Streaming Algorithms for Graph *k*-Matching with Optimal or Near-Optimal Update Time

JIANER CHEN*, QIN HUANG*, IYAD KANJ[†], QIAN LI[‡], AND GE XIA[§]

Abstract

We present streaming algorithms for the graph k-matching problem in both the insertonly and dynamic models. Our algorithms, with space complexity matching the best upper bounds, have optimal or near-optimal update time, significantly improving on previous results. More specifically, for the insert-only streaming model, we present a one-pass algorithm with optimal space complexity $O(k^2)$ and optimal update time O(1), that w.h.p. (with high probability) computes a maximum weighted k-matching of a given weighted graph. The update time of our algorithm significantly improves the previous upper bound of $O(\log k)$, which was derived only for k-matching on unweighted graphs. For the dynamic streaming model, we present a one-pass algorithm that w.h.p. computes a maximum weighted k-matching in $O(Wk^2 \cdot \text{polylog}(n))$ space¹ and with O(polylog(n)) update time, where W is the number of distinct edge weights. Again the update time of our algorithm improves the previous upper bound of $O(k^2 \cdot \operatorname{polylog}(n))$. This algorithm, when applied to unweighted graphs, gives a streaming algorithm on the dynamic model whose space and update time complexities are both near-optimal. Our results also imply a streaming approximation algorithm for maximum weighted k-matching whose space complexity matches the best known upper bound with a significantly improved update time.

keywords. streaming algorithm; graph matching; parameterized algorithm; lower bound

1 Introduction

Streaming algorithms for graph matching have been studied extensively. A graph stream S for an underlying graph G is a sequence of edge operations. In the *insert-only* streaming model, each operation is an edge-insertion, while in the *dynamic* streaming model each operation is either an edge-insertion or an edge-deletion (with a specified weight if G is weighted). Most of the previous work on the graph matching problem in the streaming model have focused on approximating a maximum matching, with the majority of the work pertaining to the (simpler) insert-only model (see, e.g., [2, 17, 19, 24, 25, 28]). More recently, streaming algorithms for the GRAPH k-MATCHING problem (i.e., constructing a matching of k edges in an unweighted graph or a maximum weighted matching of k edges in a weighted graph), in both the insert-only and the dynamic models, have drawn increasing interests [3, 5, 6, 7, 14].

The performance of streaming algorithms is measured by the limited memory (*space*) and the limited processing time per item (*update time*). For the space complexity, a lower bound of $\Omega(k^2)$

^{*}Department of Computer Science and Engineering, Texas A&M University, College Station, TX 77843, USA (chen@cse.tamu.edu, huangqin@tamu.edu).

[†]School of Computing, DePaul University, Chicago, IL 60604, USA (ikanj@depaul.edu).

[‡]Institute of Computing Technology, Chinese Academy of Sciences, Beijing, China (liqian@ict.ac.cn).

[§]Department of Computer Science, Lafayette College, Easton, PA 18042, USA (xiag@lafayette.edu).

¹We denote by polylog(n) the function $log^{O(1)} n$, where n is the size of the input.

has been known for GRAPH k-MATCHING on unweighted graphs for randomized streaming algorithms, even in the simpler insert-only model [5]. Nearly space-optimal streaming algorithms, i.e., streaming algorithms with space complexity $O(k^2 \cdot \text{polylog}(n))$, have been developed for GRAPH k-MATCHING on unweighted graphs [3, 5].

The current paper is focused on the update time of streaming algorithms for GRAPH k-MATCHING. While there has been much work pertaining to the space complexity of streaming algorithms for graph matching, much less is known about the update time complexity of the problem. Note that the update time sometimes could be even more important than the space complexity [23], since the data stream can come at a very high rate. If the update processing rate does not catch the update arrival rate, the whole system may fail (see, e.g., [1, 31]). A major contribution of the current paper is the development of a collection of streaming algorithms for GRAPH k-MATCHING that, while keeping the optimal or near-optimal space complexity, also reach optimal or near-optimal update time.

1.1 Previous work on Graph *k*-Matching

We start by reviewing the relevant previous work on the problem.

Fafianie and Kratsch [14] studied kernelization streaming algorithms in the insert-only model for the NP-hard *d*-SET MATCHING problem (among others), which for d = 2, is equivalent to the GRAPH *k*-MATCHING problem on unweighted graphs. Their result gives a one-pass (deterministic) kernelization streaming algorithm for GRAPH *k*-MATCHING on unweighted graphs. The algorithm implies a streaming algorithm in the insert-only model for the GRAPH *k*-MATCHING problem on unweighted graphs, with space complexity $O(k^2)$ and update time $O(\log k)$.

More recently, streaming algorithms in the dynamic model for the GRAPH k-MATCHING problem have been studied [3, 5, 6, 7]. Under the assumption that at *every* instant the size (i.e., the cardinality) of a maximum matching of the graph stream is bounded by k, a randomized onepass dynamic streaming algorithm is given in [7], which was refined in [6]. The algorithm w.h.p.computes a maximum matching in an unweighted graph stream, and runs in $O(k^2 \cdot \text{polylog}(n))$ space and $O(k^2 \cdot \text{polylog}(n))$ update time (see [7]).

The authors of [5] revisited the problem of constructing maximum matchings in the dynamic streaming model. Under a slightly less restricted constraint that the size of a maximum matching of the stream graph is bounded by k (we will call this constraint "the Size-k Constraint"), a sketch-based streaming algorithm is presented in [5] that w.h.p. computes a maximum matching of an unweighted graph. The algorithm retains the space complexity at $O(k^2 \cdot \text{polylog}(n))$ but has an improved update time of O(polylog(n)).

For general graph streams that may not satisfy the Size-k Constraint, a randomized approximation algorithm was given in [5] for maximum matchings in unweighted graph streams. Specifically, if the graph contains matchings of size larger than k, then for any $1 \leq \alpha \leq \sqrt{k}$ and $0 < \epsilon \leq 1$, there exists an $O(k^2/(\alpha^3 \epsilon^2) \cdot \text{polylog}(n))$ -space algorithm that w.h.p. returns a matching of size at least $(1-\epsilon)k/(2\alpha)$. The algorithm has $O(k^2 \cdot \text{polylog}(n)/(\alpha^2 \epsilon^2))$ update time (see [5], Theorem 4.1). In particular, for a graph G in the stream in which the size of a maximum matching is at least $c_0 k$ for a constant $c_0 \geq 1$, the algorithm, by properly choosing α and ϵ (e.g., if $c_0 = 4$ then let $\alpha = 1$ and $\epsilon = 1/2$), will w.h.p. construct a matching of size at least k in the graph G. This streaming algorithm has space complexity $O(k^2 \cdot \text{polylog}(n))$ but its update time is raised back to $O(k^2 \cdot \text{polylog}(n))$. When we combine this algorithm with the streaming algorithm under the Size-k Constraint (more precisely, under the Size-(c_0k) Constraint) given in [5], we will obtain a streaming algorithm in the dynamic model for GRAPH k-MATCHING on general unweighted graph streams (i.e., without the assumption of the Size-k Constraint), which

runs in space complexity $O(k^2 \cdot \text{polylog}(n))$ and update time $O(k^2 \cdot \text{polylog}(n))$. Since $\Omega(k^2)$ is a lower bound on the space complexity of streaming algorithms for the GRAPH k-MATCHING problem [7], the streaming algorithm described above for GRAPH k-MATCHING has near-optimal space complexity (i.e., optimal modulo a poly-logarithmic factor).

As described in [5], streaming algorithms for GRAPH k-MATCHING on unweighted graph streams can be extended to solve the GRAPH k-MATCHING problem on weighted graph streams (i.e., constructing a maximum weighted matching of k edges in a weighted graph stream), with space complexity increased by a factor of the number W of distinct edge weights. Thus, under the Size-k Constraint, there is a streaming algorithm in the dynamic model for the GRAPH k-MATCHING problem on weighted graph streams with space complexity $O(k^2W \cdot \text{polylog}(n))$ and update time O(polylog(n)), while without the assumption of the Size-k Constraint, there is a streaming algorithm in the dynamic model for the GRAPH k-MATCHING problem on weighted graph streams with space complexity $O(k^2W \cdot \text{polylog}(n))$ and update time $O(k^2 \cdot \text{polylog}(n))$.

The above described algorithms are the best known streaming algorithms for the GRAPH k-MATCHING problem.

1.2 Our contributions

We start by discussing our results for the insert-only model. We present a one-pass randomized streaming algorithm that constructs a maximum weighted k-matching in a weighted graph. Our algorithm runs in $O(k^2)$ space and has O(1) update time, which both are optimal. Our algorithm relies on the critical observation that there is a "compact" subgraph of size $O(k^2)$ that contains a maximum weighted k-matching in the original graph. We show that using techniques of universal hashing, w.h.p., we can identify the compact subgraph effectively, and that using the technique of interleaving executions of multiple parts of the algorithm, we can efficiently update the compact subgraph when new edges are inserted while keeping the O(1) update time.

Compared to the previous best result by Fafianie and Kratsch [14], who developed a (deterministic) kernelization streaming algorithm that implies a one-pass streaming algorithm in the insert-only model for GRAPH k-MATCHING on unweighted graphs with space complexity $O(k^2)$ and update time $O(\log k)$, our algorithm is randomized, achieving the same (optimal) space complexity, but also has optimal update time O(1). Most significantly, our streaming algorithm solves the GRAPH k-MATCHING problem on weighted graphs, which is a much more difficult problem compared to the problem on unweighted graphs.

We then study streaming algorithms for GRAPH k-MATCHING in the dynamic model. We give a one-pass randomized streaming algorithm that, for a weighted graph G containing a k-matching, w.h.p., constructs a maximum weighted k-matching of G. The algorithm runs in $O(k^2W \cdot \text{polylog}(n))$ space and has O(polylog(n)) update time, where W is the number of distinct edge weights in the graph. This result directly implies a one-pass randomized streaming algorithm for GRAPH k-MATCHING on unweighted graphs, with near-optimal space complexity $O(k^2 \cdot \text{polylog}(n))$ and near-optimal update time O(polylog(n)).

The faster update time of our streaming algorithm, while keeping the same (near-optimal) space complexity, is achieved based on a technique of randomized construction of a many-tomany mapping between a given large set U and a small integral interval. Note that this approach is different from that of previous randomized streaming algorithms, which in general *partition* the set U into disjoint subsets. Briefly speaking, for any (unknown) k-subset S of the set U we construct a small collection H^+ of $O(\log k)$ hash functions, each using $O(k \cdot \text{polylog}(n))$ space. The collection H^+ makes a many-to-many mapping between U and an integral interval I of size $O(k \cdot \text{polylog}(n))$. We show that w.h.p. there are k integers in I whose pre-images in U are pairwise disjoint and each contains exactly one element in S. Compared to the popular approach for graph matching, which uses universal hash functions for streaming algorithms, our approach uses less space and achieves O(polylog(n)) update time. This technique combined with the ℓ_0 -sampling techniques [10, 16] enables us to select a smaller subset of edges, from the vertex subsets of our construction, that w.h.p. contains the desired k-matching. From this smaller subset of edges, a maximum weighted k-matching can be extracted.

In comparison with the previous best results, Chitnis et al. [5], under the Size-k Constraint, developed a randomized streaming algorithm in the dynamic model for GRAPH k-MATCHING on unweighted graphs, that has the same space complexity $O(k^2 \cdot \text{polylog}(n))$ and update time O(polylog(n)) as our algorithm. However, our algorithm is not restricted to the Size-k Constraint, which seems a rather strong assumption on graph streams. The previous best streaming algorithm for GRAPH k-MATCHING on unweighted graphs without the assumption of the Size-k Constraint, as given in [5] and explained above, runs in space $O(k^2 \cdot \text{polylog}(n))$ and has update time $O(k^2 \cdot \text{polylog}(n))$. Compared to this algorithm, our algorithm has a much faster update time of O(polylog(n)). Similarly, compared with the best streaming algorithms in the dynamic model for GRAPH k-MATCHING on weighted graphs as given in [5], our algorithm, while keeping the space complexity matching that in [5], is applicable to a much larger class of graphs (i.e., without assuming the Size-k Constraint), and has significantly improved (and near-optimal) update time O(polylog(n)).

