Well-Quasi-Ordering Bounded Treewidth Graphs

Michael R. Fellows[1], Danny Hermelin *[2], and Frances A. Rosamond[1]

[1] School of Electrical Engineering and Computer Science,
The University of Newcastle, Calaghan NSW 2308 - Australia
{mike.fellows,frances.rosamond}@cs.newcastle.edu.au
[2] Department of Computer Science, University of Haifa,
Mount Carmel, Haifa 31905 - Israel
danny@cri.haifa.ac.il

Abstract. We show that three subclasses of bounded treewidth graphs are well-quasi-ordered by refinements of the minor order. Specifically, we prove that graphs with bounded feedback-vertex-set are well-quasi-ordered by the topological-minor order, graphs with bounded vertex-covers are well-quasi-ordered by the subgraph order, and graphs with bounded circumference are well-quasi-ordered by the induced-minor order. Our results give an algorithm for recognizing any graph family in these classes which is closed under the corresponding minor order refinement.

1 Introduction

The treewidth parameter is one of the most commonly used structural parameterizations in parameterized complexity [7, 10, 13]. The reason for this being that many natural graph problems turn out to be fixed-parameter tractable when parameterized by the treewidth of the input graph. Indeed, various algorithmic methodologies such as tree-decomposition dynamic programming [1, 2, 4] and Courcelle's Theorem [5] provide a single framework to a vast multitude of different combinatorial problems in bounded treewidth graphs.

With that being said, there are still quite a few problems which are impregnable by any of the algorithmic methodologies for bounded treewidth graphs. For instance, vertex ordering problems such as Bandwidth or Coalition Width, and partitioning problems such as Multiple Interval Number or Multiple Chordal Number, have no known fixed-parameter algorithm when the treewidth is taken as a parameter. There is thus room for more algorithmic methodologies, perhaps by imposing more structure on the input than bounded treewidth. In this paper we suggest the method of well-quasi-ordering as a means towards this aim. Using this method, we are able to prove the following algorithmic result concerning subclasses of bounded treewidth graphs:

Theorem 1. *Let k be some fixed positive integer. There is a linear-time algorithm for recognizing:*

- *Any family of graphs with vertex-cover at most k that is closed under subgraphs.*
- *Any family of graphs with feedback-vertex-set at most k that is closed under topological minors.*
- *Any family of graphs with circumference at most k that is closed under induced-minors.*

We recall that a vertex cover in a graph is a set of vertices which covers all edges in the graph, a feedback-vertex-set is a set of vertices which covers all cycles in the graph, and the circumference of a graph is the length of its maximum cycle. Bounding each of these parameters results in a bound in the treewidth as well. By closed under subgraphs (resp. topological minors, induced minors), we mean that whenever a graph belongs to the family, then either all of its subgraphs (resp. topological minors, induced-minors) also belong to the family, or all of its supergraphs (resp. topological majors,

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induced-majors[1]). We mention that the first item of Theorem 1 was already shown indirectly by Ding [6] (see Section 3).

To see how our theorem applies to fixed-parameter algorithms, lets us consider some examples. Given a graph G, the *bandwidth* of G is the minimum bandwidth of all vertex-orderings $\pi : V(G) \rightarrow \{1, \dots, |V(G)|\}$, where the bandwidth of a given vertex-ordering π is defined as $\max_{\{u,v\} \in E(G)} |\pi(v) - \pi(u)|$. The BANDWIDTH problem is the problem of computing the bandwidth of a given graph. It is known to W[t]-hard for all $t > 1$ when parameterized by the bandwidth of the input graph [3], and not known to be in FPT when parameterized by the treewidth of the graph. However, observe that for each $\ell \in \mathbb{N}$, the family of graphs with bandwidth at most ℓ is closed under subgraphs. Thus, by the first item of Theorem 1, we get:

Corollary 1. *For any $k, \ell \in \mathbb{N}$, there is an $f(k + \ell) \cdot n$ time algorithm which determines whether a given graph G with n vertices and vertex-cover at most k, has bandwidth at most ℓ.*

A *coalition* in a graph is a subset of vertices pairwise connected by vertex-disjoint paths, or in other words, a topological clique minor. Given a vertex-ordering π of G, let us denote by $G^+_\pi(v)$ the graph induced by $\{v, \pi^{-1}(\pi(v) + 1), \pi^{-1}(\pi(v) + 2), \dots, \pi^{-1}(|V(G)|)\}$, and by $N^+_\pi(v)$ the set of neighbors v has in $G^+_\pi(v)$. The *coalition-width* of a given graph G is defined as the minimum coalition-width over all vertex-orderings of G, where the coalition-width of a given vertex-ordering π is defined as $\max_{v \in V(G)} |\{|K| : K \subseteq N^+_\pi(v), K \text{ forms a coalition in } G^+_\pi(v)\}$. We do not know whether the corresponding COALITION WIDTH problem is in FPT when parameterized by the coalition-width of the input graph. However, observe that for each $\ell \in \mathbb{N}$, the family of all graphs with coalition-width at most ℓ is closed under topological minors, and so according to second item of Theorem 1 we get:

