A Probabilistic Approach to Problems Parameterized Above Tight Lower Bound

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Abstract. We introduce a new approach for establishing fixed-parameter tractability of problems parameterized above tight lower bounds. To illustrate the approach we consider three problems of this type of unknown complexity that were introduced by Mahajan, Raman and Sikdar (J. Comput. Syst. Sci. 75, 2009). We show that a generalization of one of the problems and non-trivial special cases of the other two are fixed-parameter tractable.

1 Introduction

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *main part* and k (usually an integer) is the *parameter*. Π is called *fixed-parameter tractable* (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ denotes the size of I, $f(k)$ is a computable function, and c is a constant independent of k and I (for further background and terminology on parameterized complexity we refer the reader to the monographs [4, 5, 14]). If the nonparameterized version of Π (where k is just a part of input) is NP-hard, then the function $f(k)$ must be superpolynomial provided P $\neq$ NP. Often $f(k)$ is "moderately exponential," which makes the problem practically feasible for small values of k. Thus, it is important to parameterize a problem in such a way that the instances with small values of k are of real interest.

Consider the following well-known problem: given a digraph $D = (V, A)$, find an acyclic subdigraph of D with the maximum number of arcs. We can parameterize this problem "naturally" by asking whether D contains an acyclic subdigraph with at least k arcs. It is easy to prove that this parameterized problem is fixed-parameter tractable by observing that D always has an acyclic subdigraph with at least $|A|/2$ arcs. (Indeed, consider a bijection $\alpha : V \to \{1, \ldots, |V|\}$ and the following subdigraphs of D: $(V, \{ xy \in A : \alpha(x) < \alpha(y) \})$ and $(V, \{ xy \in A : \alpha(x) > \alpha(y) \})$. Both subdigraphs are acyclic and at least one of them has at least $|A|/2$ arcs.) However, $k \leq |A|/2$ for every small value of k

and almost every practical value of $|A|$ and, thus, our "natural" parameterization is of almost no practical or theoretical interest.

Instead, one should consider the following parameterized problem: decide whether $D = (V, A)$ contains an acyclic subdigraph with at least $|A|/2 + k$ arcs. We choose $|A|/2 + k$ because $|A|/2$ is a *tight lower bound* on the size of a largest acyclic subdigraph. Indeed, the size of a largest acyclic subdigraph of a symmetric digraph $D = (V, A)$ is precisely $|A|/2$. (A digraph $D = (V, A)$ is *symmetric* if $xy \in A$ implies $yx \in A$.)

In a recent paper [13] Mahajan, Raman and Sikdar provided several examples of problems of this type and argued that a natural parameterization is one above a tight lower bound for maximization problems, and below a tight upper bound for minimization problems. Furthermore, they observed that only a few non-trivial results are known for problems parameterized above a tight lower bound [7, 8, 10, 12], and they listed several problems parameterized above a tight lower bound whose complexity is unknown. The difficulty in showing whether such a problem is fixed-parameter tractable can be illustrated by the fact that often we even do not know whether the problem is in XP, i.e., can be solved in time $O(|I|^{g(k)})$ for a computable function $g(k)$. For example, it is non-trivial to see that the digraph problem from above is in XP when parameterized above the $|A|/2$ bound.

In this paper we introduce the *Strictly Above Expectation Method*, a novel approach for establishing the fixed-parameter tractability of maximization problems parameterized above a tight lower bound. The new method is based on probabilistic arguments and utilizes certain probabilistic inequalities. We will state the equalities in the next section, and in the subsequent sections we will apply the Strictly Above Expectation Method to three open problems posed by Mahajan, Raman and Sikdar [13]. Essentially, our new method allows us to derive upper bounds on the size of NO-instances in terms of a function of the parameter k. If the size of a given instance exceeds this bound, then we know the answer is YES; otherwise, if its size is within the bound then it is already a *problem kernel* [4] and can be solved by any brute force algorithm. We indicate how our method can be extended to actually find a solution that witnesses a YES-answer.

In Section 3 we consider the LINEAR ORDERING problem, a generalization of the problem discussed above: Given a digraph $D = (V, A)$ in which each arc ij has a positive integral weight w_{ij}, find an acyclic subdigraph of D of maximum weight. Observe that $W/2$, where W is the sum of all arc weights, is a tight lower bound for LINEAR ORDERING. We prove that the problem parameterized above $W/2$ is fixed-parameter tractable and admits a quadratic kernel. Note that this parameterized problem generalizes the parameterized maximum acyclic subdigraph problem considered in [13]; thus, our result answers the corresponding open question of [13].