A byproduct of our results is a one-pass streaming approximation algorithm that, for any $\epsilon > 0$, w.h.p. computes a k-matching in a weighted graph stream that is within a factor of $1 - \epsilon$ from a maximum weighted k-matching. The algorithm runs in $O(k^2 \log R_{wt} \cdot \text{polylog}(n)/\epsilon)$ space and has O(polylog(n)) update time, where R_{wt} is the ratio of the maximum edge-weight to the minimum edge-weight in the graph. This result improves the update time complexity over the approximation result in [5], which has the same space complexity but has update time $O(k^2 \cdot \text{polylog}(n))$.

We mention that Chen et al. [4] studied algorithms for k-matching in unweighted and weighted graphs in the RAM model with limited computational resources. Clearly, the RAM model is very different from the streaming model. In order to translate their algorithm to the streaming model, it would require $\Omega(nk)$ space and multiple passes, where n is the number of vertices. However, we mention that one of the steps of our algorithm in the insert-only model was inspired by an operation for constructing a reduced graph, which was introduced in [4].

Finally, there has been work on computing matchings in special graph classes, and with respect to parameters other than the cardinality of the matching (see, e.g. [11, 21, 25, 26]).

2 Preliminaries

For a positive integer i, let [i] denote the set of integers $\{1, 2, \ldots, i\}$, and let $[i]^-$ denote the set $\{0, 1, \ldots, i-1\}$. We write "*u.a.r.*" as an abbreviation for "uniformly at random".

All graphs discussed in this paper are undirected and simple. We write V(G) and E(G) for the vertex set and edge set of a graph G, respectively, and write [u, v] for an edge with the endpoints u and v. The size of a graph G, denoted by |G|, is equal to the number of vertices plus the number of edges in the graph. A matching $M \subseteq E(G)$ is a set of edges in which no two edges share a common endpoint. A matching M is a k-matching if it consists of exactly k edges. A weighted graph G is a graph associated with a weight function $wt : E(G) \longrightarrow \mathbb{R}$; we denote the weight of an edge e by wt(e). Let M be a matching in a weighted graph G. The weight of M, wt(M), is the sum of the weights of the edges in M. A maximum weighted k-matching in a weighted graph G is a k-matching whose weight is the maximum over all k-matchings in G.

2.1 The graph streaming model

A graph stream S for an underlying graph G is a sequence of elements, each of the form (e, op), where op is an update to an edge e in G. An update could be either an insertion or a deletion of an edge (and would include the edge weight if the graph G is weighted). In the insert-only graph streaming model, a graph G is given as a stream S of elements in which each operation is an edge insertion, while in the dynamic graph streaming model a graph G is given as a stream S of elements in which the operations could be either edge insertions or edge deletions. We will assume that a graph stream always starts with an empty edge set.

Without loss of generality, we will assume that a graph G of n vertices has $[n]^-$ as its vertex set, and that the length of a stream S for G is polynomial in n. Since the graph G can have at most n(n-1)/2 edges, each edge in G can be represented as a unique number in $[n(n-1)/2]^-$.

2.2 Problem definitions

The formulation of our problem is of a multivariate nature. An instance of our problem is of the form (S, k), where S is a graph stream of some underlying graph G and k is an integer. The goal is to construct a k-matching in G (if G is an unweighted graph) or a maximum weighted k-matching in G (if G is a weighted graph). We will consider the problem in both the insert-only and the dynamic streaming models. We formally define the problems under consideration:

p-MATCHING **Given:** a graph stream S for an unweighted graph G and an integer k, **Goal:** construct a k-matching in G or report that no k-matching exists in G. p-WMATCHING **Given:** a graph stream S for a weighted graph G and an integer k, **Given:** a graph stream S for a weighted graph G and an integer k,

Goal: construct a maximum weighted k-matching in G or report that no k-matching exists in G.

We will design streaming algorithms for the above problems.

2.3 Probability

For any probabilistic events E_1, \ldots, E_r , the union bound states that $\Pr[\bigcup_{i=1}^r E_i] \leq \sum_{i=1}^r \Pr[E_i]$. For any random variables X_1, \ldots, X_r whose expectations are well-defined, the *linearity of expectation* states that $Exp[\sum_{i=1}^r X_i] = \sum_{i=1}^r Exp[X_i]$. A set of discrete random variables $\{X_1, \ldots, X_j\}$ is λ -wise independent if for any subset $J \subseteq \{1, \ldots, j\}$ with $|J| \leq \lambda$ and for any values $x_i, i \in J$, we have $\Pr[\bigwedge_{i \in J} X_i = x_i] = \prod_{i \in J} \Pr[X_i = x_i]$. A random variable is a 0-1 random variable if it only takes the values 0 and 1. The following theorem bounds the tail probability of the sum of 0-1 random variables with limited independence:

Proposition 2.1 (Theorem 2 in [30]) Given a set $\{X_1, \ldots, X_j\}$ of 0-1 random variables, let $X = \sum_{i=1}^{j} X_i$ and $\mu = Exp[X]$. For any $\delta > 0$, if $\{X_1, \ldots, X_j\}$ is $\lceil \mu \delta \rceil$ -wise independent, then

$$\Pr[X \ge \mu(1+\delta)] \le \begin{cases} e^{-\mu\delta^2/3} & \text{if } \delta < 1\\ e^{-\mu\delta/3} & \text{if } \delta \ge 1 \end{cases}$$

2.4 ℓ_0 -samplers

Let $0 < \delta < 1$ be a parameter. Let $S = (i_1, \Delta_1), (i_2, \Delta_2), \ldots, (i_p, \Delta_p), \ldots$ be a stream of updates of an underlying vector $\mathbf{x} \in \mathbb{R}^n$, where for each $j, i_j \in [n], \Delta_j \in \mathbb{R}$, and the update (i_j, Δ_j) updates the i_j -th coordinate of \mathbf{x} by setting $\mathbf{x}_{i_j} = \mathbf{x}_{i_j} + \Delta_j$. An ℓ_0 -sampler for $\mathbf{x} \neq 0$ either fails with probability at most δ , or conditioned on not failing, returns a pair (j, \mathbf{x}_j) with probability $1/||\mathbf{x}||_0$ for any non-zero coordinate \mathbf{x}_j of \mathbf{x} , where $||\mathbf{x}||_0$ is the ℓ_0 -norm of \mathbf{x} , which is the number of non-zero coordinates of \mathbf{x} . For more details, we refer to [10].

Based on the results in [10, 16], and as shown in [5], we can develop a sketch-based ℓ_0 -sampler algorithm for a dynamic graph stream that samples an edge from the stream. More specifically, the following result was shown in [5]:

Proposition 2.2 (Proof of Theorem 2.1 in [5]) Let $0 < \delta < 1$ be a parameter. There exists an ℓ_0 -sampler algorithm that, given a dynamic graph stream, either returns FAIL with probability at most δ , or returns an edge chosen u.a.r. amongst the edges of the stream that have been inserted and not deleted. This ℓ_0 -sampler algorithm can be implemented using $O(\log^2 n \cdot \log(\delta^{-1}))$ bits of space and $O(\operatorname{polylog}(n))$ update time, where n is the number of vertices of the graph stream.

2.5 Hash functions

Let U be a finite set of n elements (without loss of generality, we will assume that $U = [n]^{-}$). A hash function h from U is *perfect* w.r.t. a subset S of U if it is injective on S, i.e., $h(x) \neq h(y)$ for any two distinct x and y in S. For a family \mathcal{H} of hash functions, we write $h \xleftarrow{u.a.r.}{\mathcal{H}} \mathcal{H}$ to denote that the hash function h is chosen u.a.r. from \mathcal{H} .

Let r be an integer, $0 < r \leq |U|$. A family \mathcal{H} of hash functions, each mapping U to $[r]^-$, is called *universal* if for each pair of distinct elements $x, y \in U$, the number of hash functions $h \in \mathcal{H}$ for which h(x) = h(y) is at most $|\mathcal{H}|/r$, or equivalently, for a hash function $h \xleftarrow{u.a.r.} \mathcal{H}$, we have $\Pr[h(x) = h(y)] \leq 1/r$.

The following universal family of hash functions has been well-known (see Chapter 11, [9]):

Proposition 2.3 The collection $\mathcal{H} = \{h_{a,b,r} \mid 1 \leq a \leq p-1, 0 \leq b \leq p-1\}$ is a universal family of hash functions from U to $[r]^-$, where $p \geq |U|$ is a prime number, and $h_{a,b,r}$ is defined as $h_{a,b,r}(x) = ((ax + b) \pmod{p}) \pmod{p}$.

A perfect hash function can be retrieved from a universal family of hash functions, as given in the following proposition.

Proposition 2.4 (Theorem 11.9 in [9]) Let \mathcal{H} be a universal family of hash functions, each mapping the finite set U to $[r^2]^-$. For any set S of r elements in U, the probability that a hash function $h \xleftarrow{u.a.r.} \mathcal{H}$ is perfect w.r.t. S is larger than 1/2.

A family \mathcal{H} of hash functions mapping U to $[r]^-$ is κ -wise independent if for any κ distinct elements $x_1, x_2, \ldots, x_{\kappa}$ in U, and any (not necessarily distinct) $a_1, a_2, \ldots, a_{\kappa}$ in $[r]^-$, we have

$$\Pr_{h \leftarrow u.a.r.} \mathcal{H}[(h(x_1) = a_1) \land (h(x_2) = a_2) \land \dots \land (h(x_{\kappa}) = a_{\kappa})] = 1/r^{\kappa}.$$

Proposition 2.5 (Corollary 3.34 in [32]) For each integer $\kappa > 0$, there is a family of κ wise independent functions $\mathcal{H} = \{h : U \to [r]^-\}$ such that choosing a random function h from \mathcal{H} takes space $O(\kappa \log |U|)$. Moreover, evaluating a function h from \mathcal{H} on an element x in U, i.e., computing the value h(x), takes time polynomial in κ and $\log |U|$.²

²In the original statement of this theorem in [32], the hash functions h in the family \mathcal{H} map binary strings of

3 Streaming algorithms on the insert-only model

In this section, we give a streaming algorithm for p-WMATCHING, and hence for p-MATCHING as a special case, in the insert-only model. We start with some notations.

Let G = (V, E) be a weighted graph with a weight function $wt : E \to \mathbb{R}_{\geq 0}$, where $V = [n]^-$. We define a new function $\beta : E \to \mathbb{R}_{\geq 0} \times V \times V$ that on an edge e = [u, v] in G, where $u < v, \beta(e) = (wt(e), u, v)$. Observe that each edge in G has a distinct β -value, and that the lexicographic order w.r.t. β defines a total order of the edges in G. The *i*-th heaviest edge in an edge set E' is the edge that has the *i*-th largest β -value among all edges in E'. Because each edge has a distinct β -value, the *i*-th heaviest edge in E' is uniquely defined. Note that the "heaviness" of edges is defined in terms of the edge β -values, while the "weight" of edges, which is used to measure the weight of matchings in the graph, is defined in terms of the original edge weight function wt of the graph.

Let $f: V \to [4k^2]^-$ be a hash function. The function f partitions the vertex set V of G into a collection of subsets $\mathcal{V} = \{V_0, V_1, \ldots, V_{4k^2-1}\}$, where for each $i \in [4k^2]^-$, the subset V_i consists of the vertices v in V such that f(v) = i. A matching M in G is said to be *nice* w.r.t. f if no two vertices of M belong to the same subset V_i for any i. If the hash function f is clear from the context, we will simply say that M is nice.

For a subgraph H of the graph G, we define the *compact subgraph* of H under f, denoted $C_f(H)$, as the subgraph of H consisting of the edges e of H such that the two endpoints of e belong to two distinct subsets V_i and V_j in the collection \mathcal{V} , and that $\beta(e)$ is maximum over all edges between V_i and V_j in H.

Furthermore, we define the *reduced compact subgraph* of H under f, denoted $\mathcal{R}_f(H)$, obtained from the compact subgraph $\mathcal{C}_f(H)$ using the following procedure:

- (1) Delete the edges [u, v] in $C_f(H)$, where $u \in V_i$ and $v \in V_j$ for some $i \neq j$, if [u, v] is either not among the 2k heaviest edges incident to vertices in V_i or not among the 2k heaviest edges incident to vertices in V_j . Let the resulting graph be $\mathcal{R}'_f(H)$.
- (2) Delete all edges e in $\mathcal{R}'_f(H)$ if e is not among the $4k^2$ heaviest edges in $\mathcal{R}'_f(H)$.

