Search Algorithms

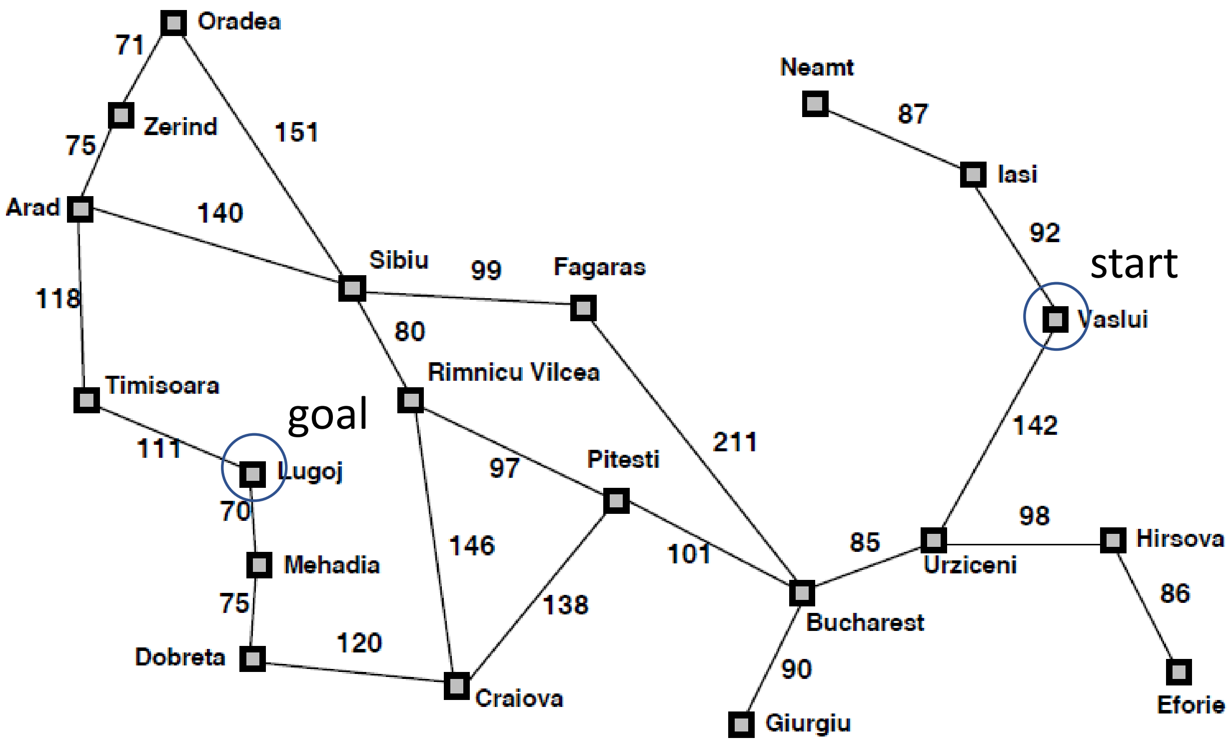
CSCE 420 – Spring 2022

read: Ch. 3

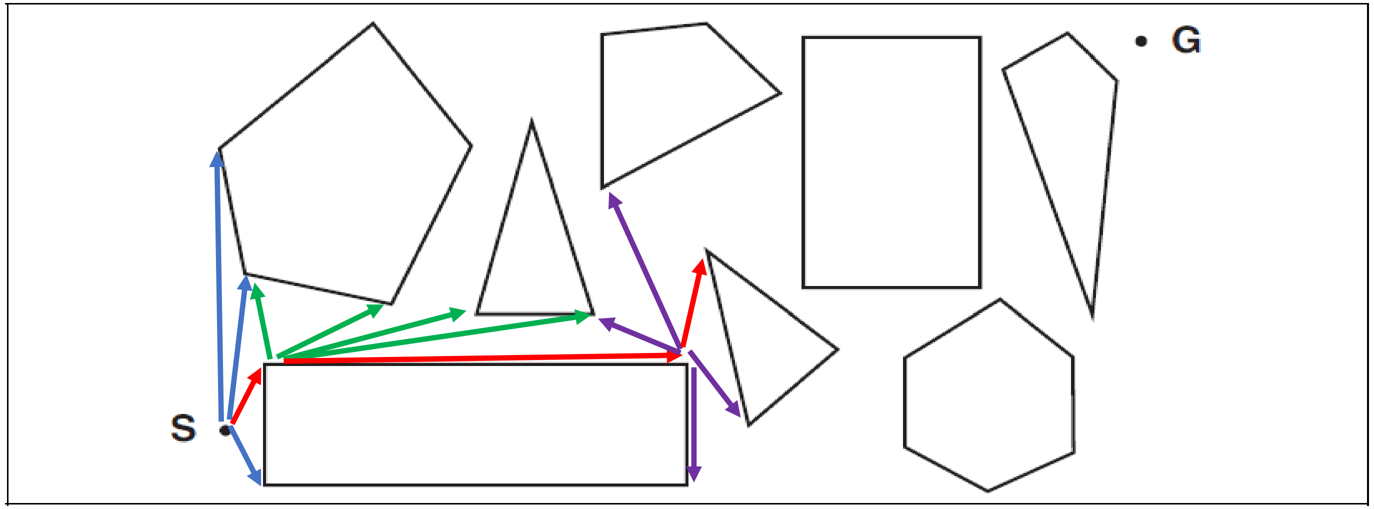
Search as a Model of Problem Solving in AI

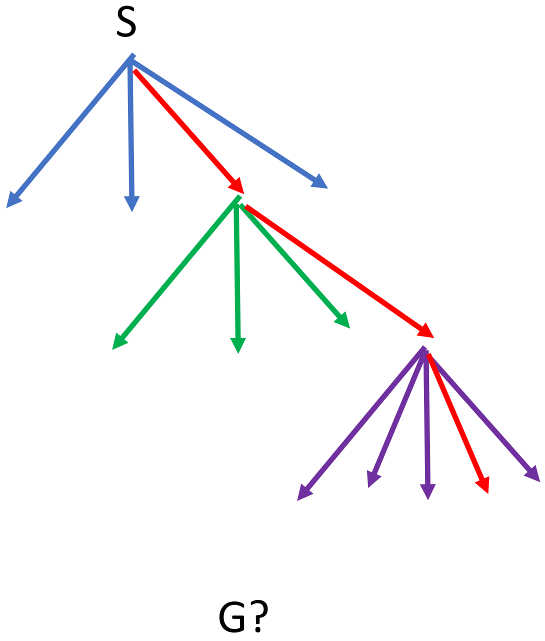
- many AI problems can be formulated as Search
- planning, reasoning, learning...
- define *discrete states* of the world, connected by possible actions
- find a path from the *current state* to a desired *goal state*, producing a *sequence of actions*
- we start by describing generic (uninformed) search algorithms (like DFS)
- then we will extend this to heuristic search algorithms (like A*) which utilize domain knowledge to make the search more efficient

Example: Navigation as Search

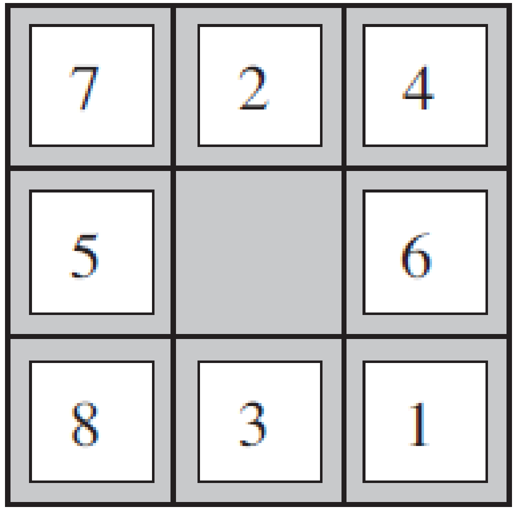


- finding a path from an initial location (start) to a desired destination (goal)
- emphasis on discrete moves (city to city, or corner to corner as way-points