In Section 4 we consider the problem MAX LIN-2: Given a system of m linear equations $e_1, \ldots, e_m$ in n variables over GF(2), and for each equation e_j a positive integral weight w_j; find an assignment of values to the n variables that maximizes the total weight of the satisfied equations. We show in Section 4 that $W/2$, where $W = w_1 + \cdots + w_m$, is a tight lower bound for Max Lin-2. The complexity of the problem parameterized above $W/2$ is open [13]. We prove that if each equation involves a constant number of variables then the problem parameterized above the $W/2$ bound is fixed-parameter tractable and admits a quadratic kernel. We also show that if we allow the weights w_j to be positive reals, the problem is NP-hard already if $k = 1$ and each equation involves two variables.

In Section 5, we consider the problem Max Exact r-SAT: given an exact r-CNF formula $\mathcal{C}$ with m clauses (i.e., a CNF formula where each clause contains exactly r distinct literals), find a truth assignment that satisfies the maximum number of clauses. Here a tight lower bound is $(1-2^{-r})m$; the complexity of the problem parameterized above $(1-2^{-r})m$ is an open question [13]. This seems to be the most difficult problem of the three considered. We obtain a quadratic kernel for a non-trivial special case of this problem.

2 Probabilistic Inequalities

In our approach we introduce a random variable X such that the answer to the problem parameterized above a tight lower bound is yes if and only if X takes with positive probability a value greater or equal to the parameter k. If X happens to be a symmetric random variable, then the following simple inequality can be useful: $\text{Prob}(\ X \geq \sqrt{\mathbb{E}(X^2)}\ ) > 0$; see Section 3 for an application of this inequality. However, often X is not symmetric. Lemma 1 provides an inequality that can be used in many such cases. This lemma was proved by Alon, Gutin and Krivelevich [1]; a weaker version was obtained by Håstad and Venkatesh [9].

Lemma 1. *Let X be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \sigma^2 > 0$ and $\mathbb{E}(X^4) \leq b\sigma^4$, respectively. Then* $\text{Prob}(\ X > \frac{\sigma}{4\sqrt{b}}\ ) \geq \frac{1}{4^{4/3}b}$.

Often this result can be used together with the next lemma which is an extension of Khinchin's Inequality by Bourgain [3].

Lemma 2. *Let $f = f(x_1, \ldots, x_n)$ be a polynomial of degree r in n variables $x_1, \ldots, x_n$ with domain $\{-1, 1\}$. Define a random variable X by choosing a vector $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\epsilon_1, \ldots, \epsilon_n)$. Then, for every $p \geq 2$, there is a constant c_p such that*

$$(\mathbb{E}(|X|^p))^{1/p} \leq (c_p)^r (\mathbb{E}(X^2))^{1/2}.$$

In particular, $c_4 \leq 2^{3/2}$.

3 Linear Ordering

Let $D = (V, A)$ be a digraph with no loops or parallel arcs in which every arc ij has a positive weight w_{ij}. The problem of finding an acyclic subdigraph of D of maximum weight, known as LINEAR ORDERING, has applications in economics [2]. Let $n = |V|$ and consider a bijection $\alpha : V \to \{1, \dots, n\}$. Observe that the subdigraphs $(V, \{ ij \in A : \alpha(i) < \alpha(j) \})$ and $(V, \{ ij \in A : \alpha(i) > \alpha(j) \})$ are acyclic. Since the two subdigraphs contain all arcs of D, at least one of them has weight at least $W/2$, where $W = \sum_{ij \in A} w_{ij}$, the *weight* of D. Thus, $W/2$ is a lower bound on the maximum weight of an acyclic subdigraph of D. Consider a digraph D where for every arc ij of D there is also an arc ji of the same weight. Each maximum weight subdigraph of D has weight exactly $W/2$. Hence the lower bound $W/2$ is tight.

LINEAR ORDERING ABOVE TIGHT LOWER BOUND (LOALB)
Instance: A digraph $D = (V, A)$, each arc ij has an integral positive weight w_{ij}, and a positive integer k.
Parameter: The integer k.
Question: Is there an acyclic subdigraph of D of weight at least $W/2+k$, where $W = \sum_{ij \in A} w_{ij}$?

Mahajan, Raman, and Sikdar [13] asked whether LOALB is fixed-parameter tractable for the special case when all arcs are of weight 1 (i.e., D is unweighted). In this section we will prove that LOALB admits a kernel with $O(k^2)$ arcs; consequently the problem is fixed-parameter tractable. Note that if we allow weights to be positive reals, then we can show, similarly to the NP-completeness proof given in the next section, that LOALB is NP-complete already for $k = 1$.

Consider the following simple reduction rule: Assume D has a directed 2-cycle iji; if $w_{ij} = w_{ji}$ delete the cycle, if $w_{ij} > w_{ji}$ delete the arc ji and replace w_{ij} by $w_{ij} - w_{ji}$, and if $w_{ji} > w_{ij}$ delete the arc ij and replace w_{ji} by $w_{ji} - w_{ij}$. It is easy to check that the answer to LOALB for a digraph D is YES if and only if the answer to LOALB is YES for a digraph obtained from D using the reduction rule as long as possible. Thus, for the rest of this section we may restrict our considerations to oriented graphs.

