \subseteq V$ is a subset of size at most k that dominates B, then some i, $1 \leq i \leq t$, satisfies that $D_i \subseteq D$ and $B_i = B \setminus N[D_i]$.

Proof. The algorithm uses the method of bounded search trees. In each step of the algorithm, B denotes the vertices that still need to be dominated. If $|B| > (4d+2)k$, then denote by R the set of all vertices that dominate at least $|B|/k$ vertices of B. Every set of size at most k that dominates B must contain a vertex from R. It follows from Lemma 1 that $|R| \leq (4d+2)k$, so we can build our search tree, by creating $|R|$ branches and checking all possible options of adding one of the vertices of R to the dominating set. For each such vertex $v \in R$, we add v to the dominating set, assign $B := B \setminus N(v)$, and remove v from the graph. We continue until $|B| \leq (4d+2)k$ in all the leaves of the search tree. The search tree can grow to be of size at most $(4d+2)^k k!$, and each subset $D \subseteq V$ of size at most k that dominates the original input set B will correspond to one of the leaves of this search tree, as needed. □

Though the dominating set problem has a polynomial time approximation scheme when restricted to a class of graphs with an excluded minor [18], for our purposes, a fast algorithm that achieves a constant approximation is required. The following combinatorial Theorem is proved in [5] (note that $\uplus$ denotes disjoint set union).

Theorem 2. *Let s be the constant from Proposition 1. Suppose that the graph $G = (B \uplus W, E)$ satisfies that W is an independent set, all vertices of W have degree at least 2, and $N(w_1) \neq N(w_2)$ for every two distinct vertices $w_1, w_2 \in W$ for which $|N(w_1)| < h-1$. If G does not contain K_h as a topological minor, then there exists a vertex $b \in B$ of degree at most $(3sh)^h$.*

This gives the following constant factor approximation algorithm.

Theorem 3. *Let s be the constant from Proposition 1. Suppose that the graph $G = (B \uplus W, E)$ does not contain K_h as a topological minor, and there is a set of size k that dominates B. There is an $O(nk)$ time algorithm that finds a set of size at most $(3sh)^h k$ that dominates B.*

Proof. Start with a solution $D := \emptyset$. Given a graph $G = (B \uplus W, E)$, remove all edges whose two endpoints are in W and all vertices of W of degree 0 or 1. As long as there are two different vertices $w_1, w_2 \in W$ with $N(w_1) = N(w_2)$, $|N(w_1)| < h-1$, remove one of them from the graph. As proved in [5], these modifications can be performed in time $O(|E|)$ and they obviously do not affect the minimum size of a set that dominates B. It follows from Theorem 2 that there is a vertex $b \in B$ of degree at most $(3sh)^h$. We assign $D := D \cup N[b]$, move the vertices of $N(N[b]) \cap B$ from B to W, and remove the vertices of $N[b]$ from the graph. The size of the optimal solution decreased by at least one, since every set that dominates b must contain at least one vertex from $N[b]$. We continue as before in the resulting graph, and after at most k steps, the algorithm will compute a dominating set as needed. □

4 Bounds on the Number of Central Vertices

For graphs with no topological K_h, the following bound applies.

Lemma 2. *Let s be the constant from Proposition 1. If the graph $G = (V, E)$ does not contain K_h as a topological minor, and $U \subseteq V$ is of size k, then for every l, $|\widehat{U}_{h-1,l}| \leq (2sh^2 l)^{hl} k$.*

Proof. Denote $d = sh^2$. To bound the size of $\widehat{U}$, we initially define the set B to be equal U, and then in a series of $1 + (h-1)(l-1)$ phases, vertices will be added to B, until eventually $\widehat{U} \subseteq B$. As proved later, after every phase i, $1 \leq i \leq 1 + (h-1)(l-1)$, the set B will be of size at most $(1 + sh^2(2l-1))^i k$. This gives the needed bound for $\widehat{U}$, by setting $i = 1 + (h-1)(l-1)$.

The following is the description of a phase. At the beginning of phase i, the set B is of size at most $(1 + sh^2(2l-1))^{i-1} k$. Consider the vertices of $V \setminus B$ in some arbitrary order. For each vertex $w \notin B$, if there exist two vertex disjoint paths of length at most l from w to two vertices $b_1, b_2 \in B$, such that b_1 and b_2 are not connected, and all the inner vertices of the two paths are not in B, then add the edge $\{b_1, b_2\}$ to G and remove the vertex w from the graph together with the two paths (the vertices b_1 and b_2 remain in the graph). Denote the resulting graph by G'. Obviously, $G'[B]$ does not contain K_h as a topological minor and therefore has at most $d|B| = sh^2|B|$ edges. The number of edges in the induced subgraph $G'[B]$ is at least the number of deleted vertices divided by $(2l-1)$, which means that at most $sh^2(2l-1)|B|$ vertices were deleted so far. All the vertices that were removed from the graph during this phase are added to the set B, and now we start the next phase with the original graph G and a new set B of size at most $(1 + sh^2(2l-1))^i k$.

