Constraint Satisfaction

CSCE 420 – Spring 2022

read: Ch. 6

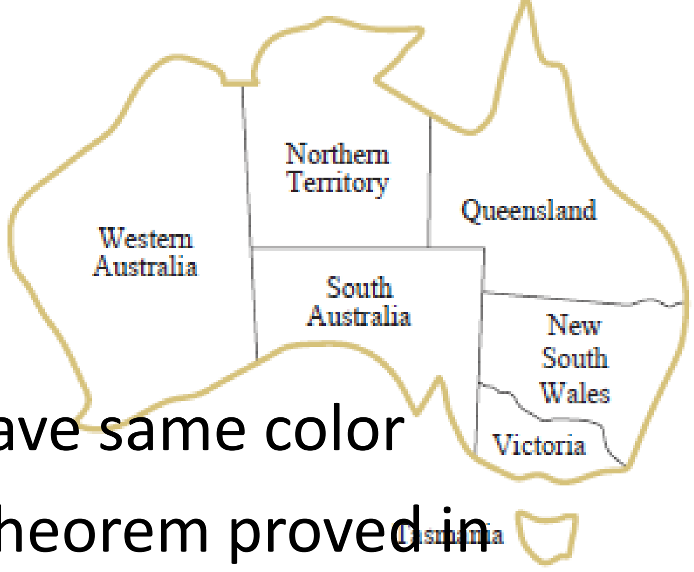
Constraint Satisfaction

- Constraint Satisfaction Problems (CSPs) are a wide class of problems can be solved with specialized search algorithms
- these types of problems typically required finding a configuration of the world that satisfies some requirements (constraints) which restrict the possible solutions
- examples:
  - limited resources that can only be used one at a time
  - satisfying precedence order constraints (e.g. taking prerequisite classes first)
  - assignments (or matching) of agents to tasks of which they are capable

Constraint Satisfaction

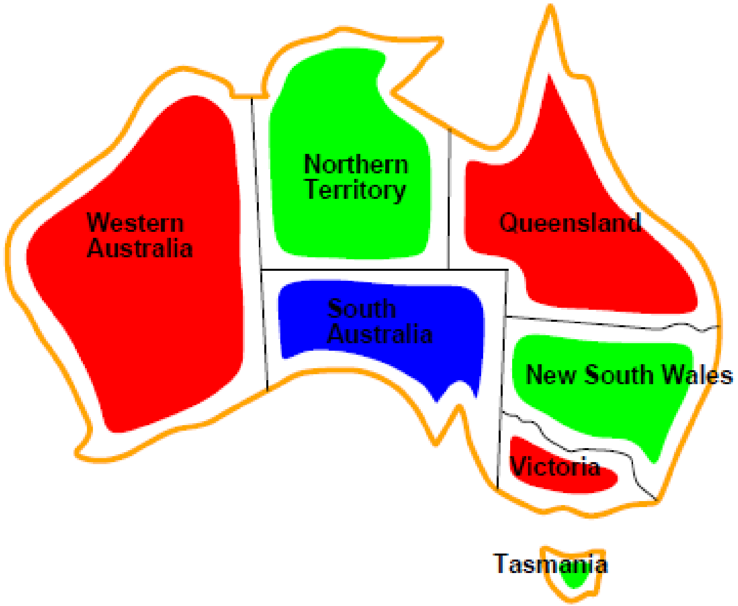
- formal framework:
  - <u>variables</u>: $\{V_i\}$
  - <u>domains</u>: $dom(V_i)=\{a_1...a_n\}$ – a *finite* set of possible values for each variable
  - <u>constraints</u>:
    - the form of constraints can be different for each problem
    - sometimes they are presented as equations
    - examples (binary constraints) : U+V=6; U and V must be opposite parity: $(U\%2)\neq(V\%2)$
    - abstractly, a constraint involving variables can be viewed as a restriction on the allowed set of tuples in the cross-product of domains:
      - constraint $C_j =\{<x_1...x_n> | x_k \in dom(V_k)\} \subset \Pi_{k=1..c}\ dom(V_k)$
      - dom(U)=dom(V)={0,1,2,3,4,5,6,7,8,9}
      - U+V=6: {<0,6>,<6,0>,<1,5>,<5,1>,<4,2>,<2,4>,<3,3>} $\subset$ {<0,0>,<0,1>,...<0,9>,<1,0>,<1,1>,<1,2>....<9,9>} (100 possible 2-tuples)
  - <u>solution</u>: a *complete variable assignment* that satisfies all constraints
    - for some CSPs, there can be multiple solutions

CSP Example: Map coloring



- no two adjacent states (sharing part of an border) can have same color
- (in general, need at most 4 colors – famous Four Color Theorem proved in 1997 with the help of a computer to enumerate all possible cases)
- Australia:
  - vars = {WA,NT,SA,Q,NSW,V,T}
  - domains: dom(S)={R,G,B}
  - constraints: WA≠NT,WA≠SA,NT≠SA,NT≠Q...

  - solution: {WA=R,NT=G,SA=B,Q=R,NSW=G,V=R,T=G}
  - also: {WA=G,NT=R,SA=B,Q=G,NSW=R,V=G,T=R}
  - and so on



CSP Example: Cryptarithmetic

- vars: {F,T,W,O,U,R}
  - and add carry bits {c1,c2}
- domains: dom(var)={0,1,2...9} (digits)
  - domain for c1 and c2 is just {0,1}
- constraints:
  - all var bindings must be distinct: F≠T, F≠W...
  - leading chars can't be 0: T≠0, T≠0
  - the math must add up correctly:
    - O+O=R - what if there is a carry? introduce c1, dom(c1)={0,1}
    - O+O=R-c1*10
    - c1+W+W=U-c2*10
    - C2+T+T=U-F*10

```
   c2 c1
    T W O
  + T W O
  -------
= F O U R
```

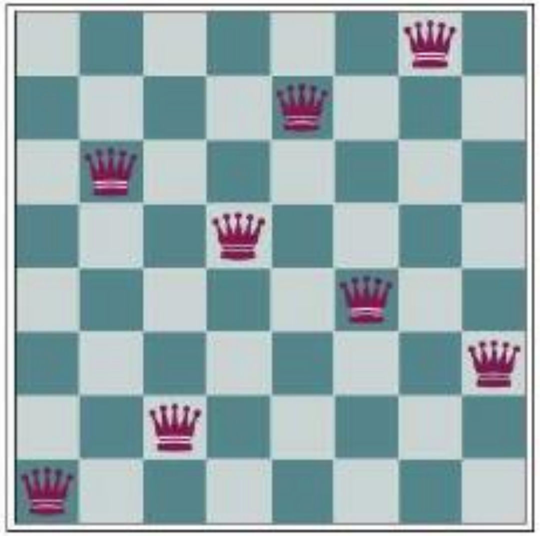
```
    7 6 5
  + 7 6 5
  -------
= 1 5 3 0
```

a solution:
F=1
T=7
W=6
O=5
U=3
R=0

are there other solutions?

```
    S E N D
  + M O R E
  ---------
= M O N E Y
```

CSP Example: 8-queens



- assume there is one queen in each column
- for each column i, what row is the queen in?
- vars: $Q_1..Q_8$
- domains: $Q_i \in \{1..8\}$
- constraints:
  - no 2 queens can be in same row: $Q_i \neq Q_j$ for all $i \neq j$
  - no 2 queens can be in same diagonal: $|Qi-Qj| \neq |i-j|$
  - equivalent representation:
    - allowed Q1-Q2 pairs: {(1,3),(1,4),(1,5)...(1,8),(2,4)...(2,8),(3,1),(3,5)...(3.8)...}
    - allowed Q1-Q3 pairs: {(1,2),(1,4),(1,5)...(1,8),(2,1),(2,3),2,5)...}