Let $D = (V, A)$ be an oriented graph, let $n = |V|$ and $W = \sum_{ij \in A} w_{ij}$. Consider a random bijection: $\alpha : V \to \{1, \dots, n\}$ and a random variable $X(\alpha) = \frac{1}{2} \sum_{ij \in A} \epsilon_{ij}(\alpha)$, where $\epsilon_{ij}(\alpha) = w_{ij}$ if $\alpha(i) < \alpha(j)$ and $\epsilon_{ij}(\alpha) = -w_{ij}$, otherwise. It is easy to see that $X(\alpha) = \sum \{ w_{ij} : ij \in A, \alpha(i) < \alpha(j) \} - W/2$. Thus, the answer to LOALB is YES if and only if there is a bijection $\alpha : V \to \{1, \dots, n\}$ such that $X(\alpha) \geq k$. Since $\mathbb{E}(\epsilon_{ij}) = 0$, we have $\mathbb{E}(X) = 0$.

Let $W^{(2)} = \sum_{ij \in A} w_{ij}^2$. We will prove the following:

Lemma 3. $\mathbb{E}(X^2) \geq W^{(2)}/12$.

Proof. Let $N^+(i)$ and $N^-(i)$ denote the sets of out-neighbors and in-neighbors of a vertex i in D. By the definition of X,

$$4 \cdot \mathbb{E}(X^2) = \sum_{ij \in A} \mathbb{E}(\epsilon_{ij}^2) + \sum_{ij, pq \in A} \mathbb{E}(\epsilon_{ij} \epsilon_{pq}), \tag{1}$$

where the second sum is taken over ordered pairs of arcs. Clearly,

$$\sum_{ij \in A} \mathbb{E}(\epsilon_{ij}^2) = W^{(2)}. \tag{2}$$

To compute $\sum_{ij,pq\in A}\mathbb{E}(\epsilon_{ij}\epsilon_{pq})$ we consider the following cases:

Case 1: $\{i,j\}\cap\{p,q\}=\emptyset$. Then ϵ_{ij} and ϵ_{pq} are independent and $\mathbb{E}(\epsilon_{ij}\epsilon_{pq})=\mathbb{E}(\epsilon_{ij})\mathbb{E}(\epsilon_{pq})=0$.

Case 2a: $|\{i,j\}\cap\{p,q\}|=1$ and $i=p$. Since the probability that $i<\min\{j,q\}$ or $i>\max\{j,q\}$ is $2/3$, $\epsilon_{ij}\epsilon_{iq}=w_{ij}w_{iq}$ with probability $2/3$ and $\epsilon_{ij}\epsilon_{iq}=-w_{ij}w_{iq}$ with probability $1/3$. Thus, $\sum_{ij,iq\in A}\mathbb{E}(\epsilon_{ij}\epsilon_{iq})=\frac{1}{3}\sum\{\,w_{ij}w_{iq}:j\neq q\in N^+(i)\,\}=\frac{1}{3}(\sum_{j\in N^+(i)}w_{ij})^2-\frac{1}{3}\sum_{j\in N^+(i)}w_{ij}^2$.

Case 2b: $|\{i,j\}\cap\{p,q\}|=1$ and $j=q$. Similarly to Case 2a, we obtain $\sum_{ij,pj\in A}\mathbb{E}(\epsilon_{ij}\epsilon_{pj})=\frac{1}{3}(\sum_{i\in N^-(j)}w_{ij})^2-\frac{1}{3}\sum_{i\in N^-(j)}w_{ij}^2$.

Case 3a: $|\{i,j\}\cap\{p,q\}|=1$ and $i=q$. Since $\epsilon_{ij}\epsilon_{pi}=w_{ij}w_{pi}$ with probability $1/3$ and $\epsilon_{ij}\epsilon_{pi}=-w_{ij}w_{pi}$ with probability $2/3$, we obtain $\sum_{ij,pi\in A}\mathbb{E}(\epsilon_{ij}\epsilon_{pi})=-\frac{1}{3}\sum\{\,w_{ij}w_{pi}:j\in N^+(i),\ p\in N^-(i)\,\}=-\frac{1}{3}\sum_{j\in N^+(i)}w_{ij}\sum_{p\in N^-(i)}w_{pi}$.

Case 3b: $|\{i,j\}\cap\{p,q\}|=1$ and $j=p$. Similarly to Case 3a, we obtain $\sum_{ij,jq\in A}\mathbb{E}(\epsilon_{ij}\epsilon_{jq})=-\frac{1}{3}\sum_{i\in N^-(j)}w_{ij}\sum_{q\in N^+(j)}w_{jq}$.