The resulting graph is the reduced compact subgraph $\mathcal{R}_f(H)$.

Lemma 3.1 The compact subgraph $C_f(H)$ has nice k-matchings if and only if the reduced compact subgraph $\mathcal{R}_f(H)$ has nice k-matchings. If this is the case, then the weight of a maximum weighted nice k-matching in $C_f(H)$ is equal to that in $\mathcal{R}_f(H)$.

Proof. From the definition, the reduced compact subgraph $\mathcal{R}_f(H)$ is a subgraph of the compact subgraph $\mathcal{C}_f(H)$. Thus, every nice k-matching in $\mathcal{R}_f(H)$ is also a nice k-matching in $\mathcal{C}_f(H)$. In particular, the weight of a maximum weighted nice k-matching in $\mathcal{R}_f(H)$ cannot be larger than that in $\mathcal{C}_f(H)$.

For the other direction, suppose that $C_f(H)$ has nice k-matchings. For convenience, we will say that an edge e is "incident" to a subset V_i if an endpoint of e is in V_i , and that a k-matching M "covers" a subset V_i if M has an edge incident to V_i . Let M_c be a maximum weighted nice k-matching in $C_f(H)$ that contains the largest number of edges in $\mathcal{R}'_f(H)$, which is the graph given in the construction of the reduced compact subgraph $\mathcal{R}_f(H)$ from the compact subgraph $C_f(H)$. We prove that M_c is entirely contained in the graph $\mathcal{R}'_f(H)$.

length n to binary strings of length m for some fixed integers n and m. Here we have simplified the expressions. Thus, by j = h(x), we really mean that j is the integer whose binary representation j_{bin} is the result of $h(x_{bin})$, where x_{bin} is the binary representation of x.

To the contrary, suppose that there is an edge e in the matching M_c in $C_f(H)$ that is not in $\mathcal{R}'_f(H)$. Then e is incident to a subset V_{i_1} but is not among the 2k heaviest edges incident to V_{i_1} in the graph $\mathcal{C}_f(H)$. Thus, there are more than 2k edges incident to V_{i_1} in the graph $\mathcal{C}_f(H)$. Since the nice k-matching M_c covers only 2k subsets in the collection \mathcal{V} , there must be an edge e_1 in $\mathcal{C}_f(H)$ among the 2k heaviest edges incident to V_{i_1} whose other end is incident to a subset V_{i_2} not covered by M_c . Note that we have $\beta(e) < \beta(e_1)$. The edge e_1 cannot be among the 2k heaviest edges incident to V_{i_2} — otherwise, e_1 would be in $\mathcal{R}'_f(H)$ and $M_c \setminus \{e\} \cup \{e_1\}$ would be a maximum weighted k-matching in $\mathcal{C}_f(H)$ that contains more edges in $\mathcal{R}'_f(H)$ than M_c does, contradicting the assumption of M_c . Using the same argument on the subset V_{i_2} , we can find an edge e_2 among the 2k heaviest edges incident to V_{i_2} whose other endpoint is in a subset V_{i_3} not covered by M_c such that e_2 is not among the 2k heaviest edges incident to V_{i_3} and that $\beta(e_1) < \beta(e_2)$. Continuing this process will produce a sequence of edges e_1 , e_2 , \cdots , with $\beta(e_h) < \beta(e_{h+1})$ for all h. Since the graph $\mathcal{C}_f(H)$ is finite, we must have some edges repeating in this sequence, i.e., we will find e_s and e_t in the sequence with s < t such that $e_s = e_t$. However, this would imply that

$$\beta(e_s) < \beta(e_{s+1}) < \dots < \beta(e_{t-1}) < \beta(e_t) = \beta(e_s),$$

which is impossible. This contradiction shows that the maximum weighted nice k-matching M_c is entirely contained in the graph $\mathcal{R}'_f(H)$. As a consequence, M_c is a maximum weighted nice k-matching in $\mathcal{R}'_f(H)$.

If $\mathcal{R}'_f(H)$ contains no more than $4k^2$ edges, then $\mathcal{R}'_f(H) = \mathcal{R}_f(H)$ and M_c is also a maximum weighted nice k-matching in $\mathcal{R}_f(H)$.

If $\mathcal{R}'_f(H)$ contains more than $4k^2$ edges, then the reduced compact subgraph $\mathcal{R}_f(H)$ contains exactly the $4k^2$ heaviest edges in $\mathcal{R}'_f(H)$. Now consider a maximum weighted k-matching M'_c in $\mathcal{R}'_f(H)$ that contains the maximum number of edges in the reduced compact subgraph $\mathcal{R}_f(H)$. If there is an edge e' in M'_c that is not in $\mathcal{R}_f(H)$, then delete all vertices in $\mathcal{R}_f(H)$ that are in the subsets of \mathcal{V} covered by $M'_c \setminus \{e'\}$. This will delete no more than $2(k-1)\cdot 2k < 4k^2$ edges in $\mathcal{R}_f(H)$, because each subset V_i is incident to at most 2k edges in $\mathcal{R}_f(H)$. Therefore, there is at least one edge e'' left in $\mathcal{R}_f(H)$, and by definition, $\beta(e'') > \beta(e')$. Thus, the matching $M'_c \setminus \{e'\} \cup \{e''\}$ would be a maximum weighted nice k-matching in $\mathcal{R}'_f(H)$ that contains more edges in $\mathcal{R}_f(H)$ than M'_c does, but this contradicts the assumption of the matching M'_c . This contradiction shows that the maximum weighted nice k-matching M'_c in $\mathcal{R}'_f(H)$ is also a maximum weighted nice k-matching in $\mathcal{R}_f(H)$. This completes the proof.

The following lemma shows how we construct the reduced compact subgraph $\mathcal{R}_f(H)$ when the subgraph H has been stored in memory.

Lemma 3.2 Let f be a hash function mapping the vertex set of a graph G to $[4k^2]^-$. There is an algorithm that, for any subgraph H of G, constructs the subgraph $\mathcal{R}_f(H)$ in time and space both bounded by $O(|H| + k^2)$.

Proof. For each $i \in [4k^2]^-$, remember that V_i is the subset of vertices in G whose image under f is i. The algorithm on a subgraph H of G works as follows. By going through the edges of the graph H and using the hash function f, the algorithm deletes, in time $O(|H| + k^2)$, all edges whose two endpoints have the same image under f, and puts each [u, v] of the remaining edges in the sets E_i and E_j , where i = f(u) and j = f(v). Now for each set E_i , the algorithm sorts, using Radix-Sort in linear time, the edges [u, v] in E_i in terms of the pairs (f(u), f(v)), identifies the heaviest edge (i.e., the edge with the maximum β -value) between the sets V_i and V_j , for

each j, and deletes all other edges between V_i and V_j . The result of this process is the compact subgraph $\mathcal{C}_f(H)$. Now using the linear-time selection algorithm [9] on each set E_i , the algorithm can identify the (2k)-th heaviest edge in the set E_i , and, by going through all edges in $\mathcal{C}_f(H)$, delete the edges [u, v] between E_i and E_j , for all $i, j \in [4k^2]^-$, if [u, v] is either not among the 2kheaviest edges in E_i or not among the 2k heaviest edges in E_j . This gives the subgraph $\mathcal{R}'_f(H)$. Finally, using the linear-time selection algorithm one more time, the algorithm can delete the edges in $\mathcal{R}'_f(H)$ that are not among the $4k^2$ heaviest, and obtain the reduced compact subgraph $\mathcal{R}_f(H)$. This shows that the running time of the algorithm is bounded by $O(|H| + k^2)$, which also bounds the space complexity of the algorithm.

We now describe our streaming algorithm in the insert-only model for the p-WMATCHING problem. Let (S, k) be an instance of the problem, where $S = \{(e_1, wt(e_1)), \ldots, (e_s, wt(e_s)), \ldots\}$ is the stream of inserting edges of a graph G. For each s, let G_s be the subgraph of G consisting of the first s edges e_1, \ldots, e_s of S, and for $r \leq s$, let $G_{r,s}$ be the subgraph of G consisting of the edges $e_r, e_{r+1}, \ldots, e_s$. Let f be a hash function mapping the vertex set of the graph G to $[4k^2]^-$. For each integer s, we denote by \hat{s} the largest multiple of $4k^2$ that is strictly smaller than s, i.e., $\hat{s} = 4k^2 \cdot i$ for an integer i and $s = \hat{s} + q$ with $1 \leq q \leq 4k^2$. Note that even when s is a multiple of $4k^2$, we still have $\hat{s} < s$.

Let $G_0^f = \emptyset$. For each s > 0, we define, recursively, a subgraph G_s^f of the graph G_s as

$$G_s^f = \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s}). \tag{1}$$

Lemma 3.3 For each $s \ge 0$, the graph G_s^f has at most $4k^2$ edges. Moreover, $G_s^f = \mathcal{R}_f(G_s)$.

Proof. The bound on the size of the graph G_s^f comes directly from the definition of the reduced compact subgraphs under the hash function f.

To prove the equality $G_s^f = \mathcal{R}_f(G_s)$, we first prove the following claim:

Claim. Let H_1 be a subgraph of H_2 and let e be an edge in H_1 . Then $e \in \mathcal{R}_f(H_2)$ implies $e \in \mathcal{R}_f(H_1)$.

Consider the edge e in H_1 that is in $\mathcal{R}_f(H_2)$. Then (1) e is the heaviest edge in H_2 between two vertex sets V_i and V_j in the partition by the hash function f (i.e., e is in the compact subgraph $\mathcal{C}_f(H_2)$); and (2) e is among the 2k heaviest edges incident to V_i and among the 2kheaviest edges incident to V_j in the graph $\mathcal{C}_f(H_2)$ (i.e., e is in the graph $\mathcal{R}'_f(H_2)$); and (3) e is among the $4k^2$ heaviest edges in $\mathcal{R}'_f(H_2)$. Since H_1 is a subgraph of H_2 and $e \in H_1$, by the conditions (1)-(3) above, it is easy to verify that (1') e is in the compact subgraph $\mathcal{C}_f(H_1)$; (2') e is in the graph $\mathcal{R}'_f(H_1)$; and (3') e is among the $4k^2$ heaviest edges in $\mathcal{R}'_f(H_1)$, i.e., the edge e must be in the reduced compact subgraph $\mathcal{R}_f(H_1)$. This proves the claim.

Now we get back to the proof of $G_s^f = \mathcal{R}_f(G_s)$. Our proof goes by induction on $s \ge 0$. The equality obviously holds true for $s \le 4k^2$, since in this case $\hat{s} = 0$ so $G_{\hat{s}} = \emptyset$. Thus, we will assume $s > 4k^2$. Note that $G_s = G_{\hat{s}} \cup G_{\hat{s}+1,s}$.

Let e be an edge in $\mathcal{R}_f(G_s)$. If e is in $G_{\hat{s}} \setminus G_{\hat{s}}^f$. Then since $e \in \mathcal{R}_f(G_s)$ and $G_{\hat{s}}$ is a subgraph G_s , by the Claim we proved above, we would have $e \in \mathcal{R}_f(G_{\hat{s}}) = G_{\hat{s}}^f$, contradicting the assumed condition. Thus, this case is not possible, and we must have $e \in G_{\hat{s}}^f \cup G_{\hat{s}+1,s}$. Since $G_{\hat{s}}^f \cup G_{\hat{s}+1,s}$ is a subgraph of G_s and $e \in \mathcal{R}_f(G_s)$, by the Claim above, $e \in \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s}) = G_s^f$. Since e is an arbitrary edge in $\mathcal{R}_f(G_s)$, this proves that $\mathcal{R}_f(G_s)$ is a subgraph of G_s^f .