Corollary 2. *For any $k, \ell \in \mathbb{N}$, there is an $f(k + \ell) \cdot n$ time algorithm which determines whether a given graph G with n vertices and feedback-vertex-set at most k, has coalition-width at most ℓ.*

The above two examples were concerned with vertex ordering problems. Let us now consider two examples of partitioning problems. The *multiple-interval-number* of a graph G is the smallest number ℓ for which there exist $G_1, \dots, G_\ell$ with $G = \bigcup_{1 \leq i \leq \ell} G_i$, and such that each G_i is an interval graph [11]. The *multiple-chordal-number* is defined similarly, except that each G_i is required to be chordal. Determining whether a graph has multiple-interval-number t for $t \geq 2$ is NP-complete [11]. However, since chordal and interval graphs are closed under induced-minors, the family of all graphs with multiple-interval-number or multiple-chordal-number at most ℓ, for any $\ell \in \mathbb{N}$, is closed under induced minors. Thus, the last item of Theorem 1 implies:

Corollary 3. *For any $k, \ell \in \mathbb{N}$, there is an $f(k + \ell) \cdot n$ time algorithm which determines whether a given graph G with n vertices and circumference at most k, has multiple-interval-number or multiple-chordal-number at most ℓ.*

The reader should observe that all algorithms implied by the corollaries above are non-uniform by nature: For each k and ℓ we get a different algorithm. However, using the techniques by Fellows and Langston [8, 9], the above results along with many other natural examples can be transformed into uniform algorithms. We refer the reader for more details also to [7].

The remainder of this chapter is devoted to proving Theorem 1. We begin by briefly reviewing the fundamentals behind the method of well-quasi-ordering, and how it applies to bounded treewidth

[1] We call a graph G a *topological-major* of a graph H if H is a topological minor of G, and an *induced-major* if H is an induced minor of G.

graphs. In Section 3, we provide the general framework for proving Theorem 1 by devising what we call well-quasi-order identification tools. The correctness of these tools is proved in Sections 4 and 5.

2 The WQO Method in Bounded Treewidth Graphs

Let us begin with some fundamental terminology from well-quasi-order theory. A *quasi-order* on a set X is a reflexive transitive subset of $X \times X$. That is, if $\preceq$ is a quasi-order on X, then $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in X$, and $x \preceq x$ for all $x \in X$. We write $x \succ y$ if $x \succeq y$ and $y \not\succeq x$. An infinite sequence $x_1, x_2, x_3 \cdots$ is called *strictly descending* if $x_1 \succ x_2 \succ x_3 \cdots$, and *good* if it a *good pair* – a pair (x_i, x_j) with $x_i \preceq x_j$ and $i < j$. A *bad* sequence is an infinite sequence which is not good. A *well-founded quasi-order* is a quasi order with no infinite strictly descending sequences. A *well-quasi-order* (*wqo*) is a well-founded quasi-order with no infinite bad sequences. Equivalently, a well-quasi-order is a well-founded quasi-order with no infinite antichain.

Given a quasi-ordered set $\langle X, \preceq \rangle$, a subset $X' \subseteq X$ is said to be *closed under* $\preceq$, if for all $x, y \in X$ we have $x \in X'$ whenever $x \preceq y$ and $y \in X'$. Closed subsets of wqo sets have a property which is very interesting in our context: Consider the set

$$\mathrm{Forb}(X') := \{y \in X \setminus X' \,:\, z \not\prec y \text{ for all } z \in X \setminus X'\}.$$

This set has the property that $x \in X'$ iff $y \not\preceq x$ for all $y \in \mathrm{Forb}(X')$, and thus it is called a *forbidden characterization* of X'. Furthermore, since $\preceq$ is wqo, $\mathrm{Forb}(X')$ is necessarily finite as it constitutes an anti-chain w.r.t $\preceq$. Thus, every closed subset of a wqo set has a finite forbidden characterization.

Theorem 2 (WQO Recognition Theorem). *Let $\langle X, \preceq \rangle$ be a quasi-ordered set. If:*

(i) $\preceq$ is a wqo on X, and
(ii) for any $x, y \in X$, one can determine whether $y \preceq x$ in $f(|y|) \cdot |x|^c$ time, for some $c \in \mathbb{N}$.

Then one can recognize in $O(n^c)$ time any subset $X' \subseteq X$ that is closed under $\preceq$.