Equations (1) and (2) and the results of the above cases imply that

$$4\cdot\mathbb{E}(X^2)=W^{(2)}+\frac{1}{3}(Q-R),$$

where $Q=\sum_{i\in V}\big(\sum_{j\in N^+(i)}w_{ij})\big)^2-\sum_{j\in N^+(i)}w_{ij}^2+\big(\sum_{j\in N^-(i)}w_{ji}\big)^2-\sum_{j\in N^-(i)}w_{ji}^2$, and $R=2\cdot\sum_{i\in V}\big(\sum_{j\in N^+(i)}w_{ij}\big)\big(\sum_{j\in N^-(i)}w_{ji}\big)$. By the inequality of arithmetic and geometric means, for each $i\in V$, we have $\big(\sum_{j\in N^+(i)}w_{ij}\big)^2+\big(\sum_{j\in N^-(i)}w_{ji}\big)^2-2\big(\sum_{j\in N^+(i)}w_{ij}\big)\big(\sum_{j\in N^-(i)}w_{ji}\big)\geq 0$. Therefore, $Q-R=-\sum_{i\in V}\sum_{j\in N^+(i)}w_{ij}^2-\sum_{i\in V}\sum_{j\in N^-(i)}w_{ji}^2=-2W^{(2)}$, and $4\cdot\mathbb{E}(X^2)\geq W^{(2)}-2W^{(2)}/3=W^{(2)}/3$, implying $\mathbb{E}(X^2)\geq W^{(2)}/12$. □

Now we can prove the main result of this section.

Theorem 1. *The problem* LOALB *admits a kernel with* $O(k^2)$ *arcs.*

Proof. It is easy to check that Prob($Y\geq\sqrt{\mathbb{E}(Y^2)}$) >0 for a symmetric random variable Y. By definition, X is symmetric. Thus, by Lemma 3, we have Prob($X\geq\sqrt{W^{(2)}/12}$) >0. Hence, if $\sqrt{W^{(2)}/12}\geq k$, there is a bijection $\alpha:V\to\{1,\dots,n\}$ such that $X(\alpha)\geq k$ and, thus, the answer to LOALB is YES. Otherwise, $|A|\leq W^{(2)}<12\cdot k^2$. □

We could prove Theorem 1 using Lemmas 1 and 2 as in the next section. For this, we could represent X as $X(\alpha)=\frac{1}{2}\sum_{ij\in A}w_{ij}\epsilon_{ij}(\alpha)$, where $\epsilon_{ij}(\alpha)=1$ if $\alpha(i)<\alpha(j)$ and $\epsilon_{ij}(\alpha)=-1$, otherwise. Thus, X is a polynomial of degree 1. However, a proof using Lemmas 1 and 2 yields the inferior bound $|A|<12288\cdot(k-1)^2$.

We close this section by outlining how Theorem 1 can be used to actually find a solution to LOALB if one exists. Let (D, k) be an instance of LOALB where $D = (V, A)$ is a directed graph with integral positive arc-weights and $k \geq 1$ is an integer. Let W be the total weight of D. As discussed above, we may assume that D is an oriented graph. If $|A| < 12k^2$ then we can find a solution, if one exists, by trying all subsets $A' \subseteq A$, and testing whether (V, A') is acyclic and has weight at least $W/2 + k$; this search can be carried out in time $2^{O(k^2)}$. Next we assume $|A| \geq 12k^2$. We know by Theorem 1 that (D, k) is a YES-instance; it remains to find a solution.

For a vertex $i \in V$ let $d_D(i)$ denote its unweighted degree in D, i.e., the number of arcs (incoming or outgoing) that are incident with i. Consider the following reduction rule: If there is a vertex $i \in V$ with $|A| - 12k^2 \geq d_D(i)$, then delete i from D. Observe that by applying the rule we obtain again a YES-instance $(D - i, k)$ of LOALB since $D - i$ has still at least $12k^2$ arcs. Moreover, if we know a solution D'_i of $(D - i, k)$, then we can efficiently obtain a solution D' of (D, k): if $\sum_{j \in N^+(i)} w_{ij} \geq \sum_{j \in N^-(i)} w_{ij}$ then we add i and all outgoing arcs $ij \in A$ to D'_i; otherwise, we add i and all incoming arcs $ji \in A$ to D'_i. After multiple applications of the rule we are left with an instance (D_0, k) to which the rule cannot be applied. Let $D_0 = (V_0, A_0)$. We pick a vertex $i \in V_0$. If i has a neighbor j with $d_{D_0}(j) = 1$, then $|A_0| \leq 12k^2$, since $|A_0| - d_{D_0}(j) < 12k^2$. On the other hand, if $d_{D_0}(j) \geq 2$ for all neighbors j of i, then i has less than $2 \cdot 12k^2$ neighbors, since $D_0 - i$ has less than $12k^2$ arcs; thus $|A_0| < 3 \cdot 12k^2$. Therefore, as above, time $2^{O(k^2)}$ is sufficient to try all subsets $A'_0 \subseteq A_0$ to find a solution to the instance (D_0, k). Let n denote the input size of instance (D, k). The reduction rule can certainly be applied in polynomial time $n^{O(1)}$, and we apply it less than n times. Hence, we can find a solution to (D, k), if one exists, in time $n^{O(1)} + 2^{O(k^2)}$.