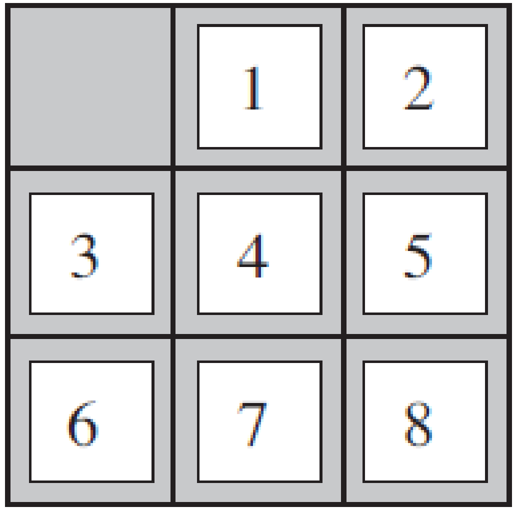




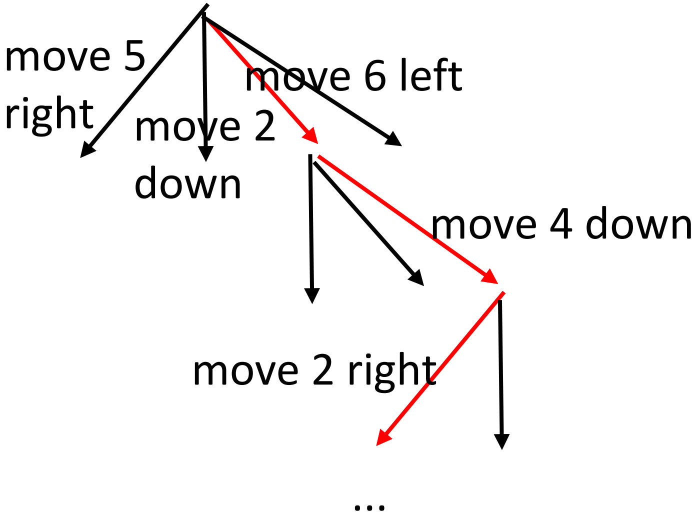
Example: Puzzles as Search



Start State



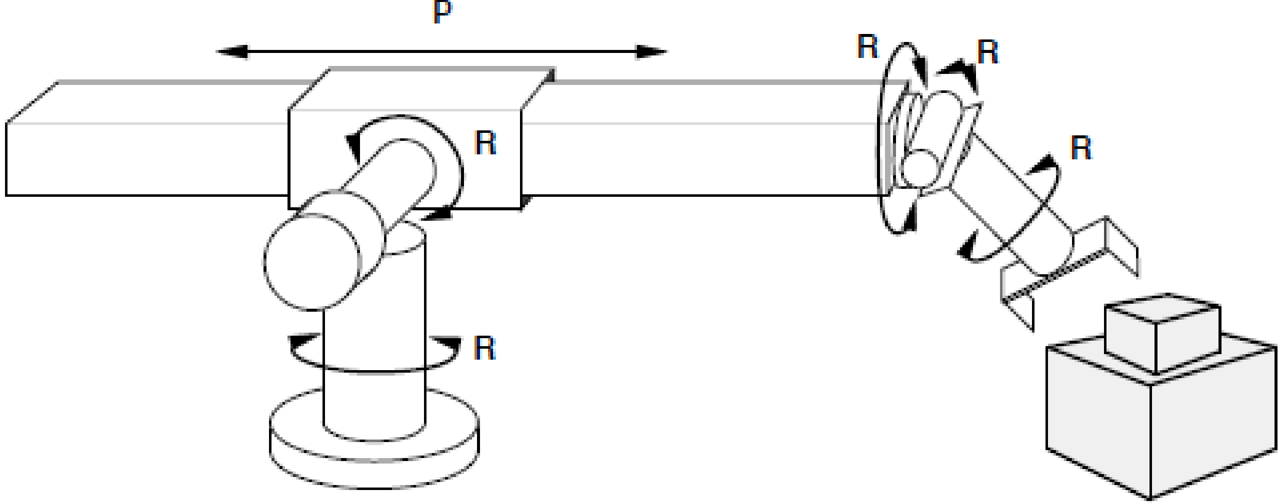
Goal State



actions = slide a tile up/down/left/right into empty space

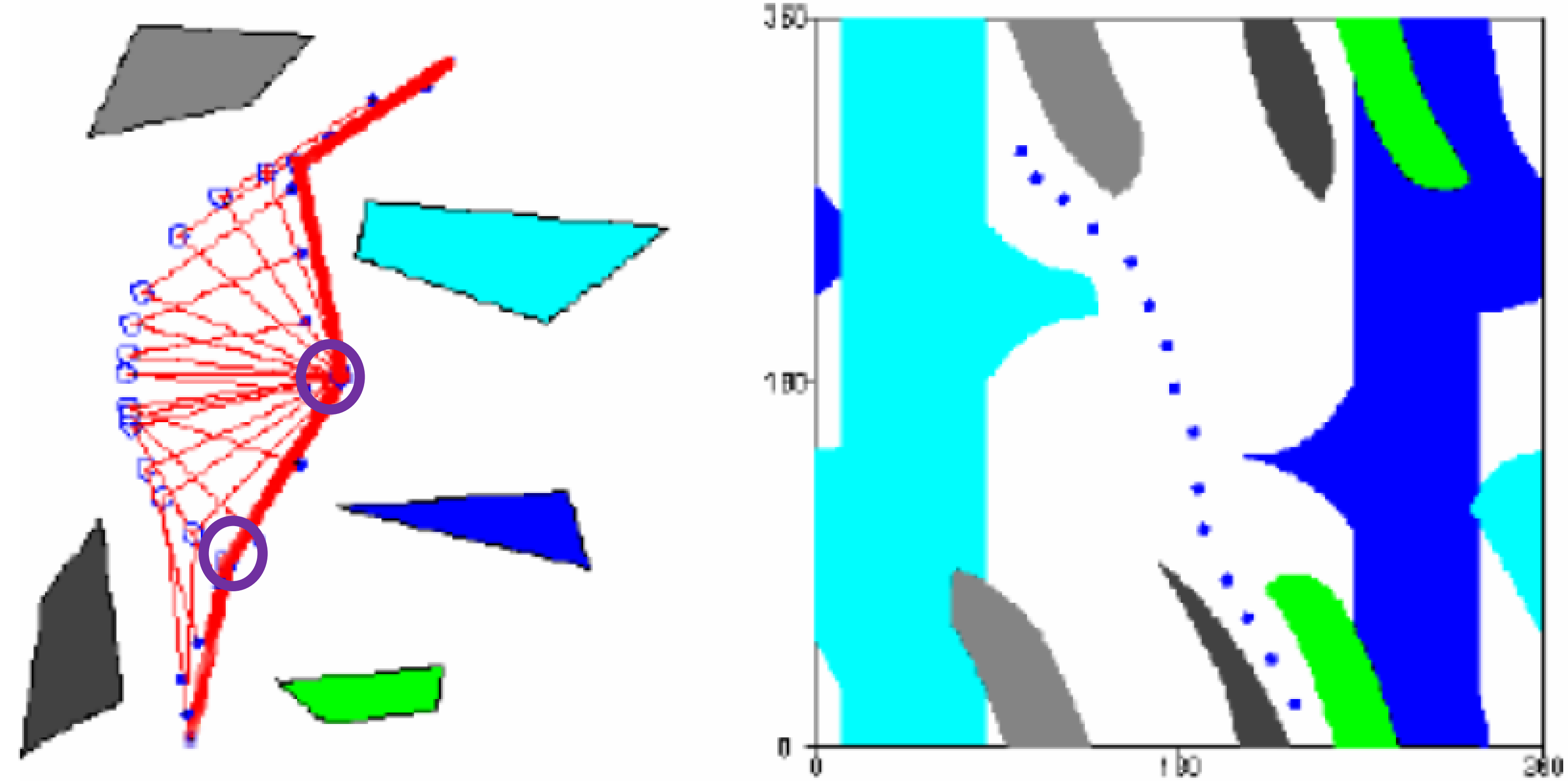
a <u>solution path</u> is <u>sequence of actions</u> that transforms start state into the goal

Example: Robot Motion Planning as Search (in configuration space)



this is the physical space of a 2-armed robot, with 2 joint angles it can adjust

many states are shown here



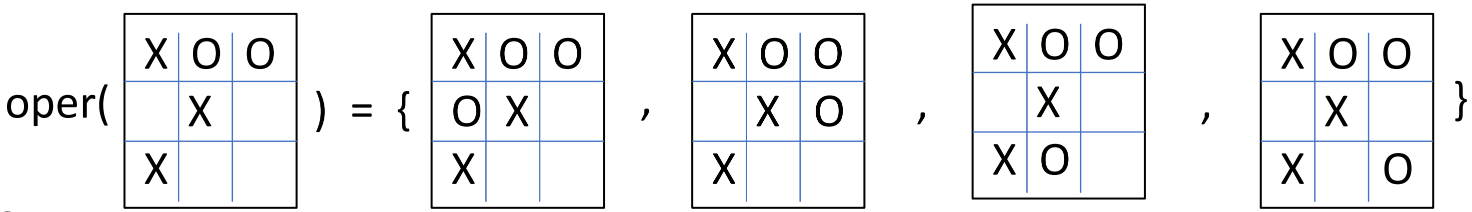
this is the abstract configuration space defined by joint angles that the robot can change

each point represents a distinct state of the robot

some of these images are taken from http://www.cs.cmu.edu/~motionplanning/lecture/Chap3-Config-Space_howie.pdf

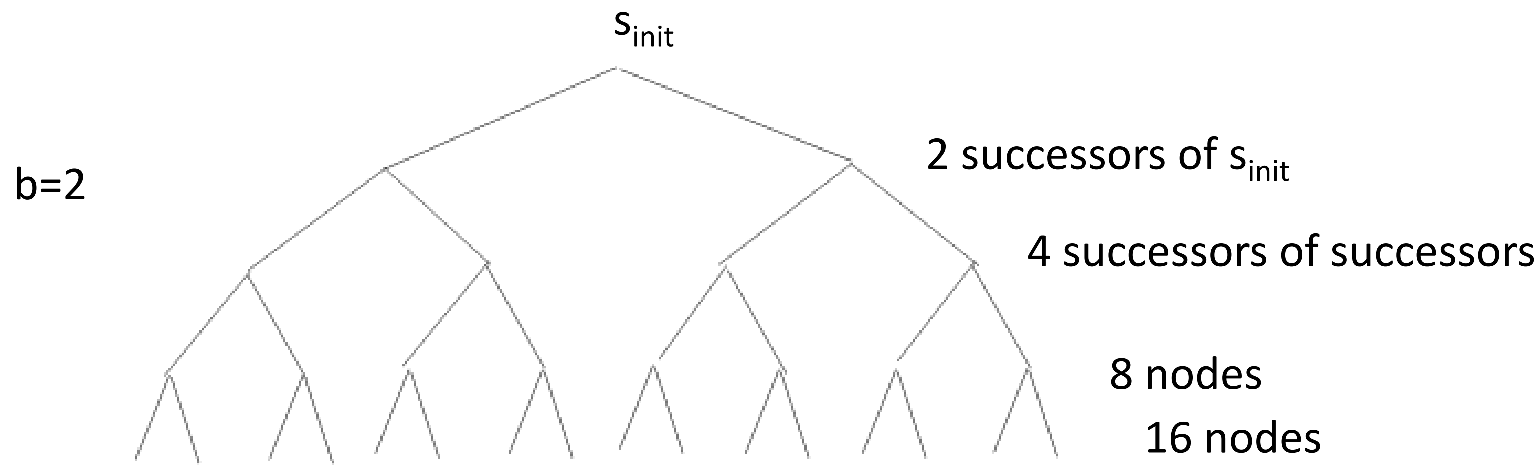
Framework for Formulating Search Problems

- states: a set of discrete representations/configurations of the world
  - this defines the State Space, $S = \{s_1, s_2 ...\}$
  - could be infinite
- *operator*: a function that generates successor states
  - $S \mid\!\rightarrow 2^S$ ... mapping from S to powerset of S, i.e. subset of states
  - $op(s_i) = \{s_j\}$
  - this encodes the legal “moves” or “actions” in the space that transform from one state to another (or possibly multiple successors, or none)
  - example: think about moves in tic-tac-toe



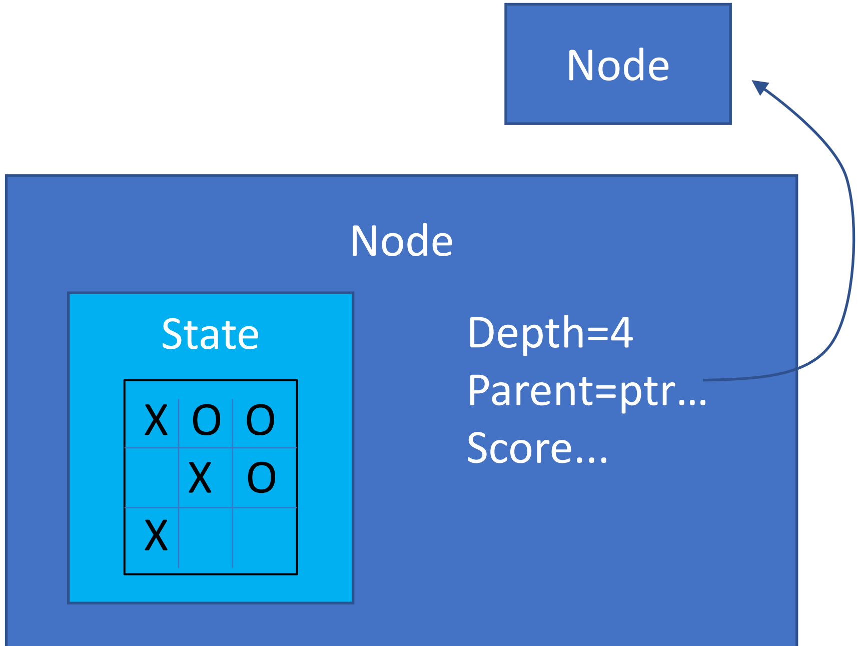
Search Framework

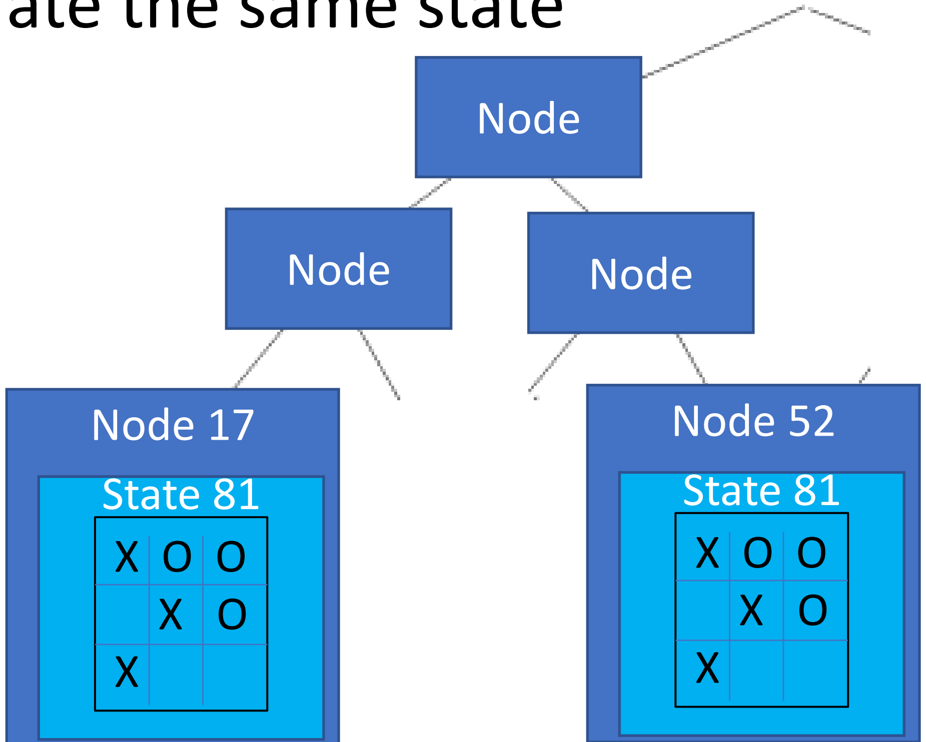
- the operator, applied recursively to the initial state, s_{init}, generates the State Space (or at least, the reachable part)
- visualize it as a tree (the search tree)
- define *b* as the 'branching factor': *average number of successors* for each state
- the size of the tree (nodes on each level) grow exponentially with b



Search Framework

- <u>nodes</u> in the search tree represent <u>states</u> in the state space
- however, they are not quite the same
- a *node* represents a particular path (sequences of actions) to a *state*
- there might be multiple paths that generate the same state