For the other direction, let e be an edge in $G_s^f = \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s}) = \mathcal{R}_f(\mathcal{R}_f(G_{\hat{s}}) \cup G_{\hat{s}+1,s}),$ where the second equality is by the inductive hypothesis, and assume that e is an edge between two vertex sets V_i and V_j in the vertex partition by the hash function f. Then, (1) e is the heaviest edge between V_i and V_j in the graph $\mathcal{R}_f(G_{\hat{s}}) \cup G_{\hat{s}+1,s}$, thus, is also such an edge in the graph $G_{\hat{s}} \cup G_{\hat{s}+1,s} = G_s$, i.e., e is an edge in $\mathcal{C}_f(G_s)$; (2) e is among the 2k heaviest edges incident to V_i in the graph $\mathcal{C}_f(G_s)$: if not, then there must be an edge e' incident to V_i in $\mathcal{C}_f(G_s)$ with $\beta(e') > \beta(e)$ and $e' \notin \mathcal{R}'_f(\mathcal{R}_f(G_{\hat{s}}) \cup G_{\hat{s}+1,s})$. However, this is impossible because e'must be in $\mathcal{C}_f(\mathcal{R}_f(G_{\hat{s}}) \cup G_{\hat{s}+1,s})$ and the edge e is among the 2k heaviest edges incident to V_i in $\mathcal{C}_f(\mathcal{R}_f(G_{\hat{s}}) \cup G_{\hat{s}+1,s})$. Similarly, e is among the 2k heaviest edges incident to V_j in $\mathcal{C}_f(G_s)$. Thus, the edge e is in $\mathcal{R}'_f(G_s)$; and (3) e must be among the $4k^2$ heaviest edges in $\mathcal{R}'_f(G_s)$: if not, there must be an edge e'' in $\mathcal{R}_f(G_s)$ with $\beta(e'') > \beta(e)$ and $e'' \notin \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s}))$. By the Claim we proved above, e'' cannot be in $G_{\hat{s}}^f \cup G_{\hat{s}+1,s}$. The remaining possibility is $e'' \in G_{\hat{s}} \setminus G_{\hat{s}}^f$, but this, plus $e'' \in \mathcal{R}_f(G_s)$ and the Claim above would give the contradiction $e'' \in \mathcal{R}_f(G_s) = G_s^f$. Therefore, the edge e must be in the graph $\mathcal{R}_f(G_s)$, proving that G_s^f is a subgraph of $\mathcal{R}_f(G_s)$.

Combining the above two cases proves that $G_s^f = \mathcal{R}_f(G_s)$.

Now we are ready to present the streaming algorithm $w-Match_{ins}$ in the insert-only model for the p-WMATCHING problem, as given in Figure 1, where $\mathcal{S} = \{e_1, e_2, \ldots, e_s, \ldots\}$ is an edge stream of a graph G (here we assume that the edge weight has been included in each edge e_s in the stream), k is the parameter, and $\epsilon > 0$ is a fixed constant that bounds the error probability. Note that the algorithm has used some notations that are used in Lemma 3.3 and its proof.

Algorithm w-Match_{ins}(S, k) INPUT: an edge stream $S = \{e_1, e_2, \dots, e_s, \dots\}$ of a weighted graph G and parameter k OUTPUT: a maximum weighted k-matching in G, or report that no k-matching is in G. \\ Preprocessing 1. let \mathcal{H}_{ϵ} be a set of $\lceil \log(1/\epsilon) \rceil$ hash functions picked *u.a.r.* from a universal family of hashing functions mapping V(G) to $[4k^2]^-$; 2. for (each hash function f in \mathcal{H}_{ϵ}) $G_0^{\dagger} = \emptyset$; 3. $G_{1,s} = \{e_1, e_2, \dots, e_s\}$, where either $s = 4k^2$, or $s < 4k^2$ and e_s is the last edge in \mathcal{S} ; \\ Updating 4. while (the edge stream S is not ended) interleave the executions of steps 4.1 and 4.2 so that the time between reading two consecutive edges in S into $G_{s+1,s'}$ is bounded by O(1): for (each hash function f in \mathcal{H}_{ϵ}) $G_s^f = \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s});$ 4.1
$$\begin{split} G_{s+1,s'} &= \{e_{s+1}, e_{s+2}, \dots, e_{s'}\}, \\ \text{where either } s' &= s + 4k^2, \text{ or } s' < s + 4k^2 \text{ and } e_{s'} \text{ is the last edge in } \mathcal{S}; \end{split}$$
4.24.3s = s';\\ Query for (each hash function f in \mathcal{H}_{ϵ}) 5. $G_f = \mathcal{R}_f(G_{\hat{s}}^f \cup G_{\hat{s}+1,s});$ 5.1construct the maximum weighted k-matching in G_f ; 5.2if (a k-matching is constructed in step 5) then return the k-matching with the largest weight constructed in step 5; else return("no k-matching exists in G").

Figure 1: A streaming algorithm for p-wMATCHING in the insert-only model

Lemma 3.4 The algorithm w-Match_{ins}(S, k) runs in space $O(k^2)$ and has update time O(1).

Proof. Since $\epsilon > 0$ is a fixed constant, the number of hash functions in the set \mathcal{H}_{ϵ} is a constant. Thus, steps 1-2 of the algorithm **w-Match**_{ins}(\mathcal{S}, k) take constant time and need constant space to store the hash functions in \mathcal{H}_{ϵ} .

Step 3 of the algorithm uses $O(k^2)$ space to store the $s \leq 4k^2$ edges in the subgraph $G_{1,s}$.

Step 4 of the algorithm works with four subgraphs: $G_{\hat{s}}^f$, $G_{\hat{s}+1,s}$, G_s^f , and $G_{s+1,s'}$, where $s' \leq s + 4k^2$. By definition, each of $G_{\hat{s}+1,s}$ and $G_{s+1,s'}$ contains at most $4k^2$ edges, and, by Lemma 3.3, each of the subgraphs $G_{\hat{s}}^f$ and G_s^f contains at most $4k^2$ edges. Moreover, since the size of the graph $G_{\hat{s}}^f \cup G_{\hat{s}+1,s}$ is bounded by $O(k^2)$, by Lemma 3.2, the computation of step 4.1 takes space $O(k^2)$. In summary, step 4 of the algorithm runs in space $O(k^2)$.

Similarly, for each hash function f in \mathcal{H}_{ϵ} , step 5.1 takes space $O(k^2)$. Since the size of the graph G_f is bounded by $O(k^2)$, step 5.2 also takes space $O(k^2)$. Since the number of hash functions in \mathcal{H}_{ϵ} is a constant, we conclude that step 5 of the algorithm runs in space $O(k^2)$.

This shows that the algorithm **w-Match**_{ins}(\mathcal{S}, k) runs in space $O(k^2)$. Now we consider the update time of the algorithm. The update time for reading each of the first $s \leq 4k^2$ edges in the stream \mathcal{S} in step 3 is obviously O(1). The rest of the edges in the stream \mathcal{S} are read in step 4. Since the size of the graph $G_{\hat{s}}^f \cup G_{\hat{s}+1,s}$ is bounded by $O(k^2)$, by Lemma 3.2, step 4.1 of the algorithm runs in time $O(k^2)$. Therefore, the execution of step 4.1 can be divided into $4k^2$ segments such that each segment takes time O(1). Now the interleaved execution of steps 4.1 and 4.2 can read an edge in the stream \mathcal{S} in step 4.2 with the execution of a segment of step 4.1, until either the set $G_{s+1,s'}$ has $4k^2$ edges or the stream end is encountered. This guarantees an update time of O(1) for reading each of the edges in the input stream \mathcal{S} .

We conclude this section with the following theorem.

Theorem 3.5 For any fixed $\epsilon > 0$, the streaming algorithm **w-Match**_{ins}(S, k), where S is a stream for a weighted graph G in the insert-only model, runs in space $O(k^2)$ and update time O(1), and (1) if G has k-matchings then the algorithm returns a maximum weighted k-matching in G with probability $\geq 1 - \epsilon$; and (2) if G has no k-matchings then the algorithm reports so.

Proof. First note that for each hash function f in \mathcal{H}_{ϵ} , the graph G_f constructed in step 5.1 is a subgraph of the graph G. Therefore, if the graph G has no k-matchings, then the graph G_f cannot have k-matchings. Thus, in this case, the algorithm reports correctly.

Now assume that the graph G has k-matchings. Let M be a maximum weighted k-matching in G. The vertex set V(M) of the matching M has 2k vertices. By Proposition 2.4, each hash function f in \mathcal{H}_{ϵ} , which is *u.a.r.* picked from a universal family of hash functions mapping V(G)to $[(2k)^2]^- = [4k^2]^-$, has probability at least 1/2 to be perfect w.r.t. V(M). Since there are $\lceil \log(1/\epsilon) \rceil$ hash functions in \mathcal{H}_{ϵ} , with probability at least $1 - 1/2^{\lceil \log(1/\epsilon) \rceil} \ge 1 - \epsilon$, there is a hash function f_0 in \mathcal{H}_{ϵ} that is perfect w.r.t. V(M). As a result, the maximum weighted k-matching M in G is a maximum weighted nice k-matching in the compact subgraph $\mathcal{C}_{f_0}(G)$. By Lemma 3.1, the reduced compact subgraph $\mathcal{R}_{f_0}(G)$ also has a k-matching M_0 whose weight is equal to that of M. Since $\mathcal{R}_{f_0}(G)$ is a subgraph of G, M_0 is also a maximum weighted k-matching in G.

By Lemma 3.3 and step 5.1 of the algorithm, we have $G_{f_0} = \mathcal{R}_{f_0}(G)$. Therefore, the matching M_0 is a (maximum weighted) k-matching in the graph G_{f_0} , and the maximum weighted k-matching constructed in step 5.2 for the graph G_{f_0} , which could be different from M_0 but must have the same weight as M_0 , is a maximum weighted k-matching in the graph G, which will be returned in step 6 of the algorithm. This completes the proof that if the graph G has

k-matchings, then with probability at least $1 - \epsilon$, the algorithm **w-Match**_{ins}(\mathcal{S}, k) returns a maximum weighted *k*-matching in the graph *G*.

Note that the algorithm **w-Match**_{ins}(S, k) given in Figure 1 queries and computes a maximum weighted k-matching at the end of the stream. It is easy to see that the algorithm can be trivially modified so that it can query and compute a maximum weighted k-matching in the graph G_s consisting of the first s edges in the stream for any $s \ge 0$ after seeing the edge e_s , keeping the same space and update time complexities.

4 Streaming algorithms on the dynamic model

In this section, we present a streaming algorithm for p-WMATCHING, thus also for p-MATCHING, on the dynamic model. For this, we first develop a hashing scheme that uses $O(k \cdot \text{polylog}(n))$ space and has a high success probability. The streaming algorithm on the dynamic model will use the hashing scheme and the ℓ_0 -sampling technique discussed in Section 2.

4.1 Perfect hashing in $O(k \cdot \text{polylog}(n))$ space with high probability

The hashing scheme developed in this subsection will be used in our streaming algorithm on the dynamic model. We believe that the result should also be useful in other applications. Let S be an (unknown) k-subset of a universal set U and suppose that we want to distinguish the k elements of S by constructing a collection of subsets of U that contains k pairwise disjoint subsets, each containing exactly one element in S. For example, by Proposition 2.4, a hash function h picked u.a.r. from a universal family of hash functions from U to $[k^2]^-$ has a probability $\geq 1/2$ to be perfect w.r.t. S, and thus distinguishes S. The hash function h uses $O(k^2)$ space, which is large and would directly impact the space complexity of our streaming algorithms. Moreover, the success probability 1/2 of h is not sufficiently large for our purposes. An O(k)-space hash function perfect w.r.t. S can be constructed using a 2-level hashing scheme (see [9], Section 11.5), in which, however, the construction of the hash functions in level 2 must know the set S and the hashing result on S in level 1. Moreover, the scheme uses multiple hash functions from universal hash families, which would significantly decrease the success probability.

We propose a hashing scheme that follows the ideas of 2-level hashing, but with a more careful selection on the hashing methods and on the hashing parameters. Our scheme uses $O(k \cdot \text{polylog}(n))$ space but has a much higher success probability, and its construction in level 2 needs to know neither the set S nor the hashing results on S in level 1. The hashing scheme is given in Figure 2.

