CSCE 222-200 Discrete Structures for Computing

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Assignment #1 Solutions

1. Prove by induction that $3^n < n!$ if n is an integer greater than 6.

Proof. We prove the inequality by induction on the integer $n > 6$.

Basis Step. Since $n > 6$, the basis case is $n = 7$. In this case, we have LHS $= 3^7 = 2187$, and RHS $= 7! = 5040$. Thus, LHS $<$ RHS. The inequality holds true.

Inductive Step. Assume inductively that the inequality holds true for all n, when $7 \leq n \leq k$.

Now consider the inequality for $n = k + 1$. We have

$$\text{LHS} = 3^n = 3^{k+1} = 3^k \cdot 3 < k! \cdot 3 < k! \cdot (k+1) = (k+1)! = n! = \text{RHS},$$

where in the first inequality, we have used the inductive hypothesis $3^k < k!$, while in the second inequality, we have used the fact that $3 < k + 1$ (because $7 \leq k$). As a result, for $n = k + 1$, we still have the inequality $3^n < n!$ hold true.

By mathematical induction, this proves that $3^n < n!$ holds true for all $n > 6$.

2. (a) Find a formula for $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ by examing the values of this expression for small values of n.

(b) Prove by induction the formula you conjectured in part (a).

Proof. (a) You may try the formula in any way you like. On the other hand, if you observe that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (for all $n \geq 1$), then the formula of the given summation becomes obvious:

$$\begin{aligned} & \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ = \; & \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ = \; & 1 - \frac{1}{n+1}. \end{aligned}$$

(b) Now we verify the above guessed equality

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

by induction on the integer $n \geq 1$.

Basis Step. For the basis case $n = 1$, we have LHS $= \frac{1}{1\cdot 2} = \frac{1}{2}$, and RHS $= 1 - \frac{1}{1+1} = \frac{1}{2}$. Thus, LHS = RHS. The equality holds true.

Inductive Step. Assume inductively that the equality holds true for all n, when $1 \leq n \leq k$.

Now consider the equality for $n = k + 1$. We have (note that $n \geq 2$ because $k \geq 1$)

$$\begin{aligned}
\text{LHS} &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}\\
&= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}\\
&= \left(1-\frac{1}{k+1}\right)+\frac{1}{(k+1)(k+2)}\\
&= \left(1-\frac{1}{k+1}\right)+\left(\frac{1}{(k+1)}-\frac{1}{(k+2)}\right)\\
&= 1-\frac{1}{(k+2)}\\
&= 1-\frac{1}{n+1}\\
&= \text{RHS},
\end{aligned}$$

where in the third equality, we have used the inductive hypothesis $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$ for $n = k$, and in the fourth equality, we have used the fact $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$. As a result, for $n = k + 1$, we also have the equality $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$ hold true.

By mathematical induction, this proves that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$ holds true for all $n \geq 1$.

3. Prove by contradiction that if x^3 is irrational, then x is irrational.

Proof. Assume the contrary that x is not irrational, i.e., x is a rational number. Then x can be written as the ratio of two integers m and n: $x = m/n$. Take the third power on both sides, we get $x^3 = (m/n)^3 = m^3/n^3$. Since m and n are integers, m^3 and n^3 are also integers. As a result, $x^3 = m^3/n^3$ would also be a rational number, contradicting the assumption that x^3 is irrational.

This contradiction proves that x must be irrational.

4. Prove by contradiction that $\sqrt{2}+\sqrt{3}$ is irrational.

Proof. Assume the contrary that $\sqrt{2}+\sqrt{3}$ is a rational number, i.e., $\sqrt{2}+\sqrt{3} = m/n$, where m and n are integers. Take square on both sides, we get

$$(\sqrt{2}+\sqrt{3})^2 = m^2/n^2.$$

Since $(\sqrt{2}+\sqrt{3})^2 = 2 + 2\cdot\sqrt{2}\cdot\sqrt{3} + 3 = 5 + 2\cdot\sqrt{6}$, we get

$$\sqrt{6} = (m^2/n^2 - 5)/2 = (m^2 - 5n^2)/(2n^2).$$

Since m and n are integers, $m^2 - 5n^2$ and $2n^2$ are also integers. As a result, $\sqrt{6} = (m^2 - 5n^2)/(2n^2)$ is a rational number, which can be written as

$$\sqrt{6} = r/q, \tag{1}$$

where r and q are integers with no common factor larger than 1. Square both sides of equality (1) then multiply both sides by q^2, we get

$$6q^2 = r^2. \tag{2}$$

Since $6q^2$ is divisible by 2, r^2 is divisible by 2, which, as we discussed in class, leads to the conclusion that r is also divisible by 2. Write $r = 2k$, where k is an integer, then $r^2 = 4k^2$. Replacing r^2 in (2) by $4k^2$, then divide both sides by 2, we get

$$3q^2 = 2k^2. \tag{3}$$

However, this implies that q^2 is divisible by 2, which, again by our discussion in class, will further leads to the conclusion that q is divisible by 2. Thus, we would derive that both r and q are divisible by 2, contradicting our assumption that r and q have no common factor larger than 1.

This contradiction proves that $\sqrt{2}+\sqrt{3}$ is irrational.

5. Let p, q, and r be the propositions:

p: You get an A on the final exam;
q: You do every exercise in this book;
r: You get an A in this class.

Write each of the propositions below using p, q, r and logical operators:

(a) You get an A in this class, but you do not do every exercise in this book.
(b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
(c) To get an A in this class, it is necessary for you to get an A on the final.
(d) You get an A on the final, but you do not do every exercise in this book; nevertheless, you get an A in this class.

(e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
(f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution.

(a) $r \wedge \neg q$. (b) $p \wedge q \wedge r$. (c) $r \to p$.
(d) $p \wedge \neg q \wedge r$. (e) $(p \wedge q) \to r$. (f) $r \leftrightarrow (p \vee q)$.

6. For each of the compound propositions below, use the conditional-disjunction equivalence to find an equivalent compound proposition that does not involve conditionals:

(a) $\neg p \to \neg q$. (b) $(p \vee q) \to \neg p$. (c) $(p \to \neg q) \to (\neg p \to q)$.

Solution. The formula for the conditional-disjunction equivalence is $p \to q \equiv \neg p \vee q$.

(a) $\neg p \to \neg q \;\equiv\; \neg(\neg p) \vee (\neg q) \;\equiv\; p \vee \neg q$.

(b) $(p \vee q) \to \neg p \;\equiv\; \neg(p \vee q) \vee (\neg p) \;\equiv\; (\neg p \wedge \neg q) \vee (\neg p)$.

(c) $(p \to \neg q) \to (\neg p \to q) \;\equiv\; (\neg p \vee \neg q) \to (p \vee q) \;\equiv\; \neg(\neg p \vee \neg q) \vee (p \vee q) \;\equiv$
$\equiv\; (p \wedge q) \vee (p \vee q) \;\equiv\; (p \vee (p \vee q)) \wedge (q \vee (p \vee q)) \;\equiv\; (p \vee q) \wedge (p \vee q) \;\equiv\; p \vee q$.