Consider a vertex $v \in \widehat{U}$ at the beginning of a phase. There are $h-1$ vertex disjoint paths of length at most l from v to a set H of $h-1$ different vertices of U. Assume that when v is considered in the arbitrary order, all the vertices of these $h-1$ paths are still in the graph. We claim that the $h-1$ vertices of H cannot all be adjacent to each other, since otherwise they form a topological K_h together with v. Thus, if v was not removed from the graph during the phase, then this can only happen in case there exists a vertex $u \notin B$ on one of the $h-1$ vertex disjoint paths, which was removed from the graph before v was considered. This vertex u was later added to B at the end of the phase. There are $h-1$ vertex disjoint paths of length at most l from v to H, and these paths contain at most $(h-1)(l-1)$ inner vertices. Thus, after at most $1+(h-1)(l-1)$ phases, the vertex v will be added to B. $\square$

Itai, Perl, and Shiloach [19] proved that given a graph G with two distinct vertices s and t, the problem of deciding whether there exist m vertex disjoint paths of length at most K from s to t is NP-complete for $K \geq 5$ and polynomially solvable for $K \leq 4$. Thus, $\widehat{U}_{r,3}$ can be efficiently computed as follows.

Lemma 3. *There is an $O(|V|^{1.5}|E|)$ time algorithm for computing $\widehat{U}_{r,3}$ for a graph $G = (V, E)$, a subset $U \subseteq V$, and an integer r.*

Proof. Suppose that $v \in V \setminus U$, and let w be a new vertex that is connected to all the vertices of U. By definition, $v \in \widehat{U}_{r,3}$ if and only if there are r vertex disjoint paths of length at most 4 from v to w. To determine this, apply the $O(|V|^{0.5}|E|)$ time algorithm of Itai et al. [19] for finding the maximum number of vertex disjoint paths of length at most 4 from v to w. □

5 Problem Kernel for Graphs with an Excluded Minor

The reduction rules described in [4] examine the neighborhood of either a single vertex or a pair of vertices. In this section we generalize these definitions to a neighborhood of a set of arbitrary size.

Definition 1. *Consider a subset of vertices $A \subseteq V$ of a given graph $G = (V, E)$. The neighborhood of A is partitioned into four disjoint sets $N_1(A)$, $N_2(A)$, $N_3(A)$, and $N_4(A)$.*

- $N_1(A) := \{u \in N(A) : N(u) \setminus N[A] \neq \emptyset\}$
- $N_2(A) := \{u \in N(A) \setminus N_1(A) : N(u) \cap N_1(A) \neq \emptyset\}$
- $N_3(A) := \{u \in N(A) \setminus (N_1(A) \cup N_2(A)) : N(u) \cap N_2(A) \neq \emptyset\}$
- $N_4(A) := N(A) \setminus (N_1(A) \cup N_2(A) \cup N_3(A))$

In the original definitions from [4], which are described in section 7.2, the neighborhood is partitioned into only three parts. Here, the definition of $N_3(A)$ is modified and $N_4(A)$ takes the role of the "inner neighborhood" of A.

Proposition 3. *Let D be a dominating set of a graph G. If $v \notin N_4(A) \cup A$, then there is a path of length at most 4 from v to a vertex of D, and the path does not contain any vertices of A.*

Proof. Since $v \notin N_4(A) \cup A$, there is a path of length at most 3 from v to a vertex $w \notin N[A]$, and the path does not contain any vertices of A. Since D is a dominating set, this vertex w is adjacent to some vertex $d \in D$. Since $w \notin N[A]$, then obviously $d \notin A$ (it could be that $d \in N(A)$). This gives a path of length at most 4 from v to d, as needed. □

We now define our two reduction rules. Rule 2 applies Rule 1 as a subroutine. Rule 1 removes a vertex u from the graph in case there are two other vertices v and w such that $\{u, v, w\}$ is an independent set and $N(u) = N(v) = N(w) \neq \emptyset$. Rule 2 examines the "inner neighborhood" $N_4(A)$ of a subset A of size k. By applying a fixed-parameter algorithm for finding dominating sets in degenerated graphs, it calculates a small set W that contains all the vertices that dominate many vertices of $N_3(A) \cup N_4(A)$. More formally, for every set D of size of at most k that dominates $N_3(A) \cup N_4(A)$, there is a subset $D' \subseteq D$, such that $D' \subseteq W$ and $(N_3(A) \cup N_4(A)) \setminus N[D'] \subseteq W$. In case $N_4(A)$ is large, many of the vertices of $N_4(A) \setminus W$ can be removed from the graph. The main goal of this section will be to analyze graphs for which Rule 2 cannot be applied anymore.

Rule 1: Let $A \subseteq V$ be an independent set of the graph $G = (V, E)$ and assume that $N(v) \neq \emptyset$ for every $v \in A$.

– Partition the set A into disjoint subsets $A_1, A_2, \ldots, A_t$ according to the neighborhoods of vertices of A. That is, every two vertices $v, w \in A_i$ satisfy $N(v) = N(w)$, whereas every two vertices $v \in A_i$ and $w \in A_j$ for $i \neq j$ satisfy $N(v) \neq N(w)$.
– For every $1 \leq i \leq t$ for which $|A_i| > 2$, let $v, w \in A_i$ be two arbitrary distinct vertices. Remove all the vertices of $A_i \setminus \{v, w\}$ from the graph.

Rule 2: Suppose that $G = (V, E)$ is d-degenerated and $A \subseteq V$ is a subset of k vertices. If $|N_4(A)| > 2^{(4dk+3k)^{k+1}}$, do the following.

– Let $\mathcal{F}$ be a family of $t \leq (4d+2)^k k!$ pairs (D_i, B_i) of subsets of V, such that $|D_i| \leq k$ and $|B_i| \leq (4d+2)k$ for every $1 \leq i \leq t$ for which the following holds. If $D \subseteq V$ is a subset of size at most k that dominates $N_3(A) \cup N_4(A)$, then some i, $1 \leq i \leq t$, satisfies that $D_i \subseteq D$ and $B_i = (N_3(A) \cup N_4(A)) \setminus N[D_i]$.
– Denote $W := A \cup \bigcup_{i=1}^{t} (D_i \cup B_i)$. Remove all edges between vertices of $(N_3(A) \cup N_4(A)) \setminus W$.
– Apply Rule 1 to the resulting graph and the independent set $N_4(A) \setminus W$.

