CSCE 222-200 Discrete Structures for Computing

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Assignment #2 Solutions

1. Determine whether each of the statements below is true or false. Give a one-sentence explanation to your solution to each statement.

(a) $\emptyset \in \{\emptyset\}$; (b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$; (c) $\{\emptyset\} \in \{\emptyset\}$; (d) $\{\emptyset\} \in \{\{\emptyset\}\}$;
(e) $\emptyset \subset \{\emptyset\}$; (f) $\emptyset \subset \{\emptyset, \{\emptyset\}\}$; (g) $\{\emptyset\} \subset \{\emptyset\}$; (h) $\{\emptyset\} \subset \{\{\emptyset\}\}$.

Solution.

(a) $\emptyset \in \{\emptyset\}$: True, $\emptyset$ is an element in the set $\{\emptyset\}$.
(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$: True, $\emptyset$ is an element in the set $\{\emptyset, \{\emptyset\}\}$.
(c) $\{\emptyset\} \in \{\emptyset\}$: False, $\{\emptyset\}$ (the set that contains a single element $\emptyset$) is not an element in the set $\{\emptyset\}$.
(d) $\{\emptyset\} \in \{\{\emptyset\}\}$: True, $\{\emptyset\}$ is an element in the set $\{\{\emptyset\}\}$.
(e) $\emptyset \subset \{\emptyset\}$: True, the empty set $\emptyset$ is a subset of every set and is a proper subset of $\{\emptyset\}$ because $\{\emptyset\}$ is not an empty set.
(f) $\emptyset \subset \{\emptyset, \{\emptyset\}\}$: True, the same reason as that for (e).
(g) $\{\emptyset\} \subset \{\emptyset\}$: False, $\{\emptyset\} \subseteq \{\emptyset\}$ but $\{\emptyset\}$ is not a proper subset of $\{\emptyset\}$.
(h) $\{\emptyset\} \subset \{\{\emptyset\}\}$: False, the element $\emptyset$ in $\{\emptyset\}$ is not an element of $\{\{\emptyset\}\}$.

2. Prove or disprove that for all sets A, B, and C, we have

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$; (b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution. Here we heavily use the fact that for any two sets S and T, we always have $S \subseteq S \cup T$.

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ holds true. The following is a proof.

Let t be an arbitrary element in $A \times (B \cup C)$. By definition, $t = (a, d)$, where $a \in A$ and $d \in B \cup C$. If $d \in B$, then $(a, d) \in A \times B \subseteq (A \times B) \cup (A \times C)$ so $(a, d) \in (A \times B) \cup (A \times C)$. Similarly, if $d \in C$, then $(a, d) \in A \times C$ so $(a, d) \in (A \times B) \cup (A \times C)$. Thus, we always have $t \in (A \times B) \cup (A \times C)$. Since t is an arbitrary element in $A \times (B \cup C)$, this proves that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Conversely, let s be an arbitrary element in $(A\times B)\cup(A\times C)$, then by definition, $s=(a',d')$ is either in $A\times B$ or in $A\times C$. Thus, we must have $a'\in A$, and d' is either in B or in C. Thus, d' must be in $B\cup C$. As a result, $s=(a',d')\in A\times(B\cup C)$. This proves that $(A\times B)\cup(A\times C)\subseteq A\times(B\cup C)$.

Summarizing the above results, we conclude that $A\times(B\cup C)=(A\times B)\cup(A\times C)$.

(b) $A\times(B\cap C)=(A\times B)\cap(A\times C)$ also holds true, as proved below.

Let t be an arbitrary element in $A\times(B\cap C)$. By definition, $t=(a,d)$, where $a\in A$ and $d\in B\cap C$, so $d\in B$ and $d\in C$. This gives $(a,d)\in A\times B$ and $(a,d)\in A\times C$. As a result, (a,d) is in $(A\times B)\cap(A\times C)$.This proves that $A\times(B\cap C)\subseteq(A\times B)\cap(A\times C)$.

Conversely, let s be an arbitrary element in $(A\times B)\cap(A\times C)$. Then by definition, s is in both $A\times B$ and $A\times C$. Thus, we must have $s=(a',d')$, where $a'\in A$ and d' is in both B and C. Thus, $d'\in B\cap C$. As a result $s=(a',d')$ is in $A\times(B\cap C)$. This proves that $(A\times B)\cap(A\times C)\subseteq A\times(B\cap C)$.

Summarizing the above results, we conclude that $A\times(B\cap C)=(A\times B)\cap(A\times C)$.

3. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for each positive integer i,

(a) $A_i=[0,i)$, that is, the set of real numbers x with $0\le x<i$;

(b) $A_i=[i,\infty)$, that is, the set of real numbers x with $x\ge i$.

Solution.

(a) If $A_i=[0,i)$ for all $i\ge 1$, then by the definition, we have $\bigcup_{i=1}^{\infty} A_i=[0,\infty)$, and $\bigcap_{i=1}^{\infty} A_i=[0,1)$.