 $\mathbf{HashScheme}(U,k)$

\\ construct a hashing scheme H^+ of $O(\log k)$ hash functions from U to $[d_4]^-$, where $d_4 = O(k \log^2 k)$. Level-1: pick a hash function f u.a.r. from a $\lceil 12 \ln k \rceil$ -wise independent family of hash functions mapping U to $[d_1]^-$, where $d_1 = 2^d$ and d is the smallest integer satisfying $k / \ln k \le 2^d$; Level-2: pick u.a.r. and independently a set $H = \{h_1, h_2, \dots, h_{d_2}\}$ of d_2 hash functions from a universal family of hash functions mapping U to $[d_3]^-$, where $d_2 = \lceil 8 \ln k \rceil, d_3 = \lceil 13 \ln k \rceil^2$; let $H^+ = \{h_1^+, h_2^+, \dots, h_{d_2}^+\}$, where each h_i^+ is defined as $h_i^+(x) = f(x)d_2d_3 + (i-1)d_3 + h_i(x)$.

Figure 2: A hashing scheme that uses smaller space with higher success probability

We give some remarks on the hashing scheme in Figure 2. Suppose that the hash function f in Level-1 partitions the universal set U into $U_0, U_1, \ldots, U_{d_1-1}$. To ensure the pairwise

disjointness of the d_1d_2 sets $h_i^+(U_j)$, $1 \le i \le d_2$, $0 \le j \le d_1 - 1$, we define $h_i^+(U_j)$ to be the set $h_i(U_j)$ plus an offset $jd_2d_3 + (i-1)d_3$. Therefore, for each $x \in U_j$, the function h_i^+ has the value

$$h_i^+(x) = jd_2d_3 + (i-1)d_3 + h_i(x) = f(x)d_2d_3 + (i-1)d_3 + h_i(x).$$

Each h_i^+ is a function mapping U to $[d_4]^-$, where $d_4 = d_1 d_2 d_3 = O(k \log^2 k)$.

Theorem 4.1 The hash function set H^+ can be constructed in space $O(\log k \log |U|)$. For each $x \in U$ and each $h_i^+ \in H^+$, the value of $h_i^+(x)$ can be computed in time polynomial in $\log |U|$.

Proof. By Proposition 2.5, constructing and storing the function f in Level-1, which is picked u.a.r. from a $\lceil 12 \ln k \rceil$ -wise independent family of hash functions from U to $\lfloor d_1 \rfloor^-$, uses space $O(\log k \log |U|)$. Each hash function h_i in Level-2 picked from the universal family of hash functions from U to $\lfloor d_3 \rfloor^-$, which is of the form given in Proposition 2.3, takes O(1) space. Since $|H^+| = d_2 = O(\log k)$, we conclude that the set $H^+ = \{h_1^+, h_2^+, \ldots, h_{d_2}^+\}$ of hash functions can be constructed and stored in space $O(\log k \log |U|)$.

By Proposition 2.5, computing the value j = f(x) takes time polynomial in $\log |U|$ and $\log k$. This, plus the O(1) time for computing the other parts of the function h_i^+ , shows that the value of $h_i^+(x)$ can be computed in time polynomial in $\log |U|$, after noting that $k \leq |U|$.

For each value $q \in [d_4]^-$, let $H^+_{inv}(q)$, i.e., the "inverse" of H^+ on q, be the set of such an element x in U such that $h^+_i(x) = q$ for some hash function h^+_i in H^+ .

Theorem 4.2 For any subset S of U with $|S| = k \ge 2$, with probability at least $1 - 4/(k^3 \ln k)$, there are k disjoint subsets $H_{inv}^+(q_1), \ldots, H_{inv}^+(q_k)$ of U such that $|H_{inv}^+(q_i) \cap S| = 1$ for all $i \in [k]$.

Proof. Recall that the function f partitions the set U into d_1 disjoint subsets $U_0, U_1, \ldots, U_{d_1-1}$. We first show that, for each U_j , with a high probability, there is a hash function in the set $H = \{h_1, h_2, \ldots, h_{d_2}\}$ that is perfect w.r.t. $S \cap U_j$.

For an element $x \in S$, and for each $j \in [d_1]^-$, let $X_{x,j}$ be the 0-1 random variable such that $X_{x,j} = 1$ if and only if f(x) = j. Let $X_j = \sum_{x \in S} X_{x,j}$, which is the number of elements in S that are hashed to j by the hash function f. Thus, $X_j = |S \cap U_j|$.

Since f is picked u.a.r. from a $\lceil 12 \ln k \rceil$ -wise independent family of hash functions, the random variables $X_{x,j}$, for $x \in S$, are $\lceil 12 \ln k \rceil$ -wise independent and $\Pr[X_{x,j} = 1] = 1/d_1$. Thus, $Exp[X_j] = |S|/d_1$. Since $k/\ln k \leq d_1 < 2k/\ln k$, so $\ln k/2 < Exp[X_j] \leq \ln k$. Applying Proposition 2.1 with $\mu = Exp[X_j]$ and $\delta = 12 \ln k/Exp[X_j] > 1$, we get

$$\Pr[X_j \ge (1+\delta) Exp[X_j]] \le e^{-\delta \cdot Exp[X_j]/3} = 1/k^4.$$

Since $Exp[X_j] \leq \ln k$ and $\delta = 12 \ln k / Exp[X_j]$, we have $(1 + \delta) Exp[X_j] \leq 13 \ln k$. Hence,

$$\Pr[X_j \ge 13 \ln k] \le \Pr[X_j \ge (1+\delta)E[X_j]] \le 1/k^4.$$

Let \mathcal{E}' denote the event that for all $j \in [d_1]^-$, $X_j < 13 \ln k$. By the union bound, we have

$$\Pr[\mathcal{E}'] \ge 1 - d_1/k^4 \ge 1 - 2/(k^3 \ln k),$$

where the last inequality holds since $d_1 < 2k/\ln k$.

Assume the event \mathcal{E}' that for all $j \in [d_1]^-$, $|S \cap U_j| < 13 \ln k$ holds for all $j \in [d_1]^-$. For each $j \in [d_1]^-$, let \mathcal{E}_j be the event that the set H does not contain a hash function perfect

w.r.t. $S \cap U_j$. Since $|S \cap U_j| < 13 \ln k$, and each hash function in H is picked independently and u.a.r. from a universal family of hash functions from U to $[d_3]^-$ with $d_3 = \lceil 13 \ln k \rceil^2$, which, by Proposition 2.4, is perfect w.r.t. $S \cap U_j$ with probability $\geq 1/2$. Since H consists of $d_2 = \lceil 8 \ln k \rceil$ such hash functions, we derive $\Pr[\mathcal{E}_j | \mathcal{E}'] \leq 1/2^{d_2} < 1/k^4$. Applying the union bound on all $j \in [d_1]^-$, we conclude that, under the event \mathcal{E}' , the probability that there is a $j \in [d_1]^-$ such that the set H contains no hash function perfect w.r.t. $S \cap U_j$ is bounded by $d_1/k^4 < 2/(k^3 \ln k)$. Now let \mathcal{E}'' be the event that, under the event \mathcal{E}' , for every $j \in [d_1]^-$, the set H contains a hash function h_{i_j} perfect w.r.t. $S \cap U_j$. Then $\Pr[\mathcal{E}''|\mathcal{E}'] \geq 1 - 2/(k^3 \ln k)$, which gives directly

$$\Pr[\mathcal{E}' \cap \mathcal{E}''] = \Pr[\mathcal{E}''|\mathcal{E}'] \cdot \Pr[\mathcal{E}'] \ge (1 - 2/(k^3 \ln k))^2 \ge 1 - 4/(k^3 \ln k).$$

Since $\mathcal{E}' \cap \mathcal{E}''$ is the event in which for every $j \in [d_1]^-$, there is a hash function h_{i_j} in the set H that is perfect w.r.t. $S \cap U_j$, which implies immediately that the hash function $h_{i_j}^+$ in the set H^+ is perfect w.r.t. $S \cap U_j$. Thus, under the event $\mathcal{E}' \cap \mathcal{E}''$, the union Q_k of the pairwise disjoint sets $h_{i_0}^+(S \cap U_0), \ldots, h_{i_{d_1-1}}^+(S \cap U_{d_1-1})$ contains exactly k values q_1, \ldots, q_k in $[d_4]^-$, such that each subset $H_{inv}^+(q_1)$ of U contains an element in S. To see the pairwise disjointness of the subsets $H_{inv}^+(q_1), \ldots, H_{inv}^+(q_k)$, observe that the d_1d_2 subsets $h_i^+(U_j)$, for $i \in [d_2]$ and $j \in [d_1]^-$, are pairwise disjoint, so each value in Q_k is in a unique subset $h_i^+(U_j)$. Thus,

(1) if q_s , q_t in Q_k , $q_s \neq q_t$, are in the same $h_i^+(U_j)$, then, since no element in U_j can have two different images under the same function h_i^+ , the subsets $H_{inv}^+(q_s)$ and $H_{inv}^+(q_t)$ are disjoint;

(2) if q_s and q_t in Q_k are in two different $h_i^+(U_j)$ and $h_{i'}^+(U_{j'})$, respectively, then we must have $j \neq j'$ because by the definition of Q_k , if j = j' then we must have $i = i' = i_j$. But this implies that $H_{inv}^+(q_s) \subseteq U_j$ and $H_{inv}^+(q_t) \subseteq U_{j'}$ so $H_{inv}^+(q_s)$ and $H_{inv}^+(q_t)$ must be disjoint. \Box

4.2 The streaming algorithm for p-WMATCHING

Now we are ready for our streaming algorithm for the p-WMATCHING problem on the dynamic model. We first give a high-level description of the algorithm. Let S be a dynamic stream of a weighted graph G = (V, E) with a weight function wt, and let k be the parameter. Let M_{max} be any fixed maximum weighted k-matching in G. We first use a hashing scheme H^+ , as given in Subsection 4.1, to hash the vertices of G into a range $[r]^-$, where $r = O(k \log^2 k)$. By Theorem 4.2, w.h.p. there is a set B of 2k values in $[r]^-$ such that the collection $H^+_{inv}(B) = \{H^+_{inv}(i) \mid i \in B\}$ consists of 2k pairwise disjoint subsets of V(G) in which each subset contains exactly one vertex in the matching M_{max} . As a result, every edge in M_{max} appears between two different subsets in $H^+_{inv}(B)$, so we only need to consider the edges in G that are between different subsets in the collection $\{H^+_{inv}(i) \mid i \in [r]^-\}$.

To handle edges between two given vertex subsets $H_{inv}^+(i)$ and $H_{inv}^+(j)$ in the graph G, we can employ an ℓ_0 -sampler algorithm (see Section 2.4), which, by Proposition 2.2, can handle dynamic edge changes and edge samplings between the two vertex subsets, efficiently in terms of both space complexity and update time. This, however, does not take edge weights into consideration, which can certainly impact the weight of the constructed k-matching. To address this issue, instead of using a single ℓ_0 -sampler for a pair of values in $[r]^-$, for each edge weight value w, and for each pair (i, j) of values in $[r]^-$, we employ an ℓ_0 -sampler $\mathbf{L}_{i,j,w}$ to handle the dynamic changes of weight-w edges between the two vertex subsets $H_{inv}^+(i)$ and $H_{inv}^+(j)$. Now, for an edge [u, v] in the maximum weighted k-matching M_{\max} , the associated ℓ_0 -sampler $\mathbf{L}_{i,j,wt(u,v)}$, where $u \in H_{inv}^+(i)$, $v \in H_{inv}^+(j)$, and $H_{inv}^+(i)$ and $H_{inv}^+(j)$. Since the subsets in the collection $H_{inv}^+(B)$ are pairwise disjoint, the $k \ 0$ -samplers associated with the k edges in $M_{\rm max}$ give a maximum weighted k-matching in G. Thus, the following collection of ℓ_0 -samplers

$$\{\mathbf{L}_{i,j,w} \mid i, j \in [r]^-, w \text{ is an edge weight value in } G\}$$

makes a sketch that is a subgraph of G containing a maximum weighted k-matching in G.

The formal description of our algorithm is given in Figure 3. Without loss of generality, we will assume that the algorithm is queried at the end of the stream S, even though the query could take place at any point in the stream.