One could call (D_0, k) a "faithful kernel" as from a solution for (D_0, k) we can construct in polynomial time a solution for (D, k).

4 Max Lin-2

Consider a system of m linear equations $e_1, e_2, \ldots, e_m$ in n variables $x_1, x_2, \ldots, x_n$ over GF(2), and suppose that each equation e_j has a positive integral weight w_j, $j = 1, 2, \ldots, m$. The problem MAX LIN-2 asks for an assignment of values to the variables that maximizes the total weight of the satisfied equations. Let $W = w_1 + \cdots + w_m$.

To see that the total weight of the equations that can be satisfied is at least $W/2$, we assign values to the variables $x_1, \ldots, x_n$ sequentially and simplify the system as we go along. When we are about to assign a value to x_i, we consider all equations reduced to the form $x_i = b$, for a constant b, and we choose a value for x_i satisfying at least half (in the weighted sense) of these equations. To see that the lower bound $W/2$ for the total weight of the satisfied equations is tight, consider a system consisting of pairs of equations of the form $\prod_{i \in I} x_i = 1$ and $\prod_{i \in I} x_i = 0$ where both equations have the same weight.

The parameterized complexity of MAX LIN-2 parameterized above the tight lower bound $W/2$ was stated by Mahajan, Raman and Sikdar [13] as an open question:

> MAX LIN-2 PARAMETERIZED ABOVE TIGHT LOWER BOUND (LINALB)
> *Instance:* A system S of m linear equations $e_1, \ldots, e_m$ in n variables $x_1, x_2, \ldots, x_n$ over GF(2), each equation e_i with a positive integral weight w_i, $i = 1, 2, \ldots, m$, and a positive integer k.
> *Parameter:* The integer k.
> *Question:* Is there an assignment of values to the variables $x_1, \ldots, x_n$ such that the total weight of the satisfied equations is at least $W/2 + k$, where $W = \sum_{i=1}^{m} w_i$?

In LinALB, we assume that every variable appears at most once in an equation. Let r_j be the number of variables in equation e_j, and let $r(S) = \max_{i=1}^{m} r_j$. We are not able to determine whether LINALB is fixed-parameter tractable or not, but we can prove that the special case where r is constant is fixed-parameter tractable and admits a quadratic kernel.

Notice that in our formulation of LINALB it is required that each equation has a positive integral weight. In a relaxed setting in which an equation may have any positive real number as its weight, the problem is NP-complete even for $k = 1$ and each $r_j = 2$. Indeed, let each linear equation be of the form $x_u + x_v = 1$. Then the problem is equivalent to MAXCUT, the problem of finding a cut of total weight at least L in an undirected graph G, where $V(G)$ is the set of variables, $E(G)$ contains (x_u, x_v) if and only if there is a linear equation $x_u + x_v = 1$, and the weight of an edge (x_u, x_v) equals the weight of the corresponding linear equation. The problem MAXCUT is a well-known NP-complete problem. Let us transform an instance I of MAXCUT into an instance I' of the "relaxed" LINALB by replacing the weight w_i by $w_i' := w_i/(L - W/2)$. We may assume that $L - W/2 > 0$ since otherwise the instance is immediately seen as a YES-instance. Observe that the new instance I' has an assignment of values with total weight at least $W'/2 + 1$ if and only if I has a cut with total weight at least L. We are done.

Consider the following simple reduction rule for LINALB. If we have, for a subset I of $\{1, 2, \ldots, n\}$, the equation $\prod_{i \in I} x_i = 0$ with weight w', and the equation $\prod_{i \in I} x_i = 1$ with weight w'', then we replace this pair by the equation whose weight is bigger, modifying its new weight to be the difference of the two old ones. If the resulting weight is 0, we omit the equation from the system. Note that the reduction rule preserves bounds on the values of r_j. If this reduction rule is not applicable to a system we call the system *irreducible*. Repeated applications of the reduction rule will result in an irreducible system. The problem LINALB for the original system and the reduced system have the same answer; thus, we may assume that the system under consideration is irreducible, i.e., the left hand sides of the equations are pairwise distinct.

Let $I_j \subseteq \{1, 2, \ldots, n\}$ be the set of indices of the variables participating in equation e_j, and let $b_j \in \{0, 1\}$ be the right hand side of e_j. Define a random variable $X = \sum_{j=1}^m X_j$, where $X_j = (-1)^{b_j} w_j \prod_{i \in I_j} \epsilon_i$ and all the ϵ_i are independent uniform random variables on $\{-1, 1\}$ (X was first introduced in [1]).

We set $x_i = 0$ if $\epsilon_i = 1$ and $x_i = 1$, otherwise, for each i. Observe that $X_j = w_j$ if e_j is satisfied and $X_j = -w_j$, otherwise. Thus, X is the difference between the weights of satisfied and non-satisfied equations. Therefore, the weight of the satisfied equations equals $(X + W)/2$, and it is at least $W/2 + k$ if and only if $X \geq 2k$.