The next two Lemmas prove the correctness of these rules.

Lemma 4. *Let $A \subseteq V$ be an independent set of the graph $G = (V, E)$. Applying Rule 1 to G and A does not change the domination number.*

Proof. It is enough to prove that if $\{x, y, z\}$ is an independent set, such that $N(x) = N(y) = N(z) \neq \emptyset$, then $\gamma(G - z) = \gamma(G)$. To prove that $\gamma(G) \leq \gamma(G - z)$, let D be a dominating set of $G - z$. If $D \cap N(x) = \emptyset$, then $\{x, y\} \subseteq D$, and therefore $(D \setminus \{y\}) \cup \{u\}$ is a dominating set of G, for any $u \in N(x)$.

To prove that $\gamma(G - z) \leq \gamma(G)$, let D be a *minimum* dominating set of G. It cannot be the case that $\{x, y, z\} \subseteq D$, since adding one of the vertices of $N(x)$ to $D \setminus \{y, z\}$ results in a smaller dominating set. We can assume, without loss of generality, that $z \notin D$, and therefore D is a dominating set of $G - z$. □

Lemma 5. *Suppose that $G = (V, E)$ is d-degenerated and $A \subseteq V$ is a subset of k vertices. In case Rule 2 is applied to G and A, then at least one vertex is removed from the graph, whereas the domination number does not change.*

Proof. Using the notations of Rule 2, denote by G' the graph obtained from G by removing all edges between vertices of $(N_3(A) \cup N_4(A)) \setminus W$, just before Rule 1 is applied. It follows from Lemma 4 that in order to verify that Rule 2 does not change the domination number, it is enough to prove that $\gamma(G') = \gamma(G)$. It is obvious that $\gamma(G') \geq \gamma(G)$, since removing edges cannot decrease the domination number. We now prove that $\gamma(G') \leq \gamma(G)$. Let D be a *minimum* dominating set of G, and let $D' \subseteq D$ be a subset of *minimum* size that dominates $N_3(A) \cup N_4(A)$. This implies that $D' \subseteq A \cup N_2(A) \cup N_3(A) \cup N_4(A)$ and $N[D'] \subseteq N[A]$. Obviously $|D'| \leq k$, since otherwise $(D \setminus D') \cup A$ would be a smaller dominating set of G. Thus, from Theorem 1, some i, $1 \leq i \leq t$, satisfies that $D_i \subseteq D'$ and $B_i = (N_3(A) \cup N_4(A)) \setminus N[D_i]$. To prove that D is also a dominating set of G', we need to show that the vertices of $(N_3(A) \cup N_4(A)) \setminus W$ are dominated by D in

G', since the neighborhood of all other vertices remained the same. Assume that $v \in (N_3(A) \cup N_4(A)) \setminus W$. Since $B_i \subseteq W$, it follows that $v \notin B_i$, and therefore v is dominated in G by some vertex $d \in D_i$. This means that v is still dominated by d in G', since $D_i \subseteq W$. This completes the proof that Rule 2 does not change the domination number.

We now prove that when Rule 2 is applied, at least one vertex of $N_4(A) \setminus W$ is removed from the graph G'. First, note that $(N_3(A) \cup N_4(A)) \setminus W$ is an independent set, and therefore $N_4(A) \setminus W$ is also independent. Given a vertex $v \in N_4(A) \setminus W$, obviously $N(v) \subseteq A \cup N_3(A) \cup N_4(A)$ and $N(v) \neq \emptyset$, since it is adjacent to at least one vertex of A. The important property of v is that it is adjacent in G' only to vertices of W, since all other edges incident at v were removed. Since $W = A \cup \bigcup_{i=1}^{t}(D_i \cup B_i)$, it follows that $|W| \leq k + (4d+2)^k k!(k + (4d+2)k)) = (4d+3)k(4d+2)^k k! + k$. It is easy to verify that $2 \cdot 2^{|W|} + |W| \leq 2^{|W|+2} \leq 2^{(4dk+3k)^{k+1}} < N_4(A)$. Thus, $|N_4(A) \setminus W| \geq |N_4(A)| - |W| > 2 \cdot 2^{|W|}$. By the pigeonhole principle, we conclude that there are three distinct vertices $x, y, z \in N_4(A) \setminus W$, such that $N(x) = N(y) = N(z) \neq \emptyset$. One of these three vertices will be removed by Rule 1. ◻

The following Lemma is useful for showing that given a graph with an excluded minor and a dominating set D of size k, there exists a subset of vertices U whose size is linear in k, such that all vertices not in $D \cup U$ belong to the "inner neighborhood" $N_4(A)$ of a subset $A \subseteq D \cup U$ of constant size.

Lemma 6. *Let D be a dominating set of the graph $G = (V, E)$. If $r \geq 1$ and $v \notin D \cup \widehat{D}_{r+1,4}$, then there exists a subset $A \subseteq D \cup \widehat{D}_{r+1,3}$ of size at most $40r^5$, such that $v \in N_4(A)$.*

Proof. To simplify the notation, the symbol $\widehat{D}$ will refer to $\widehat{D}_{r+1,3}$. Let q be the maximum number of disjoint paths of length 4 from v to q different vertices of D. Since $v \notin D \cup \widehat{D}_{r+1,4}$, it follows from the definition of $\widehat{D}_{r+1,4}$ that $q \leq r$. Construct q such paths, whose total length is the minimum possible. Denote by B the set of all vertices that appear in these q paths and call the inner vertices of these paths $B' := B \setminus (D \cup \{v\})$. Assign $t := 3r(r + r^2 + r^4) + 1$, and assume, by contradiction, that $v \notin N_4(A)$ for all subsets $A \subseteq D \cup \widehat{D}$ of size at most $4(r + t - 1)$. Note that $4(r + t - 1) \leq 40r^5$.