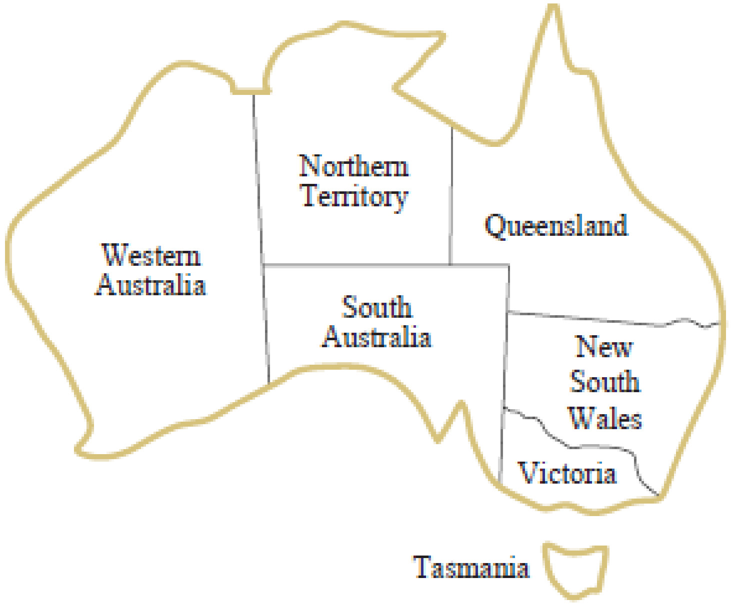
CSP Example: scheduling

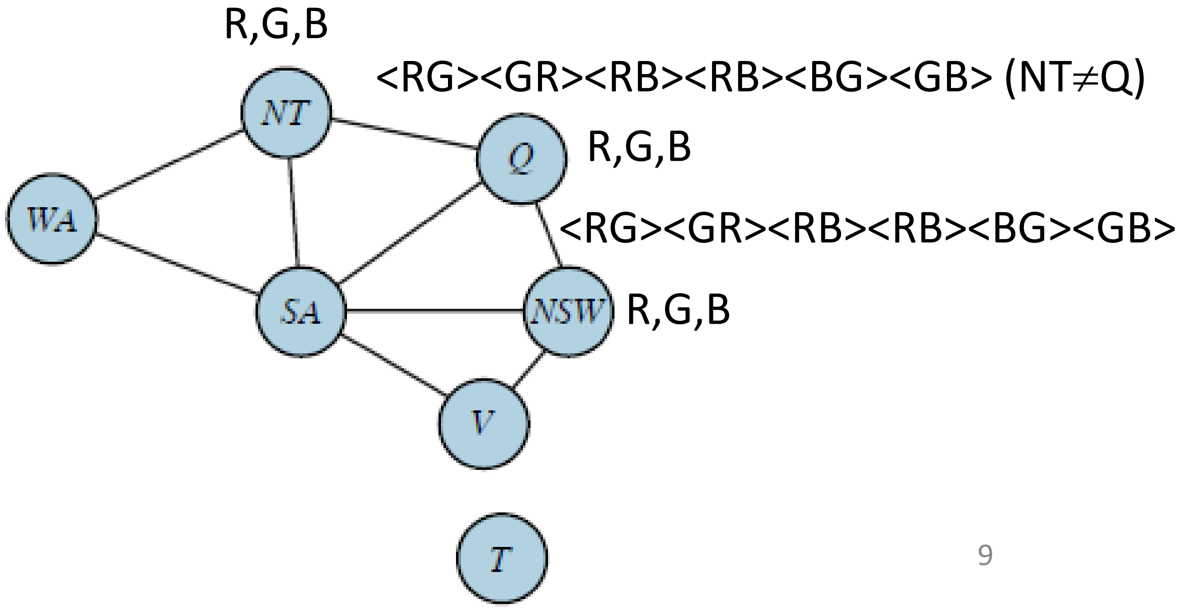
- Job Shop scheduling
  - car assembly tasks: install axles (2), install wheels (4), tighten bolts (4), put on hubcaps(4), inspection (1)
  - variables: time steps for each task (integers): T_{axleF} , T_{axleR} , $T_{wheelFR}$ ...$\in$[1..20] (time limit)
  - precedence constraints: $T_{axleF}<T_{wheelFR}<T_{nutFR}< T_{inspection}$
  - (we could also model task durations)
  - solution: assignment of time slot for each step
    - $T_{axleF}=1$, $T_{wheelFR}=2$, $T_{wheelFL}=3$, $T_{axleR}=4$, ...$T_{inspection}=15$
- you can do the same thing with undergrad courses:
  - CSCE 313 is needed to graduate
  - CSCE 312 is a prerequisite for CSCE 313
  - only want to take at most 5 courses per semester
  - can you figure out a solution (assignment of courses to semesters) that satisfies all prereqs and will enable you to graduate in 4 yrs?

- note: Scheduling is a big field of computer science, and there are many variants of scheduling problems
- often, we want to know more that just whether there is a feasible solution: we want to find a schedule of minimum length (make-span)
- this goes beyond CSPs

Constraint Graphs

- nodes=vars (label with domain, possible values)
- edges=constraints
  - easy for binary constraints
  - label edges with pairs of consistent values from each domain

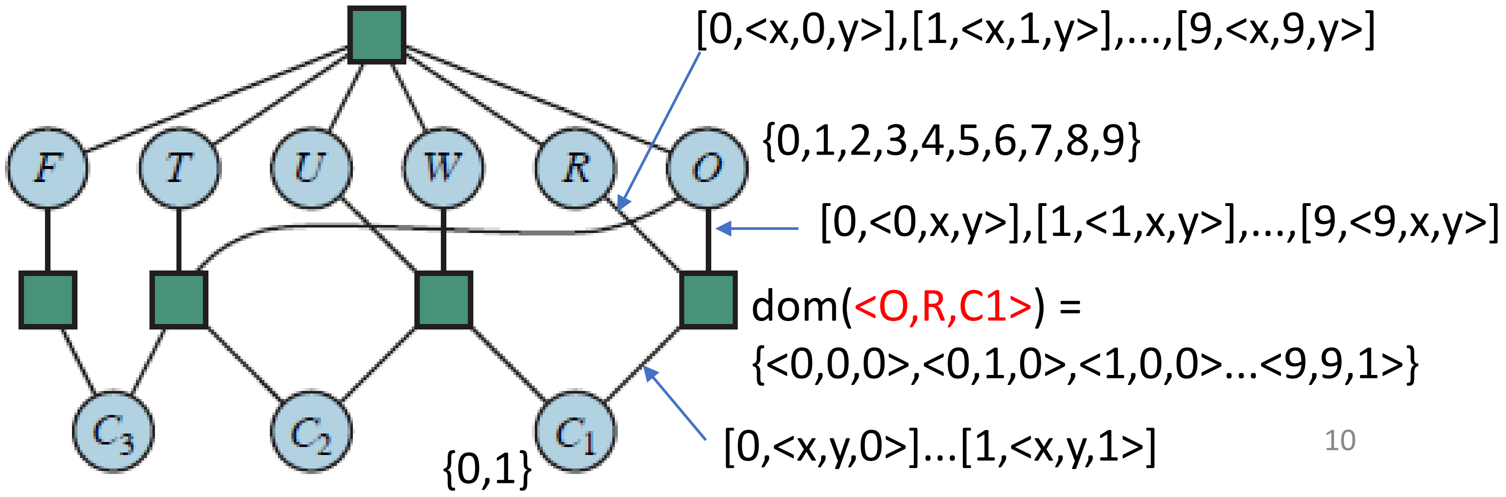




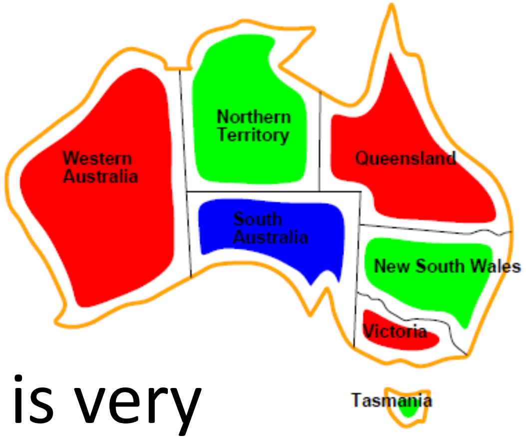
Constraint Graphs

- for ternary constraints (3 or more variables), e.g. O+O=R-c1*10
  - creates a “hypergraph” with special edges that connect ≥3 nodes (hard to draw)
  - convert to a binary graph:
    - create new nodes (green) for each constraint
    - label the new nodes with all possible tuples based on cross-product of domains
    - connect the new nodes to the constrained variables
    - label the edges to enforce consistency of variable assignment with position in tuple