Proof. We describe an algorithm for recognizing an arbitrary subset $X' \subseteq X$ that is closed under $\preceq$. Since X' is closed under $\preceq$, the set $\mathrm{Forb}(X')$ defined above is a forbidden characterization of X'. According to the first condition in the theorem, this set is finite, and so our algorithm can have all elements of $\mathrm{Forb}(X')$ "hardwired" into it. On input $x \in X$, our algorithm checks whether $y \preceq x$ for each $y \in \mathrm{Forb}(X')$, using the order testing procedure promised by the second condition in the theorem. It determines that $x \in X'$ iff $y \not\preceq x$ for all $y \in \mathrm{Forb}(X')$. Correctness of this algorithm follows from the fact $\mathrm{Forb}(X')$ is a forbidden characterization of X'. Furthermore, its running-time can be bounded by $O(n^c)$, $n := |x|$, since the number and sizes of the elements in $\mathrm{Forb}(X')$ depends only on X', and is constant with respect to $|x|$. □

The WQO Recognition Theorem encapsulates the two main ingredients behind the method of well-quasi-ordering. Probably the best known application of this method is the astonishing result implied by Robertson and Seymour's graph minor project: For any graph family $\mathcal{G}$ closed under minors, there is an $O(n^3)$ time algorithm for recognizing $\mathcal{G}$. This result is proved in an ongoing series of over twenty papers, where the two items of the theorem above are shown to apply to graph minors: The set of all graphs is wqo by the minor order, and one can test whether a k-vertex graph is a minor of an n-vertex graph in $f(k) \cdot n^3$ time. Combining these two extremely complex results together gives one of the deepest result in graph theory yielding polynomial-time algorithms for many problems previously not known to be even decidable.

The minor order on graphs is typically defined via graph operations: A graph H is *minor* of a graph G if H can be obtained in G (via isomorphism) by vertex and edge deletions, and by edge contractions. A *contraction* of an edge $\{u, v\}$ in a graph G is the operation that replaces u and v by a new vertex which is adjacent to all neighbors of u and v (and removing all resulting multiple edges and self loops). We can therefore consider orders that are defined by a subset of these operations, or by applying restrictions on them. For example, a graph H is *topological-minor* of a graph G if H can be obtained in G by vertex and edge deletions, and by topological contractions, where a *topological contraction* (or *subdivision removal*) is a contraction of an edge incident to at least one vertex of degree 2. The *induced minor* order is similar to the minor order but without edge-deletions, and the *subgraph order* is the well-known order defined by vertex and edge deletions alone.

In contrast to the graph minor order, none of the above orders is a wqo, not even in the very restrictive universe of bounded treewidth graphs [14]. However, in bounded treewidth graphs we have Courcelle's Theorem [5] which states that for any monadic-second-order formula ϕ there is an $f(|\phi| + k) \cdot n$ time algorithm for determining whether a given graph on n vertices and treewidth at most k satisfies ϕ. Since order testing for any of the above refinements of the graph minor order can be expressed in monadic-second-order logic, we get the second ingredient in the WQO method for free in bounded treewidth graphs, due to Courcelle's Theorem.

Lemma 1 (Bounded Treewidth Order-Testing Lemma). *For any graph H on ℓ vertices, there is an $f(\ell + k) \cdot n$ time algorithm for determining whether H is a subgraph (resp. topological-minor, induced-minor) of a given graph G on n vertices and treewidth at most k.*

The setting should be clear by now. In order to obtain the recognition algorithm promised in Theorem 1, we need to show that each subclass of bounded treewidth graphs in the theorem is wqo by its corresponding order. This along with the Bounded Treewidth Order-Testing Lemma will give us both conditions in the WQO Recognition Theorem, which in turn will give us Theorem 1 as a direct corollary. In the following sections of this chapter we will prove that:

Lemma 2. *Let k be any fixed positive integer. Then:*

- *The set of all graphs with vertex-cover at most k is wqo by subgraphs.*
- *The set of all graphs with feedback-vertex-set at most k is wqo by topological-minors.*
- *The set of all graphs with circumference at most k is wqo by induced-minors.*

Proof (of Theorem 1 assuming Lemma 2). Consider a family of graphs $\mathcal{G}$ with vertex-cover at most k. Then according to Lemma 2, $\mathcal{G}$ satisfies the first requirement of the WQO Recognition Theorem, and since each graph in $\mathcal{G}$ has treewidth at most $f(k)$ for some function $f()$, the second condition is satisfied according to the Bounded Treewidth Order Testing Lemma. Thus, any subset of $\mathcal{G}$ that is closed under subgraphs can be recognized in $O(n)$ time. The second and third items of the theorem can be proven similarly. $\square$

Thus, what remains to prove is Lemma 2. For this we develop in the next section two tools for determining when a given graph order is a wqo on a specific graph class.