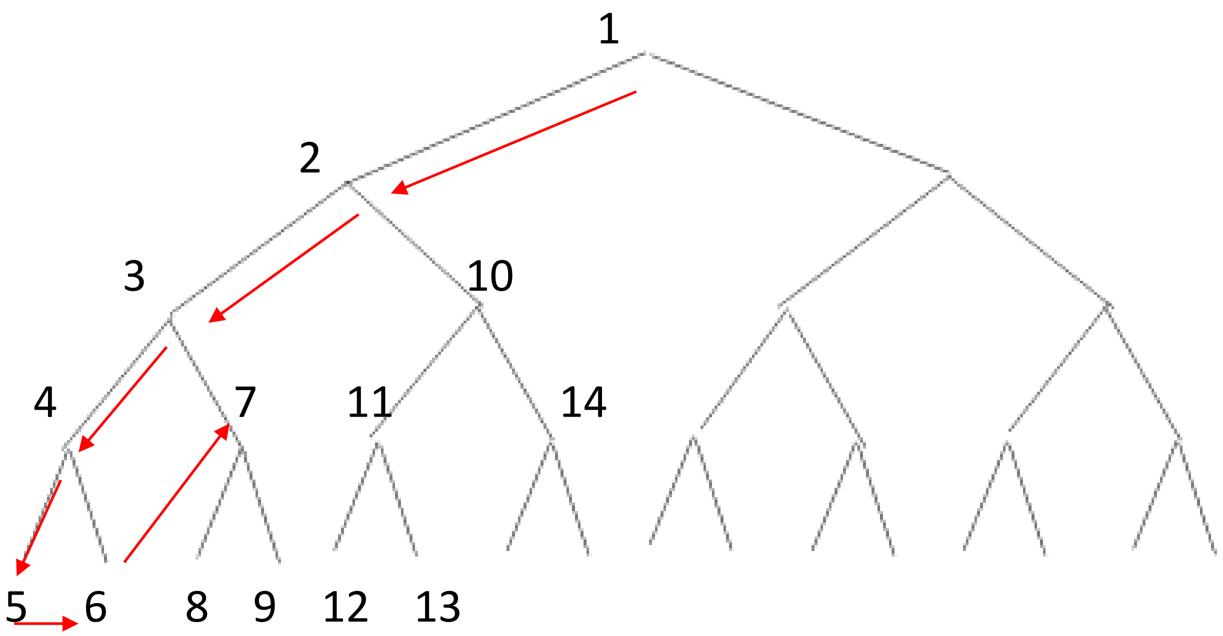
Search Framework

- goals: often specified in a domain-specific way as a set of requirements
  - example: "winning states in tic-tac-toe have 3 X's in a row or column or diagonal"
  - abstractly: we can think of goals as a *subset of states* in the State Space, i.e. $G=\{s_j\} \subset S$
- for many AI problems, we would be happy to find any goal node
  - (doesn't matter which one)
  - we are interested in the path, which is the sequence of actions that transforms the initial state s_{init} into the goal s_{goal}
- in some cases, we might prefer the shortest path (fewest actions required)
- in other cases, if each operator has a different cost, we might be interested in finding the solution with the least path cost
- example: deciding to take a bus instead of a cab as part of a trip in order to minimize cost

$$\text{cost}(s_1..s_n) = \sum_{i=1..n} c(op_i)$$ where s_1=init, s_n=goal, and $s_{i+1} \in op(s_i)$

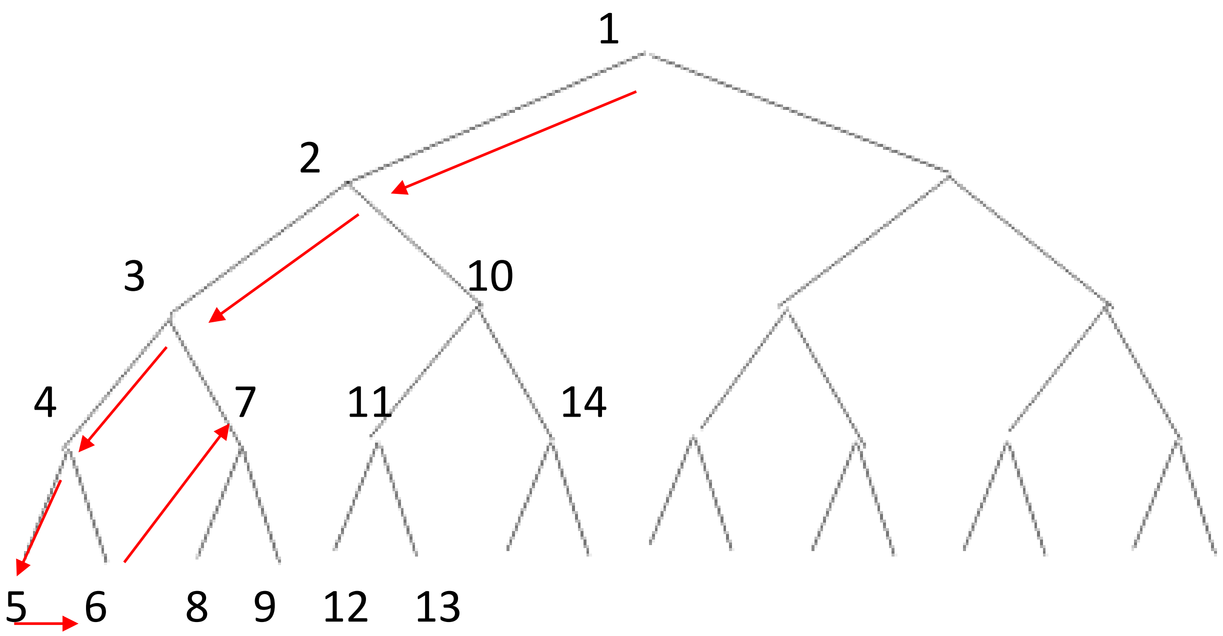
Uninformed Search (‘Weak’ Methods)

- Depth-first Search (DFS) – expand children of children before siblings

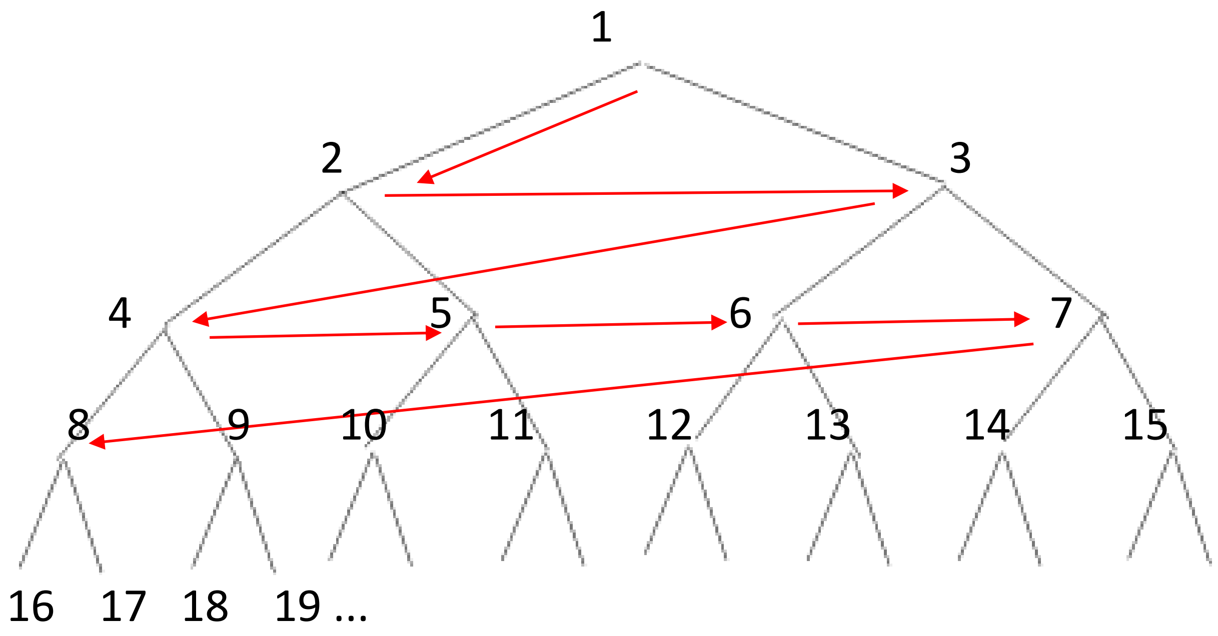


Uninformed Search (‘Weak’ Methods)

- Depth-first Search (DFS) – expand children of children before siblings

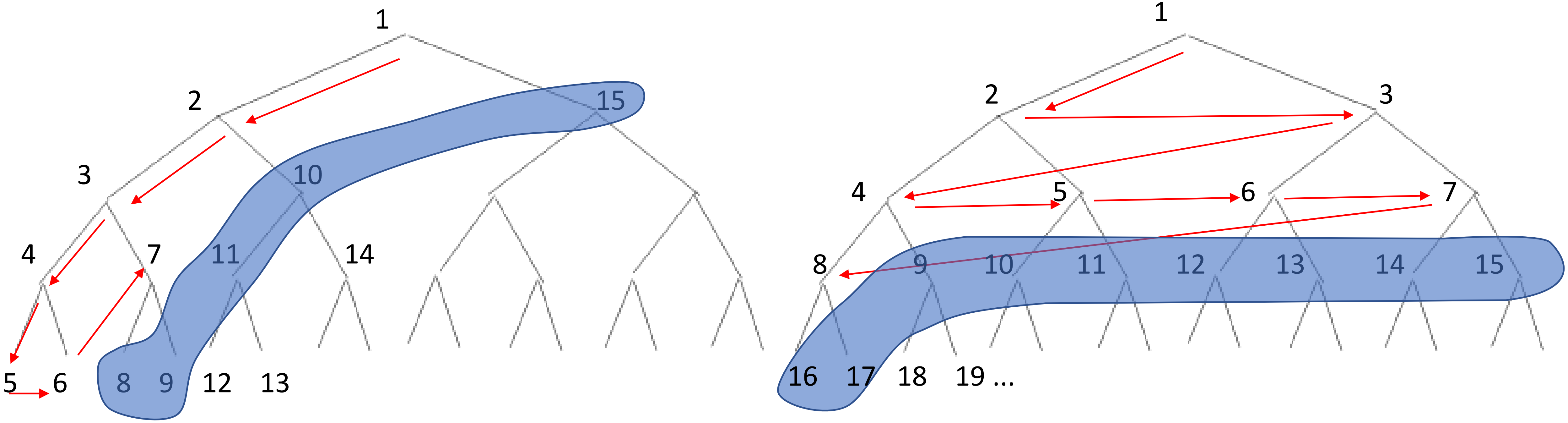


- Breadth-first Search (BFS) – expand children of children AFTER siblings



Uninformed Search (‘Weak’ Methods)

- the ‘frontier’ or ‘agenda’ is the set of nodes that have been expanded but not yet explored, where expanded means it is a child of a visited node and explored means goal-tested

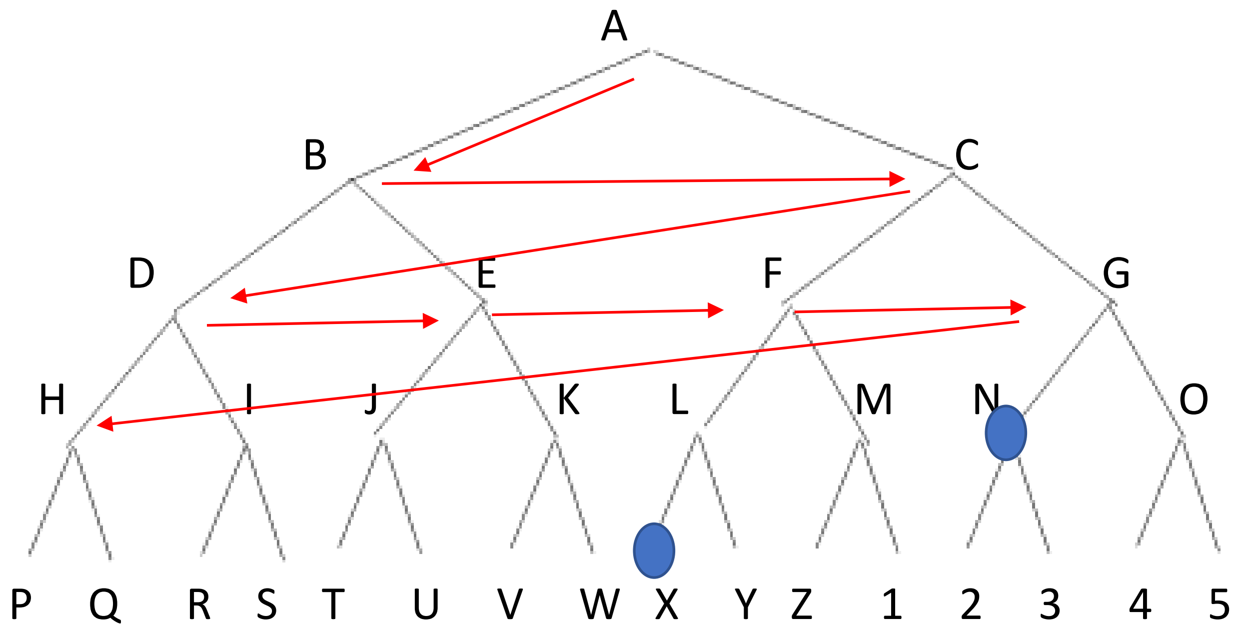


A Unified Search Algorithm

- although it is easy to write pseudo-code for DFS and BFS separately, they can be unified in an iterative procedure using a data structure to hold the nodes in the frontier
- BFS: frontier = queue (FIFO)
- DFS: frontier = stack (LIFO)

```
function BREADTH-FIRST-SEARCH(problem) returns a solution node or failure
  node ← NODE(problem.INITIAL)
  if problem.IS-GOAL(node.STATE) then return node
  frontier ← a FIFO queue, with node as an element
  reached ← {problem.INITIAL}
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if problem.IS-GOAL(s) then return child
      if s is not in reached then
        add s to reached
        add child to frontier
  return failure
```