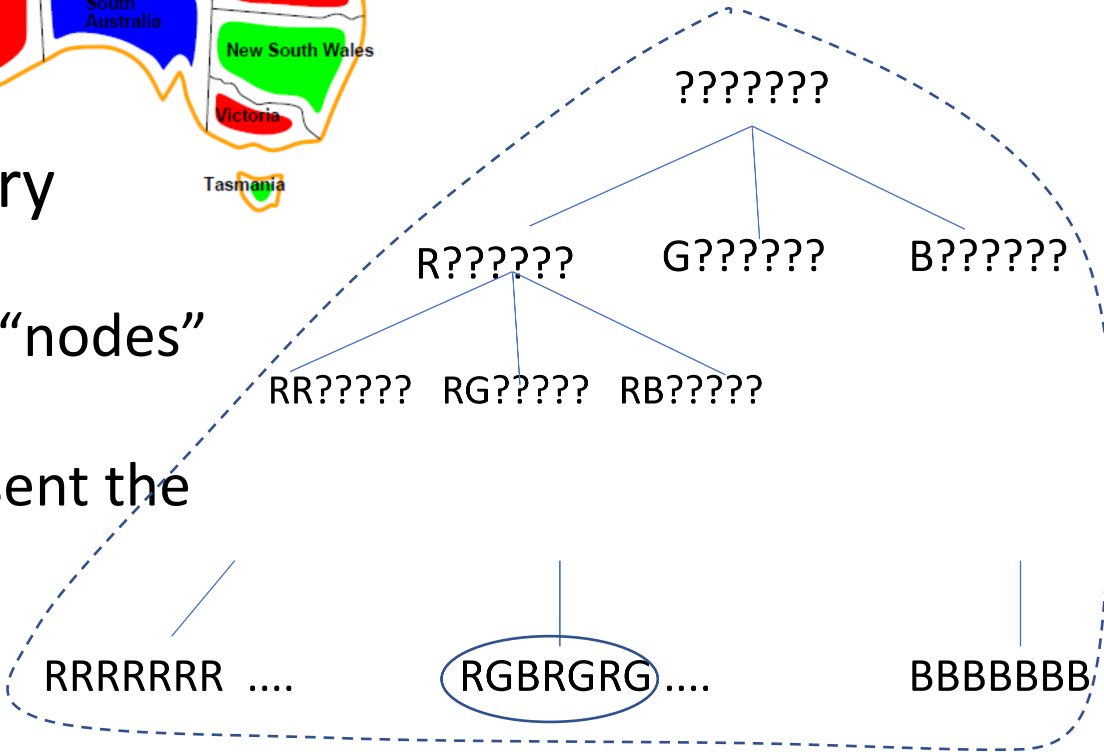
$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



Back-tracking



vars: WA,NT,SA,Q,NSW,V,T
states: <c1,c2,c3,c4,c5,c6,c7>
where ci∈{R,G,B,?}

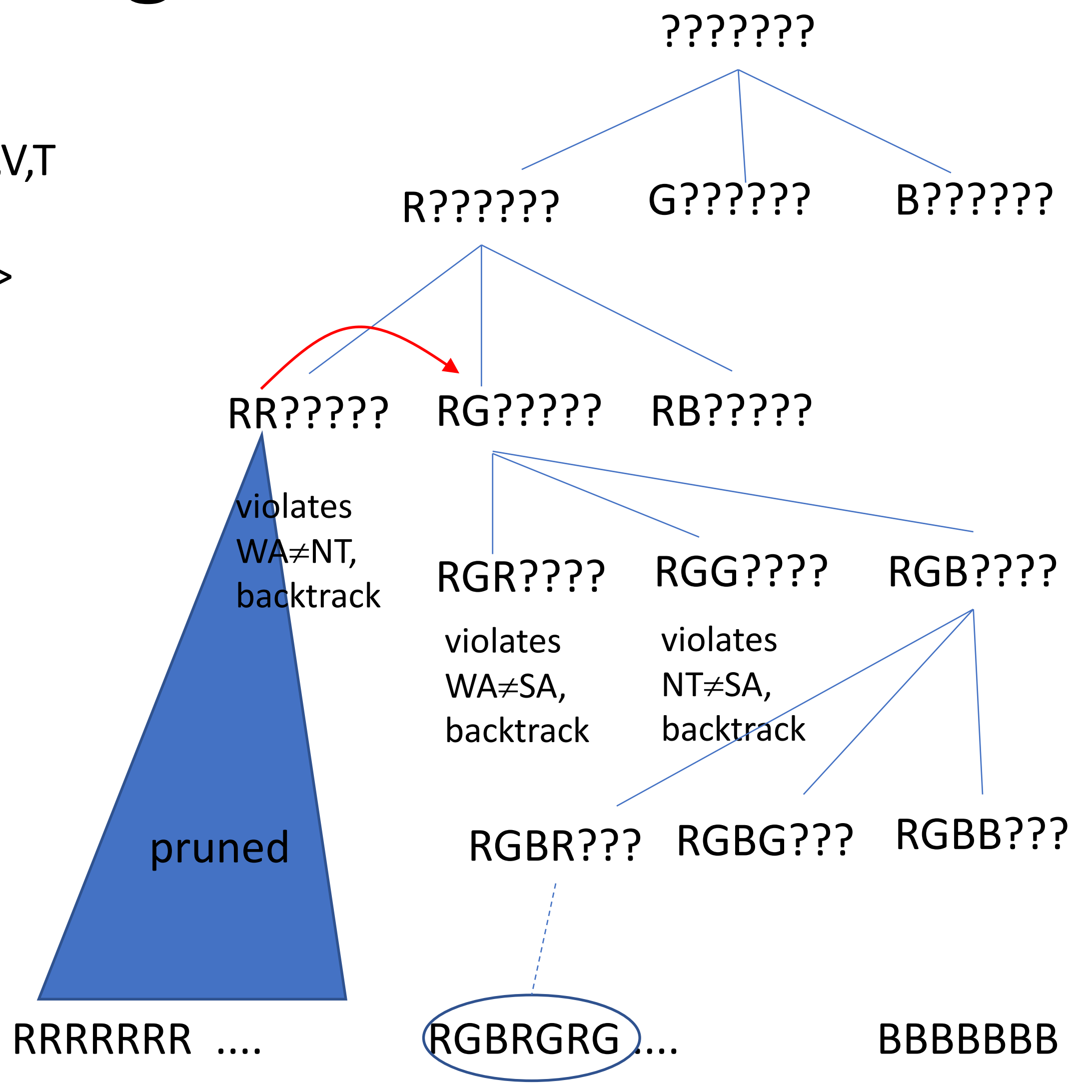


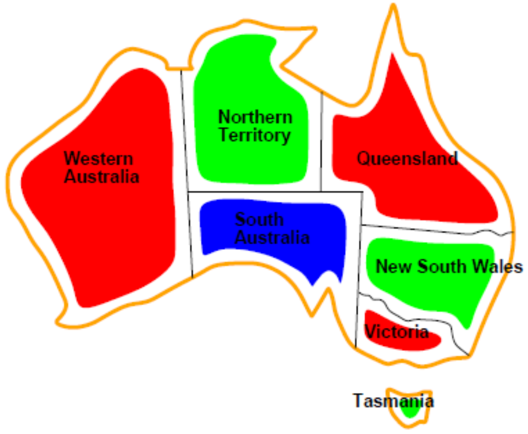
how many leave are there?

- the basic search algorithm for CSPs is very similar to DFS
  - variable assignments represent "states" or "nodes"
  - the root node is the <u>empty assignment</u>
  - for a selected variable, the branches represent the choices from the domain
  - each level assigns one more variable
- there are two important differences:
  - tree depth is uniform (# vars), and <u>all goals occur at the fringe</u>
  - as soon as assigning any variable at an internal node causes inconsistency with a constraint, prune that subtree and <u>backtrack</u> immediately

Back-tracking

vars: WA,NT,SA,Q,NSW,V,T
state representation:
<c1,c2,c3,c4,c5,c6,c7>
where ci∈{R,G,B,?}





```
function BACKTRACKING-SEARCH(csp) returns a solution or failure
  return BACKTRACK(csp, { })

function BACKTRACK(csp, assignment) returns a solution or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp, assignment)
  for each value in ORDER-DOMAIN-VALUES(csp, var, assignment) do
      if value is consistent with assignment then
        add {var = value} to assignment

          result ← BACKTRACK(csp, assignment)
          if result ≠ failure then return result

        remove {var = value} from assignment
  return failure
```