3 Two Tools for Identifying WQOs

In this section we develop two tools that will help us in proving Lemma 2. These tools allow us to reduce the question of whether a given graph family is wqo by a particular order, to the question of whether a simpler family is wqo by some "colored variant" on that order. To specify these colored variants precisely, it will be convenient to speak of the graph orders we study in terms of embeddings. Let H and G be two given graphs:

- A *subgraph embedding* of H onto G is an injection $f : V(H) \to V(G)$ with $\{u, v\} \in E(H) \Rightarrow \{f(u), f(v)\} \in E(G)$.
- A *topological-minor embedding* of H onto G is an injection $f : V(H) \to V(G)$ where there exist vertex disjoint paths in G between $f(u)$ and $f(v)$ for every $\{u, v\} \in E(H)$.
- An *induced-minor embedding* of H onto G is a injective mapping $f : V(H) \to 2^{V(G)}$ with $f(v)$ connected in G for all $v \in V(H)$, $f(u) \cap f(v) = \emptyset$ for all $u \neq v \in V(H)$, and $\{u, v\} \in E(H) \iff \exists x \in f(u)$ and $\exists y \in f(v)$ with $\{x, y\} \in E(G)$.

We write $H \subseteq G$ (resp. $H \trianglelefteq G$, $H \sqsubseteq G$) if there exists a subgraph (resp. topological-minor, induced-minor) embedding of H onto G. It is easy to see that $H \subseteq G$ (resp. $H \trianglelefteq G$, $H \sqsubseteq G$) iff H is a subgraph (resp. topological-minor, induced-minor) of G as defined in the previous section. We will also use $H \subseteq^* G$ to denote that there is an induced-subgraph embedding of H onto G, where an *induced subgraph embedding* is an injection $f : V(H) \to V(G)$ with the condition that $\{u, v\} \in E(H) \Leftrightarrow \{f(u), f(v)\} \in E(G)$. Finally, we write $H \cong G$ to denote that H and G are *isomorphic*, *i.e.* that $H \subseteq G$ and $G \subseteq H$.

We will speak of graph universes, where by a universe $\mathcal{U}$ we mean an infinite set of graphs which is closed under vertex deletions, *i.e.* $G \in \mathcal{U} \implies G - V \in \mathcal{U}$ for all $V \subseteq V(G)$. Let $\mathcal{U}$ be some graph universe. A *labeling* of $\mathcal{U}$ is a set $\{\sigma_G : G \in \mathcal{U}\}$, where each σ_G is a *labeling* of the vertices of G by a set of labels Σ_G, *i.e.* $\sigma_G : V(G) \to \Sigma_G$. The set $\Sigma = \bigcup_{G \in \mathcal{U}} \Sigma_G$ is the set of labels assigned by σ to $\mathcal{U}$. If Σ is wqo by some quasi order $\preceq$, we say that σ is a *wqo labeling w.r.t* $\preceq$. Well-quasi-ordered labelings of $\mathcal{U}$ allow us to refine the subgraph, topological minor, and induced minor orders on $\mathcal{U}$ in a natural manner. Given a wqo labeling $\sigma = \{\sigma_G : G \in \mathcal{U}\}$ w.r.t $\preceq$, and a pair of graphs $H, G \in \mathcal{U}$, we will write $H \subseteq_\sigma G$ (resp. $H \subseteq^*_\sigma G$, $H \trianglelefteq_\sigma G$) if there is a subgraph (topological-minor) embedding of H onto G with $\sigma_H(v) \preceq \sigma_G(f(v))$ for all $v \in V(H)$. We write $H \cong_\sigma G$ whenever $H \subseteq_\sigma G$ and $H \subseteq_\sigma G$. Also, we extend this definition to the induced-minor order, and write $H \sqsubseteq_\sigma G$ whenever there exists an induced-minor embedding of H onto G where for each $v \in V(H)$ there is some $x \in f(v)$ with $\sigma(v) \preceq \sigma(x)$.

Let us next give two important examples of well-quasi-ordered graph families, that we will use later on. The first is due to Kruskal, and is known as the famous Labeled Forests Theorem, and the second is due to Ding:

Theorem 3 (Kruskal's Labeled Forests Theorem [12]). *The universe of all forests is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ.*

Theorem 4 (Ding's Bounded Paths Theorem [6]). *For any $k \in \mathbb{N}$, the universe of all graphs with no paths of length greater than k is wqo by $\subseteq^*_\sigma$, for any wqo labeling σ.*

We are now in position to describe our first wqo identification tool. This tool is especially suited for universes consisting of graphs which have a small subset of vertices whose removal leaves a very simple structured graph, *e.g.* graphs with bounded vertex-cover or bounded feedback-vertex-set. Given a graph universe $\mathcal{U}$, and a natural k, let us denote by $\mathcal{U}_k$ is the universe of all graphs G which have a subset of k vertices V with $G - V \in \mathcal{U}$.