(b) If $A_i=[i,\infty)$ for all $i\ge 1$, then by the definition, we have $\bigcup_{i=1}^{\infty} A_i=[1,\infty)$, and $\bigcap_{i=1}^{\infty} A_i=\emptyset$ (please make sure that you understand this solution).

4. Give an example of a function from $\mathbf{N}$ to $\mathbf{N}$ (where $\mathbf{N}$ is the set of natural numbers, i.e., $\mathbf{N}=\{0,1,2,3,\ldots\}$) that is

(a) one-to-one but not onto;

(b) onto but not one-to-one;

(c) both onto and one-to-one;

(d) neither one-to-one nor onto.

Solution.

(a) The function $f_1(n)=n+1$ for all $n\in\mathbf{N}$ is an example of such a function: f_1 is obviously one-to-one because $n_1\neq n_2$ implies $f_1(n_1)\neq f_1(n_2)$, but is not onto because 0 is not in the range of f_1.

(b) The function $f_2(n)=\lfloor n/2\rfloor$ for all $n\in\mathbf{N}$ is an example of such a function: f_2 is onto because for each integer k in the co-domian $\mathbf{N}$, $f_2(2k)=k$, but is not one-to-one because, for example, $f_2(2)=f_2(3)=1$.

(c) The trivial function $f_3(n)=n$ is obviously both onto and one-to-one.

(d) The trivial function $f_4(n)=0$ is obviously neither onto nor one-to-one.

5. Suppose that g is a function from A to B and f is a function from B to C.

(a) Prove that if $f \circ g$ is onto, then f must be onto;

(b) Prove that if $f \circ g$ is one-to-one, then g must be one-to-one;

(c) Prove that if both f and g are one-to-one, then $f \circ g$ must be one-to-one.

Proof. By the definition, $f \circ g$ is a function from A to C.

(a) We prove this by contraposition. Assume the contrary that f is not an onto function from B to C, that is, there is an element c in C such that for all elements b in B, $f(b) \neq c$. Since g is a function from A to B, for all a in A, $g(a)$ is an element in B. Thus, for all $a \in A$, we must have $f \circ g(a) = f(g(a)) \neq c$, i.e., $f \circ g$ is not onto from A to C, contradicting the assumption of subquestion (a) that $f \circ g$ is onto. This contradiction proves that if $f \circ g$ is onto, then f must be onto.

(b) Again we prove by contraposition. Assume the contrary that g is not a one-to-one function from A to B, that is, there are two distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. However, this would lead to $f \circ g(a_1) = f(g(a_1)) = f(g(a_2)) = f \circ g(a_2)$, i.e., the function $f \circ g$ is not one-to-one, contradicting the assumption of subquestion (b) that $f \circ g$ is one-to-one. This contradiction proves that if $f \circ g$ is one-to-one, then g must be one-to-one.

(c) Suppose that both f and g are one-to-one. Then, since g is one-to-one, for any two distinct elements a_1 and a_2 in A, $g(a_1)$ and $g(a_2)$ are both in B and $g(a_1) \neq g(a_2)$. Now the fact that f is also one-to-one (from B to C), plus $g(a_1) \neq g(a_2)$, shows that $f \circ g(a_1) = f(g(a_1)) \neq f(g(a_2)) = f \circ g(a_2)$. Since a_1 and a_2 are two arbitrary elements in A, this shows that $f \circ g$ is one-to-one from A to C.

6. Prove the following statements.

(a) If n is an integer, then $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$;

(b) For all integers n, $\lceil n/2 \rceil \cdot \lfloor n/2 \rfloor = \lfloor n^2/4 \rfloor$.

Proof. First note that if n is even, then $\lceil n/2 \rceil = \lfloor n/2 \rfloor = n/2$, while if n is odd, then $\lceil n/2 \rceil = (n+1)/2$ and $\lfloor n/2 \rfloor = (n-1)/2$.

(a) If n is even, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n/2 + n/2 = n$. On the other hand, if n is odd, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = (n+1)/2 + (n-1)/2 = 2n/2 = n$. Thus, regardless of the parity of n, we always have $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$. This proves proposition (a).

(b) If n is even, then $\lceil n/2 \rceil \cdot \lfloor n/2 \rfloor = (n/2) \cdot (n/2) = n^2/4 = \lfloor n^2/4 \rfloor$. The last equality is because when n is even, n^2 is divisible by 4, i.e., $n^2/4$ is an integer.

Now suppose that n is odd, i.e., $n = 2k + 1$ for an integer k. Then, $n^2 = 4k^2 + 4k + 1$, and $(n^2/4) - 0.25 = k^2 + k$ is an integer that is obviously the largest integer that is not larger than $n^2/4$. This gives $\lfloor n^2/4 \rfloor = (n^2/4) - 0.25$. Thus

$$\lfloor n^2/4 \rfloor = (n^2/4) - 0.25 = (n^2 - 1)/4 = ((n+1)/2) \cdot ((n-1)/2) = \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor.$$

This proves proposition (b) for odd n. Thus, regardless of the parity of n, we always have $\lceil n/2 \rceil \cdot \lfloor n/2 \rfloor = \lfloor n^2/4 \rfloor$. This proves proposition (b).