We will now construct t paths of length at most 4 and a series of t subsets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_t$ of size at most $4(r+t-1)$. Let $A_1 := B \cap (D \cup \widehat{D})$. For each i from 1 to t, do the following. According to our assumption $v \notin N_4(A_i) \cup A_i$, which means by Proposition 3 that there is a path of length at most 4 from v to a vertex of D, and this path does not contain any vertices from A_i. Denote by P_i the vertices of a minimum length path, which satisfies these properties. Define $A_{i+1} := A_i \cup (P_i \cap (D \cup \widehat{D}))$ and proceed to the next iteration to construct P_{i+1}.

Note that $|A_1| \leq 4r$ and $|A_{i+1}| \leq |A_i| + 4$. Thus, all the sets A_i are of size at most $4r + 4(t-1) = 4(r+t-1)$. After completing this process, we get t paths of length at most 4 that start at v. Note that a vertex $u \in D \cup \widehat{D}$ can participate in at most one of these paths, since once it appears in a path P_i, it is immediately added to A_{i+1}. Because of the maximality of q, each path P_i must contain a vertex of B'. From now on, we will consider the last appearance of a vertex from B' in a path P_i as the starting point of the path. This means that all the paths P_i start at a vertex of B' and are of length at most 3. Since $|B'| \leq 3q \leq 3r$ and the number of paths is $t = 3r(r + r^2 + r^4) + 1$, by the pigeonhole principle there must be a vertex $b \in B'$ that is a starting point of $r + r^2 + r^4 + 1$ paths of length at most 3. We now prove that $b \in \widehat{D}$. There are three possible cases.

Case 1: The vertex b starts at least $r+1$ paths of length 1. This means that b is adjacent to $r+1$ vertices of D and therefore $b \in \widehat{D}$.

Case 2: The vertex b starts at least $r^2 + 1$ paths of length 2. It follows from the construction that all these paths are from b to a different vertex of D. A vertex u cannot be the middle vertex of more than r of these paths, since this would imply that $u \in \widehat{D}$, but as mentioned before, vertices of $\widehat{D}$ can appear in at most one path. Thus, there are at least $r+1$ middle vertices that are part of $r+1$ vertex disjoint paths of length 2 from b to D, which implies that $b \in \widehat{D}$.

Case 3: The vertex b starts at least $r^4 + 1$ paths of length 3. The vertex b is the first vertex of these paths, whereas the fourth vertex is always a different vertex from D. Denote by U_2 and U_3 the vertices that appear as a second and third vertex on one of these paths, respectively. Recall that when creating the paths P_i, we always chose a path of minimum length that leads to a vertex of D. This implies that $U_2 \cap U_3 = \emptyset$. As before, vertices of U_2 and U_3 can belong to at most r^2 and r paths, respectively. The total number of paths is $r^4 + 1$, and therefore $|U_2| \geq r^2 + 1$. Since a vertex of U_3 belongs to at most r paths, we can find $r+1$ vertices of U_2 that can be matched to $r+1$ different vertices of U_3 in a way which would give $r+1$ vertex disjoint paths of length 3 from b to $r+1$ different vertices of D. This implies that $b \in \widehat{D}$.

In all three cases $b \in \widehat{D}$, which means that $b \in A_1$. Thus, b cannot belong to any path P_i, and we get a contradiction. □

Theorem 4. *For every fixed h, given a graph G with n vertices that does not contain K_h as a topological minor, there is an $O(n^{3.5} + k^{O(1)})$ time algorithm that constructs a subgraph G' of G, such that if $\gamma(G) = k$, then $\gamma(G') = k$ and G' has at most k^c vertices, where c is a constant that depends only on h.*

Proof. Let s be the constant from Proposition 1. Suppose that the graph G contains no K_h as a topological minor and $\gamma(G) = k > 1$. To construct the kernel, we perform at most n iterations, as follows. The iteration starts by applying the $O(nk)$ time approximation algorithm described in Theorem 3 in order to compute a dominating set D of size at most $(3sh)^h k$. It followed from Lemmas 2 and 3 that the set $\widehat{D}_{h-1,3}$ is of size at most $(6sh^2)^{3h}|D|$, and can be computed in time $O(n^{2.5})$. In case there is a subset $A \subseteq D \cup \widehat{D}_{h-1,3}$ of size $40(h-2)^5$, for which the conditions of Rule 2 are satisfied, then the rule is applied. It follows from Lemma 5 that at least one vertex is removed from the graph and the domination number does not change. We continue to the next iteration with the resulting graph. Upon termination, this process computes a kernel G' with $\gamma(G') = k$, and a dominating set D of size at most $(3sh)^h k$.