Theorem 5 (WQO Identification Tool 1). *If a universe $\mathcal{U}$ is wqo by $\subseteq_\sigma$ (resp. $\trianglelefteq_\sigma$) for any finite labeling σ, then $\mathcal{U}_k$ is wqo under $\subseteq$ (resp. $\trianglelefteq$).*

Our second identification tool is concerned with the induced-minor order and 2-connected graphs. In general, a connected graph G is called *2-connected* if it has at least three vertices, and no removal of less than two vertices leaves G disconnected. The second identification tool is especially suited for graph universes which have 2-connected graphs with very simple structure:

Theorem 6 (WQO Identification Tool 2). *If the subset of all 2-connected graphs in some universe $\mathcal{U}$ is wqo by $\sqsubseteq_\sigma$ for any wqo labeling σ, then $\mathcal{U}$ itself is wqo by $\sqsubseteq$.*

The next two sections of this chapter are devoted each to proving Theorem 5 and Theorem 6. But for now, let us next see how these two identification tools easily imply Lemma 2 of the previous section:

Proof (of Lemma 2 assuming Theorem 5 and Theorem 6). We prove the first two items of the lemma using Theorem 5, and the last item using Theorem 6:

- Let $\mathcal{U}$ denote the set of all graphs with no edges. Then for any $k \in \mathbb{N}$, $\mathcal{U}_k$ is the universe of all graphs with vertex-cover at most k by definition. According to Kruskal's Labeled Forests Theorem, we know that $\mathcal{U}$ is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ, since $\mathcal{U}$ includes only forests. Moreover, if H is a topological-minor of G for $H, G \in \mathcal{U}$, then H is also a subgraph of G, since graphs in $\mathcal{U}$ have no edges. This implies that $\mathcal{U}$ is also wqo by $\subseteq_\sigma$ for any wqo labeling σ. Plugging this into Theorem 5 gives us that graphs with vertex-cover at most k are wqo by subgraphs.
- For graphs with bounded feedback-vertex-set the argument is similar to the above. Note that if $\mathcal{U}$ is the set of all forests, then for any $k \in \mathbb{N}$, $\mathcal{U}_k$ is the universe of all graphs with feedback-vertex-set at most k by definition. Due to Kruskal's Labeled Forests Theorem we get by Theorem 5 that graphs with feedback-vertex-set at most k are wqo by topological-minors.
- Let $\mathcal{U}$ denote the set of all graphs with circumference at most k, and let $\mathcal{U}'$ denote the subset of 2-connected graphs in $\mathcal{U}$. Since any two vertices in a 2-connected graph are connected by at least two paths, and thus belong together to some cycle, we get that 2-connected graphs in $\mathcal{U}'$ have no paths of length greater than k, due to the bound on the circumference of graphs in $\mathcal{U}$. Therefore, according to Ding's Theorem, we get that $\mathcal{U}'$ is wqo by $\subseteq^*_\sigma$ for any wqo labeling σ, and so it is also wqo by $\sqsubseteq^*_\sigma$ for any wqo labeling σ. Plugging this into Theorem 6 gives us that graphs with circumference at most k are wqo by induced-minors.

□

4 Correctness of the First Tool

In this section we prove the correctness of Theorem 5. We will specify only the proof for the $\subseteq$ order, as the proof for the $\trianglelefteq$ order follows the same lines. To start with, we will assume we have a positive integer k, and a graph universe $\mathcal{U}$ which is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ. We will show that any infinite sequence of graphs in $\mathcal{U}_k$ is good, *i.e.* it has graph which is a topological-minor of another graph succeeding it in the sequence.

Let $\{G_i\}_{i=1}^{\infty}$ be any infinite sequence in $\mathcal{U}_k$. By definition, each graph G_i in this sequence has a subset of k vertices U_i with $G_i - U_i \in \mathcal{U}$. Let V_i denote the subset of vertices $V(G_i) \setminus U_i$. We construct a labeling $\sigma = \{\sigma_i : i \in \mathbb{N}\}$ on $\{G_i : i \in \mathbb{N}\}$ in a way that codifies the adjacency of vertices in U_i with vertices of V_i, for each $i \in \mathbb{N}$. For this, σ_i first assigns each vertex $u \in U_i$ an arbitrary distinct label $\sigma_i(u) \in \{1, \ldots, k\}$, and then it assigns a label in $2^{\{1,\ldots,k\}}$ to each $v \in V_i$ by

$$\sigma_i(v) := \left\{\, x \quad : \quad \exists u \in U_i \text{ with } \{u, v\} \in E(G_i) \text{ and } \sigma_i(u) = x \,\right\}$$

(see Fig. 1 for an example). Observe that since the set of labels Σ assigned by σ is finite, it is wqo by equality, and σ is a wqo labeling on $\mathcal{U}$ with respect to $=$.

