CSCE 222-200 Discrete Structures for Computing

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Notes #1. Proof by Mathematical Induction

Let $P(n)$ be a mathematical statement (i.e., a *proposition*) on the integer n.

A proof by induction that the proposition $P(n)$ holds true for all integers $n \geq c$, where c is a fixed integer, consists of two major steps:

Basis Step. directly verify that the proposition $P(c)$ holds true for the integer c;

Inductive Step. For an arbitrary integer k, (inductively) assume that the proposition $P(i)$ holds true for all integers i, where $c \leq i \leq k$. Then use these assumptions (and strictly valid mathematics reasoning) to prove that the proposition $P(k+1)$ holds true for the integer $k+1$.

A successful completion of the above basis step and inductive step gives a valid proof that the proposition $P(n)$ holds true for all integers $n \geq c$.

Example. (the sentences in bold should be included in every of your proofs by induction)

Proposition $P(n)$: $1 + 2 + \cdots + n = n(n+1)/2$.

To prove by induction that Proposition $P(n)$ holds true for all integers $n \geq 1$, we have

Basis Step.

Verify the proposition $P(n)$ for the case of $n = 1$:
LHS-of-$P(1) = 1$; and
RHS-of-$P(1) = 1 * (1+1)/2 = 1$.
Thus, LHS-of-$P(1)$ = RHS-of-$P(1)$, so $P(1)$ holds true. **This completes the basis step.**

Inductive Step. Inductively, assume Proposition $P(i)$ holds true for all $1 \leq i \leq k$.

We prove that $P(k+1)$ holds true, using the above assumptions. We have

$$\begin{aligned} \text{LHS-of-}P(k+1) &= 1 + 2 + \cdots + k + (k+1) \\ &= (1 + 2 + \cdots + k) + (k+1) \\ &= \text{LHS-of-}P(k) + (k+1) \qquad (1) \end{aligned}$$

By the inductive hypothesis, Proposition $P(k)$ holds true. Thus,

$$\text{LHS-of-}P(k) = \text{RHS-of-}P(k) = k(k+1)/2.$$

Replacing "LHS-of-$P(k)$" in (1) with "$k(k+1)/2$", we get

$$\text{LHS-of-}P(k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2 = (k+1)[(k+1)+1]/2 = \text{RHS-of-}P(k+1).$$

Thus, Proposition $P(k+1)$ holds true. **This completes the inductive step.**

By mathematical induction, we conclude that Proposition $P(n)$ holds true for all $n \geq 1$.

Practice Exercises

Use mathematical induction to prove the following propositions, where $n!$ (n *factorial*) is defined to be the product of the first n positive integers 1, 2, …, n.

1. Prove that $3^n < n!$ if n is an integer greater than 6.
2. Prove that $2^n > n^2 + n$ whenever n is an integer greater than 4.
3. Find an integer N such that $2^n > n^4$ whenever n is an integer greater than N. Prove that your result is correct using mathematical induction.
4. For which positive integers n is $n + 6 < (n^2 - 8n)/16$? Prove your answer using mathematical induction.
5. Prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.
6. Prove that 3 divides $n^3 + 2n$ whenever n is positive integer.
7. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.
8. Prove that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ whenever n is a positive integer.
9. Prove that $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}$ whenever n is a positive integer.
10. (a) Find a formula for $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$ by examing the values of this expression for small values of n.
    (b) Prove by induction the formula you conjectureed in part (a).