As for the kernel size, Lemma 2 implies that $|\widehat{D}_{h-1,4}| = O(k)$, whereas from Lemma 6 we know that if $v \notin D \cup \widehat{D}_{h-1,4}$, then there exists a subset $A \subseteq D \cup \widehat{D}_{h-1,3}$ of size at most $40(h-2)^5$, such that $v \in N_4(A)$. The number of such subsets A is $k^{O(1)}$ and it follows from Lemma 5 that each subset A satisfied that $N_4(A) = O(1)$, since Rule 2 cannot be applied anymore. We conclude that the number of vertices not in $D \cup \widehat{D}_{h-1,4}$ is $k^{O(1)}$, and the theorem is proved. □

Theorem 5. *There is an $O(n^{3.5}+2^{O(\sqrt{k})})$ time algorithm for finding a dominating set of size at most k in an H-minor-free graph with n vertices that contains such a set.*

Proof. Construct a problem kernel G' using Theorem 4 and apply the $2^{O(\sqrt{k})}n^c$ time algorithm of Demaine et al. [11] on the graph G'. □

6 Concluding Remarks and Open Problems

- The dominating set problem is fixed-parameter tractable for degenerated graphs. An interesting open problem, stated in a preliminary version of this paper [6], is to decide whether there is a polynomial size kernel in this case. This problem has been very recently resolved by Philip et al. [24], who exhibited a polynomial kernel in $K_{i,j}$-free and degenerated graphs. In their reduction, the kernel constructed is not a subgraph of the input graph, and therefore the property of being H-minor-free is not preserved. This is why the construction of Philip et al. cannot be used for obtaining Theorem 5.
- Another challenging question is to characterize the families of graphs for which the dominating set problem admits a linear kernel. We cannot rule out the possibility that a linear kernel can be obtained for graphs with any fixed excluded minor.

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7 Appendix

7.1 Bounds on the Number of Central Vertices

The following Lemma from [5] gives an upper bound on the number of cliques of a prescribed fixed size in a degenerated graph.

Lemma 7. *If a graph G with n vertices is d-degenerated, then for every $k \geq 1$, G contains at most $\binom{d}{k-1}n$ copies of K_k.*

Given a graph with no topological $K_{m,h}$ and a specific set M of m vertices, we would like to bound the number of vertices that have m vertex disjoint paths of *total* length at most l to the m vertices of M.

Lemma 8. *Suppose that $G = (V, E)$ does not contain $K_{m,h}$ as a topological minor and $M \subseteq V$ is a set of $m > 1$ vertices. The number of vertices $v \notin M$ that have m vertex disjoint paths of total length at most l to the m vertices of M is less than $h(m(h-1))^{l-m}(l-m+1)!$.*

Proof. Define $f(m,h,l) := h(m(h-1))^{l-m}(l-m+1)!$. The lemma is proved by induction on l. For $l = m$, the lemma bounds the number of vertices that are adjacent to all the vertices of M, and the claim follows from the fact that G does not have a subgraph isomorphic to $K_{m,h}$. Suppose that the lemma is true for $l \geq m$. We prove it for $l+1$, as follows. Given a set M of m vertices, denote by Q the set of vertices $v \notin M$ that have m vertex disjoint paths of total length at most $l+1$ to the m vertices of M, and denote by P_v the set of all vertices on these m paths, except for the vertices of M. Thus, $|P_v| \leq l+2-m$ and $v \in P_v$.

We first prove that each vertex $w \notin M$ belongs to at most $m \cdot f(m,h,l) - m + 1$ of the sets $\{P_v\}_{v \in Q}$. Suppose, by contradiction, that there is a vertex w that belongs to more than $m \cdot f(m,h,l) - m + 1$ of the sets P_v. Since possibly $w \in P_w$, there are more than $m(f(m,h,l)-1)$ vertices v for which $w \in P_v$ and w is an inner vertex of one of the m paths from v to M. Thus, there is some vertex $u \in M$, such that there are at least $f(m,h,l)$ vertices v for which $w \in P_v$ and w appears in P_v on the path from v to u. This means that there are $f(m,h,l)$ vertices v for which there are m vertex disjoint paths of total length at most l to the m vertices $(M \setminus \{u\}) \cup \{w\}$. This contradicts the induction hypothesis.

We showed that every vertex w belongs to at most $m \cdot f(m,h,l) - m + 1$ of the sets P_v. To prove the lemma for $l+1$, assume, by contradiction, that $|Q| \geq f(m,h,l+1)$. Assign $W := Q$. We do the following $h-1$ times, for each i, $1 \leq i \leq h-1$. Let w_i be an arbitrary element of W. Note that $|P_{w_i}| \leq l+2-m$. Remove from W all the vertices $v \in W$ such that $P_v \cap P_{w_i} \neq \emptyset$, and continue to choose an arbitrary $w_{i+1} \in W$. At each step, at most $(l+2-m)(m \cdot f(m,h,l) - m + 1)$ vertices are removed from W, so after $h-1$ steps, at most $(h-1)(l+2-m)(m \cdot f(m,h,l) - m + 1) < f(m,h,l+1)$ vertices will be removed from W. Thus, the $h-1$ iterations can indeed be performed, and at the end, the set W is not empty, so we can choose the last arbitrary element $w_h \in W$. The vertices of M together with the h vertices $w_1, \ldots, w_h$ form a topological $K_{m,h}$. This is a contradiction and the lemma is proved. □

The following result bounds the number of central vertices in graphs with no topological $K_{m,h}$.

Lemma 9. *Let s be the constant from Proposition 1. If $G = (V, E)$ does not contain $K_{m,h}$ as a topological minor, and $U \subseteq V$ is of size k, then for every l, $|\widehat{U}_{m,l}| \le (smhl)^{2m^2l^2}k$*

Proof. Denote $d = s(m + h)^2$. The proof is similar to that of Lemma 2, so we highlight only the modifications needed. Initially we set B to be equal to U. During a phase, if there is still a vertex $w \notin B$ for which there are two vertex disjoint paths of length at most l from w to two vertices $b_1, b_2 \in B$, such that b_1 and b_2 are not connected, and all the inner vertices of the two paths are not in B, then add the edge $\{b_1, b_2\}$ and remove the vertex w from the graph together with the two paths. Denote the resulting graph by G'. It follows from the analysis of Lemma 2 that at most $sh^2(2l-1)|B|$ vertices are removed during the phase. All the removed vertices are later added to B.