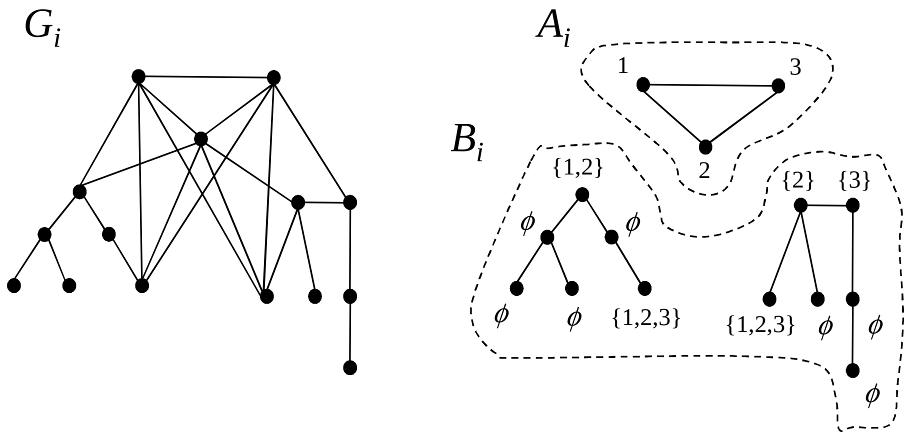

Fig. 1. The labeling used in proving the first identification tool.

Now, for each $i \in \mathbb{N}$, let A_i denote the graph $G_i - V_i$, and let B_i denote $G_i - U_i$. Then $B_i \in \mathcal{U}$ for all $i \in \mathbb{N}$. Since there are only finitely many graphs A_i under isomorphism, and only finitely many ways to label the vertices of these graphs with distinct labels in $\{1, \ldots, k\}$, there must be an infinite subsequence $G_{i_1}, G_{i_2}, \ldots$ in $\{G_i\}_{i=1}^{\infty}$ with $A_{i_1} \cong_\sigma A_{i_2} \cong_\sigma \cdots$. By our assumption, the family of graphs $\{B_{i_j} : j \in \mathbb{N}\}$ is wqo by $\subseteq_\sigma$, and as this set is infinite, there must be a pair B_{i_x} and B_{i_y}, $x < y$, with $B_{i_x} \subseteq_\sigma B_{i_y}$. Write $i = i_x$ and $j = i_y$. We argue that:

$$A_i \cong_\sigma A_j \text{ and } B_i \subseteq_\sigma B_j \text{ implies } G_i \subseteq G_j$$

Let f_A denote the isomorphic embedding showing that $A_i \cong_\sigma A_j$, and let f_B denote the isomorphic embedding of B_i onto B_j. We argue that the mapping $g = f_A \cup f_B$ is an isomorphic embedding of G_i onto G_j. Clearly, for all edges $\{u, v\} \in E(G_i)$ with either $u, v \in A_i$ or $u, v \in B_i$, we have $\{g(u), g(v)\} \in E(G_i)$ by our assumptions on f_A and f_B. For $\{u, v\} \in E(G_i)$ with $u \in U_i$ and $v \in V_i$, we have $\sigma_i(u) = \sigma_j(g(u))$ and $\sigma_i(v) = \sigma_j(g(v))$. Thus, by construction of σ, we get $\{u, v\} \in E(G_i) \implies \sigma_i(u) \in \sigma_i(v) \implies \sigma_j(g(u)) \in \sigma_j(g(v)) \implies \{g(u), g(v)\} \in E(G_j)$.

It follows that $\{G_i\}_{i=1}^{\infty}$ is a good sequence, and as this sequence was chosen arbitrarily, this implies that $\mathcal{U}_k$ does not contain any bad sequences. This completes the proof of Theorem 5.

5 Correctness of the Second Tool

In this section we prove the correctness of Theorem 6. We will need some additional terminology: A *rooted graph* is a pair (G, v) where G is a graph and v is a single distinguished vertex v of G referred to as its *root*. Thus two rooted graphs with the same vertex and edge set, but with different roots, are considered different. Apart from the following definition, we will omit the parentheses notation and simply state that G is a rooted graph with $root(G) = v$.