Since $X_j = w_j$ if e_j is satisfied and $X_j = -w_j$ otherwise, and since each of the two events has probability 1/2, it follows that $\mathbb{E}(X_j) = 0$. Hence $\mathbb{E}(X) = 0$ by linearity of expectation. Moreover,

$$\mathbb{E}(X^2) = \sum_{j=1}^{m} \mathbb{E}(X_j^2) + \sum_{1 \leq j \neq q \leq m} \mathbb{E}(X_j X_q) = \sum_{j=1}^{m} w_j^2 > 0.$$

Since X is a polynomial of degree at most r, it follows by Lemma 2 that $\mathbb{E}(X^4) \leq 2^{6r}\mathbb{E}(X^2)^2$. This inequality and the results in the previous paragraph show that the conditions of Lemma 1 are satisfied and, thus,

$$\text{Prob}\left(X > \frac{\sqrt{\sum_{j=1}^m w_j^2}}{4 \cdot 8^r} \right) > 0, \quad \text{implying} \quad \text{Prob}\left(X > \frac{\sqrt{m}}{4 \cdot 8^r} \right) > 0.$$

Consequently, if $2k-1 \leq \sqrt{m}/(4 \cdot 8^r)$, then there is an assignment of values to the variables $x_1, \ldots, x_n$ which satisfies equations of total weight at least $W/2+k$. If, on the other hand, $2k-1 > \sqrt{m}/(4 \cdot 8^r)$, and $r = O(1)$, then $m < 16(2k-1)^2 64^r$, and we can check whether there is an assignment which satisfies equations of total weight at least $W/2 + k$ by exhaustive search. Thus, we have proved the following:

Theorem 2. *The special case of* LinALB *where each equation involves a constant number of variables is fixed-parameter tractable and admits a quadratic kernel.*

Consider an instance (S, k) of LinALB over n variables with $r(S) = O(1)$, $|S| = m \geq 16(2k-1)^2 64^r =: f(k, r)$. We can find a solution to for (S, k) (i.e., an assignment satisfying at least $W/2 + k$ equations) in polynomial time by using the observation from [1] that the random variables ϵ_i are $4r$-wise independent and, thus, one can use an $O(n^{2r})$-size sample space to support each ϵ_i (for more details and the sample space construction, see [1]).

Alternatively, we can use a modification of the approach given earlier for LOALB to obtain in polynomial time a faithful kernel. We consider two reduction rules.

Rule 1: If there is some variable x of S that occurs in at most $|S| - f(k, r)$ equations, then remove all the equations from S in which x occurs.

Rule 2: If there are two variables x, y occurring in exactly the same equations of S, then remove x and y from all equations and add instead a new variable z

such that an equation contains z positively if before the change it contained x and y with the same parity, and an equation contains z negatively if before the change it contained x and y with opposite parities.

Let S' be the system obtained from S by applying Rules 1 and 2 as long as possible. Using a similar argument as in the previous section, we can show that (S', k) is still a YES-instance, and that we can efficiently transform a solution for (S', k) to a solution for (S, k). To show that the number of equations in S' is polynomially bounded in k, let x be a variable of S' that occurs in the smallest number of equations of S', and let $S' = S'_1 \cup S'_2$ where all equations in S'_1 contain x and no equations in S'_2 contains x. Since S' is reduced under Rule 1, $|S'_2| < f(k, r) = O(k^2)$; thus S'_2 involves at most $r \cdot f(k, r) = O(k^2)$ variables. However, since S' is also reduced under Rule 2, and by the choice of x, x is the only variable that occurs in S'_1 but does not occur in S'_2. Hence $|S'_1| = O((r \cdot f(k, r))^{r-1}) = O(k^{2r-2})$.

5 Max Exact r-SAT

Let $x = (x_1, x_2, \ldots, x_n)$ be a vector of Boolean variables. We assume that each variable x_i attains 1 or -1 (meaning TRUE and FALSE, respectively). We will denote the negation of a variable x_i by $-x_i$ and we let $L = \{x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n\}$ be the set of literals over $x_1, \ldots, x_n$. Let $r \geq 2$ be a fixed integer, and let $C = \{C_1, C_2, \ldots, C_m\}$ be a set of clauses, each involving exactly r literals from L such that for no pair y, z of the literals we have $y = -z$. Then $\mathcal{C}$ is an *exact r-CNF formula*.

Following [11, 15] we say that a pair of distinct clauses Y and Z has a *conflict* if there is a literal $p \in Y$ such that $-p \in Z$.