(ignore reached for now;
It is for GraphSearch,
see slides below)



- frontier (queue) for BFS:
  - A // [front | A | end]
  - B C // pop A, push children on end
  - // pop B from front
  - // push children D E on end
  - C D E ←
  - D E F G
  - E F G H I // start adding next level
  - F G H I J K
  - G H I J K L M
  - ...

```
function BREADTH-FIRST-SEARCH(problem) returns a solution node or failure
  node ← NODE(problem.INITIAL)
  if problem.IS-GOAL(node.STATE) then return node
  frontier ← a FIFO queue, with node as an element
  reached ← {problem.INITIAL}
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if problem.IS-GOAL(s) then return child
      if s is not in reached then
        add s to reached
        add child to frontier
  return failure
```

to change it to do DFS, all you have to do is replace the frontier with a stack (LIFO):

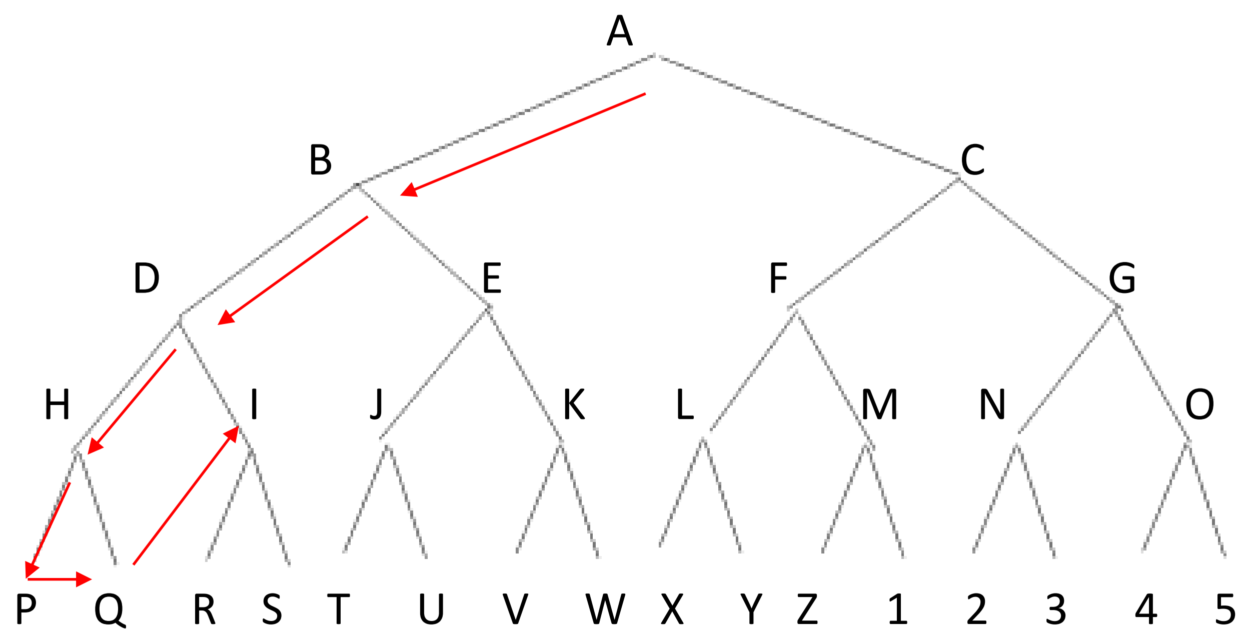
frontier ← **stack,** initialized with start node as first element

function Depth-First Search (*problem*) **returns** a solution node or *failure*

```
node ← NODE(problem.INITIAL)
if problem.IS-GOAL(node.STATE) then return node
frontier ← a LIFO queue, with node as an element
reached ← {problem.INITIAL}
while not IS-EMPTY(frontier) do
  node ← POP(frontier)
  for each child in EXPAND(problem, node) do
    s ← child.STATE
    if problem.IS-GOAL(s) then return child
    if s is not in reached then
      add s to reached
      add child to frontier
return failure
```

to change it to do DFS, all you have to do is replace the frontier with a stack (LIFO): i.e.

frontier ← **stack,** initialized with start node as first element



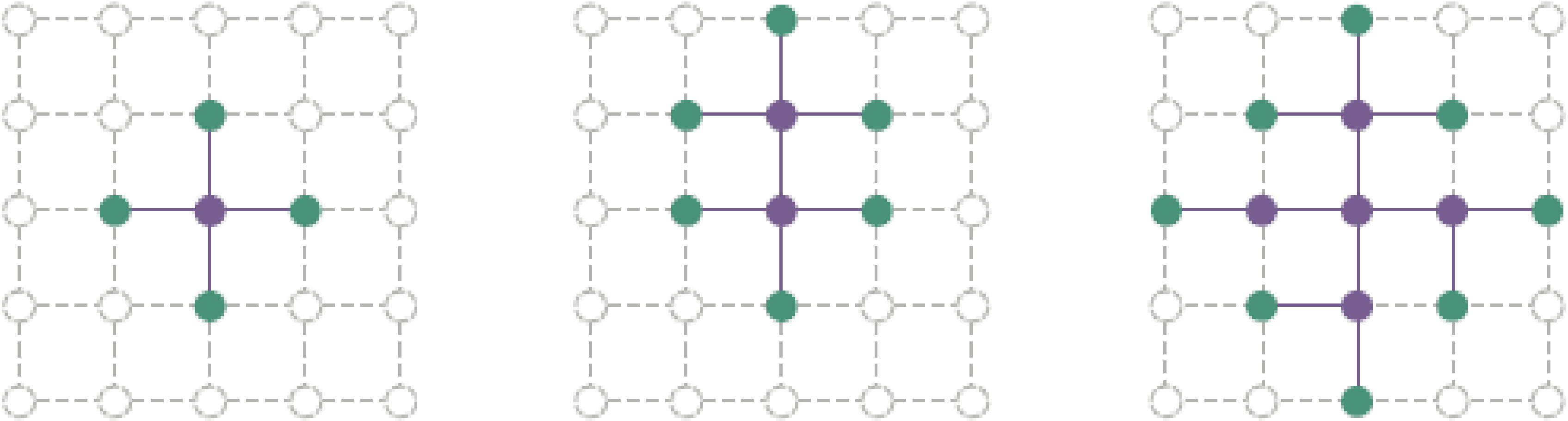
- frontier (stack) for DFS:
  - A
  - // pop A, push children B and C
  - B C
  - // pop B, push D and E on front
  - D E C
  - H I E C // pop D, push H and I
  - P Q I E C // pop H, push P and Q
  - Q I E C // pop P
  - I E C // pop Q
  - R S E C // go to I, push R and S
  - ....

note: when you expand a node, the order in which you push the children makes a difference
In this example, I am pushing the children in *reverse* order, e.g. C before B (as children of A)
what would the search order look like if we pushed the children in alphabetical order?

Graph Search

- in some Search Trees, there are multiple paths to the same state
- example: reversible operators (move, then move back); or think of a map; or think of circular moves in the tile puzzle
- detecting repeated (visited) states can greatly reduce redundancy in the search space
  - if you have already explored children beneath node n, there is no need to do it again
- exception: if you find a shorter/cheaper path to n, you might want to keep track of the best such path found
- ‘reached’: you need a data structure (like a hash table) to keep track of these states

- Graph Search
  - in BFS on a grid, how badly would the size of the search tree scale up if we didn't keep track of reached states?
  - Assume each node has 4 neighbors, so b=4 (branching factor, wrst cs) (b_{avg}=~3)
  - level 0=1 node (initial state, at the center)
  - level 1=4 nodes
  - level 2=16 nodes
  - level 3=64 nodes
  - level 4=256 nodes
  - ...
  - level i: 4^i nodes



  - and yet, there are only 25 distinct states in this space!

Graph Search (=BFS+checking for visited states)

reached is a data structure (e.g. hash table) for keeping track of expanded states to avoid repeating the search

note: that we check reached *before* putting nodes into the frontier, not as we pull them out

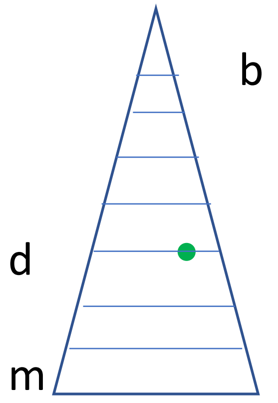
```
function BREADTH-FIRST-SEARCH(problem) returns a solution node or failure
  node ← NODE(problem.INITIAL)
  if problem.IS-GOAL(node.STATE) then return node
  frontier ← a FIFO queue, with node as an element
  reached ← {problem.INITIAL}
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if problem.IS-GOAL(s) then return child
      if s is not in reached then
        add s to reached
        add child to frontier
  return failure
```

*If s *has* been reached before,*
you might want to see if a shorter/cheaper path has been discovered and keep track of that…

Computational Complexity

- analysis of computational properties for comparison of DFS and BFS
- **time-complexity**: number of nodes goal-tested (# of loop iterations)
- **space-complexity**: maximum size to which the frontier grows
- **completeness**: if a goal exists, does ALGO guarantee to find it?
- **optimality**: does ALGO guarantee to find the goal node with the minimum path cost?

Computational Complexity of BFS



- time-complexity: number of nodes goal-tested (# of loop iterations)
  - if the shallowest node occurs at depth d, and branching factor is b,
  - then nodes checked (worst case) will be all levels up to and including b
  - $1+b+b^2+....b^d$ = $O(b^{d+1})$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

$$\sum_{i=0}^{n} b^i = \left(\frac{b^{n+1} - 1}{b - 1}\right)$$

- space-complexity: maximum size to which the frontier grows
  - in worst case, have to store all children at level below goal, $O(b^{d+1})$
- completeness: if a goal exists, does ALGO guarantee to find it?
  - yes (because every goal exists at a finite depth, and BFS explores each level)
- optimality: does ALGO guarantee to find the goal node with the minimum path cost?
  - yes (assuming all operator have equal cost) (but no in general)
  - in this case, the goal with least path cost is shallowest, and BFS will find it first, because it explores level-by-level)

Computational Complexity of DFS

- time-complexity: number of nodes goal-tested (# of loop iterations)
  - if the maximum depth of the tree is m,
  - the worst case is when goal at depth d is on the right-most branch
  - the nodes checked will be almost all in the tree (even deeper than d): $O(b^m)$
- space-complexity: maximum size to which the frontier grows
  - each time we expand a node, we pop 1 and push b children, (b-1)m = O(bm)
- completeness: if a goal exists, does ALGO guarantee to find it?
  - no, in general (i.e. if any branch has infinite depth)
  - yes, only in finite search spaces
- optimality: does ALGO guarantee to find the goal node with the minimum path cost?
  - no (since it is not complete)

Comparison of BFS and DFS

- so which is better? when would we prefer to use one over the other?
- although time-complexity could be exponentially worse for DFS ($O(b^m) >> O(b^d)$), DFS has linear space-complexity
- in practice, the size of the frontier is what limits AI search
- given modern CPU clock cycles, I can easily search a billion (10^9) nodes (10 μs per loop iteration=17 min), but storing a billion nodes takes too much memory (~100 bytes per node=100 Gb)

	BFS	DFS
time-complexity	$O(b^{d+1})$	$O(b^m)$
space-complexity	$O(b^{d+1})$	$O(bm)$