In addition to the vertices that were added to B in the way described, we would like to add more vertices to B, as follows. Consider a vertex $v \in \widehat{U}$ at the beginning of the phase. There are m vertex disjoint paths of length at most l from v to a set M of m different vertices. Assume that none of the vertices on these m paths were removed during the phase. This means that if v was not removed either, then this can only happen in case $G'[M]$ is a clique of size m. In this case we also add v to B. We now count the number of vertices v of this type. Since G' does not contain $K_{m,h}$ as a topological minor, we get from Lemma 8 that there can be at most $h(m(h-1))^{lm-m}(lm-m+1)!$ vertices v with m vertex disjoint paths of length at most l from v to the m vertices of M. It follows from Lemma 7 that there are at most $\binom{d}{m-1}|B| \le (s(m+h)^2)^{m-1}|B|$ cliques of size m in $G'[B]$, which means that at most $(s(m+h)^2)^{m-1}h(m(h-1))^{lm-m}(lm-m+1)!|B|$ vertices of $\widehat{U}$ were not accounted for. The total number of vertices that are added to B during a phase is therefore less than $(smhl)^{2ml}|B|$, whereas the number of phases is at most $m(l-1)+1$. This gives the needed bound for $|\widehat{U}|$. □

7.2 Problem Kernel for Graphs with no Topological $K_{3,h}$

All graphs considered in this section contain no $K_{3,h}$ as a topological minor, for some fixed h. Whenever using the big Oh notation, the hidden constant depends only on h. We use the following definitions from [4] concerning the neighborhood of a single vertex and the neighborhood of a pair of vertices.

Definition 2. *Consider a vertex $v \in V$ of a given graph $G = (V, E)$. The neighborhood of v is partitioned into three disjoint sets $N_1(v)$, $N_2(v)$, and $N_3(v)$.*

- $N_1(v) := \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\}$
- $N_2(v) := \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\}$
- $N_3(v) := N(v) \setminus (N_1(v) \cup N_2(v))$

Definition 3. *Consider two distinct vertices $v, w \in V$ of a given graph $G = (V, E)$. The neighborhood of the two vertices is partitioned into three disjoint sets $N_1(v, w)$, $N_2(v, w)$, and $N_3(v, w)$.*

- $N_1(v, w) := \{u \in N(v, w) : N(u) \setminus N[v, w] \neq \emptyset\}$
- $N_2(v, w) := \{u \in N(v, w) \setminus N_1(v, w) : N(u) \cap N_1(v, w) \neq \emptyset\}$
- $N_3(v, w) := N(v, w) \setminus (N_1(v, w) \cup N_2(v, w))$

Here is a simple observation that follows immediately from the previous definitions.

Proposition 4. *Let D be a dominating set of a graph G. If $u \notin N_3(v) \cup \{v\}$, then there is a path of length at most 3 from u to a vertex of D, and the path does not contain v. If $u \notin N_3(v, w) \cup \{v, w\}$, then there is a path of length at most 3 from u to a vertex of D, and the path contains neither v nor w.*

The following is a simplified presentation of the two reduction rules of Alber et al. [4]. As proved there, these reduction rules do not change the domination number of the graph. Unlike the original rules, in which new vertices can be added to the graph, in our formulation the only modifications made to the graph are the removal of vertices and edges. Another useful property of the following formulation is that in case a rule is applied, at least one vertex is removed from the graph. Unlike the original rules, vertices that belong to $N_2(v)$ or $N_2(v, w)$ are not removed in our rules. We note that Alber et al. specifically proved that their rules preserve planarity, and this might imply that the authors did not notice the fact that the reduction rules actually construct a subgraph of the input graph, and therefore all monotone properties are preserved.

Rule 3: Given a graph $G = (V, E)$ and a vertex $v \in V$, if $|N_3(v)| > 1$, then do the following. Let v' be some arbitrary vertex of $N_3(v)$. Remove all the vertices of $N_3(v) \setminus \{v'\}$ and all the edges incident at v', except for $\{v, v'\}$.

Rule 4: Let v and w be two distinct vertices of the graph $G = (V, E)$. If $|N_3(v, w)| > 2$ and $N_3(v, w)$ cannot be dominated by a single vertex from $N_2(v, w) \cup N_3(v, w)$, then do the following.

- If both v and w dominate $N_3(v, w)$, then let z and z' be two arbitrary distinct vertices of $N_3(v, w)$. Remove all the vertices of $N_3(v) \setminus \{z, z'\}$ and all the edges incident at z and z', except for the edges $\{v, z\}, \{w, z\}, \{v, z'\}, \{w, z'\}$.
- If v dominates $N_3(v, w)$ but w does not dominate it, then let v' be some arbitrary vertex of $N_3(v, w)$. Remove all the vertices of $N_3(v) \setminus \{v'\}$ and all the edges incident at v', except for the edge $\{v, v'\}$. The case that only w dominates $N_3(v, w)$ is handled in a symmetric manner.
- If neither v nor w dominate $N_3(v, w)$, then let v' and w' be two arbitrary distinct vertices of $N_3(v, w)$ such that v' is adjacent to v and w' is adjacent to w. Remove all the vertices of $N_3(v) \setminus \{v', w'\}$ and all the edges incident at v' and w', except for the edges $\{v, v'\}, \{w, w'\}$.