Consider an exact r-CNF formula $\mathcal{C}$ with m clauses and a random truth assignment for x. Since the probability of a clause of $\mathcal{C}$ to be satisfied is $1 - 2^{-r}$, the expected number of satisfied clauses in $\mathcal{C}$ is $(1 - 2^{-r})m$. Thus, there is a truth assignment for x that satisfies at least $(1 - 2^{-r})m$ clauses. This bound is tight as can be seen by considering a complete r-CNF formula that contains all 2^r possible clauses over the same r variables, or by considering a disjoint union of several complete r-CNF formulas.

Mahajan, Raman and Sikdar [13] stated the complexity of the following problem as an open question.

> EXACT r-SAT ABOVE TIGHT LOWER BOUND (r-SATALB)
> *Instance:* An exact r-CNF formula $\mathcal{C}$ and a positive rational number k with denominator 2^r.
> *Parameter:* The number k.
> *Question:* Is there a truth assignment $(x'_1, x'_2, \ldots, x'_n)$ satisfying at least $(1 - 2^{-r})m + k$ clauses of $\mathcal{C}$?

Mahajan, Raman and Sikdar [13] require k to be a positive integer, but since $(1 - 2^{-r})m$ is a positive rational number with denominator 2^r, our setting for k seems more natural.

We will prove that the problem has a quadratic kernel for a wide family of instances.

Theorem 3. *The problem r-SATALB restricted to exact r-CNF formulas with m clauses and at most $(2^{r-1}-1)m$ conflicts admits a quadratic kernel.*

To establish the theorem, consider an exact r-CNF formula $\mathcal{C}$ with m clauses and at most $(2^{r-1}-1)m$ conflicts, and consider a clause Z of $\mathcal{C}$. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ be the variables corresponding to the literals of Z and let $x^0_{i_1}, x^0_{i_2}, \ldots, x^0_{i_r}$ be the unique truth assignment not satisfying Z. Define a random variable X_Z as follows: Let $V = \{-1,1\}^r - \{(x^0_{i_1}, x^0_{i_2}, \ldots, x^0_{i_r})\}$ and

$$X_Z(x_1, x_2, \ldots, x_n) = \sum_{(v_1,\ldots,v_r)\in V} \frac{\prod_{j=1}^r (1 + x_{i_j} v_j)}{2^r} - \frac{2^r-1}{2^r}.$$

Let $X = \sum_{Z\in\mathcal{C}} X_Z$.

We study some properties of X in the following two lemmas.

Lemma 4. *Let $x' = (x'_1, x'_2, \ldots, x'_n)$ be a truth assignment. Then the value of X at x' equals $m(x') - (1-2^{-r})m$, where $m(x')$ is the number of clauses in $\mathcal{C}$ satisfied by x'. Thus, the answer to r-SATALB is* YES *if and only if $X(x'') \geq k$ for some truth assignment x''. We also have $\mathbb{E}(X) = 0$, and if $\mathbb{E}(X^2) > 0$, then $\mathbb{E}(X^4) \leq 2^{6r}\mathbb{E}(X^2)^2$.*

Proof. The value of X_Z at a truth assignment $x' = (x'_1, x'_2, \ldots, x'_n)$ is 2^{-r} if x' satisfies Z, and it is $2^{-r}-1$, otherwise. Thus, we have $X(x') = m(x')2^{-r} + (2^{-r}-1)(m - m(x')) = m(x') - (1-2^{-r})m$. Hence, $m(x') \geq (1-2^{-r})m + k$ if and only if $X(x') \geq k$.

Observe that the probability of X_Z being satisfied (not satisfied) is $1-2^{-r}$ (2^{-r}). Thus, the expectation of X_Z is zero, and $\mathbb{E}(X) = 0$ by linearity of expectation. Suppose that $\mathbb{E}(X^2) > 0$. Since X is a polynomial of degree at most r in $x_1, x_2, \ldots, x_n$, it follows, by Lemma 2, that $\mathbb{E}(X^4) \leq 2^{6r}\mathbb{E}(X^2)^2$. □

To apply Lemma 1, we can use the last inequality of Lemma 4 provided we know a lower bound on $\mathbb{E}(X^2)$. We obtain such a bound in the following:

Lemma 5. *We have $\mathbb{E}(X^2) \geq m4^{-r}$.*

Proof. Observe that $\mathbb{E}(X^2) = \sum_{Z\in\mathcal{C}} \mathbb{E}(X_Z^2) + \sum_{Y\neq Z\in\mathcal{C}} \mathbb{E}(X_Y X_Z)$. We will compute $\mathbb{E}(X_Z^2)$ and $\mathbb{E}(X_Y X_Z)$ separately.

By the proof of Lemma 4, X_Z equals 2^{-r} with probability $1-2^{-r}$ and $2^{-r}-1$ with probability 2^{-r}. Thus, X_Z^2 equals 2^{-2r} with probability $1-2^{-r}$ and $(2^{-r}-1)^2$ with probability 2^{-r}. Hence, $\mathbb{E}(X_Z^2) = 2^{-r} - 4^{-r}$.

