CSCE-433 Formal Languages & Automata
CSCE-627 Theory of Computability

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Solutions to Assignment #1

1. Prove by induction: $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = (4n^3 - n)/3$.

Proof. For $n = 1$, LHS $= 1^2 = 1$, and RHS $= (4 \cdot 1^3 - 1)/3 = 1$. The equality holds true.

Inductively, assume that the equality holds true for all postive integers n not larger than k, where $k \geq 1$.

Now consider $n = k+1$. We have

$$\begin{aligned}
\text{LHS for } n = k+1 &= 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 \\
&= 1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 \\
&= 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 \\
&= (4k^3 - k)/3 + (2(k+1)-1)^2 \\
&= (4k^3 - k)/3 + (2k+1)^2 \\
&= (4k^3 - k)/3 + 4k^2 + 4k + 1 \\
&= (4k^3 - k + 12k^2 + 12k + 3)/3 \\
&= (4(k^3 + 3k^2 + 3k + 1) - (k+1))/3 \\
&= (4(k+1)^3 - (k+1))/3 \\
&= \text{RHS for } n = k+1,
\end{aligned}$$

where the fourth equality has used the inductive hypothesis. The above derivation shows that the equality given in the problem also holds true for $n = k+1$, which completes the inductiive proof of the equality. □

2. Prove by induction: for every integer $n \geq 0$, $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

Proof. For $n = 0$, LHS $= 5^{2\cdot 0+1} + 2^{2\cdot 0+1} = 7$, the statement holds true.

Inductively, assume the statement holds true for all integers $n \geq 0$ that are not larger than k, where $k \geq 0$.

Now consider $n = k + 1$. We have

$$
\begin{aligned}
& 5^{2n+1} + 2^{2n+1} \\
= \; & 5^{2(k+1)+1} + 2^{2(k+1)+1} \\
= \; & 5^{(2k+1)+2} + 2^{(2k+1)+2} \\
= \; & 25 \cdot 5^{(2k+1)} + 4 \cdot 2^{(2k+1)} \\
= \; & 21 \cdot 5^{(2k+1)} + 4(5^{(2k+1)} + 2^{(2k+1)}). \qquad (1)
\end{aligned}
$$

By the inductive hypothesis, $5^{(2k+1)} + 2^{(2k+1)}$ is divisible by 7. Thus, the second term in (1), i.e., $4(5^{(2k+1)} + 2^{(2k+1)})$, is divisible by 7. Moreover, the first term $21 \cdot 5^{(2k+1)}$ in (1) is also divisible by 7 because of the factor 21. As a result, the value in (1), which is equal to $5^{2(k+1)+1} + 2^{2(k+1)+1}$ is divisible by 7. This shows that the statement also holds true for $n = k + 1$, which completes the inductive proof of the statement. □

3. Prove by contradiction: for any integers a, b, c, if $a^2 + b^2 = c^2$, then at least one of a and b is an even number;

Proof. Assume the contrary that both a and b are odd numbes, and that $a^2 + b^2 = c^2$ for an integer c. Then we can write $a = 2n + 1$ and $b = 2m + 1$, where both n and m are integers. Bringing this to $a^2 + b^2 = c^2$, we get

$$(2n + 1)^2 + (2m + 1)^2 = 4n^2 + 4n + 1 + 4m^2 + 4m + 1 = 4(n^2 + n + m^2 + m) + 2 = c^2.$$

Since $4(n^2 + n + m^2 + m) + 2$ is an even number, c^2, thus c, must be an even number. Let $c = 2p$, where p is an integer, then we have $c^2 = 4p^2$. So from $4(n^2 + n + m^2 + m) + 2 = c^2$, we get

$$2 = 4p^2 - 4(n^2 + n + m^2 + m) = 4(p^2 - n^2 - n - m^2 - m).$$

Since all p, n, and m are integers, the above equality shows that the integer 2 is divisible by 4, which is impossible. This contradiction shows that at least one of a and b is an even number. □

4. Prove by contradiction:

(a) For any integer n, if n^2 is divisible by 6, then n is also divisible by 6;

(b) $\sqrt{6}$ is an irrational number.

Both (a) and (b) should be proved by contradiction.

Proof. **(a)** Assume that n^2 is divisible by 6 but n is not divisible by 6. Then n can be written as $n = 6m + p$, where both m and p are integers, and $1 \leq p \leq 5$.

We have $n^2 = 36m^2 + 12mp + p^2$. By the assumption, n^2 is divisible by 6. Since both 36 and 12 are divisible by 6, we must have p^2 divisible by 6. However, since p is one of 1, 2, 3, 4, and 5, whose square p^2 can be only one of 1, 4, 9, 16, and 25, of which none is divisible by 6. This derives a contradiction, and the contradiction proves statement (a).

