CSCE 222-200 Discrete Structures for Computing

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Assignment #5 Solutions

1. Find a big-O estimate for the function $f(n)$ that satisfies the recurrence relation

$$f(2) = 1; \quad \text{and} \quad \text{for } n > 2,\ f(n) = 2f(\sqrt{n}) + \log_2 n.$$

(*Hint*: Make the substitution $m = \log_2 n$. You can assume that m is a power of 2.)

Solution. Let $m = \log_2 n$, i.e., $n = 2^m$. Define $F(m) = f(n) = f(2^m)$. We have

$F(1) = f(2^1) = f(2) = 1$, and

for $m > 1$,

$F(m) = f(2^m) = f(n) = 2f(\sqrt{n}) + \log_2 n = 2f(\sqrt{2^m}) + m = 2f(2^{m/2}) + m$
$= 2F(m/2) + m.$

Solving $F(1) = 1$ and for $m > 1$, $F(m) = 2F(m/2) + m$ is not difficult, as we have done similar recurrence relations. For a general $k \le m$, we can verify

$$F(m) = 2^k F(m/2^k) + k \cdot m.$$

Letting $k = \log_2 m$ gives

$$F(m) = m \cdot F(1) + \log_2 m \cdot m = m + m \log_2 m.$$

Recall that $m = \log_2 n$ and $n = 2^m$. We have, for a general $n \ge 2$,

$f(n) = f(2^m) = F(m) = m + m \log_2 m = \log_2 n + \log_2 n \cdot \log_2 \log_2 n$
$= O(\log_2 n \cdot \log_2 \log_2 n).$

2. A coin is flipped n times where each flip comes up either heads or tails. How many possible outcomes (assuming that n is even and that $k \le n$)

(a) are there in total?
(b) contain exactly k heads?
(c) contain at least k heads?
(d) contain the same number of heads and tails?

Give an explanation to your answer to each of the questions.

Solution. If we treat head H as the bit 0 and tail T as the bit 1, then each outcome of n coin-flippings uniquely corresponds to a binary string of length n. This gives a one-to-one correspondence between the set of outcomes of n coin-flippings and the set of binary strings of length n.

(a) As we know, the total number of binary strings of length n is 2^n. Thus, there are totally 2^n possible outcomes for n coin-flippings.

(b) Each outcome with exactly k heads corresponds uniquely to k positions in the binary string of length n at which the bit is 0 (and all other bits are 1). Since there are $\binom{n}{k}$ different ways to pick k positions in a binary string of length n, the total number of outcomes that contain exactly k heads is equal to $\binom{n}{k}$.

(c) As given in (b), for each h, $k \le h \le n$, the number of outcomes that contain exactly h heads is equal to $\binom{n}{h}$. The total number of outcomes that contain at least k heads is equal to the sum of the numbers of outcomes that contain exactly h heads for all $k \le h \le n$. That is, the total number of outcomes that contain at least k heads is equal to $\sum_{h=k}^{n} \binom{n}{h}$.

(d) Since n is even, $n/2$ is an integer. An outcome containing the same number of heads and tails contains exactly $n/2$ heads. As a consequence of (b), the total number of outcomes that contain the same number of heads and tails is equal $\binom{n}{n/2}$.

3. Suppose that k are n are integers with $1 \le k \le n$. Prove the *hexagon identity*:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

Solution. We prove the equality using the formula $\binom{n}{k} = n!/(k! \cdot (n-k)!)$.

$$\begin{aligned}
\text{LHS} &= \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} \\
&= \frac{(n-1)!}{(n-k)!(k-1)!} \cdot \frac{n!}{(n-k-1)!(k+1)!} \cdot \frac{(n+1)!}{(n-k+1)!k!} \\
&= \frac{(n-1)! \cdot n! \cdot (n+1)!}{(k-1)! \cdot k! \cdot (k+1)! \cdot (n-k-1)! \cdot (n-k)! \cdot (n-k+1)!},
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} \\
&= \frac{(n-1)!}{(n-k-1)!k!} \cdot \frac{n!}{(n-k+1)!(k-1)!} \cdot \frac{(n+1)!}{(n-k)!(k+1)!} \\
&= \frac{(n-1)! \cdot n! \cdot (n+1)!}{(k-1)! \cdot k! \cdot (k+1)! \cdot (n-k-1)! \cdot (n-k)! \cdot (n-k+1)!}.
\end{aligned}$$

Thus, we have LHS = RHS. The equality is proved.