Iterative Deepening

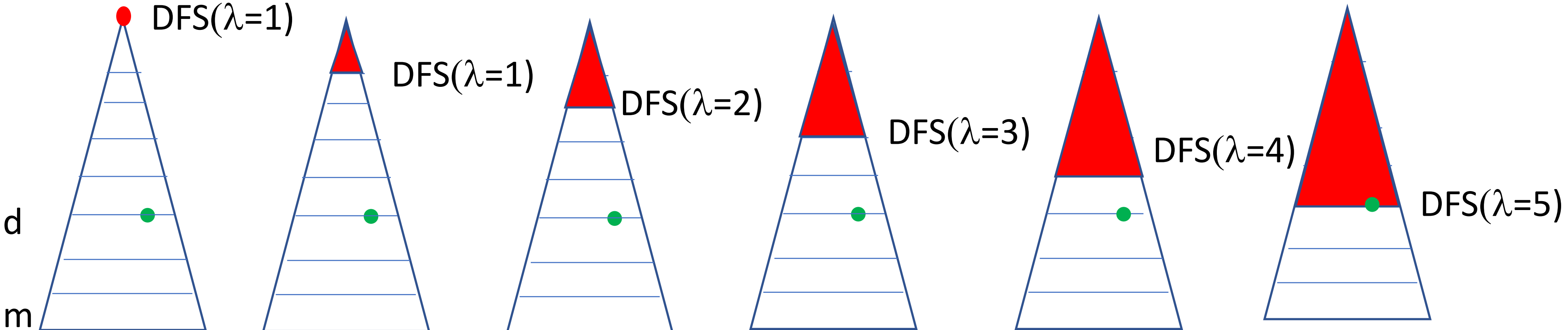
- Is there a way to get the benefits of both BFS and DFS?
- how can we maintain a *linear* frontier size like DFS while still searching level-by-level like BFS?
- how can you maintain the linear space-complexity of DFS while avoiding descending infinitely deep down any single branch?
- answer: depth-limited search
  - do DFS down to depth=1
  - if goal not found, do DFS down to depth=2
  - if goal not found, do DFS down to depth=3
  - ...

```
function ITERATIVE-DEEPENING-SEARCH(problem) returns a solution node or failure
  for depth = 0 to ∞ do
    result ← DEPTH-LIMITED-SEARCH(problem, depth)
    if result ≠ cutoff then return result

function DEPTH-LIMITED-SEARCH(problem, ℓ) returns a node or failure or cutoff
  frontier ← a LIFO queue (stack) with NODE(problem.INITIAL) as an element
  result ← failure
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    if DEPTH(node) > ℓ then
      result ← cutoff
    else if not IS-CYCLE(node) do
      for each child in EXPAND(problem, node) do
        add child to frontier
  return result
```

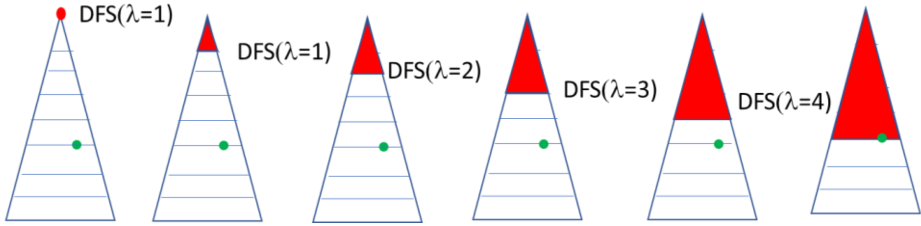
Iterative Deepening

- Complexity analysis:
- since using DFS, the frontier should never get bigger than (b-1)d, hence O(bd)
- and it should be complete and optimal (for equal operator costs)
- what about time complexity?
  - it seems wasteful because you have to re-generate the top part of the search tree each iteration



Iterative Deepening

- time complexity?
  - $1+(1+b)+(1+b+b^2)+(1+b+b^2+b^3)+...+(1+b+...+b^d)$
  - $\leq (1+b+...+b^d)+(1+b+...+b^d)+...\ (1+b+...+b^d)$
  - $\leq d(1+b+...+b^d) \leq d\Sigma b^i \leq d(b^{d+1}-1)/(b-1) = O(b^{d+1})$



- it seems wasteful because you have to re-generate the top part of the search tree each iteration
- why not just "save" the part of the tree generated so far?
- because it will grow exponentially as depth limit increases, negating the benefit of the linear size of the frontier – you have to throw them away
- so it is a <u>tradeoff</u>: you spend a little more time computing (expanding nodes), but you save memory (linear frontier size)

Uniform Cost Algorithm

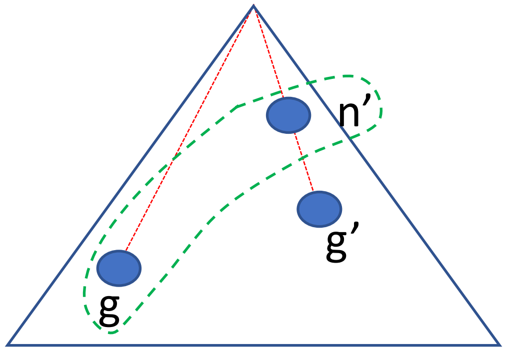
- suppose we want to find the goal node with the least path cost, when operators have different costs?
- the shortest path (number of actions) is not necessarily the cheapest path (sum of operator costs)
- in this case, BFS is not optimal
- however, we can use the same iterative search algorithm, but change the frontier to a *priority queue*
- keep the expanded-but-unexplored nodes sorted in order of increasing path cost
- nodes must keep track of cost; update when generating successors:
  - cost(child) = cost(parent)+cost(op_i)

Uniform Cost Algorithm

```
function BEST-FIRST-SEARCH(problem, f) returns a solution node or failure
  node ← NODE(STATE=problem.INITIAL)
  frontier ← a priority queue ordered by f, with node as an element
  reached ← a lookup table, with one entry with key problem.INITIAL and value node
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
        reached[s] ← child
        add child to frontier
  return failure

function EXPAND(problem, node) yields nodes
  s ← node.STATE
  for each action in problem.ACTIONS(s) do
    s′ ← problem.RESULT(s, action)
    cost ← node.PATH-COST + problem.ACTION-COST(s, action, s′)
    yield NODE(STATE=s′, PARENT=node, ACTION=action, PATH-COST=cost)
```

Uniform Cost Algorithm



- assumption: all operators have positive costs: $cost(op_i)>0 \geq \varepsilon > 0$
- Lemma: UC explores nodes in order of increasing total path cost
  - sure, every node you pull out of the PQ has costs less than all other in the PQ
  - but when you reach a goal g, how do you know there is not another cheaper goal g' out there?
  - a) there is always some node n on the path to each node that is in the PQ (even it is the initial state/root node)
  - b) the nodes along each path always increase in cost (since all ops have pos cost)
  - c) if n' was on path to g' and pathcost(g')<pathcost(g) (by assumption), then pathcost(n')<pathcost(g') and n' would have been pulled out of PQ before g

// from https://en.wikipedia.org/wiki/Dijkstra’s_algorithm

Uniform Cost Algorithm

- comparison to Djikstra’s Algorithm
  - UC and Djikstra both solve the single-source shortest-path problem
  - however, an important difference is that Djikstra is based on Dynamic Programming (DP)
  - it uses a data structure (array) to maintain partial path distances from the source to all vertices V in the graph
  - you can’t do this for most AI problems, especially if they have exponentially large or infinite State Spaces

```
function Dijkstra(Graph, source):
    create vertex set Q

    for each vertex v in Graph:
        dist[v] ← INFINITY
        prev[v] ← UNDEFINED
        add v to Q
    dist[source] ← 0

    while Q is not empty:
        u ← vertex in Q with min dist[u]
        remove u from Q

        for each neighbor v of u:
            alt ← dist[u] + length(u, v)
            if alt < dist[v]:
                dist[v] ← alt
                prev[v] ← u

    return dist[], prev[]
```

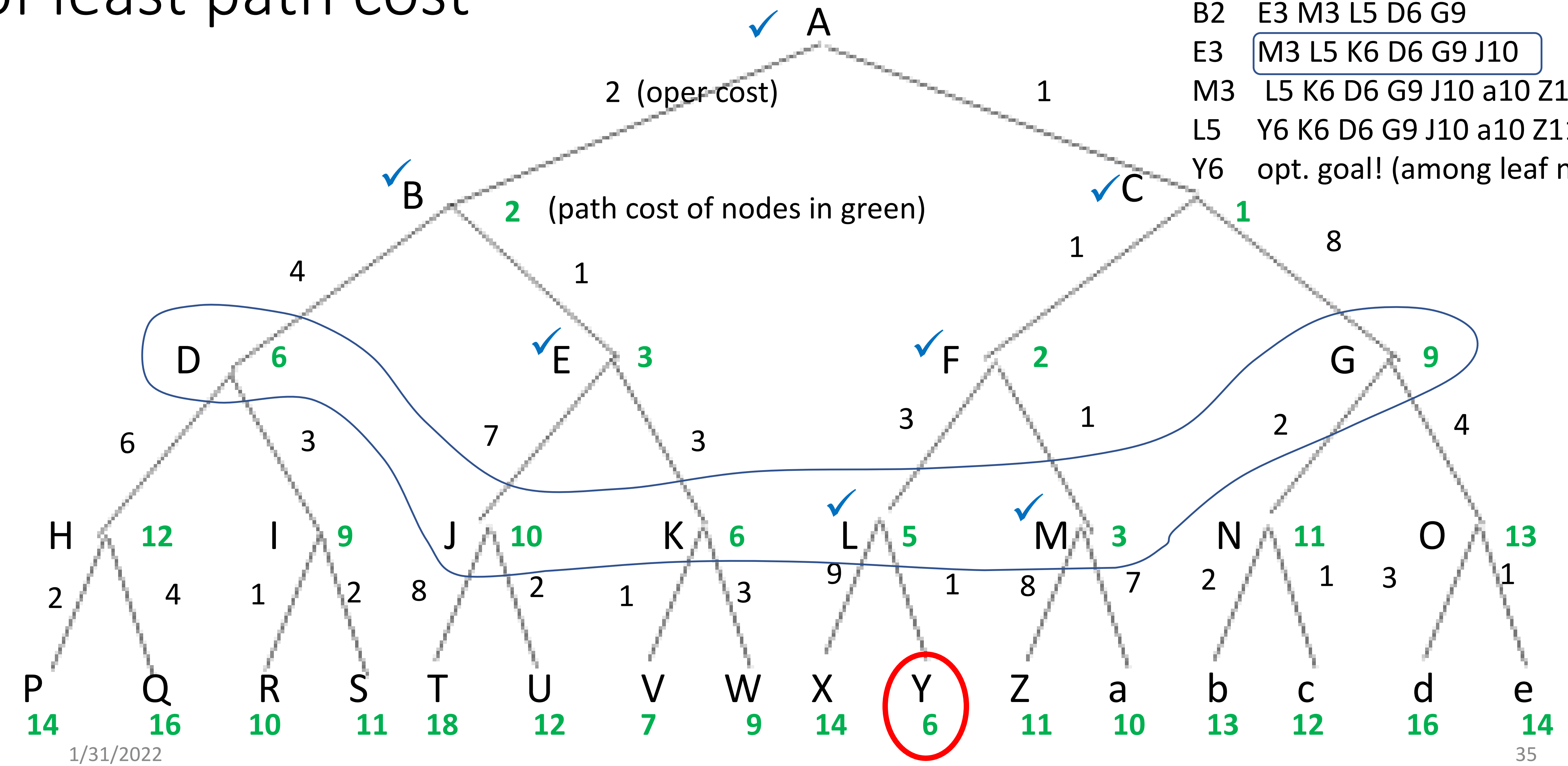
Uniform Cost Algorithm

- Computational properties of UC
  - time-complexity: $O(b^{(1+C^*/\varepsilon)})$
  - where C* is the total path cost of the cheapest solution
  - why? because each step costs at least ε, so goal occurs at depth C^*/ε in the worst case
  - space-complexity: $O(b^{(1+C^*/\varepsilon)})$
  - completeness: yes
  - optimality: yes!