Algorithm w-Match_{dyn}(S, k) INPUT: a dynamic edge stream $\mathcal S$ of a weighted graph G and a parameter kOUTPUT: a maximum weighted k-matching in G, or report no k-matching is in G. \\ Preprocessing 1. $\mathcal{L} = \emptyset$; $\backslash \backslash \mathcal{L}$ is the collection of ℓ_0 -samplers; 2. $\delta = 1/(20k^4 \ln(2k));$ \\ the parameter for the ℓ_0 -samplers; 3. $H^+ = \{h_1^+, \dots, h_{d_2}^+\} =$ **HashScheme**(V(G), 2k), where $d_2 = \lceil 8 \ln(2k) \rceil$. \\ Updating 4. while (the edge stream S is not ended) get the next element ([u, v], wt(u, v), op) in the stream S; \setminus assuming u < v4.1 $H^{+}(u) = \{h_{1}^{+}(u), \dots, h_{d_{2}}^{+}(u)\}; H^{+}(v) = \{h_{1}^{+}(v), \dots, h_{d_{2}}^{+}(v)\};$ 4.24.3 for (each $i \in H^+(u)$ and each $j \in H^+(v)$) if $(\mathbf{L}_{i,j,wt(u,v)} \notin \mathcal{L})$ create the ℓ_0 -sampler $\mathbf{L}_{i,j,wt(u,v)}$, and add it to \mathcal{L} ; 4.3.1run the ℓ_0 -sampler $\mathbf{L}_{i,j,wt(u,v)}$ with update $\langle [u,v], op \rangle$ and the parameter δ . 4.3.2\\ Query 5. $E_0 = \emptyset;$ 6. for (each $\mathbf{L}_{i,j,w}$ in \mathcal{L}) apply the ℓ_0 -sampler $\mathbf{L}_{i,j,w}$ with the parameter δ to sample an edge e; 6.16.2if $(\mathbf{L}_{i,j,w} \text{ does not fail})$ add *e* returned by $\mathbf{L}_{i,j,w}$ to E_0 ; 7. return a maximum weighted k-matching in the induced subgraph $G(E_0)$.

Figure 3: A streaming algorithm for p-wMATCHING in the dynamic model.

Lemma 4.3 If the graph G contains k-matchings, then, with probability $\geq 1 - 11/(20k^3 \ln(2k))$, the algorithm **w-Match**_{dyn}(S, k) returns a maximum weighted k-matching of the graph G.

Proof. Let $M_{\max} = \{[u_1, v_1], \ldots, [u_k, v_k]\}$ be a maximum weighted k-matching in the graph G = (V, E), where $u_j < v_j$ for all j. From the algorithm **HashScheme**(V(G), k), as given in Figure 2, for a vertex u in the graph G, each value in the set $H^+(u)$ is in the range $[r]^-$, where $r = O(k \log^2 k) = O(k \cdot \text{polylog}(n))$. Recall that for each $i \in [r]^-$, $H^+_{inv}(i)$ is the set of vertices u in G such that $i \in H^+(u)$. By Theorem 4.2, with probability at least $1 - 4/((2k)^3 \ln(2k)) = 1 - 1/(2k^3 \ln(2k))$, there are 2k pairwise disjoint subsets $H^+_{inv}(i_1), H^+_{inv}(i'_1), \ldots, H^+_{inv}(i_k), H^+_{inv}(i'_k)$ such that $u_j \in H^+_{inv}(i_j)$ and $v_j \in H^+_{inv}(i'_j)$, for $1 \leq j \leq k$. Let \mathcal{E}' be this event. Then $\Pr[\mathcal{E}'] \geq 1 - 1/(2k^3 \ln(2k))$. Under the event \mathcal{E}' , for each j, $1 \leq j \leq k$, step 4.3.1 of the algorithm **w-Match**_{dyn} will feed an edge of weight $wt(u_j, v_j)$ into the ℓ_0 -sampler $\mathbf{L}_{i_j,i'_j,wt(u_j,v_j)}$.

Now consider the sampling success probability for the ℓ_0 -samplers, under the condition of the event \mathcal{E}' . For each edge $[u_j, v_j]$ in the matching M_{\max} , we call the ℓ_0 -sampler $\mathbf{L}_{i_j,i'_j,wt(u_j,v_j)}$ in step 6.1 of the algorithm **w-Match**_{dyn} to sample an edge between the two subsets $H^+_{inv}(i_j)$ and $H^+_{inv}(i'_j)$. Let \mathcal{E}'' be the event that for all $j, 1 \leq j \leq k$, an edge e_j is sampled successfully by the ℓ_0 -sampler $\mathbf{L}_{i_j,i'_j,wt(u_j,v_j)}$. Note that e_j may not be the edge $[u_j, v_j]$ in the maximum weighted matching M_{\max} , but it must be an edge of weight $wt(u_j, v_j)$ between the two subsets $H_{inv}^+(i_j)$ and $H_{inv}^+(i'_j)$. By Proposition 2.2, for each j, the ℓ_0 -sampler $\mathbf{L}_{i_j,i'_j,wt(u_j,v_j)}$ fails with probability at most δ , which gives, by the union bound, $\Pr[\mathcal{E}''] \geq 1 - k \cdot \delta \geq 1 - 1/(20k^3 \ln(2k))$, because $\delta = 1/(20k^4 \ln(2k))$. Since the 2k subsets $H_{inv}^+(i_1)$, $H_{inv}^+(i'_1)$, \dots , $H_{inv}^+(i_k)$, $H_{inv}^+(i'_k)$ are pairwise disjoint under the event \mathcal{E}' , the k edges e_1, \ldots, e_k sampled by the k ℓ_0 -samplers $\mathbf{L}_{i_j,i'_j,wt(u_j,v_j)}$, $1 \leq j \leq k$, respectively, share no common endpoints, i.e., the edge set $\{e_1,\ldots,e_k\}$ is actually a maximum weighted k-matching in G. As a consequence, under the event $\mathcal{E}' \cap \mathcal{E}''$, the subgraph $G(E_0)$ of the graph G induced by the edge set E_0 constructed in step 6.2 of the algorithm **w-Match**_{dyn} contains a maximum weighted k-matching in the graph G. Finally, because $\Pr[\mathcal{E}'] \geq 1 - 1/(2k^3 \ln(2k))$ and $\Pr[\mathcal{E}''] \geq 1 - 1/(20k^3 \ln(2k))$, by the union bound:

$$\Pr[\mathcal{E}' \cap \mathcal{E}''] \ge 1 - 1/(2k^3 \ln(2k)) - 1/(20k^3 \ln(2k)) = 1 - 11/(20k^3 \ln(2k)).$$

In conclusion, under the event $\mathcal{E}' \cap \mathcal{E}''$, step 7 of the algorithm **w-Match**_{dyn} returns a maximum weighted k-matching in the graph G. The lemma is proved.

Now we arrive at our conclusion for the algorithm **w-Match**_{dyn}(\mathcal{S}, k).

Theorem 4.4 Let $(\mathcal{S}.k)$ be a stream of a weighted graph G in the dynamic model. Then

- (1) if G contains a k-matching then, with probability at least $1-11/(20k^3 \ln(2k))$, the algorithm w-Match_{dyn}(\mathcal{S}, k) returns a maximum weighted k-matching in G; and
- (2) if G does not contain a k-matching then the algorithm w-Match_{dun}(\mathcal{S}, k) reports so.

Moreover, the algorithm **w-Match**_{dyn}(S, k) uses $O(k^2W \cdot \text{polylog}(n))$ space and has O(polylog(n))update time, where W is the number of distinct edge weight values in the graph G.

Proof. First observe that the graph $G(E_0)$ in step 7 induced by the edge set E_0 constructed by step 6.2 of the algorithm **w-Match**_{dyn} is a subgraph of G. Therefore, statement (2) in the theorem clearly holds true. Statement (1) follows from Lemma 4.3.

To analyze the complexities of the algorithm **w-Match**_{dyn}(\mathcal{S}, k), first recall that for a vertex v in the graph G, the set $H^+(v)$ is a subset of $[r]^-$, where $r = d_1 d_2 d_3$, $d_1 = O(k/\ln k)$, $d_2 = O(\ln k)$, and $d_3 = O(\ln^2 k)$ (see the algorithm **HashScheme** in Figure 2). Thus, $r = O(k \ln^2 k)$.

Consider the update time of the algorithm. The update on elements in the stream S is processed by steps 4.2-4.3. By Theorem 4.1, step 4.2 takes time polynomial in $\log |V|$ (note $d_2 = O(\ln k)$). For step 4.3, since each of the subsets $H^+(u)$ and $H^+(v)$ contains $d_2 = O(\ln k)$ values, step 4.3 examines $O(\ln^2 k)$ pairs of the form (i, j). We can organize all the ℓ_0 -samplers $\mathbf{L}_{i,j,w}$ in a 2-dimensional array C[1..r, 1..r], where the element C[i, j] is a balanced search tree for the weights of the edges between the subsets $H^+_{inv}(i)$ and $H^+_{inv}(j)$, which supports searching, insertion, and deletion in logarithmic time per operation [9]. The array C[1..r, 1..r] of space $O(r^2W)$ supports searching a given ℓ_0 -sampler $\mathbf{L}_{i,j,w}$ in step 4.3.1 in time $O(\log W)$. This, plus the time for updating the ℓ_0 -sampler $\mathbf{L}_{i,j,w}$ in steps 4.3.1-4.3.2 (which is O(polylog(n)) by Proposition 2.2), shows that steps 4.3.1-4.3.2 take time O(polylog(n)). As a result, step 4.3 takes time $d_2^2 \cdot O(\text{polylog}(n)) = O(\text{polylog}(n))$ because $d_2 = O(\ln k)$. In conclusion, the update time of the algorithm **w-Match**_{dyn}(S, k) on each element in the stream S, as given in steps 4.1-4.3 of the algorithm, is O(polylog(n)).

We analyze the space complexity of the algorithm. By Theorem 4.1, the space taken by steps 1-3 of the algorithm is $O(\log k \log |V|)$, which is used to initialize certain values and to construct and store the hash functions in H^+ . For step 4, as we described above, we can use a 2-dimensional array C[1..r, 1..r] to store the $r^2W \ell_0$ -samplers $\mathbf{L}_{i,j,w}$. Moreover, since $\delta = 1/(20k^4 \ln(2k))$, by Proposition 2.2, each ℓ_0 -sampler $\mathbf{L}_{i,j,w}$ uses $O(\log^2 |V| \cdot \log k)$ space. As a result, step 4 of the algorithm totally takes space $O(r^2W \log^2 |V| \log k)$. Now consider steps 5-7 of the algorithm. The space complexity of steps 5-7 is dominated by the space used to store the $r^2W \ell_0$ -samplers $\mathbf{L}_{i,j,w}$, which is bounded by $O(r^2W \log^2 |V| \log k)$ as analyzed above, and the space used to store the induced subgraph $G(E_0)$. Since at most one edge is sampled from each ℓ_0 -sampler and there are $r^2W \ell_0$ -samplers, the number of edges in the set E_0 , thus the size of the graph $G(E_0)$, is $O(r^2W)$. Finally, step 7 of the algorithm uses space $O(|E_0|) = O(r^2W)$ to construct a maximum weighted k-matching in the induced subgraph $G(E_0)$ [15]. Summarizing all the discussions above, we conclude that the space complexity of the algorithm **w-Match**_{dyn} is $O(r^2W \log^2 |V| \log k) = O(k^2W \cdot \text{polylog}(n))$, thus, complete the proof of the theorem.

Remark. When W = 1, Theorem 4.4 gives a streaming algorithm in the dynamic model for the p-MATCHING problem, i.e., the k-matching problem on unweighted graphs. The algorithm runs in $O(k^2 \cdot \text{polylog}(n))$ space and O(polylog(n)) update time, and has success probability at least $1 - 11/(20k^3 \ln(2k))$. The known lower bounds for p-MATCHING, as we will discuss in the next section, show that both the space complexity and update time of this algorithm are nearly optimal, i.e., differing from the corresponding optimal bounds by at most a poly-logarithmic factor.

The space complexity $O(k^2W \cdot \text{polylog}(n))$ of the streaming algorithm **w-Match**_{dyn} in the dynamic model for weighted graphs is large if the number W of distinct edge weights is large. Unfortunately, as we will prove in the appendix, the term W in space complexity for streaming algorithms in the dynamic model for the p-WMATCHING problem is actually unavoidable. On the other hand, as suggested in [5], by rounding the edge weights, algorithms such as **w-Match**_{dyn} can be used to develop streaming approximation algorithms in the dynamic model for the p-WMATCHING problem.