For a clause Y of $\mathcal{C}$, let $\mathrm{vars}(Y)$ denote the sets of variables in Y and let $\mathrm{lits}(Y)$ be the set of literals in Y. To evaluate $\mathbb{E}(X_Y X_Z)$, we consider the following three cases:

Case 1: $\mathrm{vars}(Y) \cap \mathrm{vars}(Z) = \emptyset$. Then X_Y and X_Z are independent random variables and, thus, $\mathbb{E}(X_Y X_Z) = \mathbb{E}(X_Y)\mathbb{E}(X_Z) = 0$.

Case 2: Y and Z have a conflict. Let $|\text{vars}(Y) \cap \text{vars}(Z)| = t > 0$ and $|\text{lits}(Y) \cap \text{lits}(Z)| < t$. Without loss of generality, assume that $\text{vars}(Y) = \{x_1, \ldots, x_t, x_{t+1}, \ldots x_r\}$ and $\text{vars}(Z) = \{x_1, \ldots, x_t, x_{r+1}, \ldots x_{2r-t}\}$. Further, without loss of generality, assume that $\text{lits}(Y) = \text{vars}(Y)$ and $-x_1 \in \text{lits}(Z)$. Clearly, Y is not satisfied only if $(x_1, \ldots, x_r) = (-1, \ldots, -1)$. But if $x_1 = -1$, then Z is satisfied. Thus, $X_Y X_Z$ equals $2^{-r}(2^{-r} - 1)$ with probability 2^{-r+1}, and $X_Y X_Z$ equals 2^{-2r} with probability $1 - 2^{-r+1}$. Hence, $\mathbb{E}(X_Y X_Z) = -4^{-r}$.

Case 3: $|\text{vars}(Y) \cap \text{vars}(Z)| = t > 0$ and $|\text{lits}(Y) \cap \text{lits}(Z)| = t$. Since $Y \neq Z$, we have $t < n$. Without loss of generality, assume that $\text{lits}(Y) = \{x_1, \ldots, x_t, x_{t+1}, \ldots x_r\}$ and $\text{lits}(Z) = \{x_1, \ldots, x_t, x_{r+1}, \ldots x_{2r-t}\}$. Thus, $X_Y X_Z$ equals $(2^{-r} - 1)^2$ with probability 2^{t-2r}, $2^{-r}(2^{-r} - 1)$ with probability $(2^{r-t+1} - 1)/2^{2r-t}$, and 2^{-2r} with probability $1 - 2^{r-t+1}/2^{2r-t}$. Hence, $\mathbb{E}(X_Y X_Z) = 2^{t-2r}(1 - 2^{-r} - 2^{-t}) \geq 0$.

It follows from Cases 1–3 that $\mathbb{E}(X_Y X_Z) < 0$ if and only if Y and Z have a conflict. Thus,

$$\mathbb{E}(X^2) \geq \sum_{Z \in \mathcal{C}} \mathbb{E}(X_Z^2) + \sum_{Y \sim Z \in \mathcal{C}} \mathbb{E}(X_Y X_Z) = (2^{-r} - 4^{-r})m - 2 \cdot \text{conf}(\mathcal{C}) \cdot 4^{-r} \geq m4^{-r},$$

where $Y \sim Z$ denotes that Y and Z have a conflict and $\text{conf}(\mathcal{C})$ denotes the number of pairs of clauses of $\mathcal{C}$ that have a conflict. □

Now we can complete the proof of Theorem 3. By Lemmas 1, 4 and 5, $\text{Prob}(\ X > \sqrt{m}/(2^r \cdot 4 \cdot 8^r)\ ) > 0$. Thus, if $\sqrt{m}/(2^r \cdot 4 \cdot 8^r) \geq k$, there is a truth assignment x' such that $X(x') \geq k$, i.e., the answer to the instance of r-SATALB is YES. Otherwise, $m < 16 \cdot 256^r k^2$. Thus Theorem 3 is established.

Consider a YES-instance $(\mathcal{C}, k)$ of r-SATALB with n variables, $m \geq 16 \cdot 256^r k^2$ clauses, and at most $(2^{r-1} - 1)m$ conflicts. As in the previous section, we can find in polynomial time a solution for $(\mathcal{C}, k)$ using the facts that each random variable x_i is $4r$-wise independent and there is an $O(n^{2r})$-size sample space to support each x_i.

6 Discussion

We have introduced the Strictly Above Expectation Method of proving that some parameterized problems are fixed-parameter tractable. Moreover, in some cases our method allowed us to obtain quadratic kernels. We were able to prove that LOALB is fixed-parameter tractable and some non-trivial special cases of LinALB and r-SATALB are also fixed-parameter tractable. It would be interesting to obtain applications of our method to other problems parameterized above tight lower bound. In our forthcoming paper [6] we prove that 2-SATALB is fixed-parameter tractable using a different approach based on weighted graphs and matching theory. We find it unlikely, though, that this approach can be used to prove fixed-parameter tractability of r-SATALB for each $r > 3$.

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