Trace of UC - visit nodes in order of least path cost

at	queue
A0	C1 B2
C1	F2 B2 G9
F2	B2 M3 L5 G9
B2	E3 M3 L5 D6 G9
E3	M3 L5 K6 D6 G9 J10
M3	L5 K6 D6 G9 J10 a10 Z11
L5	Y6 K6 D6 G9 J10 a10 Z11 X14
Y6	opt. goal! (among leaf nodes)

suffixes are path costs



Summary of Computational Properties of Search Algorithms

read for yourself

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening	Bidirectional (if applicable)
Complete?	Yes[1]	Yes[1,2]	No	No	Yes[1]	Yes[1,4]
Optimal cost?	Yes[3]	Yes	No	No	Yes[3]	Yes[3,4]
Time	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(b^m)$	$O(b^\ell)$	$O(b^d)$	$O(b^{d/2})$
Space	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(bm)$	$O(b\ell)$	$O(bd)$	$O(b^{d/2})$

or $O(b^{d+1})$

if $cost(op_i)$=constant for all operators

except for finite search spaces

or $O(b^{d+1})$

Heuristic Search

- since AI search problems usually have exponential search spaces, the main focus is on how we can exploit *domain knowledge* to improve the efficiency of the search
- *domain knowledge* refers to anything we know about solving these types of problems
  - rules of thumb, common solutions, way to decompose the problem into subgoals, useful sequences of actions, interactions/dependencies between operators...
- in this context, domain knowledge will be encapsulated in a *heuristic function*, h(n)
- it is a 'scoring' function that maps every node (or state) to a real number
- the advantage is using any knowledge we have to *guide* the search toward the goal, and avoid searching 'unproductive' parts the search space

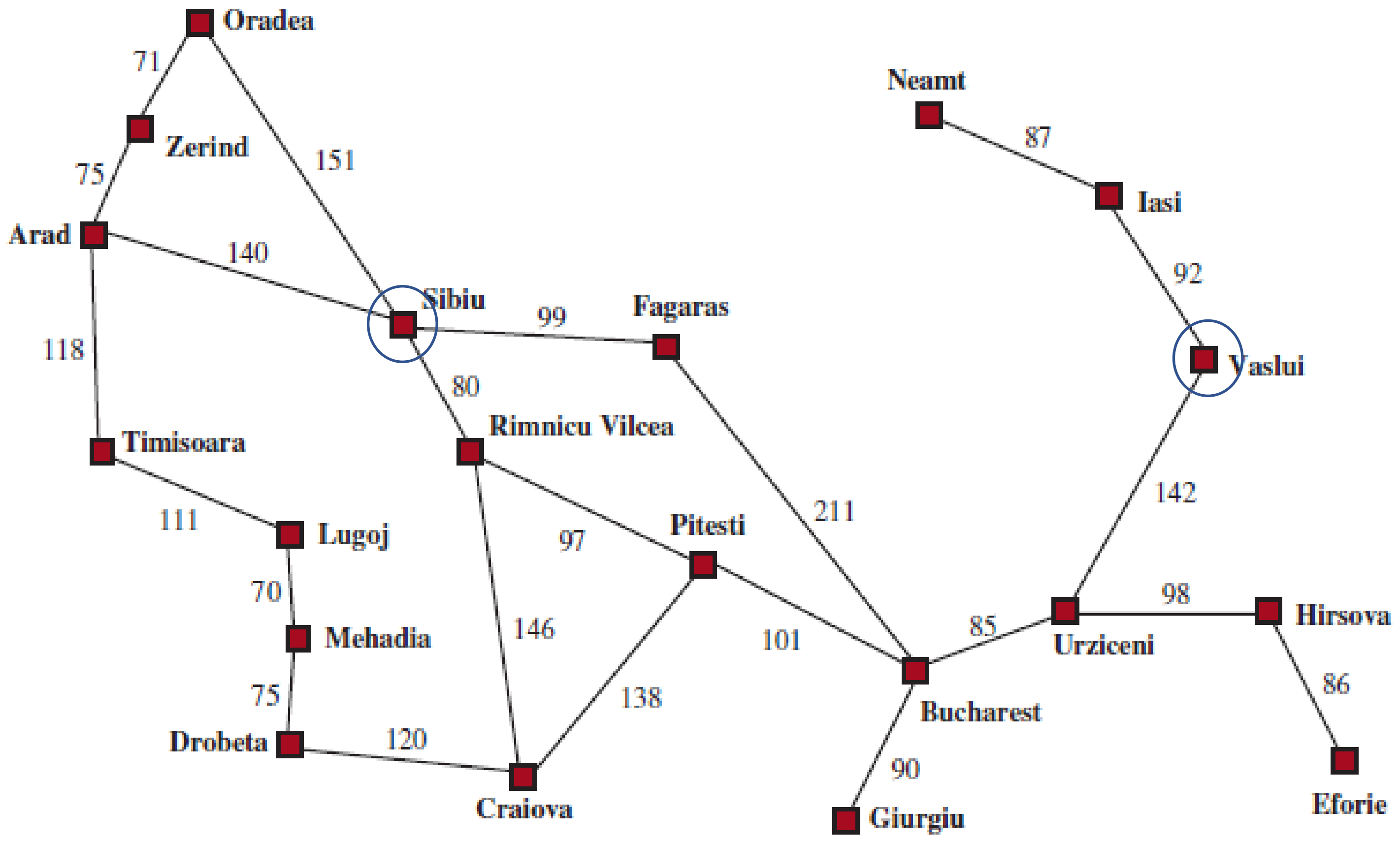
Heuristic Search

- a heuristic function h(n) is an *estimate of the distance (path cost) remaining from n to the closest goal*
- hence it is a mapping from S ⊢> R (State Space to real numbers)
- generally, h(n)≥0, and h(n)=0 for goals
- abstractly, it is a quantification of how close a state is to being solved (higher is farther away)

Heuristic Functions

- Example 1: h_{SLD} for navigation

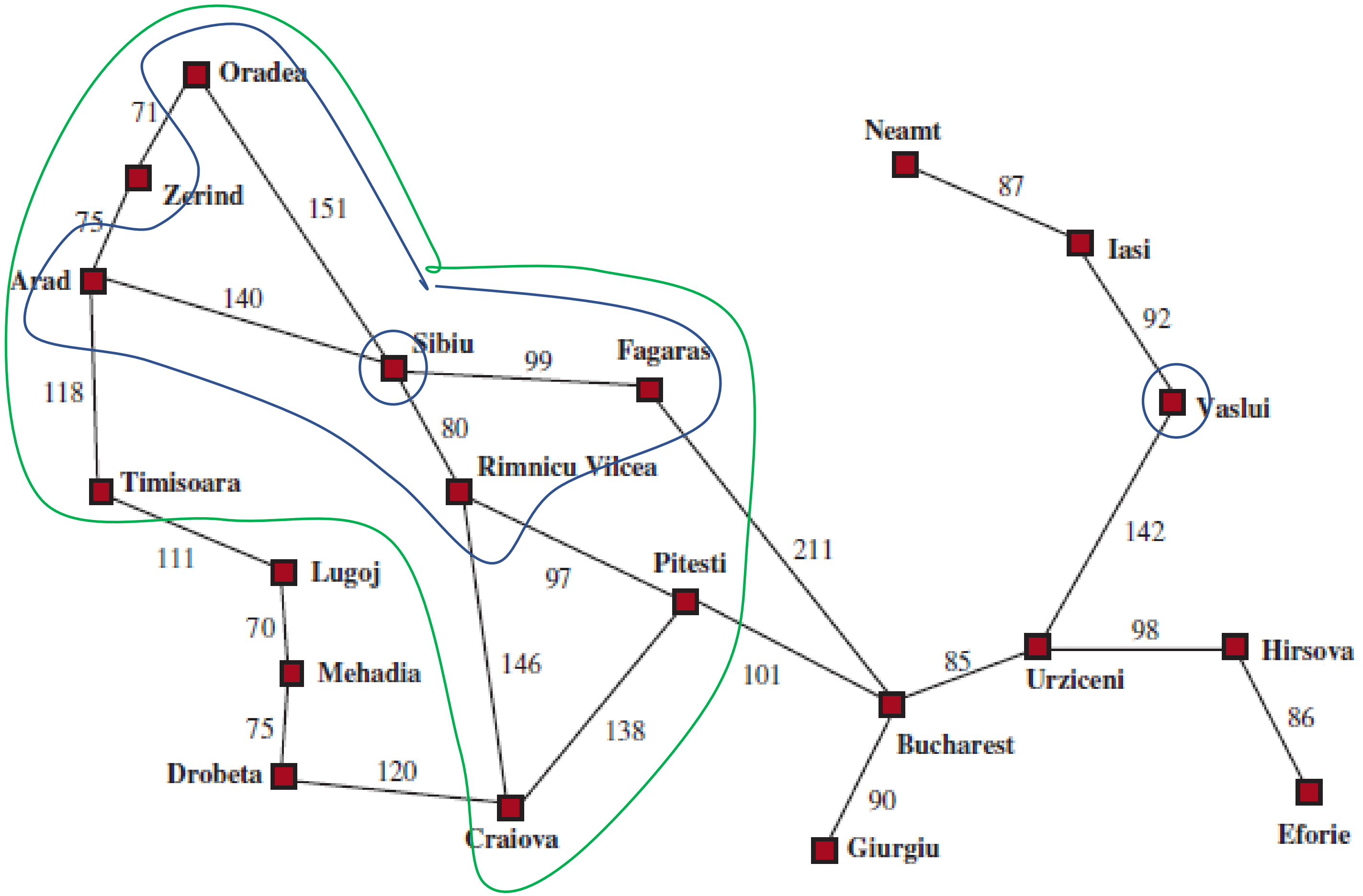
- suppose our goal was to find a route from Sibiu to Vasliu
- compare DFS vs BFS (assuming children are processed in counter-clockwise order)



Heuristic Functions

- Example 1: h_{SLD} for navigation

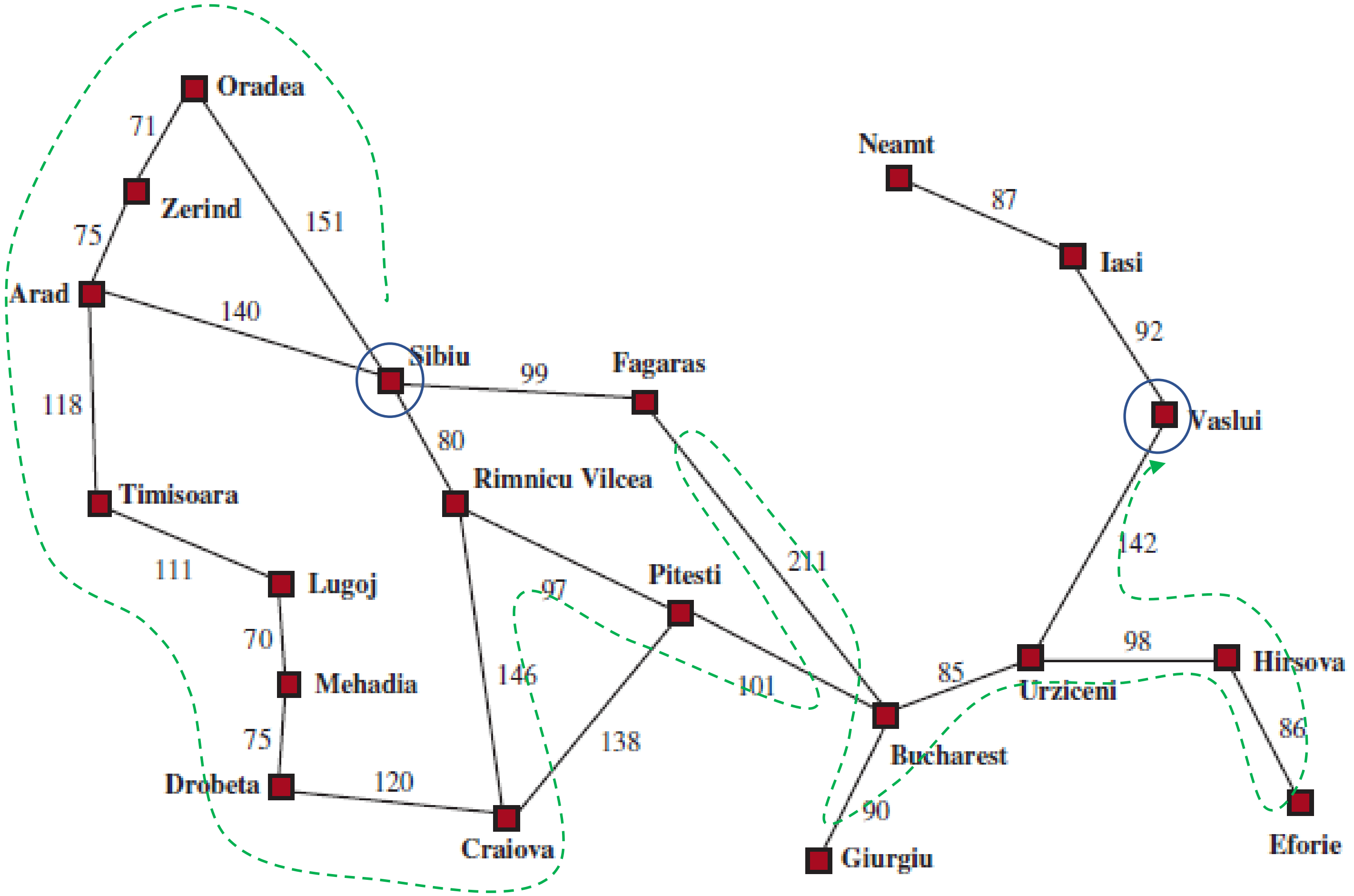
- suppose our goal was to find a route from Sibiu to Vasliu
- compare DFS vs BFS (assuming children are processed in counter-clockwise order)
- **BFS** (FIFO): (expand in levels)
  - frontier at each pass:
  - S | O,A,R,F | Z,T,C,P,B | L,D,U,G | M,H,V