(a) Assume the contrary that $\sqrt{6}$ is a rational number n/m, where n and m are integers with no common factor other than 1. Then $(n/m)^2 = (\sqrt{6})^2 = 6$, which gives $n^2 = 6m^2$, so n^2 is divisible by 6. By statement (a), n is also divisible by 6, so can be written as $n = 6p$ for an integer p. This plus $n^2 = 6m^2$ gives $6p^2 = m^2$, which shows that m^2 is divisible by 6. By statement (a) again, we derive that m is also divisible by 6. However, this would lead to the conclusion that 6 is a common factor of n and m, contradicting the assumption that n and m have no common factor other than 1. This contradiction proves statement (b). $\square$

5. [CSCE-433 Students only] Suppose that n straight lines are drawn in the plane such that no two lines are parallel and no three lines go through the same point. These lines divide the plane into r_n regions. Prove: $r_n = 1 + n(n+1)/2$.

Proof. We prove the statement by induction on n. For $n = 0$, i.e., if we draw no line in the plane, then there is only 1 region in the plane, which is the entire plane. Thus, $r_0 = 1 = 1+0(0+1)/2$. The statement holds true for $n = 0$.

Assuem inductively that the statement holds true for all $n \leq k$, where $k \geq 0$.

Now we prove the statement for $n = k+1$. Let the $k+1$ lines be $\{l_1, l_2, \cdots, l_k, l_{k+1}\}$. First consider the drawing of the first k lines $L_k = \{l_1, l_2, \cdots, l_k\}$ in the plane. These k lines divide the plane into r_k regions. Let the set of these r_k regions be R_k. Since the line l_{k+1} is not in parallel with any line in L_k, neither hits a point that is an intersection point of two lines in L_k, l_{k+1} intersects the k lines in L_k at exactly k points, each for a line in L_k. Now image that we traverse on the line l_{k+1}, starting from the infinity ∞. The semi-line of l_{k+1} from ∞ to the first intersection point of l_{k+1} and L_k divides a region in R_k into 2 regions. After that, each segment on l_{k+1} between two consecutive intersection points of l_{k+1} and L_k divides a region in R_k into 2 regions, and finally, the semi-line on l_{k+1} from the last intersection point of l_{k+1} and L_k to ∞ divides a region in R_k into 2 regions. Since that are exactly n such intersection points of l_{k+1} and L_k, adding the line l_{k+1} to the drawing of $L_k = \{l_1, l_2, \cdots, l_k\}$ increases the number of regions by $k+1$. Therefore, $r_{k+1} = r_k + (k+1)$. By the inductive hypothesis, $r_k = 1+k(k+1)/2$. Thus,

$$r_{k+1} = r_k + (k+1) = 1 + k(k+1)/2 + (k+1) = 1 + (k+1)((k+1)+1)/2.$$

This proves that the statement also holds true for the case $n = k+1$, thus completing the inductive proof. $\square$

6. [CSCE-627 Students only] Prove that every simple graph (i.e., a graph with no self-loop and multiple edges) with two or more vertices contains two vertices that have equal degrees.

Proof. This proof uses both proof by induction and proof by contradiction. We prove the statement by induction on the number n of vertices in the graph G, where $n \geq 2$.

For a graph G_2 of $n = 2$ vertices v_1 and v_2, either there is no edge in G_2 so that both vertices v_1 and v_2 have degree 0, or there is a single edge between v_1 and v_2 so that both v_1 and v_2 have degree 1. In both cases, the statement holds true.

Inductively, assume that the statement holds true for graphs of n vertices for all $n \leq k$, where $k \geq 2$.

Now consider a graph G_{k+1} of $k+1$ vertices. Here we use proof by contradiction to show that G_{k+1} has two vertices of equal degrees. Suppose to the contrary that there are no two vertices in G_{k+1} that have equal degrees. Thus, there are $k+1$ different vertex degrees in G_{k+1}. Since each vertex in G_{k+1} can be incident to at most k edges, the vertex degrees in G_{k+1} must be integers between 0 and k. Thus, the different $k+1$ vertex degrees in G_{k+1} must be exactly the $k+1$ integers 0, 1, 2, ..., k. This implies that there is a vertex v_{k+1} of degree 0. Now if we remove the vertex v_{k+1} from the graph G_{k+1}, we get a graph G_k of k vertices. Since the vertex v_{k+1} has degree 0, every vertex in G_k and its corresponding vertex in G_{k+1} have the same degree. However, by the inductive hypothesis, there are two vertices v_s and v_t in the graph G_k that have equal degrees. But this would imply that the two vertices v_s and v_t in G_{k+1} also have equal degrees, contradicting the assumption that all vertices in G_{k+1} have different degrees. This contradiction shows that there must be two vertices in G_{k+1} that have equal degrees.

Now we get back to the proof by induction. The above discussion proves that the graph G_{k+1} of $k+1$ vertices has two vertices of equal degrees. Thus, the statement also holds true for $n = k+1$, which completes the inductive proof.

As a result, we have proved that the statement holds true for all $n \geq 2$. □