Definition 1 (Rooted Closure). *The rooted closure of a universe $\mathcal{U}$, denoted $\mathcal{U}_r$, is defined as the universe of rooted graphs $\mathcal{U}_r = \{(G, v) : G \in \mathcal{U}, v \in V(G)\}$.*

We say that a minor-embedding of a rooted graph H onto a rooted graph G *preserves roots* if $root(G) \in f(root(H))$, and we will write $H \sqsubseteq G$ (and say that H is an induced minor of G) only when there exists a root-preserving minor-embedding of H onto G. Our main interest in rooted graphs lies in the above refinement of minor embeddings, and in the obvious fact that $\mathcal{U}$ is wqo under $\sqsubseteq$ whenever $\mathcal{U}_r$ is wqo under $\sqsubseteq$.

Another important notion we need to introduce before beginning the proof of Theorem 6 is the notion of minimal bad sequences, a notion first introduced by Nash-Williams [**?**], and also used by Kruskal in proving his Labeled Forests Theorem:

Definition 2 (Minimal Bad Sequence). *A bad sequence $G_1, G_2, \ldots$ is* minimal *if for every bad sequence $H_1, H_2, \ldots$, whenever $|V(H_j)| < |V(G_j)|$ for some j, there is always some $i < j$ such that $|V(G_i)| < |V(H_i)|$.*

Thus, a minimal bad sequence can be "constructed" by selecting a graph G_1 that has the smallest number of vertices among all graphs that begin bad sequences, then selecting a graph G_2 with the smallest number of vertices among all graphs which appear second in bad sequences beginning with graphs of size $|V(G_1)|$, and so forth.

Let $\mathcal{U}$ be a universe of graphs whose subset of 2-connected graphs are wqo by $\sqsubseteq_\sigma$, for any wqo labeling σ. W can assume w.l.o.g. that all graphs in $\mathcal{U}$ are connected, since if $\mathcal{U}$ contains a disconnected graph, it also contains each of its connected components by the definition of a universe. Aiming towards a contradiction, we will assume that $\mathcal{U}$ is not wqo by $\sqsubseteq$, and arrive at a contradiction by showing that $\mathcal{U}_r$, the rooted-closure of $\mathcal{U}$, is wqo by $\sqsubseteq$.

For this, let $G_1, G_2, \ldots$ be a minimal bad sequence in $\mathcal{U}_r$. For each $i \in \mathbb{N}$, select a block A_i in G_i which contains $root(G_i)$, and let C_i denote the set of cutvertices of G_i that are included in A_i. For each cutvertex $c \in C_i$, let B^i_c denote the connected component in $G_i - (V(A_i) \setminus C_i)$ including the vertex c and made into a rooted graph by setting $root(B^i_c) = c$ (see Fig. **??**). Observe that for any $c \in C_i$, we have $B^i_c \sqsubseteq G_i$ by the induced-minor root-preserving embedding f that maps every non-root vertex $v \neq c$ of B^i_c to itself, and has $f(c) = A_i \ni root(G_i)$. We argue that:

$$\text{The family of rooted graphs } \mathcal{B} = \{B^i_c : c \in C_i,\ i \in \mathbb{N}\} \text{ is wqo by } \sqsubseteq .$$

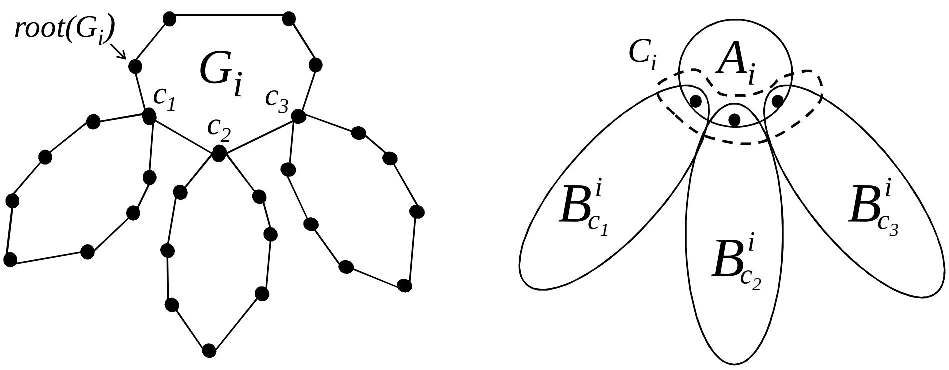

Fig. 2. The notation used in proving the second identification tool.