Heuristic Functions

- Example 1: h_{SLD} for navigation

- suppose our goal was to find a route from Sibiu to Vasliu
- compare DFS vs BFS (assuming children are processed in counter-clockwise order)
- BFS (FIFO):
  - frontier at each pass:
  - S | O,A,R,F | Z,T,C,P,B | L,D,U,G | M,H,V
- **DFS** (LIFO): (follows a single path)
  - sequence of states visited:
  - S,O,Z,A,T,L,M,D,C,R,P,B,F,G,U,H,E,V



Heuristic Functions

- Example 1: h_{SLD} for Navigation

- suppose our goal was to find a route from Sibiu to Vasliu
- compare DFS vs BFS (assuming children are processed in counter-clockwise order)
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  - frontier at each pass:
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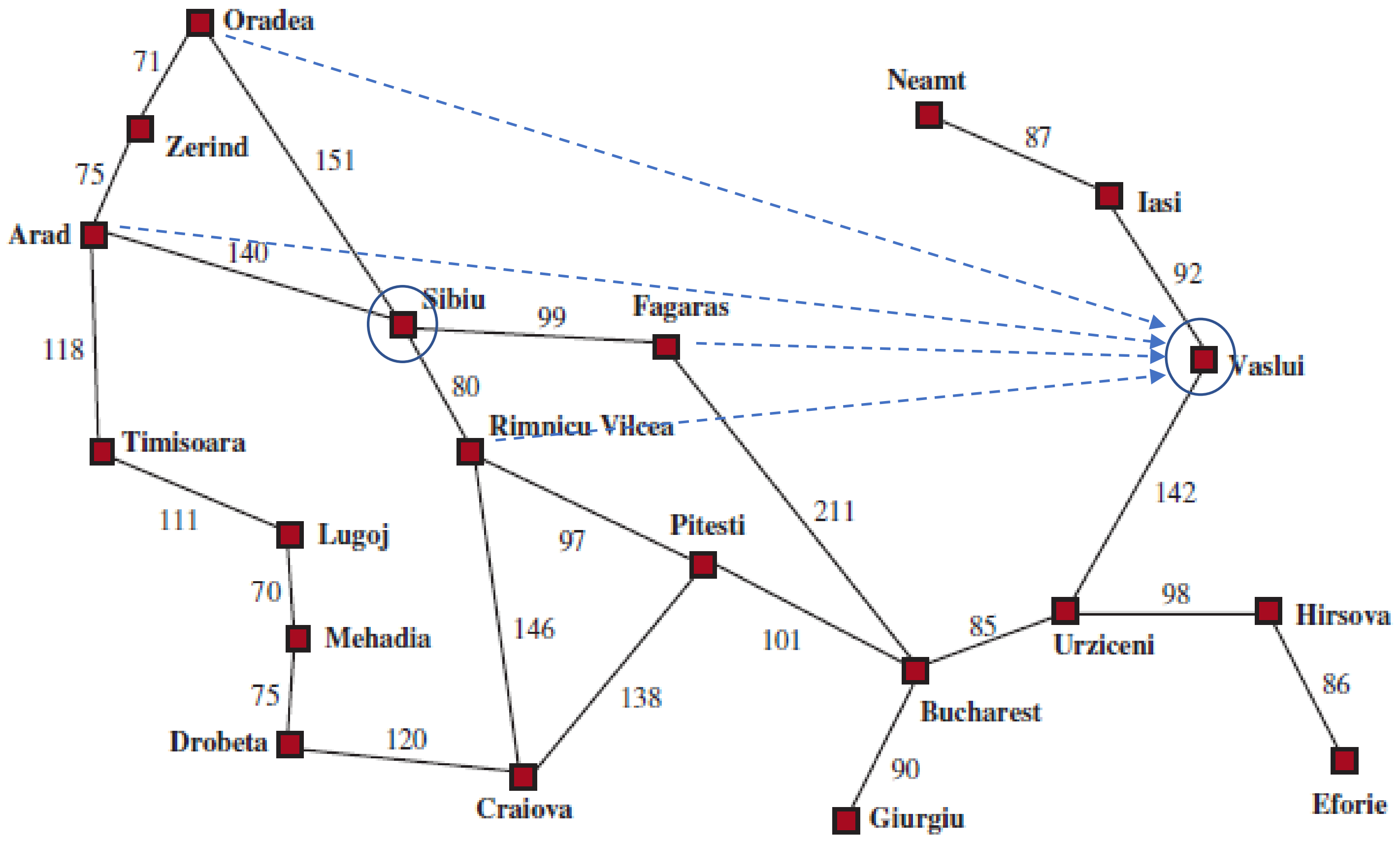
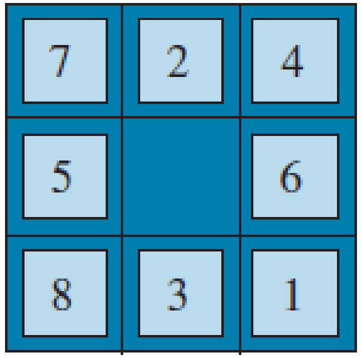
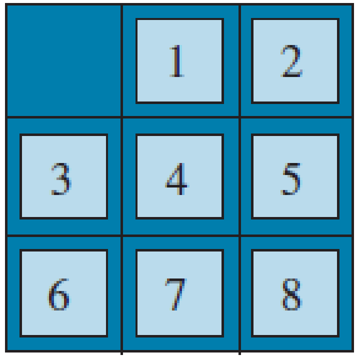
- DFS (LIFO):
  - sequence of states visited:
  - S,O,Z,A,T,L,M,D,C,R,P,B,F,G,U,H,E,V
- **h_{SLD}**: prioritize nodes in frontier based on straight-line distance to goal
  - sequence of states visited: S, F, B, U, V



Heuristic Functions



Start State

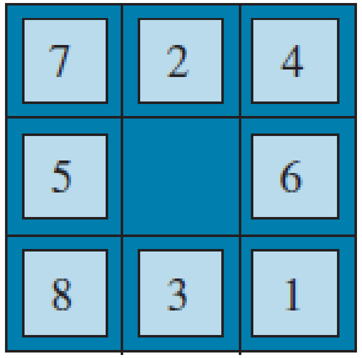


Goal State

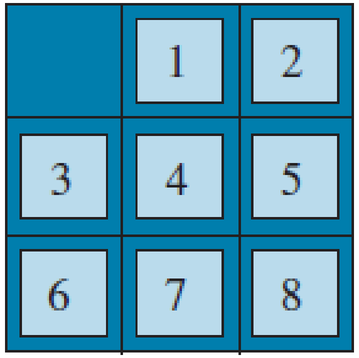
- Example 2: heuristic functions for the Tile Puzzle
  - how close is any given state to being solved?
  - $h_1(n)$: # tiles out of place
    - this is an under-estimate because it will take more than move to put each tile in its proper place
    - still, it differentiates states that are almost solved for those that are very jumbled
    - even if 1 block is out of place, it might be close or very far away
  - $h_2(n)$: Manhattan distance
    - for each tile out of place, count number of rows and columns it needs to move
    - still an under-estimate of total moves because moving one tiles can put others out of place
    - ironically, it can also be an over-estimate, because a sequence of moves could put multiple tiles in place

$$h_2(n) = \sum_{i=1}^{9} |currRow(T_i) - goalRow(T_i)| + |currCol(T_i) - goalCol(T_i)|$$

Heuristic Functions



Start State



Goal State

7	2	4
5		6
8	3	1

h1 = 8
the 1 needs to move 3 steps
the 2 needs to move 1 step
the 3 needs to move 2 steps
...
h2 = 3+1+2+2+2+3+3+2 = 18

1	2	
3	4	5
6	7	8

h1 = 2
h2 = 2

	1	6
3	4	5
2	7	8

h1 = 2
h2 = 8

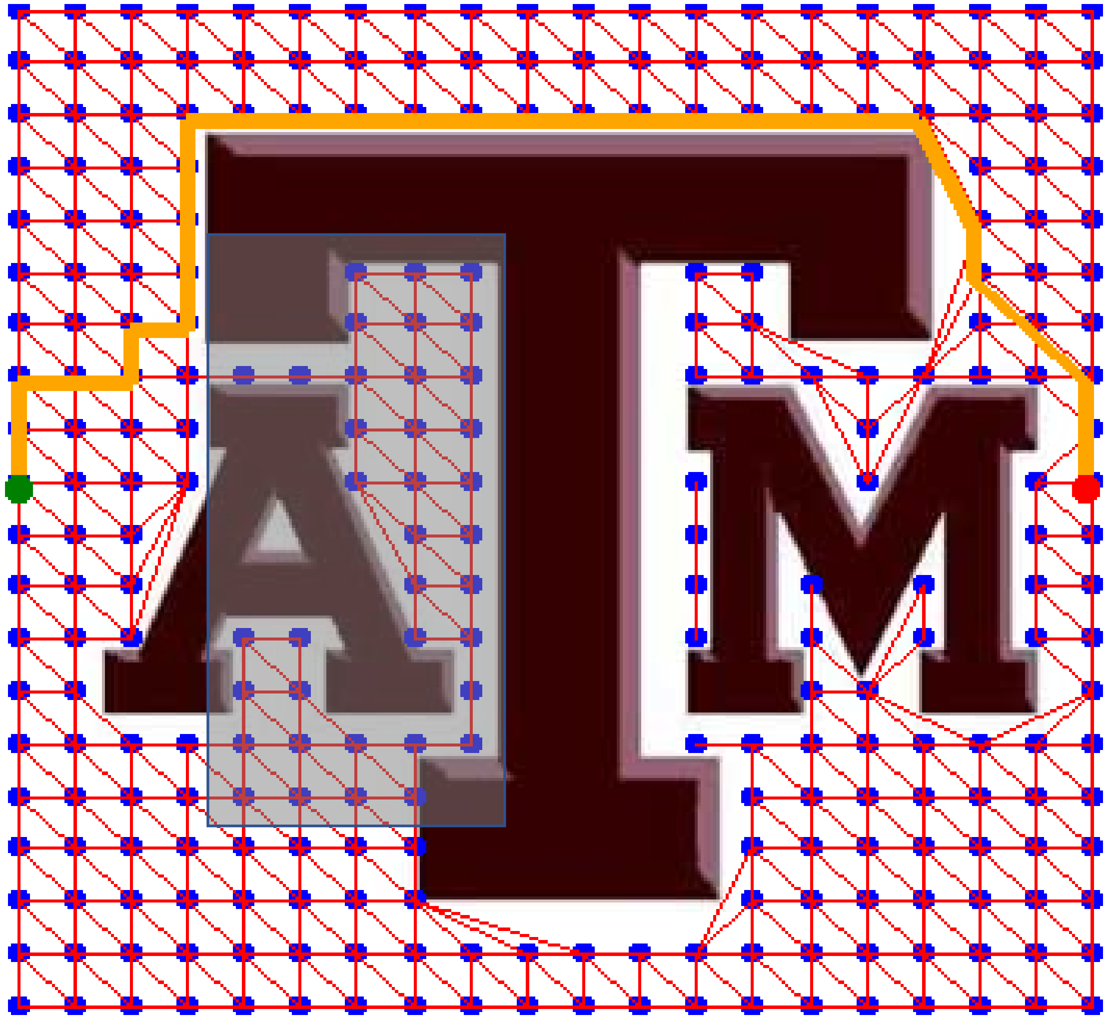
Greedy Search (best-first search with h(n))

- extending the iterative search algorithm to use a heuristic
- use a *priority queue* for frontier; sort nodes based on h(n)

```
function BEST-FIRST-SEARCH(problem, f) returns a solution node or failure
  node ← NODE(STATE=problem.INITIAL)
  frontier ← a priority queue ordered by f, with node as an element      where f is h(n)
  reached ← a lookup table, with one entry with key problem.INITIAL and value node
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
        reached[s] ← child
        add child to frontier
  return failure
```