Under the Size-k Constraint, an approximation streaming algorithm for the p-WMATCHING problem was presented in [5]. The algorithm approximates p-WMATCHING to within ratio $1 + \epsilon$, for any $\epsilon > 0$, and has space complexity $O(k^2 \log R_{wt} \cdot \text{polylog}(n)/\epsilon)$ and update time O(polylog(n)), where R_{wt} is the ratio of the maximum edge weight over the minimum edge weight. Approximation streaming algorithms for p-WMATCHING in the dynamic model with no assumption of Size-k Constraint were also studied and developed in [5], which are able to keep the same space complexity but have to worsen the update time to $O(k^2 \cdot \text{polylog}(n))$.

Using Theorem 4.4, and following the same approach in [5], we obtain the following approximation streaming algorithm of ratio $1 + \epsilon$ for the p-wMATCHING problem in the dynamic model. The algorithm has space complexity $O(k^2 \log R_{wt} \cdot \text{polylog}(n)/\epsilon)$ and update time O(polylog(n)), and does not need to assume the Size-k Constraint.

Theorem 4.5 For any $0 < \epsilon < 1$, there is an algorithm for the p-WMATCHING problem in the dynamic model that on a stream (S, k) for a weighted graph G:

- (1) returns a k-matching of weight at least (1ϵ) of that of a maximum weighted k-matching in G if G contains k-matchings; and
- (2) reports no k-matching if G does not contain a k-matching.

Moreover, the algorithm uses $O(k^2 \log R_{wt} \cdot \text{polylog}(n)/\epsilon)$ space and has O(polylog(n)) update time, where R_{wt} is the ratio of the maximum edge weight over the minimum edge weight.

Proof. For each edge e in the graph G, we assign e a new weight wt'(e) = t, where t is the integer satisfying $(1 + \epsilon)^{t-1} < wt(e) \le (1 + \epsilon)^t$. Thus, under the new edge weights, the graph G

has $O(\log R_{wt}/\epsilon)$ distinct edge weights. Now we run the algorithm **w-Match**_{dyn} on the graph G with the new edge weights. By Theorem 4.4, the algorithm returns a k-matching M in G in space $O(k^2 \log R_{wt} \cdot \text{polylog}(n)/\epsilon)$ and update time O(polylog(n)), with success probability at least $1 - 11/(20k^3 \ln(2k))$.

We prove that in terms of the original edge weight function $wt(\cdot)$ of the graph G, the weight wt(M) of the k-matching M returned by the above algorithm is at least $(1 - \epsilon)$ of the weight $wt(M_{\max})$ of a maximum weighted k-matching $M_{\max} = \{e_1, \ldots, e_k\}$ in the graph G. Consider the algorithm **w-Match**_{dyn} on the graph G with the new edge weights. As proved in Lemma 4.3, with probability at least $1 - 11/(20k^3 \ln(2k))$, the induced subgraph $G(E_0)$ constructed by step 6 of the algorithm **w-Match**_{dyn} contains a k-matching $M' = \{e'_1, \ldots, e'_k\}$, where for each s, the edge e'_s in M' and the edge e_s in the maximum weighted k-matching M_{\max} are processed by the same ℓ_0 -sampler $\mathbf{L}_{i,j,w}$. Therefore, under the new edge weights of the graph G, the edges e'_s and e_s have the same weight $wt'(e'_s) = wt'(e_s) = w$, which, by the definition of the new edge weights, immediately gives the relation $wt(e_s)/wt(e'_s) \leq 1+\epsilon$ on the original edge weights of the graph G. Summarizing this over all $1 \leq s \leq k$, we get

$$\frac{wt(M_{\max})}{wt(M')} = \frac{\sum_{s=1}^{k} wt(e_s)}{\sum_{s=1}^{k} wt(e'_s)} \le 1 + \epsilon.$$

Therefore, with probability $\geq 1 - 11/(20k^3 \ln(2k))$, the induced subgraph $G(E_0)$ constructed by step 6 of the algorithm **w-Match**_{dyn} contains the k-matching M' whose weight wt(M') is at least $wt(M_{\max})/(1+\epsilon)$. As a result, the k-matching M returned by the algorithm **w-Match**_{dyn}, which is a maximum weighted k-matching in $G(E_0)$ in terms of the new edge weights, has its weight wt(M) at least $wt(M_{\max})/(1+\epsilon) \geq (1-\epsilon)wt(M_{\max})$. This completes the proof of the theorem.

5 Conclusions and final remarks

In this paper, we presented streaming algorithms for the fundamental graph k-matching problem, for both unweighted graphs and weighted graphs, and in both the insert-only and dynamic streaming models. While matching the best space complexity of known algorithms, our algorithms have much faster update times, significantly improving previous known results. In fact, our algorithms are optimal or near-optimal for many cases for the graph k-matching problem. We give below a brief discussion on the optimality of our algorithms when they are applied in various cases of the graph k-matching problem.

A lower bound $\Omega(k^2)$ on the space complexity for the p-MATCHING problem in the insert-only streaming model to construct a k-matching in an unweighted graph has been developed in [5], which shows that for any randomized streaming algorithm for the problem, there are instances of size n and parameter k, such that the algorithm takes space of $\Omega(k^2)$ bits. The more difficult problem p-WMATCHING in the insert-only streaming model is to construct a maximum weighted k-matching in a weighted graph, for which the $\Omega(k^2)$ space lower bound certainly holds true. By Theorem 3.5, our streaming algorithm **w-Match**_{ins} given in section 3 solves the p-WMATCHING problem in the insert-only model in space $O(k^2)$ and update time O(1). The optimality of the update time O(1) of the algorithm **w-Match**_{ins} is obvious. Thus, the streaming algorithm **w-Match**_{ins} solves the p-WMATCHING problem in the insert-only model in both optimal space complexity and optimal update time. To the authors' best knowledge, this is the first streaming algorithm for the p-WMATCHING problem that achieves optimality in both space complexity and update time complexity. Similarly, the p-MATCHING problem in the dynamic streaming model is at least as hard as the problem in the insert-only streaming model, so the space lower bound $\Omega(k^2)$ also holds true for the p-MATCHING problem in the dynamic model. As we remarked in the paragraph following Theorem 4.4, our streaming algorithm **w-Match**_{dyn} given in section 4 has space complexity $O(k^2 \cdot \text{polylog}(n))$ and update time O(polylog(n)) when it is applied in solving the p-MATCHING problem in the dynamic model. This presents the first streaming algorithm in the dynamic model that solves the p-MATCHING problem in both near-optimal space complexity and near-optimal update time complexity, where by "near-optimal", we mean that the complexity bounds of the algorithm differ from the corresponding optimal bounds only by a poly-logarithmic factor.

By Theorem 4.4, when the algorithm **w-Match**_{dyn} is applied to solve the p-WMATCHING problem in the dynamic model, it still keeps the near-optimal update time O(polylog(n)), but increases its space complexity to $O(k^2W \cdot \text{polylog}(n))$, which will be quite significant if the number W of distinct edge weight values is large (note that W can be as large as the number of edges in the input graph). Unfortunately, the dependency of the space complexity on the value W for streaming algorithms solving the problem p-WMATCHING is actually unavoidable: in the appendix, we give a proof that any randomized streaming algorithm in the dynamic model that solves the p-WMATCHING problem has space complexity $\Omega(\max\{W \log W, k^2\})$. Readers who are interested in space lower bounds for streaming algorithms are referred to [13, 18] for more recent developments.

We believe that the hash scheme we developed in subsection 4.1 is of independent interests. Different from the standard hashing techniques that *partition* the universal set U into pairwise disjoint subsets, our hash scheme is actually a many-to-many mapping from the universal set U to a set whose size is smaller than that for standard hashing. Therefore, our hash scheme constructs a collection of (unnecessarily disjoint) subsets of U, but ensures that a subcollection of disjoint subsets distinguishes a subset of k elements in U. Compared with the standard hashing techniques, this approach uses less space and has a higher success probability. We believe that the results and techniques can have wider applications in other fundamental graph problems.

When applied to the problem p-MATCHING for unweighted graphs, the space complexity and update time of our streaming algorithm **w-Match**_{dyn} in the dynamic model are nearoptimal, which still differ from the corresponding proven lower bounds by a poly-logarithmic factor. When the algorithm is applied to the problem p-WMATCHING for weighted graphs in the dynamic model, the gap between the upper bound provided by the algorithm **w-Match**_{dyn} and the proven lower bound is still quite significant. It will be interesting to study how much we can further narrow down or even close the gaps between the upper bounds and the lower bounds. In particular, is it possible to have streaming algorithms in the dynamic model for the p-MATCHING problem with space complexity $O(k^2)$ and update time O(1)? This question is also related to the space lower bounds on streaming approximation algorithms for maximum matching in the dynamic model [13, 20].

Another interesting problem that deserves further study is the trade-off between the space complexity and the update time of streaming algorithms. The O(1) update time of our streaming algorithm **w-Match**_{ins} for the p-wMATCHING problem in the insert-only model used a technique of interleaving the process of updating a sketch, which is the subgraph G_s^f of the input graph G in our algorithm, with the process of reading the next input stream segment $G_{s+1,s'}$ (see Figure 1). To make the time for reading the new input stream segment to "cover" that for updating the sketch, smaller memory space for storing the (thus, shorter) new input stream segment would require longer update time per element in the segment, while faster update time per element in the stream would result in reading a longer new input stream segment that requires larger memory space for storing the segment. Moreover, longer new input stream segment would make the sketch updating more time consuming to include the information brought in by the longer new segment. It would be interesting to study the interaction/relation between these parameters in streaming algorithms.

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A Lower Bounds

In this appendix, we study lower bounds on the space complexity of randomized streaming algorithms for the problems p-MATCHING and p-WMATCHING, in both the insert-only model and the dynamic model. These lower bound results, in conjunction with the algorithms given in this paper, show that in many cases, the space complexity achieved by our algorithms is optimal or near-optimal (i.e., optimal modulo a poly-logarithmic factor in the input size). Note that the update times of our algorithms, which are O(1) in the insert-only model and O(polylog(n)) in the dynamic model, are already optimal or near-optimal.

A lower bound $\Omega(k^2)$ on the space complexity for p-MATCHING in the insert-only model has been developed in [5], which shows that for any randomized streaming algorithm for the problem, there are instances of size n and parameter k, where $n = \Theta(k^2)$, such that the algorithm takes space of $\Omega(n) = \Omega(k^2)$ bits. This result does not seem to address the following issues that are special for parameterized streaming algorithms in which (1) the graph size n and the parameter k in general are relatively independent, and (2) the graph size n can be much larger than the parameter k.

In the following, we introduce a new definition for lower bounds for streaming problems, which tries to address the above issues that are special for parameterized streaming algorithms. The definition is given in terms of space complexity, but can be extended to other complexity measures.

Definition A parameterized streaming problem Q has a space complexity lower bound $\Omega(g(k))$ if for any streaming algorithm A for Q, there are infinitely many parameter values k such that for each such parameter value k and for any integer $n \ge k$ (n does not depend on k), there are instances of parameter k and size larger than n on which the algorithm A runs in space $\Omega(g(k))$.

The definition above is consistent with the standard ones. In particular, a lower bound in terms of this definition implies the same lower bound in terms of the standard definition.

As in the previous work such as [5], we will use the *one-way communication model* to prove lower bounds on the space complexity of randomized streaming algorithms for p-MATCHING and p-WMATCHING. In this model, there are two parties, Alice and Bob, each receiving x and y, respectively, who wish to compute f(x, y). Alice is permitted to send Bob a single message M, which only depends on x and Alice's random coins. Then Bob outputs b, which is his guess of f(x, y). Here, b only depends on y, M, and Bob's random coins. We say the protocol computing f with success probability $1 - \delta$ if $\Pr[b = f(x, y)] \ge 1 - \delta$ for every x and y.

A.1 Lower bounds for p-MATCHING

We will use the lower bound on the communication complexity of the following problem:

The INDEX problem: Alice has an *m*-bit string $x \in \{0, 1\}^m$ and Bob has an integer $z \in [m]$. The goal is to compute the *z*-th bit of *x*.

It is known [22] that any randomized communication protocol solving the INDEX problem with success probability $\geq 2/3$ has communication complexity of $\Omega(m)$ bits. The constant 2/3 can be improved to any constant strictly greater than 1/2 [29].

Theorem A.1 Any randomized streaming algorithm for the p-MATCHING problem in the insertonly model with success probability at least 2/3 uses $\Omega(k^2)$ bits of space.