To see this, let $\{H_j\}_{j=1}^{\infty}$ be any sequence in $\mathcal{B}$, and for every $j \in \mathbb{N}$, choose an $i(j)$ for which $H_j = B^i_c$ for some $c \in C_i$. Pick a j with smallest $i(j)$, and consider the sequence

$$G_1, \ldots, G_{i(j)-1}, H_j, H_{j+1}, \ldots$$

Then this sequence is good by the minimality of $\{G_i\}_{i=1}^{\infty}$, and by our selection of j, and so it contains a good pair (G, G'). Now, G cannot be among the first $i(j)-1$ elements of this sequence, since otherwise $G' = H_{j'}$ for some $j' \geq j$, and we will have

$$G \sqsubseteq G' = H_{j'} = B_c^{i(j')} \sqsubseteq G_{i(j')},$$

implying that $(G, G_{i(j')})$ is a good pair in the bad sequence $\{G_i\}_{i=1}^{\infty}$. Thus, (G, G') must be a good pair in $\{H_j\}_{j=1}^{\infty}$, and so $\{H_j\}_{j=1}^{\infty}$ is good.

We next use the above to show that $\{G_i\}_{i=1}^{\infty}$ has a good pair, bringing us to our desired contradiction. For this, we will label the graph family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ so that each cutvertex c of a graph A_i gets labeled by their corresponding connected component B_c^i of G_i, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling σ_i that assigns a pair of labels $(\sigma_i^{(1)}(v), \sigma_i^{(2)}(v))$ to every vertex $v \in V(G_i)$, where the labelings $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are defined by:

– $\sigma_i^{(1)}(v) = 1$ if $v = root(G_i)$, and otherwise $\sigma^{(1)}(v) = 0$.
– $\sigma_i^{(2)}(v) = B_v^i$ if $v \in C_i$, and otherwise $\sigma_i^{(2)}(v) = 0$.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. We define a quasi-ordering $\preceq$ on the set of labels Σ assigned by σ. For two labels $(\varsigma_a^{(1)}, \varsigma_a^{(2)}), (\varsigma_b^{(1)}, \varsigma_b^{(2)}) \in \Sigma$, we define

$$(\varsigma_a^{(1)}, \varsigma_a^{(2)}) \preceq (\varsigma_b^{(1)}, \varsigma_b^{(2)}) \iff \varsigma_a^{(1)} = \varsigma_b^{(1)} \text{ and } \varsigma_a^{(2)} \sqsubseteq \varsigma_b^{(2)}.$$

Observe that the $\sqsubseteq$ order above is between rooted graphs. Also, we allow 0 to be $\sqsubseteq$-comparable only to itself. It is not difficult to see that since $\sqsubseteq$ is a wqo on $\mathcal{B}$, the $\preceq$ order is wqo on Σ. Thus, σ is wqo labeling on $\mathcal{A}$ w.r.t. σ. By the assumptions in the theorem, we know that $\mathcal{A}$ is wqo by $\sqsubseteq_\sigma$. It follows that there is a pair of graphs $A_i, A_j \in \mathcal{A}$ with $A_i \sqsubseteq_\sigma A_j$. To complete the proof we will show that:

$$A_i \sqsubseteq_\sigma A_j \Rightarrow G_i \sqsubseteq G_j.$$

To see this, let f be the induced-minor embedding of A_i onto A_j. Then for each cutvertex $c \in C_i$, $f(c)$ contains a vertex $d \in C_j$ with $B_c^i \sqsubseteq B_d^j$. Let f_c denote the induced-minor root-preserving embedding of B_c^i onto B_d^j. We construct an embedding $g : V(G_i) \to 2^{V(G_j)}$ defined by

$$g(v) = \begin{cases} f(v) & : \quad v \text{ is a vertex of } A_i \text{ and } v \notin C_i, \\ f_c(v) & : \quad v \text{ is a vertex of } B_c^i \text{ and } v \neq c, \\ f(v) \cup f_v(v) & : \quad v \in C_i. \end{cases}$$

We argue that g is an induced minor embedding of G_i onto G_j.

To see this, first note that by definitions of f and each f_c, we have $g(u) \cap g(v) = \emptyset$ for any pair of distinct vertices u and v in G_i. Moreover, for any edge $\{u, v\}$ of G_i there is a vertex $x \in g(u)$ and a vertex $y \in g(v)$ with $\{x, y\}$ and edge in G_j. Thus what remains to be shown is that $g(u)$ is connected in G_j for every vertex u of G_i. This is obviously true when $u \notin C_i$, again by the definitions of f and each f_c. If $u \in C_i$, then $f(u)$ contains a vertex $v \in C_j$ for which $B_u^i \sqsubseteq B_v^j$, and v is also contained in $f_v(v)$ since f_v preserves roots. Thus, $g(u)$ is connected also when $u \in C_i$. Noting also that the labeling σ ensures that $root(G_j) \in g(root(G_i))$, we establish that $G_i \sqsubseteq G_j$.

Thus, the sequence $\{G_i\}_{i=1}^{\infty}$ is good, and so we have obtained our desired contradiction. This completes the proof of the Theorem 6.

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