4. Prove that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

Proof. Recall that E and F are independent means that $\mathbf{Pr}[E \cap F] = \mathbf{Pr}[E] \cdot \mathbf{Pr}[F]$. Thus, the question asks to prove $\mathbf{Pr}[\overline{E} \cap \overline{F}] = \mathbf{Pr}[\overline{E}] \cdot \mathbf{Pr}[\overline{F}]$. We have

$$\begin{aligned}
\mathbf{Pr}[\overline{E} \cap \overline{F}] &= \mathbf{Pr}[\overline{E \cup F}] \\
&= 1 - \mathbf{Pr}[E \cup F] \\
&= 1 - (\mathbf{Pr}[E] + \mathbf{Pr}[F] - \mathbf{Pr}[E \cap F]) \\
&= 1 - (\mathbf{Pr}[E] + \mathbf{Pr}[F] - \mathbf{Pr}[E] \cdot \mathbf{Pr}[F]) \\
&= 1 - \mathbf{Pr}[E] - \mathbf{Pr}[F] + \mathbf{Pr}[E] \cdot \mathbf{Pr}[F] \\
&= (1 - \mathbf{Pr}[E]) \cdot (1 - \mathbf{Pr}[F]) \\
&= \mathbf{Pr}[\overline{E}]) \cdot \mathbf{Pr}[\overline{F}].
\end{aligned}$$

The first equality used De Morgan's law (note that E and F are sets), the second and seventh equalities used the formula for the probability of event complements, the third equality used the formula for the probabililty of union of non-disjoint events, and the fourth equality used the given condition that E and F are independent events.

This proves that the events $\overline{E}$ and $\overline{F}$ are also independent.

5. Suppose that we roll a fair die until a 6 comes up.

(a) What is the probability that a 6 comes up in our n-th rolling?

(b) What is the expected number of times we roll the die? (*Hint*: You need to find the value for the sum $1 + 2(5/6) + 3(5/6)^2 + \cdots + k(5/6)^{k-1} + \cdots$.)

Solution. Note that when rolling a fair die, the probability that 6 comes up is 1/6, and the probability that 6 does not come up is 5/6.

(a) According to the question statement, the game will stop when 6 comes up. Thus, that 6 comes up in the n-th rolling implies that 6 did not come up in the first $(n-1)$st rollings. Thus, the probability that a 6 comes up in the n-th rolling is $(5/6)^{n-1}(1/6)$.

(b) For each integer $n \geq 1$, define a random variable X_n such that $X_n = n$ if a 6 comes up in the n-th rolling, and $X_n = 0$ otherwise. By (a), the probability that $X_n = n$ is equal to $(5/6)^{n-1}(1/6)$, so the probability that $X_n = 0$ is $1-(5/6)^{n-1}(1/6)$. Therefore, the expected value of the random variable X_n is

$$\begin{aligned}
\mathbf{Ex}[X_n] &= n \cdot \mathbf{Pr}[X_n = n] + 0 \cdot \mathbf{Pr}[X_n = 0] \\
&= n \cdot (5/6)^{n-1}(1/6) + 0 \cdot (1 - (5/6)^{n-1}(1/6)) \\
&= n \cdot (5/6)^{n-1}(1/6).
\end{aligned}$$

Define a new random variable $X = \sum_{n=1}^{\infty} X_n$. Since in every case, there is at most one X_n that is not equal 0 (i.e., when a 6 comes up in the n-th roll, we have $X_n = n$, and for all $i \neq n$, $X_i = 0$), X is the number of times we roll the die to have a 6 comes up. Therefore, question (b) is asking the expected value of X. We have

$$\begin{aligned}\mathbf{Ex}[X] &= \mathbf{Ex}\left[X = \sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \mathbf{Ex}[X_n] \\ &= \sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}(1/6) = (1/6)\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}, \end{aligned} \tag{1}$$

where the second equality has used the linearity of expectations.

To get the final value of $\mathbf{Ex}[X]$, we need to find the value of

$$\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1} = 1 + 2(5/6) + 3(5/6)^2 + \cdots + n(5/6)^{n-1} + \cdots.$$

For this, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1} &= 1 + 2(5/6) + 3(5/6)^2 + 4(5/6)^3 + \cdots + n(5/6)^{n-1} + \cdots \\ &= 1 + (5/6) + (5/6)^2 + (5/6)^3 + \cdots + (5/6)^{n-1} + \cdots \\ &\quad + (5/6) + 2(5/6)^2 + 3(5/6)^3 + \cdots + (n-1)(5/6)^{n-1} + \cdots \\ &= 1 + (5/6) + (5/6)^2 + (5/6)^3 + \cdots + (5/6)^{n-1} + \cdots \\ &\quad + (5/6)\left(\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}\right) \end{aligned} \tag{2}$$

By the formula for the summation of geometric sequences, we have

$$1 + (5/6) + (5/6)^2 + (5/6)^3 + \cdots + (5/6)^{n-1} + \cdots = 1/(1-(5/6)) = 6.$$

Therefore, from the above equations in (2), we have

$$\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1} = 6 + (5/6)\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}.$$

Solving this for $\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}$, we get

$$\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1} = 36.$$

Bringing this into (1) gives

$$\mathbf{Ex}[X] = (1/6)\sum_{n=1}^{\infty} n \cdot (5/6)^{n-1} = (1/6) \cdot 36 = 6.$$