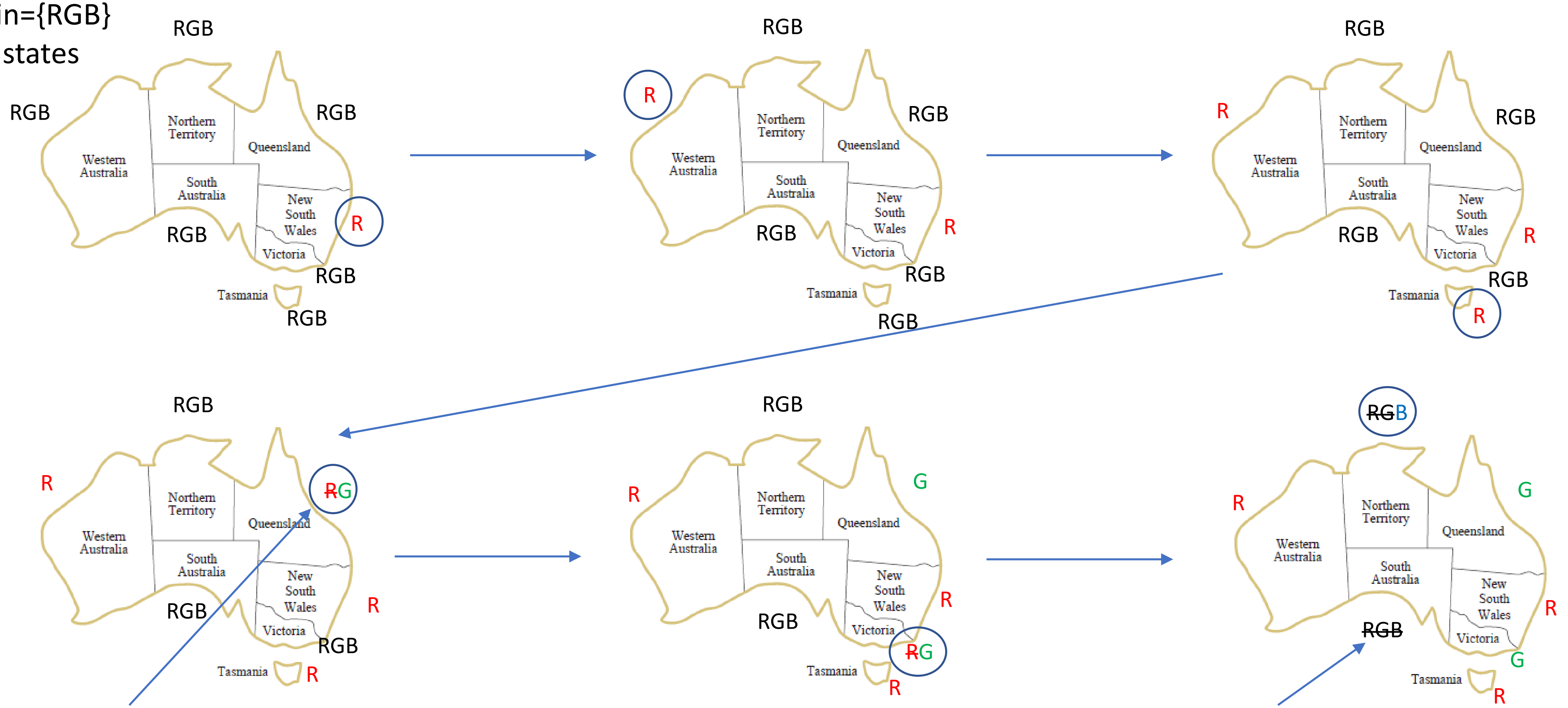
ignore inferences for now

recursion: bind more variables...

Tracing Backtracking

suppose the order of vars is given as: NSW, WA, T, Q, V, NT, SA

initially,
domain={RGB}
for all states

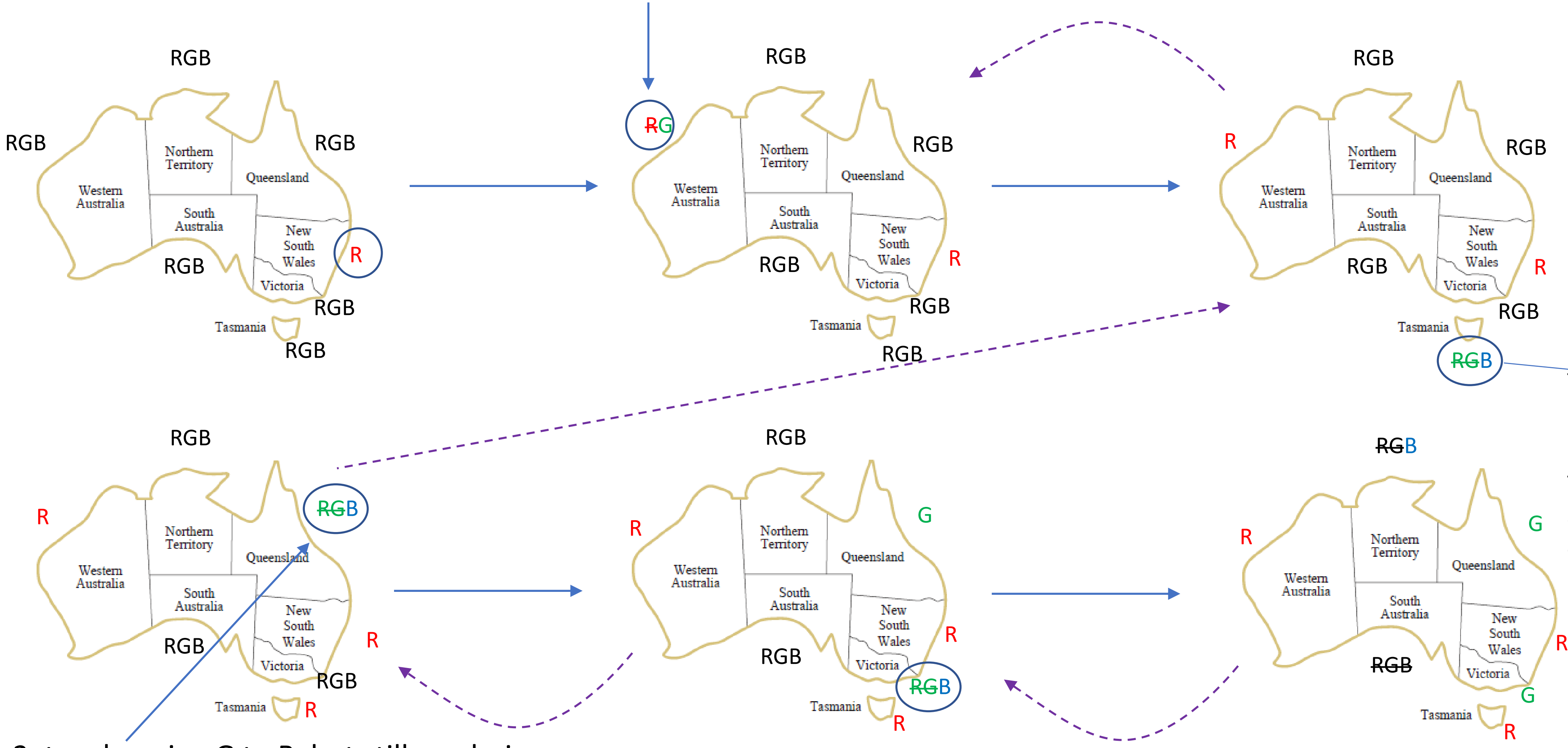


this is the first time we violate a, constraint, but only change R to G

crisis: no values remain for SA;
must back-track to WA (ultimately) and change it to G,
after trying all combinations of V, Q, and T

Tracing Backtracking

4. ultimately have to
change this to G, and resume search



3. try G...
then retry all
subsequent
choices;
then try B **and all**
subsequent
choices; still
no choices for SA

2. try changing G to B, but still no choices
remain that lead to a consistent solution

1. no other choices remain for NT, so back
track to V and try changing G to B; but
NT is still B and SA still has no values

instead of choosing next **var** arbitrarily (in order given), or we could use **MRV heuristic** to choose more intelligently...

function BACKTRACKING-SEARCH(*csp*) **returns** a solution or *failure*
  return BACKTRACK(*csp*, { })

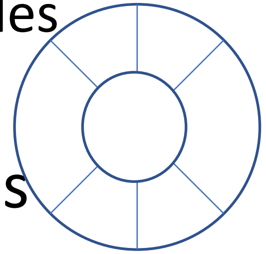
function BACKTRACK(*csp*, *assignment*) **returns** a solution or *failure*
  if *assignment* is complete **then return** *assignment*
  var ← SELECT-UNASSIGNED-VARIABLE(*csp*, *assignment*)
  for each *value* **in** ORDER-DOMAIN-VALUES(*csp*, *var*, *assignment*) **do**
    if *value* is consistent with *assignment* **then**
      add {*var* = *value*} to *assignment*
      result ← BACKTRACK(*csp*, *assignment*)
      if *result* ≠ *failure* **then return** *result*
    remove {*var* = *value*} from *assignment*
  return *failure*

instead of choosing next **value** arbitrarily (in domain order), or we could use **LCV heuristic** to choose more intelligently...