Proof. The proof follows the ideas of [5], with modifications to meet the additional conditions in the lower bound definition given above. Let \mathbf{A}_{match} be any randomized streaming algorithm for p-MATCHING in the insert-only model with success probability at least 2/3. We show how to use the algorithm \mathbf{A}_{match} to construct a communication protocol for the INDEX problem. Let (x, z) be an instance of the INDEX problem, where $x \in \{0, 1\}^m$ and $z \in [m]$. Define a subset of [m] as $S = \{i \mid \text{the } i\text{-th bit of } x \text{ is } 1\}$. The task is to decide whether or not $z \in S$.

Let $k_1 = \lceil \sqrt{m} \rceil$. Fix an injection $\chi : [m] \longrightarrow [k_1] \times [k_1]$. Suppose $\chi(z) = (p_z, q_z)$. Construct a graph G whose vertex set contains two disjoint subsets of $2k_1$ vertices: $V_L = \{l_i, l_i^* \mid i \in [k_1]\}$ and $V_R = \{r_i, r_i^* \mid i \in [k_1]\}$. The edge set of G contains the following edge subsets:

- (1) $E_S = \{ [l_s^*, r_t^*] \mid \text{if } (s, t) = \chi(y) \text{ for some } y \in S \}; \text{ and }$
- (2) $E_L = \{ [l_s, l_s^*] \mid s \neq p_z \}, \text{ and } E_R = \{ [r_t, r_t^*] \mid t \neq q_z \}.$

To make G a graph of at least n vertices for any $n > 4k_1$, we add to the graph G a disjoint star of $n - 4k_1$ vertices given by the edge set $E_{star} = \{[v_0, v_i] \mid 1 \le i \le n - 4k_1 - 1\}$. This completes the structure of the graph G, which has n vertices, where $n > 4k_1$ can be any integer. It is not difficult to verify that the graph G has a $(2k_1)$ -matching if and only if $z \in S$. Now construct an instance (S, k) for p-MATCHING, where $k = 2k_1$ and S is a stream for the graph G that first inserts the edges in the set $E_S \cup E_{star}$, then inserts the edges in the set $E_L \cup E_R$.

A communication protocol for the instance (x, z) for the INDEX problem works as follows: (1) Alice runs the streaming algorithm \mathbf{A}_{match} for p-MATCHING on the instance (S, k) for the elements in S that insert the edges in $E_S \cup E_{star}$ (Alice can generate these elements because she knows S). After processing all edges in $E_S \cup E_{star}$, Alice sends the memory contents of her computation to Bob. Bob then uses the memory contents of Alice's computation and continues running the streaming algorithm \mathbf{A}_{match} on the rest of the elements in the stream S that insert the edges in $E_L \cup E_R$ (Bob can generate these elements because he knows z). As we observed, Bob claims $z \in S$ if and only if the algorithm \mathbf{A}_{match} on (S, k) returns a k-matching in G. Thus, by the assumptions on the algorithm \mathbf{A}_{match} , this is a randomized communication protocol solving the INDEX problem with probability $\geq 2/3$. By [22], the communication complexity of the protocol is of $\Omega(m)$ bits. Since the communication complexity of the protocol is equal to the size of the message Alice passed to Bob, which is the space used by Alice when she runs the algorithm \mathbf{A}_{match} , we conclude that the algorithm \mathbf{A}_{match} on the instance (S, k) uses space of $\Omega(m) = \Omega(k^2)$ bits. The theorem is completed because \mathbf{A}_{match} is an arbitrary algorithm for p-MATCHING, and the size of the stream \mathcal{S} can be arbitrarily larger than the parameter k. \Box

Obviously, the space lower bound given by Theorem A.1 also applies to streaming algorithms for the p-MATCHING problem in the dynamic model.

A.2 Lower bounds for p-WMATCHING

We start with the necessary background in information theory. For more details, see [12]. For a random variable X, the (Shannon) entropy H(X) of X is defined as

$$H(X) = -\sum_{x} \Pr[X = x] \cdot \log(\Pr[X = x]).$$

The binary entropy function H(q) is H(X) for a 0-1 random variable X with $\Pr[X = 1] = q$. For two random variables Z_1 and Z, the conditional entropy $H(Z_1 \mid Z)$ of Z_1 given Z is

$$H(Z_1 \mid Z) = \sum_{z} H(Z_1 \mid Z = z) \cdot \Pr[Z = z],$$

and the mutual information $I(Z_1; Z)$ is

$$I(Z_1; Z) = H(Z) - H(Z \mid Z_1).$$

For random variables Z_1 , Z_2 , and Z, the conditional mutual information $I(Z_1; Z_2 | Z)$ of Z_1, Z_2 given Z is

$$I(Z_1; Z_2 | Z) = H(Z_1 | Z) - H(Z_1 | Z_2, Z).$$

Random variables X, Y, Z form a Markov chain in that order (denoted by $X \to Y \to Z$) if the conditional distribution of Z depends only on Y and is conditionally independent of X.

Proposition A.2 (Theorem 2.6.4, [12]) For any random variable X, $H(X) \leq \log |\mathcal{X}|$, where \mathcal{X} is the range of X, with equality if and only if X has a uniform distribution over \mathcal{X} .

Proposition A.3 (Theorem 2.5.2, [12]) For random variables $Z_1, Z_2, ..., Z_n$, and Z, we have the following chain rule: $I(Z_1, Z_2, ..., Z_n; Z) = \sum_{i=1}^n I(Z_i; Z \mid Z_{i-1}, Z_{i-2}, ..., Z_1)$.

Proposition A.4 (Fano's Inequality, [12], p. 39) Given a Markov chain $X \to Y \to X'$, and let $p = \Pr[X \neq X']$, then $H(X \mid Y) \leq H(p) + p \cdot \log(|\mathcal{X}| - 1)$, where \mathcal{X} is the range of X.

Now we are ready for the lower bound for p-WMATCHING. Consider the following problem:

PARTIAL MAXIMIZATION: Alice has a sequence $A = \langle a_1, a_2, \ldots, a_m \rangle$ of numbers in $[m^2]$, and Bob is given a set $P_B = \{(i, a_i) \mid i \in B\}$ of pairs, where B is a subset of [m]. The goal is to compute $\max\{a_t \mid t \notin B\}$, i.e., to compute the largest number a_t in the sequence A that is not given to Bob.

Theorem A.5 For any constant δ , $0 \leq \delta < 1$, any randomized one-way communication protocol for the PARTIAL MAXIMIZATION problem with success probability at least $1-\delta$ has communication complexity of $\Omega(m \log m)$ bits, where m is the length of the sequence given to Alice. The lower bound holds even when we assume that all numbers in the sequence given to Alice are distinct. Proof. The proof is similar to that for the AUGMENTED INDEXING problem [27, 8]. For each $j \in [m]$, consider the random variable X_j that picks its value uniformly at random from $\{(j - 1)m + 1, \ldots, j \cdot m\}$. Then $X_1 < X_2 < \cdots < X_m$ - so all numbers in the sequence given to Alice are distinct, and $H(X_j) = \log m$ for all j. For each $j \in [m]$, let $B_j = \{j + 1, \ldots, m\}$ and let X'_j be Bob's guess of $\max_{i \notin B_j} X_i = X_j$. Let M be the message sent from Alice to Bob. Since $\Pr[X'_j = X_j] \ge 1 - \delta$ and $X_j \to (M, X_{j+1}, X_{j+2}, \ldots, X_m) \to X'_j$ is a Markov chain, by Proposition A.4, for all $j \in [m]$, we have

$$H(X_j \mid M, X_{j+1}, \dots, X_m) \le \delta \cdot \log m + 1.$$

Hence,

$$I(X_{j}; M \mid X_{j+1}, \dots, X_{m}) = H(X_{j} \mid X_{j+1}, \dots, X_{m}) - H(X_{j} \mid M, X_{j+1}, \dots, X_{m})$$

= $H(X_{j}) - H(X_{j} \mid M, X_{j+1}, \dots, X_{m})$
 $\geq (1 - \delta) \log m - 1,$

where the second equality holds because $X_j, X_{j+1}, \ldots, X_m$ are mutually independent. By the definition of the mutual information and using the chain rule (Proposition A.3),

$$H(M) \geq I(X_1, X_2, \dots, X_m; M)$$

=
$$\sum_{j=1}^m I(X_j; M \mid X_{j+1}, \dots, X_m)$$

=
$$\Omega(m \log m).$$

Finally, by Proposition A.2, $\log |\mathcal{M}| \geq H(M) = \Omega(m \log m)$, where \mathcal{M} is the range of the message M, i.e., the message M has at least $2^{\Omega(m \log m)}$ possibilities. As a result, the length of the longest possible message M sent from Alice to Bob is $\Omega(m \log m)$.

A space lower bound for streaming algorithms of the p-wMATCHING problem in the dynamic model now can be derived by reducing PARTIAL MAXIMIZATION to p-wMATCHING.

Theorem A.6 Let $0 \leq \delta < 1$ be any constant. Any randomized streaming algorithm in the dynamic model for the p-WMATCHING problem that, with probability $\geq 1 - \delta$, computes a maximum weighted 1-matching has space complexity of $\Omega(W \log W)$ bits, where W is the number of distinct edge weights in the graph stream.

Proof. Let \mathbf{A}_{match} be any randomized streaming algorithm for p-WMATCHING that, with probability $\geq 1 - \delta$, computes a maximum weighted 1-matching. We show how to use \mathbf{A}_{match} to develop a communication protocol for PARTIAL MAXIMIZATION. Let (A, P_B) be an instance of PARTIAL MAXIMIZATION, where $A = \langle a_1, a_2, \ldots, a_m \rangle$ is the sequence given to Alice, in which all numbers are distinct, and $P_B = \{(i, a_i) \mid i \in B\}$ is the set of pairs given to Bob, $B \subseteq [m]$.

Let S be a dynamic graph stream that first inserts m arbitrary but distinct edges $\{e_1, \ldots, e_m\}$, where for each i, the edge e_i has weight a_i , then deletes the edges e_i for $i \in B$. Obviously, a maximum weighted 1-matching in the graph G of the stream S has its weight equal to max $\{a_t \mid t \notin B\}$, which is the solution to the instance (A, P_B) of PARTIAL MAXIMIZATION.

The communication protocol for PARTIAL MAXIMIZATION on the instance (A, P_B) works as follows: (1) Alice runs the streaming algorithm \mathbf{A}_{match} on the instance (S, 1) of p-WMATCHING for the first m elements of edge insertions in the stream S (Alice can generate these elements because she knows the sequence A), then sends the memory contents of her computation to Bob; (2) After receiving the message from Alice, Bob uses the memory contents from Alice and continues running the algorithm \mathbf{A}_{match} on (S, 1) for the rest elements in the stream S, which are edge deletions (Bob can generate these elements because he knows the subset B), to get the maximum weighted 1-matching, thus the solution to the instance (A, P_B) of the PARTIAL MAXIMIZATION problem. Under the assumption of the algorithm \mathbf{A}_{match} , this gives a randomized communication protocol with success probability $\geq 1 - \delta$ for PARTIAL MAXIMIZATION. By Theorem A.5, the size of the message sent from Alice to Bob, which is not larger than the space complexity of the algorithm \mathbf{A}_{match} on (S, 1), has $\Omega(m \log m)$ bits. Moreover, by our assumption on the sequence A, m is equal to the number W of distinct edge weights in the stream S. As a conclusion, the algorithm \mathbf{A}_{match} on the instance (S, 1) uses space of $\Omega(W \log W)$ bits.

Note that here the number W of distinct edge weights is the parameter for which we have derived a lower bound. To make the lower bound hold true for graphs of size larger than W, i.e., to meet the additional conditions in the lower bound definition given at beginning of this section, we can simply let Alice add (many) more elements in the stream S that insert edges of weight equal to the smallest value in the sequence A.

Since the space lower bound $\Omega(k^2)$ for streaming algorithms for the p-MATCHING problem in the insert-only model certainly applies for the p-WMATCHING problem in the dynamic model, we obtain the following corollary.

Corollary A.7 Any randomized streaming algorithm for the p-WMATCHING problem with success probability $\geq 2/3$ in the dynamic model uses $\Omega(\max\{W \log W, k^2\})$ bits of space, where W is the number of distinct edge weights in the graph stream.