A graph is called *reduced* in case Rules 3 and 4 cannot be applied to it anymore. The following definitions are specific to this section.

Definition 4. *Let D be a dominating set of the graph $G = (V, E)$.*

- *Denote by $\widetilde{D}$ the set of vertices in $V \setminus D$ that have at least two neighbors from D.*
- *Suppose that $d_1, d_2 \in D$ are two distinct vertices. Denote by $Inner(d_1, d_2)$ the set of all inner vertices of paths of length 3 of the type $d_1 - x - y - d_2$, such that $x, y \in N_3(d_1, d_2) \setminus (D \cup \widehat{D}_{3,3} \cup \widetilde{D})$. Denote $Inner(D) := \bigcup_{d_1,d_2 \in D, d_1 \neq d_2} Inner(d_1, d_2)$*

Lemma 10. *For a fixed $h \geq 2$, suppose that $G = (V, E)$ is a reduced graph that contains no $K_{3,h}$ as a topological minor. If D is a dominating set of size k, then $|\widetilde{D}| = O(k)$.*

Proof. Assume that $v \in \widetilde{D}$. This means that v is adjacent to at least 2 vertices of D, so we distinguish between three cases.

Case 1: The vertex v is adjacent to at least 3 vertices of D. Thus, by definition $v \in \widehat{D}_{3,3}$, and it follows from Lemma 9 that $|\widehat{D}_{3,3}| = O(k)$.

Case 2: The vertex v is adjacent to exactly 2 vertices $d_1, d_2 \in D$ and $v \notin N_3(d_1, d_2)$. It follows from proposition 4 that there is a path of a length at most 3 from v to a vertex of D, and the path does not use the vertices d_1 and d_2. This implies that $v \in \widehat{D}_{3,3}$ and we proceed as in the previous case.

Case 3: The vertex v is adjacent to exactly 2 vertices $d_1, d_2 \in D$ and $v \in N_3(d_1, d_2)$. The number of pairs $d_1, d_2 \in D$ for which there is a vertex $v \notin D$ such that $N(v) \cap D = \{d_1, d_2\}$ is $O(k)$. To see this, just connect each such pair d_1, d_2, in case they were not connected before. Denote the resulting graph by G'. The number of edges in $G'[D]$ is at least the number of pairs we are counting. Since $G'[D]$ does not contain $K_{3,h}$ as a topological minor, it has $O(k)$ edges.

For two distinct vertices $d_1, d_2 \in D$, denote by Q the set of vertices $v \in N_3(d_1, d_2)$ that are adjacent to both d_1 and d_2. It is now enough to prove that $|Q| \leq h$. By contradiction, assume that $|Q| > h \geq 2$. Since the graph is reduced, there is a vertex $w \in N_2(d_1, d_2) \cup N_3(d_1, d_2)$ that dominates $N_3(d_1, d_2)$. Note that w can possibly belong to $N_3(d_1, d_2)$. This implies that d_1,d_2, and w together with $Q \setminus \{w\}$ form a $K_{3,h}$. This is a contradiction, and the claim is proved. □

Corollary 1. *For a fixed $h \geq 2$, let D be a dominating set of size k of a reduced graph $G = (V, E)$ that contains no $K_{3,h}$ as a topological minor. If a subset $U \subseteq V$ of size m satisfies that $D \cap U = \emptyset$, then $|N[U]| = O(k + m)$.*

Proof. The set $D \cup U$ is obviously a dominating set. A vertex $v \in N[U] \setminus (D \cup U)$ is adjacent to a vertex of U and also to a vertex of D, since D is a dominating set. This means that v is adjacent to at least two vertices of $D \cup U$. The result now follows from Lemma 10. □

Lemma 11. *Suppose that $G = (V, E)$ is a reduced graph that contains no $K_{3,h}$ as a topological minor. If D is a dominating set of size k, then there are $O(k)$ pairs $d_1, d_2 \in D$ for which $Inner(d_1, d_2) \neq \emptyset$.*

Proof. Consider the pairs $d_1, d_2 \in D$ for which $Inner(d_1, d_2) \neq \emptyset$ in some arbitrary order. For each such pair d_1, d_2, there are two vertices $x, y \in N_3(d_1, d_2) \setminus (D \cup \widehat{D}_{3,3} \cup \widetilde{D})$ that appear on the path $d_1 - x - y - d_2$. We claim that both x and y do not belong to any other pair $Inner(d'_1, d'_2)$. To see this, suppose by contradiction that $x \in Inner(d_1, d_2) \cap Inner(d'_1, d'_2)$ for $\{d'_1, d'_2\} \neq \{d_1, d_2\}$. Since $x \notin \widetilde{D}$, it has only one neighbor in D, so assume, without loss of generality, that x is adjacent to $d_1 = d'_1$ and x appears on the two paths $d_1 - x - y - d_2$ and $d_1 - x - z - d'_2$. This implies that $x \in \widehat{D}_{3,3}$, a contradiction, and the claim is proved.