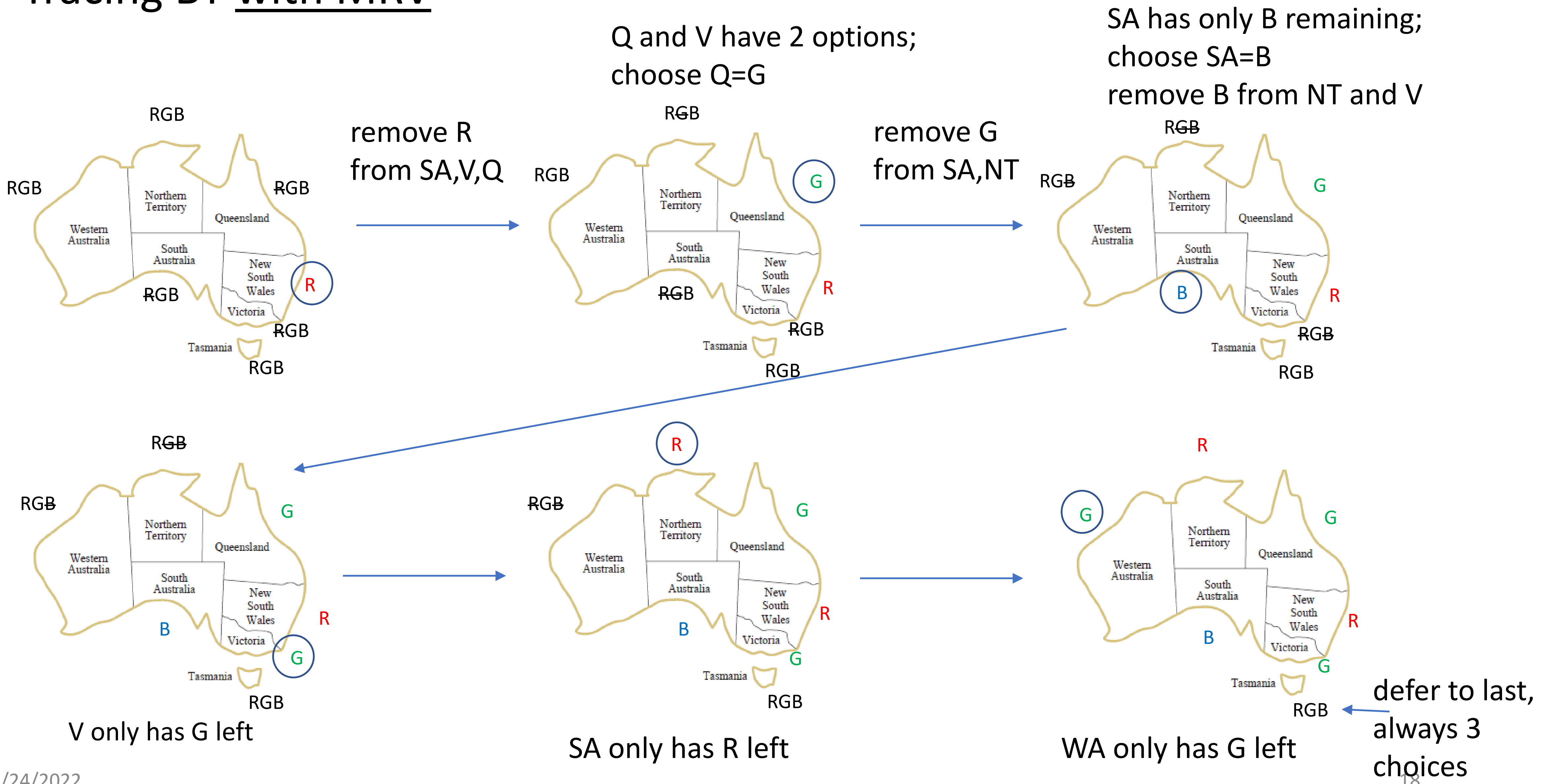
CSP Heuristics

- MRV – select var based on Minimum Remaining Values
  - in current partial assignment, some variable bindings might preclude choices in domains for unbound variables based on constrains
  - for each unbound variable, rule out values that are inconsistent with curr. assignment
  - choose variable with fewest choices
    - the best case: if there is a variable with just 1 choice left, choose it!
    - forces back-tracking to happen sooner
- LCV – select value for var based on Least Constraining Value
  - once a var is chosen, can we try the values in an intelligent order?
  - pick value that would remove the fewest (leave the most) choices for other variables
  - this will tend to delay back-tracking to happen later
- degree heuristic: if all domains are equal-sized, choose the variable that is involved in the most constraints (connected to the most other vars)

<u>Food for thought:</u>
How much would MRV help in coloring the map of USA, compared to doing BT on 50 states in alphabetical order?



- # Tracing BT with MRV



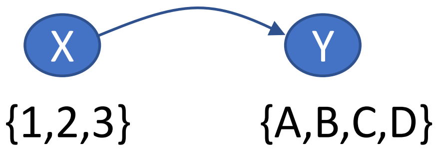
No back-tracking! notice how choices tend to *propagate* to neighbors

Forward-checking (FC)

- MRV is very similar to forward-checking
  - technically, MRV is passive; in each iteration, it re-calculates how many consistent values remain in domain of each unbound var
  - FC is active: every time you choose a value for a var, you remove inconsistent values in domains of other vars (like “propagation”)
  - almost identical, except... if making a choice at var X causes domain for var Y to become empty, back-track immediately and try another value for X (don’t have to wait till Y is selected to see that it’s domain is empty)

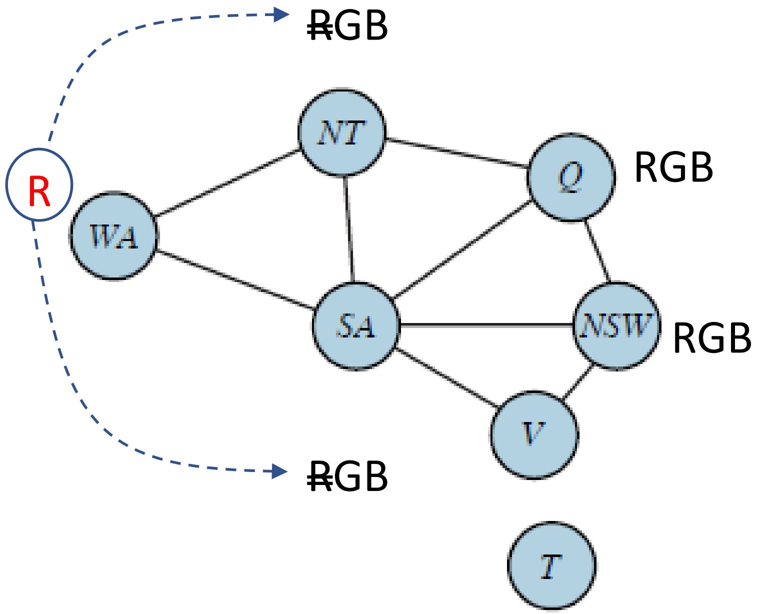
Constraint Propagation

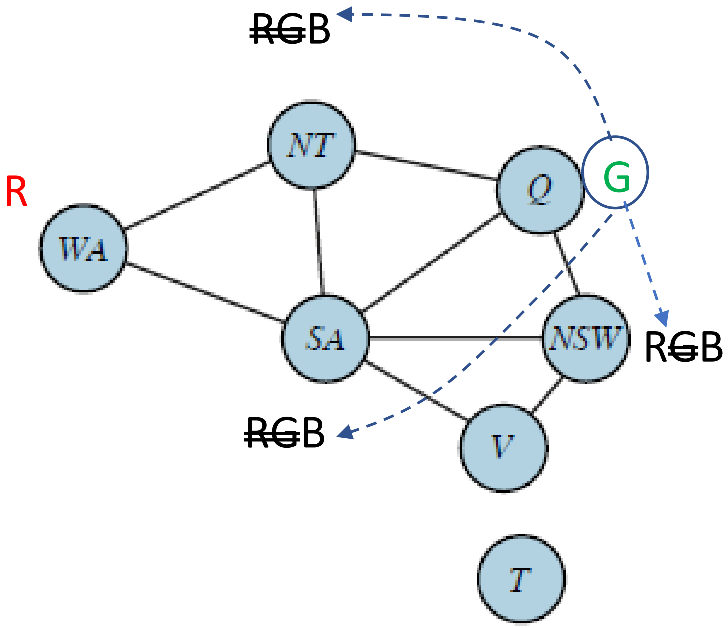
- we can generalize the idea of FC
- whenever we make a choice at one node in the constraint graph, propagate the consequences to neighboring nodes
  - remember, edges are determined by constraints
- sometimes, a choice has no effect on domains of neighbors
- sometimes, choice at node X removes some options from domain of neighbor Y
- sometimes, choice at X removes all but one option at Y
  - if so, make this choice at Y, and propagate consequences to its neighbors...
- sometimes, choice at X reduces the domain of neighbor Y to empty, forcing back-tracking