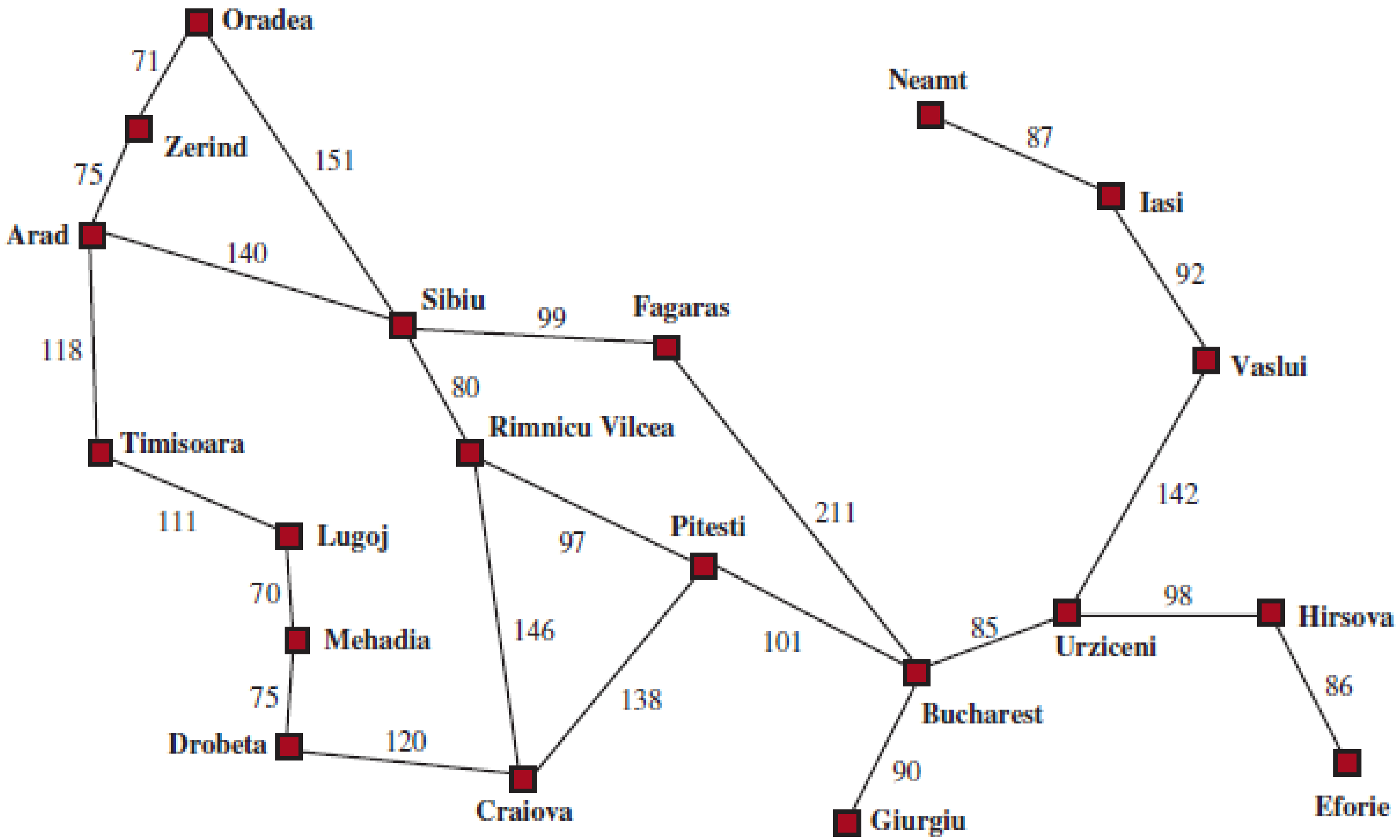
- (go back and review the slide on finding a route from Sibiu to Vasliu using h_{SLD})

Greedy Search



- The problem with Greedy Search is that it can be 'misled' by the heuristic to go in the wrong direction and waste time searching unproductive regions of the search space
- This is known as the "garden path" problem
- Greedy Search would search the gray-boxed region first, before discovering it has to go around the T to get the goal(red)

- how sub-optimal can it be? (in terms of cities expanded that are not actually on the solution path)
- what's the worst garden-path pair of cities for Romania?
- is there a pair of cities that would force Greedy search to visit every node?



A* algorithm

- one of the most widely used and practical AI search algorithms
- essentially Best-first search (with priority queue), where nodes in frontier are sorted based on f(n)=g(n)+h(n)
  - where g(n)=path cost so far (from root to n)
  - and h(n)=heuristic estimate of remaining path cost (from n to closest goal)
  - so f(n) is an <u>estimate of total path cost</u> going through n to goal

A* algorithm

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      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
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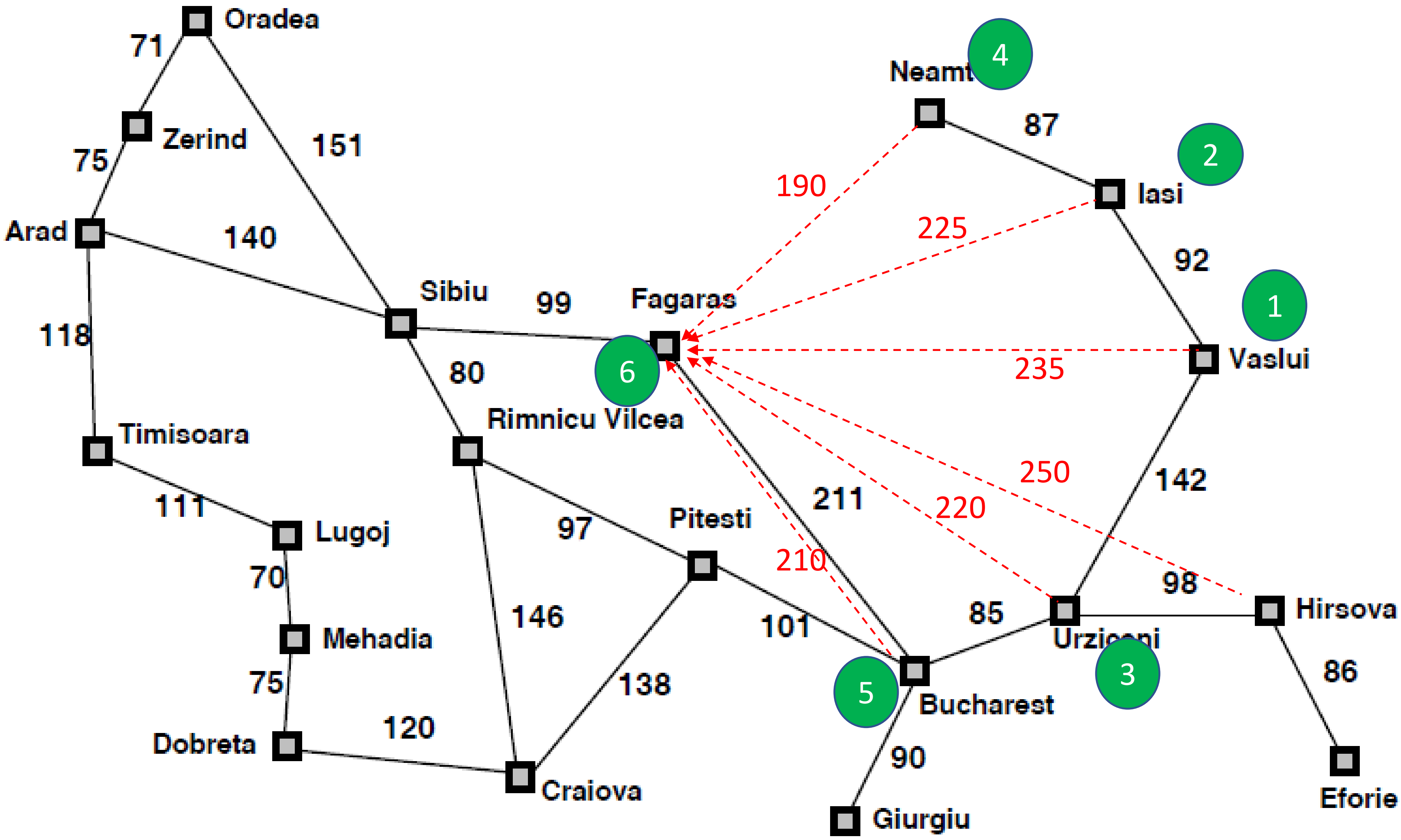
        add child to frontier
  return failure
```

A* search of Vasiliu to Fagaras:



frontier:
V(0+235=235)
<V^{235}>

1. pop V, push I and U
I(92+225=317)
U(142+220=362)
<I^{317} , U^{362}>

2. pop I; push N
N(92+87+190=369)
<U^{362}, N^{369}>

3. pop U; push B, H
B(142+85+210=437)
H(142+98+250=490)
<N^{369},B^{437},H^{490}>

4. pop N; <B^{437},H^{490}>

5. popB; push F$^{438=142+85+211}$
<F^{438},H^{490}>

6. pop F

notice how f(n) for popped nodes keeps increasing:
V(235), I(317), U(362), N(369), B(437), F(438)

Where Do Heuristics Come From?

- Heuristics encode knowledge you have about the problem
  - rules of thumb
  - common solutions that are often used
  - way to decompose the problem into subgoals
  - useful sequences of actions
  - interactions/dependencies between operators...
- This knowledge has to be formulated into a scoring function *h(n)* that estimates the distance of any state to the goal
- common strategy: approximate how many steps it would take to solve if we could *relax the constraints*
  - counting tiles out of place implies we can fix them in 1 move
  - Manhattan distance implies we can “slide tiles over each other”
  - for navigation, straight-line distance is shorter than any road, but still useful

Computational Properties A* Search

- what guarantees about completeness and optimality can we make?
- remember that h(n) could be inaccurate!
  - it could tell us that many nodes down path are getting closer and closer, when in fact there is no way to reach the goal, and back-tracking is required
- first, we need to make an assumption...
- h(n) is *admissible*
  - h(n) <u>never over-estimates the true distance to the goal</u> for any node n
  - $0 \leq h(n) \leq c^*(n)$ for all states in the State Space

Computational Properties A* Search

- Theorem: A* is optimal (finds a goal with minimum path cost)
  - although this sounds obvious because the PQ is sorted on f(n), it is deceptive because it only applies to nodes in the frontier, but not all states in the space
  - suppose the optimal goal is g* but greedy returns g first, where c(g)>c(g*)
  - let n* be a node on the optimal path to g* that is in the frontier at same time
  - $f(n^*)=g(n^*)+\underline{h(n^*)} \leq cost(n_0..n^*)+\underline{cost(n^*..g^*)} = cost(n_0..g^*) = c(g^*)$
  - because of admissibility
  - therefore, n* should have been dequeued before g (and so on, down the path to g*)
- Important point: Eventhough admissibility is desirable, it is not necessary: A* search can be made more efficient with a heuristic even if it is not admissible (however, the solution path found might not be minimal)

Computational Properties A* Search

- Lemma: f(n) scores increase *monotonically* down any path from root
  - if a path is <n_0..n_i..g>, then $f(n_0) \leq f(n_i) \leq f(g)$
  - in any step $n_i \rightarrow n_{i+1}$, $h(n_i)$ includes a guess of the cost of op_i, whereas $g(n_{i+1})$ has the actual cost of that step, which could only be higher (by admissibility)
  - also requires *consistency* of heuristic, which is slightly stronger than admissibility (see book)
  - remember that at a goal node, f(g)=c*(g) for any goal because f(g)=g(g)+h(g)=c(*g)+0
  - so f(n) could be an <u>underestimate of total path length </u>early in a path, but converges to c(g*) as you get closer to the goal
- Theorem: A* explores states in order of increasing f(n) (see Fig 3.20)

Computational Properties A* Search

- analysis of time complexity
  - efficiency of A* is complicated because it depends on accuracy of the heuristic
  - generally speaking, **the more accurate the heuristic is, the faster the search**
    - boundary case 1: h(n)=0 – no help, exponential time like Uniform Cost, $O(b^{1+C^*/\varepsilon})$
    - boundary case 2: h(n)=c(n) – a heuristic that perfectly predicts the true distance to the goal for any node will lead A* right to it (in time linear in the path length)
  - if the inaccuracy of the heuristic is bounded, search will be sub-exponential
    - define "relative error" $\Delta = |h-h^*|/h^*$ (max over all nodes in the State Space)
    - then time complexity of A* is $O(b^{\Delta \cdot L(g)})$ where L is the path length to the goal g
    - if $|h-h^*|=O(\log(h^*))$ for all n, then A* will search a sub-exponential number of nodes before finding the optimal goal
    - however, this is rarely achievable in practice