In each case as above, we delete the vertices x and y, and add an edge between d_1 and d_2, assuming this edge does not exist. Denote the resulting graph by G'. Obviously, $G'[D]$ does not contain $K_{3,h}$ as a topological minor and therefore has at most $O(k)$ edges. The number of edges in the induced subgraph $G'[D]$ is at least the number of pairs for which $Inner(d_1, d_2) \neq \emptyset$, as claimed. □

Lemma 12. *Let D be a dominating set of a reduced graph $G = (V, E)$ that contains no $K_{3,h}$ as a topological minor. Every two distinct vertices $d_1, d_2 \in D$ satisfy $|Inner(d_1, d_2)| \leq 2h^2$.*

Proof. By contradiction, assume that $|Inner(d_1, d_2)| \geq 2h^2 + 1$. This implies that $|N_3(d_1, d_2)| > 2$, and since the graph is reduced, there is a vertex $v \in N_2(d_1, d_2) \cup N_3(d_1, d_2)$ that dominates $N_3(d_1, d_2)$. Let q be the maximum number of vertex disjoint paths of the type $d_1 - x - y - d_2$, such that $x, y \in Inner(d_1, d_2)$, and denote by W the $2q$ inner vertices of these paths. Note that v can possibly belong to W. We must have that $q \leq h$, since otherwise d_1,d_2, and v would be part of a topological $K_{3,h}$. Since $|W| = 2q \leq 2h$, there are at least $2h(h-1)+1$ vertices of $Inner(d_1, d_2) \setminus W$ that appear on a path of the type $d_1 - x - y - d_2$ together with one of the vertices of W. Thus, there is a vertex $w \in W$ that belongs to at least h of these paths. Assuming, without loss of generality, that w is adjacent to d_1, there are $h+1$ different paths of length 2 from w to d_2, and the inner vertices of these paths are from $Inner(d_1, d_2)$. Thus, w,d_2, and v are part of a $K_{3,h}$. This is a contradiction, and the claim is proved. □

Lemma 13. *Suppose that the reduced graph $G = (V, E)$ contains no $K_{3,h}$ as a topological minor. If D is a dominating set of size k, then $|Inner(D)| = O(k)$.*

Proof. Follows immediately from Lemmas 11 and 12. □

Lemma 14. *Suppose that the reduced graph $G = (V, E)$ contains no $K_{3,h}$ as a topological minor. If D is a dominating set of size k, then the number of vertices that appear on a path of length 3 between two vertices of D is $O(k)$.*

Proof. We examine the inner vertices of paths of the form $d_1 - v - x - d_2$, such that $d_1, d_2 \in D$. It follows from Lemmas 9 and 10 that $|\widehat{D}_{3,3} \cup \widetilde{D}| = O(k)$, which means that it remains to count the number of vertices not in $D \cup \widehat{D}_{3,3} \cup \widetilde{D}$. Assume that $v \notin D \cup \widehat{D}_{3,3} \cup \widetilde{D}$. Since $v \notin \widetilde{D}$, it is adjacent to exactly one vertex of D, and therefore $x \notin D$. If $x \in \widehat{D}_{3,3} \cup \widetilde{D}$, then $v \in N[\widehat{D}_{3,3} \cup \widetilde{D}]$, but it follows from

Corollary 1 that $|N[\widehat{D}_{3,3} \cup \widetilde{D}]| = O(k)$. If either v or x do not belong to $N_3(d_1, d_2)$, then this implies that $x \in \widehat{D}_{3,3}$, but this case has already been addressed. The only remaining case is that $v, x \in N_3(d_1, d_2) \setminus (D \cup \widehat{D}_{3,3} \cup \widetilde{D})$, which means that $v \in Inner(D)$, and we know from Lemma 13 that $|Inner(D)| = O(k)$. □

We can now state the main result of this section.

Theorem 6. *For every fixed h, given a graph G that does not contain $K_{3,h}$ as a topological minor, there is a polynomial time algorithm that constructs a subgraph G' of G, such that if $\gamma(G) = k$, then $\gamma(G') = k$ and G' has at most ck vertices, where c is a constant that depends only on h.*

Proof. Suppose that G contains no $K_{3,h}$ as a topological minor and $\gamma(G) = k$. As long as the conditions of Rules 3 and 4 are satisfied, apply these rules to get a reduced subgraph G'. Alber et al. [4] proved that $\gamma(G') = k$, so let D be a dominating set of G' of size k. It follows from Lemma 10 that $|\widetilde{D}| = O(k)$, so we need to count the number of vertices not in $D \cup \widetilde{D}$. Assume $v \notin D \cup \widetilde{D}$ is adjacent to $d_1 \in D$. If $v \in N_3(d_1)$, then in a reduced graph $|N_3(d_1)| \leq 1$, which means that there could be at most k vertices of this type. Assume now that $v \notin N_3(d_1)$, so by Proposition 4 there is a path of length at most 3 from v to a vertex $d_2 \in D$, and d_1 is not part of this path. We examine a shortest path p from d_1 to d_2, in which v is the second vertex of the path. Since $v \notin \widetilde{D}$, it is adjacent to only one vertex of D, so the path p can be of length either 3 or 4.

In case p is of length 3, then it follows from Lemma 14 that there are at most $O(k)$ vertices of this type. If p is of length 4, denote it by $d_1 - v - x - y - d_2$, where $x, y \notin D$. The vertex x is adjacent to some vertex of D. It cannot be adjacent to d_2, since a path p on minimum length was chosen. If x is adjacent to a vertex of $D \setminus \{d_1, d_2\}$, then $x \in \widehat{D}_{3,3}$ and $v \in N[\widehat{D}_{3,3}]$, but it follows from Corollary 1 that $|N[\widehat{D}_{3,3}]| = O(k)$. The remaining case is that d_1 is the only vertex in D that is adjacent to x. Since $x \notin D$ is on a path of length 3 from d_1 to d_2, it follows from Lemma 14 and Corollary 1 that the number of vertices v of this type is also $O(k)$. □