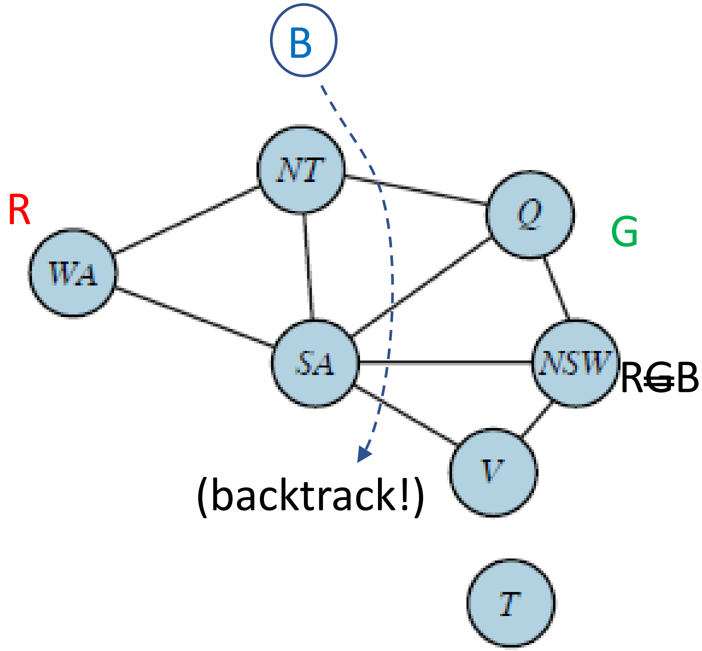


Constraint Propagation

suppose we assign WA=R, and then Q=G,
and we are doing Forward checking...



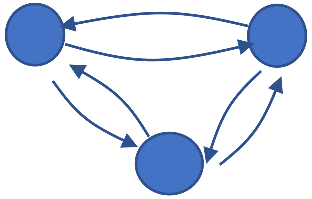




why shouldn't we be able to propagate one *more* step and see that NT is forced to be B, leaving no choices for SA? (or vice versa)

AC-3

- formalization of constraint propagation as a *graph algorithm*
- let (V,E) be the constraint graph (assume all constraints are binary)
- define *arc-consistency*:
  - a graph is arc-consistent if for every variable X, for every value a in dom(X), for every variable Y it is connected to (by a constraint), there is a value b for Y that is consistent with X=a
  - for all edges (X,Y), $\forall\, a \in dom(X)\ \exists\, b \in dom(Y)$ s.t. X=a and Y=b are consistent
- ensure the initial graph is arc-consistent
- after making a choice for an initial var, it might rule out some choices in domains of neighbors, so must check that its neighbors are arc-consistent...
- put *edges* to be checked in a *queue*



function AC-3(csp) **returns** false if an inconsistency is found and true otherwise
  $queue \leftarrow$ a queue of arcs, initially all the arcs in csp

  while $queue$ is not empty **do**
    $(X_i, X_j) \leftarrow$ POP($queue$)
    if REVISE(csp, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** $false$
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add (X_k, X_i) to $queue$
  return $true$

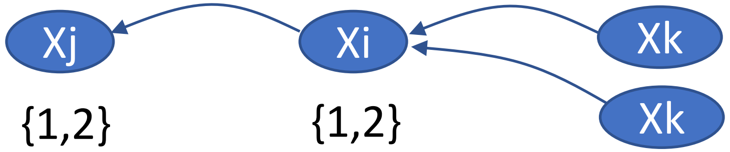
initialize queue with all directed edges between nodes

Revise() returns true if dom(Xi) was updated

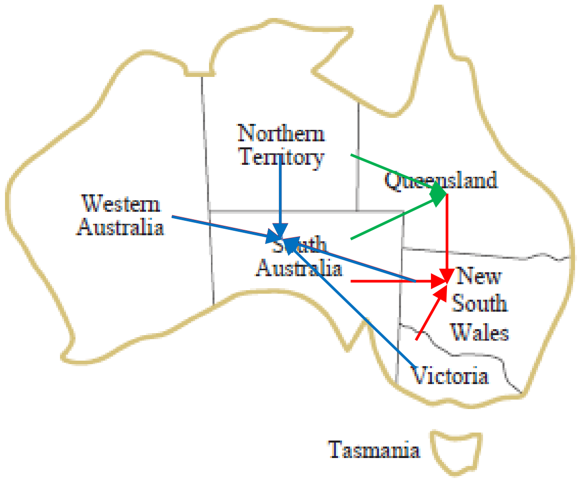
every time we delete a value from the domain of Xi, put the connected edges in the queue; note the reverse order: (X_k, X_i) – list the neighbors first

function REVISE(csp, X_i, X_j) **returns** true iff we revise the domain of X_i
  $revised \leftarrow false$
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      $revised \leftarrow true$
  return $revised$

suppose the sum of Xi and Xj must be odd, and we remove 2 from dom(Xj)



Tracing AC-3



- suppose we start by choosing NSW=R
  - all edges connected to NSW must be checked for arc-consistency
- queue: {<Q,NSW>,<SA,NSW>,<V,NSW>}
  - pop <Q,NSW>,
  - R$\in$dom(Q) has no consistent value in dom(NSW)={R} so remove R from dom(Q);
  - but G,B$\in$dom(Q) each are consistent with R$\in$dom(NSW)
  - push neighbors of Q: <NT,Q>,<SA,Q> // note the reverse order
- queue: {<SA,NSW>,<V,NSW>, <NT,Q>,<SA,Q> }
  - pop <SA,NSW>, check each choice in dom(SA)={RGB} for a consistent choice in dom(NSW)={R}; remove R from dom(SA)
  - push neighbors of SA: <WA,SA>,<NT,SA>,<V,SA>,<NSW,SA>
- queue: {<V,NSW>, <NT,Q>,<SA,Q>, <WA,SA>,<NT,SA>,<V,SA>,<NSW,SA>}

Maintaining Arc Consistency

- often, the initial graph is arc-consistent, so nothing to do
- after making first choice, run AC-3 till it quiesces
- usually the problem is not solved
  - a problem is solved when every node has just 1 value remaining
  - if some vars still have multiple values in their domains, we must make more choices
  - if any domain is empty, must back-track to previous choice point and try another value, followed by calling AC-3 to propagate consequences by reducing domains
- thus MAC is a *wrapper algorithm* around AC-3 that iteratively makes another choice and calls AC-3, till one of these two conditions is met

Maintaining Arc Consistency

```
MAC(graph G)
 if every node has exactly 1 val: return solution (complete assignment)
 if some node has no val, return fail (backtrack)
 choose a node V that still has multiple values in its domain
 for each value a in dom(V):
   G′ = G{V=a} // set node V to the value a
   G′′ = AC3(G′) // make graph arc-consistent based on this choice
   result = MAC(G′′) // recurse, try to extend this to a complete solution
   if result!=fail: return result
 return fail
```

Complexity of AC-3

- what is the time-complexity of AC-3?
- assume there are c edges (num. of constraints, $c \leq n^2$), and d is the max domain size: $d = max(|dom(V_i)|)$
- an edge is only put in the queue whenever a value is deleted from the domain of a var
- so all edges will be processed at most cd times in total (calls to Revise())
- Revise() takes up to d^2 loop iterations to check for arc-consistency
- so AC-3 is $O(cd^3) = O(n^2 d^3)$

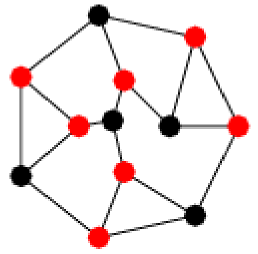
```
function AC-3(csp) returns false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (Xi, Xj) ← POP(queue)
    if REVISE(csp, Xi, Xj) then
      if size of Di = 0 then return false
      for each Xk in Xi.NEIGHBORS - {Xj} do
        add (Xk, Xi) to queue
  return true

function REVISE(csp, Xi, Xj) returns true iff we revise the domain of Xi
  revised ← false
  for each x in Di do
    if no value y in Dj allows (x,y) to satisfy the constraint between Xi and Xj then
      delete x from Di
      revised ← true
  return revised
```

Computational Complexity of CSPs

- Theorem: Solving CSPs is NP-hard.
  - one can check whether a given variable assignment satisfies all constraints in polynomial time
- Theorem: Determining whether CSPs have a solution is NP-complete.
  - Proof: Graph Coloring can be reduced to CSP (CSP $\leftarrow$ graph 3-coloring $\leftarrow$ graph clique $\leftarrow$ 3-Sat)
  - we have already shown that graph-coloring can be transformed into a CSP in polynomial size
- thus *many* discrete problems can be encoded as CSPs
- *food for thought*: how would you encode Vertex Cover as a CSP?
  - does there exists a subset of k nodes that touches every edge?



Computational Complexity of CSPs

- how can CSPs be NP-complete if AC-3 runs in polynomial time, $O(n^2d^3)$?
  - we might have to call it an exponential number of times from MAC before we find a complete and consistent solution
- relation to Linear Programming (LP)
  - Linear Programs are like CSPs except they use continuous variables instead of discrete domains, and linear constraints
  - example:

    maximize $5x+3y-z$
    subject to $8x-7y\leq12$, $y+2z\leq1$, $0\leq x\leq2$, $0\leq y\leq10$, $0\leq z\leq2$

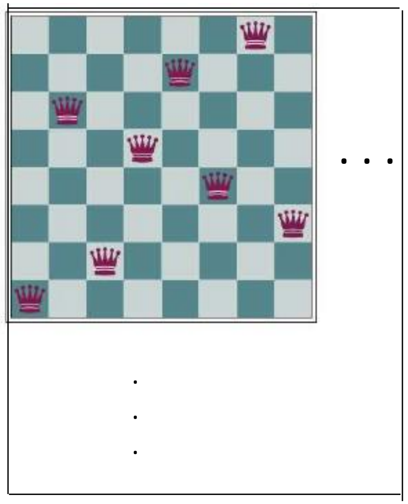
  - there exist polynomial time algorithms for LPs (e.g. Simplex Algorithm)
  - Mixed Integer-Linear Programs (MIPs): some variables are restricted to integers
  - Integer Programs (IPs) have all discrete values and can encode CSPs: IPs $\leftrightarrow$ CSPs
  - discrete values makes solving constraints HARDER computationally
    - Linear Programming is in P
    - Mixed Integer Programming is in NP (actually NP-hard)

Min-Conflicts Algorithm

```
function MIN-CONFLICTS(csp, max_steps) returns a solution or failure
  inputs: csp, a constraint satisfaction problem
          max_steps, the number of steps allowed before giving up

  current ← an initial complete assignment for csp
  for i = 1 to max_steps do
     if current is a solution for csp then return current
     var ← a randomly chosen conflicted variable from csp.VARIABLES
     value ← the value v for var that minimizes CONFLICTS(csp, var, v, current)
     set var = value in current
  return failure
```

- Local Search for CSPs
  - start by choosing a random variable assignment (which probably violates lots of constraints)
  - pick a variable at random and change its values to something that causes less conflicts
  - repeat until it "plateaus" (number of conflicts stops decreasing)
  - note: this is NOT guaranteed to find a complete and consistent solution!
  - but it works surprisingly well in practice
  - MinConflicts can solve the ***million-queens*** problem (on a 10^6x10^6 chess board) in a few minutes (!)

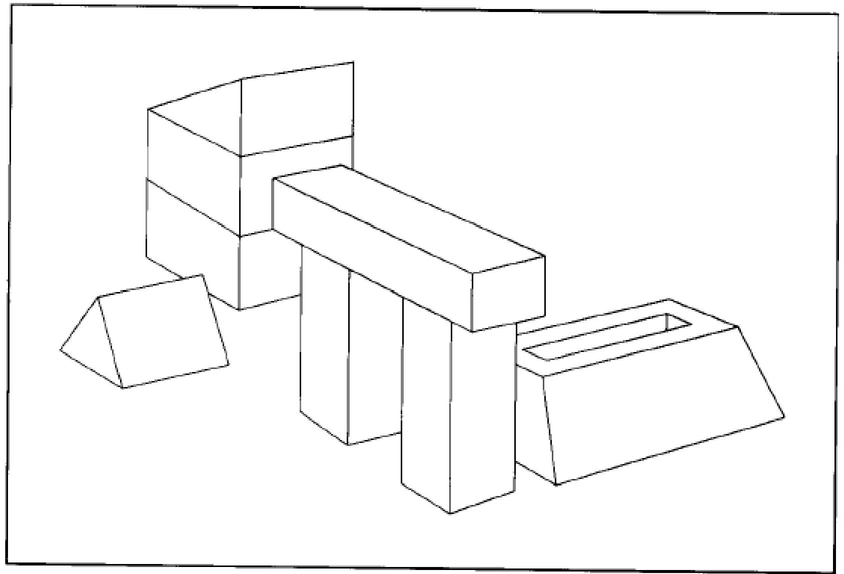


Application of CSP to Computer Vision

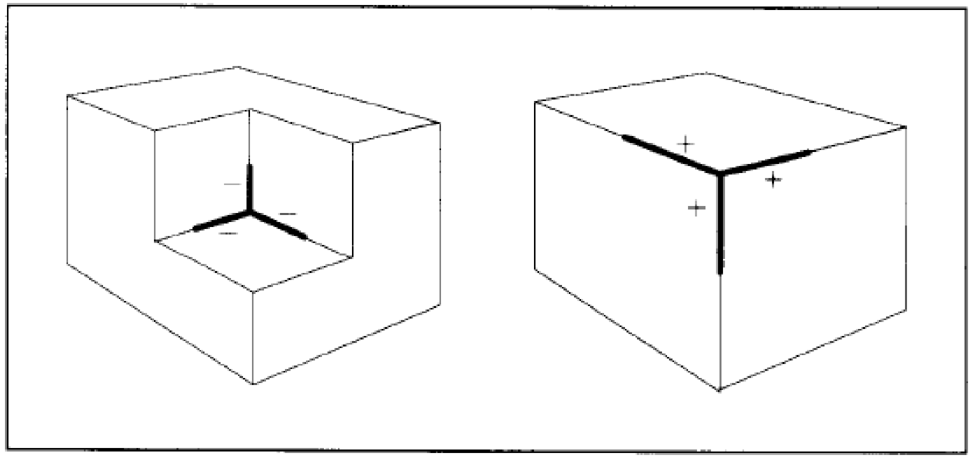
- 2D edge-detection → 3D object interpretation
- Waltz Constraint Propagation algorithm
- edges can be ambiguous – which side is part of object, vs background (or another object behind, i.e. occluded?)
- for any intersection of edge, there are only a finite number of possible labeling (for realistic 3D images)
- some 3-way intersections can be interpreted as corners

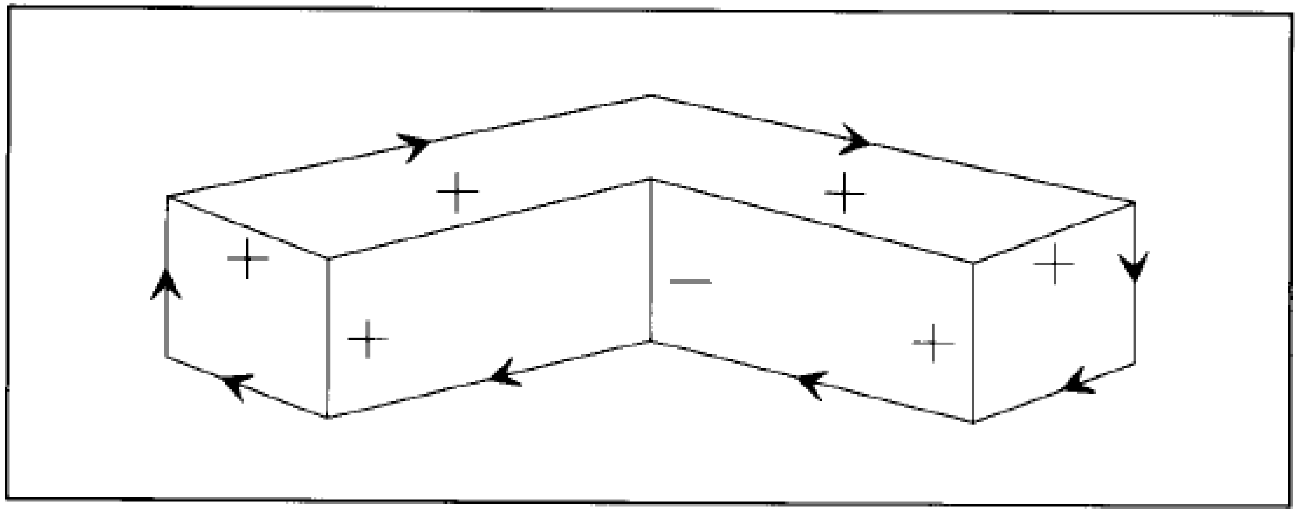
3D Image Interpretation and Waltz Propagation

- from Ch. 12 in Patrick Winston (1984). Artificial Intelligence.
  - http://courses.csail.mit.edu/6.034f/ai3/ch12.pdf
- 2D image pre-processing: edge detection
  - Gaussian filter + segmentation
- how can you infer the 3D objects from line segments?
- how many object are there in this image?

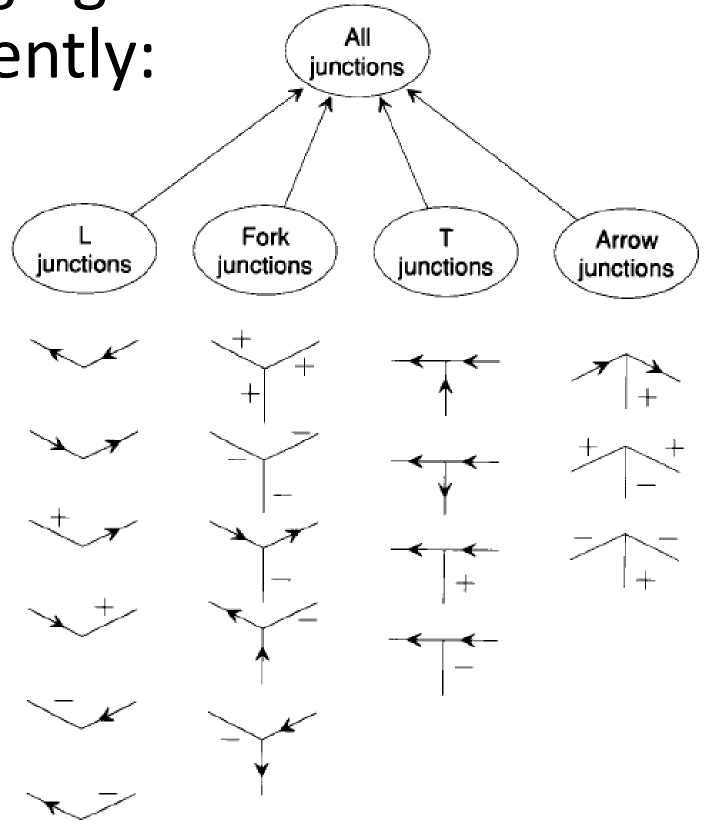


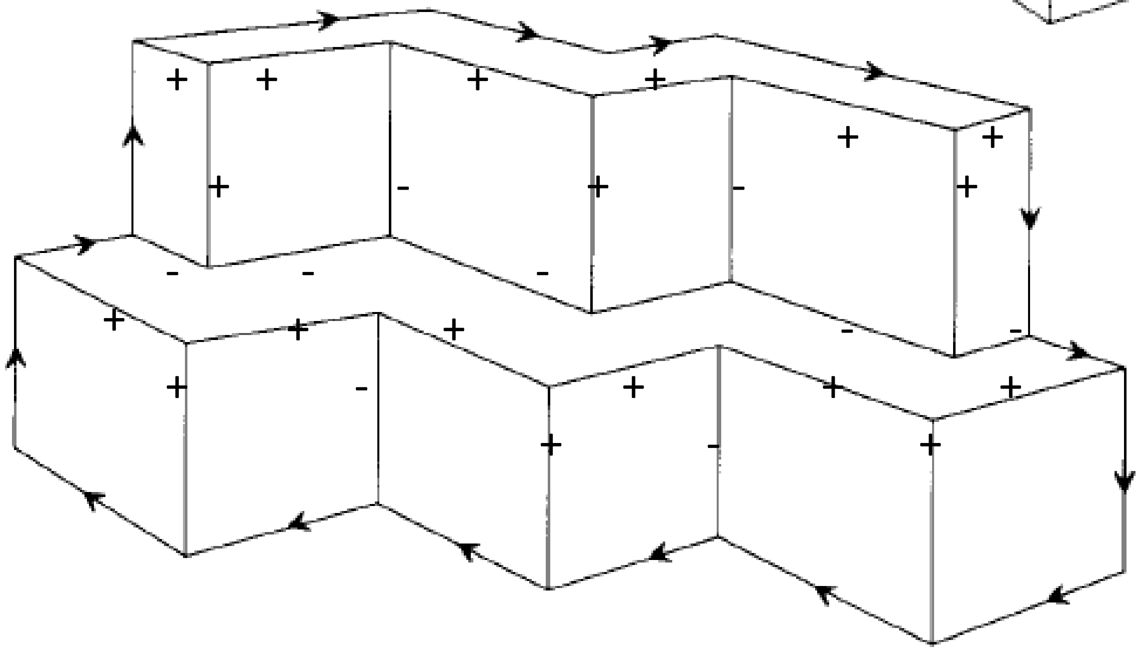
- lines are CSP variables with discrete labels:
  - + = convex
  - - = concave
  - ->- = boundary (between foreground and background; right-hand rule)

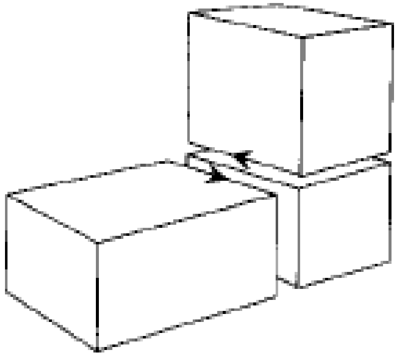


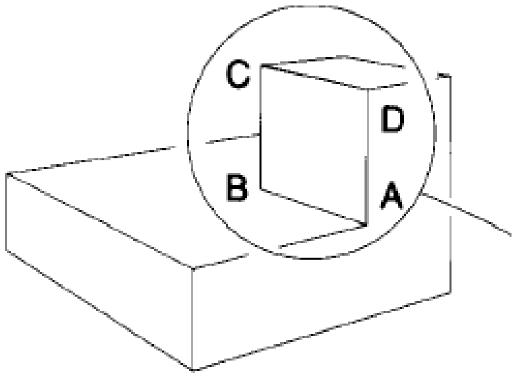


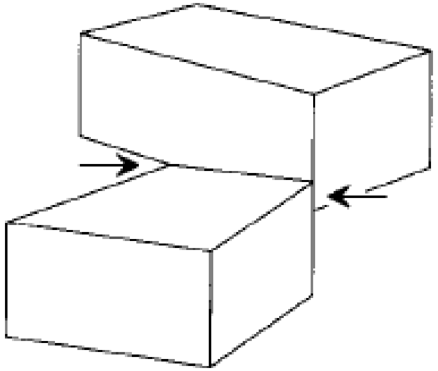
- junctions acts as constraints; converging lines must be labeled consistently:

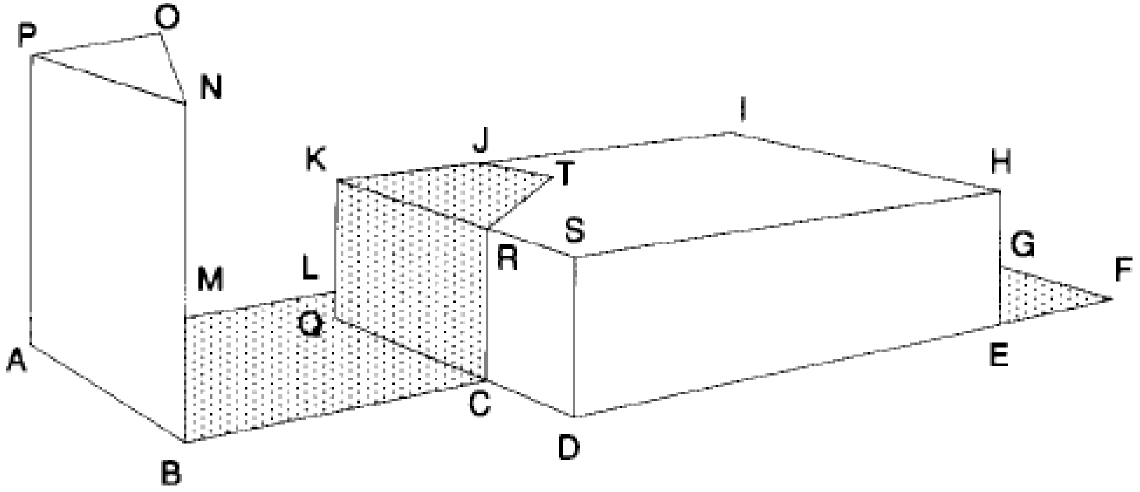












- The Waltz Propagation algorithm is a predecessor of modern Constraint Propagation, which can label these diagrams and extract 3D objects.
- Shadows, cracks, and coincident boundaries are challenges.